Astérisque

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Almost symmetric spaces

Astérisque, tome 163-164 (1988), p. 221-246
<http://www.numdam.org/item?id=AST_1988__163-164__221_0>

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Société Mathématique de France
Astérisque 163-164 (1988), p.221-246.

ALMOST SYMMETRIC SPACES

by *Maung Min-Oo*

§1. *INTRODUCTION*.

In Differential Geometry we study spaces with geometric structure. The oldest geometric structure is that of Euclidean space. However, there exist other geometries, like the spherical and hyperbolic spaces. According to F.Klein, a geometry is defined by a transitive transformation group and hence the model spaces for a geometer are homogeneous spaces. Global homogeneity, however, is in general too rigid and static. A more dynamic notion of space, as was used by A.Einstein in his general theory of relativity, was first introduced by B.Riemann, who defined the concept of a manifold carrying a metric which is Euclidean only at an infinitesimal scale. If we take instead of Euclidean geometry a more general type of geometry defined by a model homogeneous space to hold at the infinitesimal level we arrive at the notion of what I call *an almost homogeneous structure*. After the introduction of an infinitesimal geometric structure the next basic step is to study the integrability problem and hence to study the local geometry of a space. The global shape is then determined essentially by the monodromy, i.e., the fundamental group. The fundamental local invariant for integrability is curvature (or rather the torsion in certain situations), since it measures the deviation of the local geometry of a given space from that of a model space. The relevant concept for almost homogeneous spaces is the notion of a Cartan connection and its curvature. In this paper we will investigate compact almost symmetric spaces with small Cartan curvature, i.e., we study the problem of deforming an "almost" integrable structure to an integrable one.

Since not every "almost" solution to a problem is necessarily near an exact solution, we show first that the infinitesimal obstruction to a deformation lies in a certain cohomology group which is then shown to vanish for most symmetric spaces of non-compact type. These vanishing results for cohomology groups associated to co-compact discrete subgroups of semi-simple Lie groups play an essential role in our proof and are explained in §3. The final part of the proof, described in §4, is analytical and consists of proving that the infinitesimal deformations can be integrated. This paper basically reproduces the main result of [15] with an attempt to put it into a more general framework. The new technical ingredient introduced here, is to apply the heat equation method in the style of R.S.Hamilton [11] which gives a different approach to the analytical aspects of [15].

§2. *ALMOST HOMOGENEOUS STRUCTURES AND CARTAN CONNECTIONS.*

A differentiable manifold is defined by its atlas of coordinate charts and their smooth transition functions. By restricting the admissible family of the class of allowable transition functions one obtains geometric structures, which could then be used to classify manifolds. The classical example is the uniformization theorem for Riemann surfaces. The first derivatives of the transition functions define an infinitesimal linear structure, namely the tangent bundle, and as a first order approximation of a geometric structure we could define a restriction for the structural group of the tangent bundle and obtain what is aptly called a G-structure.

Definitions:

Let $pr: Gl(M) \to M$ denote the principal $Gl(n)$-bundle of all linear frames of a smooth n-dimensional manifold, and let $G \subset Gl(n)$ be a closed subgroup.

A *G-structure* is a reduction $P \subset Gl(M)$ of the frame bundle to the subgroup G. Equivalently, it is a section of the quotient bundle $Gl(M)/G$. (We refer to [8] and [22] for examples and motivation).

A diffeomorphism $\phi: M_1 \to M_2$ induces a bundle map $d\phi: Gl(M_1) \to Gl(M_2)$. Two G-structures P_1 and P_2 defined over M_1, resp. M_2 are said to be *equivalent* if there exists a diffeomorphism mapping P_1 isomorphically onto P_1.

They are said to be *locally equivalent* at the points $x_1 \in M_1$ and $x_2 \in M_2$, if there are open neighbourhoods U_1 and U_2 of x_1 resp. x_2, such that P_1 restricted to U_1 is equivalent to P_2 restricted to U_2.

The canonical parallelism of the vector space $\mathbb{R}^n$ defined by translations induces the standard trivialization of the frame bundle $Gl(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{R}^n \times Gl(n)$. For any $G \subset Gl(n)$, we call the trivial G-structure $\mathbb{R}^n \times G \subset \mathbb{R}^n \times Gl(n)$ induced by the inclusion, the *flat canonical G-structure*.

A G-structure P is said to be *integrable* iff it is locally equivalent to the flat canonical structure, i.e. if we can find at every point local coordinates $x: U \to M$ so that the induced framing $dx: U \to Gl(M)$ factors through $P \subset Gl(M)$.

On the frame bundle $pr: Gl(M) \to M$, there exists an $\mathbb{R}^n$-valued 1-form, called the *canonical 1-form*: $\bar{A}_2: TGl(M) \to \mathbb{R}^n$ defined by: $u(\bar{A}_2(X)) = d\,pr(X)$, where $X \in TGl(M)$, $u: \mathbb{R}^n \to T_xM$ and $x = pr(u)$. $\bar{A}_2$ is *horizontal* , i.e., it vanishes on vectors tangent to the fibres, and it is *equivariant* under the natural action of $Gl(n)$. (In the literature $\bar{A}_2$ is usually denoted by θ.)

Let $pr: P \to M$ be a G-structure. A subspace $H_u \subset T_uP$ is called *horizontal* if $\bar{A}_2$ restricted to H_u is an isomorphism $\bar{A}_2: H_u \simeq \mathbb{R}^n$. H_u is then complementary to the *vertical space* V_u , spanned by the vectors tangent to the fibres. A distribution of such horizontal subspaces define a *connection* if in addition it is equivariant under the action of G. There is a natural identification: $X \mapsto X^*$ of the Lie algebra $\mathfrak{G}$ of G and V_u at each u. The projection along the horizontal space $A_1 : T_uP \to V_u \simeq \mathfrak{G}$ is called the *connection form*.

If H is any horizontal distribution, connection or not, on P, then the restriction of $d\bar{A}_2$ to horizontal vectors define a linear map: $c_H(u) : \mathbb{R}^n \wedge \mathbb{R}^n \to \mathbb{R}^n$ on identifying $\mathbb{R}^n$ with H_u via $\bar{A}_2$. The map: $c_H : P \to \mathrm{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)$ is called the *torsion* of H. (If H is a connection this is just the usual torsion of a linear connection.)

If H' is a different choice of horizontal subpaces, then H and H' differ at each point $u \in P$ by a linear map $S_u : \mathbb{R}^n \to \mathfrak{G}$, defined by $(S_u(\bar{X}))^* = X' - X \in V_u$, where $X \in H_u$, $X' \in H'_u$ with $\bar{A}_2(X) = \bar{A}_2(X') = \bar{X} \in \mathbb{R}^n$ and the difference in torsion at that point is then given by $c_H(u,v) - c_{H'}(u,v) = S(u)(v) - S(u)(v)$. Thus if we define $d_2 : \mathrm{Hom}(\mathbb{R}^n, \mathfrak{G}) \to \mathrm{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)$ by $d_2 S(u,v) = S(u)(v) - S(u)(v)$, then $c_H = c_{H'}$ modulo the image of d_2 .

The function $c : P \to \mathrm{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n) / \mathrm{Im}(d_2)$ is therefore independent of the choice of horizontal spaces and is called the *torsion of the G-structure*.

The kernel of d_2 is also of interest and is called the *first prolongation* of $\mathfrak{G}$. (We refer to [22] for more details about these concepts, where c is called the structure function.)

The torsion transforms naturally under equivalence of G-structures and since the canonical flat G-structure has obviously $c \equiv 0$, a necessary condition for the integrability of a G-structure is the vanishing of its torsion. This is obviously equivalent to the condition that the structure admits a torsion free connection, since given any connection H' with $c_{H'} \equiv 0 \bmod (\mathrm{Im}\, d_2)$ then $c_{H'}$ is an <u>equivariant</u> map:$P \to \mathrm{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)$ with $c_{H'} = d_2 S$ for some equivariant map $S : P \to \mathrm{Hom}(\mathbb{R}^n, \mathfrak{G})$ and hence the modified connection $H = H' + S$ is torsion free. However, in general, this necessary condition is far from being sufficient and one has to study the torsion of higher order structures obtained by prolongations.

Examples:

(i) $G = \{e\}$ the trivial group. An $\{e\}$-structure is an absolute parallelism for M defined by everywhere linearly independent vector fields $\{X_1,\dots,X_n\}$. The torsion c is given by the Lie bracket of the vector fields: $[X_i,X_j] = c_{ij}{}^k X_k$. One of *Lie's Fundamental Theorems* states that c is a constant function on M iff M is locally a Lie group.

(ii) $G = O(n)$. Here $c = 0$ for all O(n)-structures, and the first prolongation of $\mathcal{O}(n)$ is trivial since d_2: $\mathrm{Hom}(\mathbf{R}^n,\mathcal{O}(n)) \simeq \mathrm{Hom}(\mathbf{R}^n \wedge \mathbf{R}^n, \mathbf{R}^n)$ is an isomorphism. This proves the well known fact that every Riemannian metric admits a unique torsion free connection, the Levi-Civita connection and that there exist coordinates which are flat up to first order, namely the Riemannian normal coordinates.

Now let H be a closed subgroup of a Lie group G and let us denote by $\bar{M}$ the homogeneous space G/H. We will always assume that the linear isotropy representation of H in the tangent space of $\bar{M}$, $H \to Gl(T_oM) \simeq Gl(n)$ is injective, since we would be dealing only with first order structures.

Definitions:

An *almost-homogeneous structure of type G/H* for a manifold M with dim(M) = dim(G/H) is an H-structure for M, where H is imbedded in Gl(n) via the linear isotropy representation of the homogeneous space G/H.

The projection $G \to G/H$ defines an almost-homogeneous structure for the model space $\bar{M} = G/H$, which we call the *standard homogeneous structure* of type G/H. Our main concern is to deform nearly integrable almost-homogeneous structures into the standard structure.

Example

(iii) Let G/K be an irreducible symmetric space of rank > 1 . Then a remarkable rigidity theorem of M.Berger [4] and J.Simons [21] can be interpreted in our language as follows:

Theorem

An almost-homogeneous structure of type G/K of rank > 1 on a manifold M has vanishing torsion, i.e., $c \equiv 0$, if and only if it is *locally* equivalent to the standard structure $G \to G/K$ or to the standard structure of the dual symmetric space $(G/K)^*$ or to the canonical flat structure of $\mathbb{R}^n$.

In order to explain the above formulation in the context of the original papers, we first recall some relevant definitions from [21].

According to a well known theorem of Ambrose and Singer the infinitesimal holonomy algebra at a point p of a Riemannian manifold M is generated by the set of all endomorphisms: $\gamma^*R(u,v): T_pM \to T_pM$, where u,v runs over all tangent vectors at p, and γ runs over all paths emanating from p. Here $\gamma^*R(u,v) = a^{-1} \circ (R(a(u),a(v))) \circ a$, where a denotes the parallel translation along γ. Motivated by this, we define:

An (abstract) *holonomy system* (V,R,K) is a connected group $K \subset O(V)$ of isometries of a Euclidean vector space V together with a tensor $R: V \wedge V \to V \wedge V$ satisfying all the algebraic identities of a curvature tensor and with the property that $R(u,v) \in \mathfrak{K}$ for all $u,v \in V$.

K operates in a natural manner on R via $a^*R(u,v) = a^{-1}(R(a(u),a(v)))a$ and a holonomy system is said to be *symmetric* if $a^*R = R$ for all $a \in K$.

A holonomy system is said to be *irreducible* if K operates irreducibly on V. It is not required that the action of K^R on V is irreducible.

If $\mathfrak{K}^R$ is the subalgebra of $\mathfrak{K}$ spanned by the elements $a^*R(u,v)$ for all $u,v \in V$ and $a \in \mathfrak{K}$, then the corresponding subgroup $K^R \subset K$ is called the *holonomy reduction*.

Theorem. (J.Simons [21,Thm.4])

If (V,R,K) is an irreducible holonomy system such that K^R does *not* act transitively on the unit sphere of V, then (V,R,K) is symmetric.

If G/K is an irreducible symmetric space with Cartan decomposition $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{M}$ and curvature tensor $\bar{R}$, then $(\mathfrak{M},\bar{R},K)$ is a symmetric irreducible holonomy system with $K = K^{\bar{R}}$ operating transitvely on the unit sphere of $\mathfrak{M}$ iff $\mathrm{rank}(G/K) = 1$. Moreover, if $\mathrm{rank}(G/K) \geq 2$, then R is uniquely determined up to a constant in the sense that for any such $(\mathfrak{M},R,K)$, $R = c.\bar{R}$. (This is Thm.6 and its corollary in [21]).

A torsion free connection on an K—principal bundle with K represented orthogonally as the isotropy representation of a symmetric space of rank ≥ 2 is a metric connection whose curvature tensor at each point $R:\mathcal{O}(n) \to \mathcal{O}(n)$ factors through the representation $\rho:\mathfrak{K} \to \mathcal{O}(n) \simeq \mathcal{O}(\mathfrak{M})$ and hence defines a holonomy system with group K acting non-transitively on the unit sphere of $\mathfrak{M}$. Therefore $R = c.\bar{R}$, for some constant c at each point of the manifold. By the second Bianchi identity c must be a constant function. Since the torsion of a K-structure is zero if and only if the structure admits a torsion free connection our interpretation of the rigidity theorem of Berger and Simons now follows.

Definition.

A Cartan connection of type G/H on a H-principal bundle P over a manifold M with $\dim(M) = \dim(G/H)$ is a $\mathfrak{G}$-valued 1-form $A:TP \to \mathfrak{G}$ with the following properties:

C1 $A(X^*) = X$ for all $X \in \mathfrak{H}$, where X^* is the induced vertical vector field.

C2 $R_a{}^*A = \mathrm{ad}(a^{-1})A$, for all $a \in H$, where R_a denotes the right action of a on P.

C3 $A(X) \neq 0$ for all $X \neq 0$. This means that A defines an absolute parallelism for P.

The *curvature* of a Cartan connection is defined to be the $\mathfrak{G}$-valued 2-form :

(2.1) $\quad F = dA + [A,A]$,

or more precisely by: $F(X,Y) = X(A(Y)) - Y(A(X)) - A([X,Y]) + [A(X),A(Y)]$.

The standard structure $G \to G/H$ admits an obvious Cartan connection of zero curvature defined by the Maurer-Cartan form $\bar{A}$ of the Lie group G.

Although a Cartan connection is not a connection in the usual sense, for the H-bundle P, it induces an ordinary type of connection on a larger bundle, namely the G-bundle $Q = P \times_H G$ It can also be thought of as a connection for the associated fibre bundle $B = Q \times_G G/H$. This bundle admits a tautoligical section which gives the reduction $P \subset Q$.

In case G/H is reductive, i.e., if $\mathfrak{G}$ admits an H-invariant decomposition: $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{M}$, then the tangent bundle of an almost homogeneous space M of type G/H can be expressed as $TM = P \times_H \mathfrak{M}$, and a Cartan connection A splits as $A = A_1 + A_2$, where A_1 is now an $\mathfrak{H}$-valued 1-form on the H-bundle P defining a connection in the usual sense, and A_2, which is an $\mathfrak{M}$-valued horizontal 1-form on P, can be interpreted as a gauge transformation $A_2: TM \to TM$. ($\bar{A}_2$ is the identity map.)

For the standard structure of a reductive homogeneous space with the flat Cartan connection $\bar{A}$, $\bar{A}_1$ is the canonical connection with parallel torsion and curvature, and $\bar{A}_2$ is the canonical 1-form, usually denoted by θ in the literature, corresponding to the identity gauge transformation. In particular for Riemannian symmetric spaces we obtain the Levi-Civita connection.

Given a representation $\rho: G \to Gl(V)$, we can restrict it to H and construct the vector bundle $E_\rho = P \times_H V$ on the almost homogeneous manifold M. Denoting also by ρ the representation on the Lie algebra level, its restriction to $\mathfrak{M}$ defines a bundle map $\rho: TM \to End(E_\rho)$. If we denote by ∇ the covariant differentiation for E_ρ induced by the H-connection form A_1, then the covariant differentiation D of the Cartan connection can be written as:

$$D_X = \nabla_X + \rho(A_2(X)) \tag{2.2}$$

§3. VANISHING THEOREMS FOR COHOMOLOGY GROUPS ASSOCIATED TO SIMPLE LIE ALGEBRAS

In order to deform almost integrable into integrable ones, we need to prove certain vanishing theorems. In particular, we will study here the cohomology groups of discrete subgroups of semi-simple real Lie groups with values in the adjoint representation, using bundle valued harmonic forms.

Let G be a non-compact simple Lie group with maximal compact subgroup K and associated symmetric space $\tilde{\tilde{M}} = G/K$. Let $\Gamma \subset G$ be a discrete subgroup acting without fixed points on $\tilde{\tilde{M}}$ and with a compact quotient $\bar{M} = {}_{\Gamma}\backslash G/K$. For any representation ρ of G in a vector space V, we can define cohomology groups of Γ with values in V, to be denoted by $H^*(\Gamma;\rho)$. Since $\bar{M}$ is a $K(\Gamma,1)$-space, these cohomology groups are the same as the deRham cohomology defined by exterior forms on $\bar{M}$ with values in a flat, but non-trivial bundle defined by the representation. The bundle is $E_\rho = {}_{\Gamma}\backslash G \underset{\rho}{\times} V$ equipped with the standard flat Cartan connection D coming from the Maurer-Cartan form $\bar{A}$ of G. The corresponding exterior derivative on forms will be denoted by d^D. Besides the flat connection D, there is another natural connection ∇ defined on E, which is the one induced by the Levi-Civita connection of the symmetric metric. ∇ is of course not flat and does not define cohomology groups. We will denote the covariant exterior derivative of ∇ on E_ρ-valued forms by d_1. Then:

$$(3.1) \qquad d^D = d_1 + d_2$$

where d_2 is an algebraic operator given by:

$$(3.2) \qquad d_2\alpha(v_0,\dots,v_p) = \sum_{i=0}^{p} (-1)^i \rho(v_i).\alpha(v_0,\dots,\hat{v}_i,\dots v_p)$$

for an E_ρ-valued p-form α. Here we interpret ρ restricted to $\mathfrak{M}$ as a bundle map $\rho: TM \to \mathrm{End}(E_\rho)$. We note that $d_1 \circ d_1$ and $d_2 \circ d_2$ are *not* 0.

To define the Laplacian, we introduce a positive definite metric $<\ ,\ >$ in the representation space V with the following 2 properties:

(i) $<\ ,\ >$ is K-invariant;

(ii) $\rho(v)$ is *symmetric* w.r.t. $<\ >$ for all $v \in \mathfrak{M}$,

where $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{M}$ is the Cartan decomposition of G/K.

Such a metric, called *admissible*, always exists. (See for example [13,Lemma3.1]). In the case of the adjoint representation we just change the sign of the Killing form on $\mathfrak{K}$. With respect to an admissible metric on V and the metric on $\bar{M}$, we can now define the adjoint of d^D to be $\delta^D = \delta_1 + \delta_2$, where the algebraic part δ_2 is given by:

$$(3.3) \qquad \delta_2 \alpha(v_1,\ldots,v_p) = + \sum_{k=1}^{n} \rho(e_k).\alpha(e_k,v_1,\ldots,v_p)$$

The Laplacian is now defined as $\Delta^D = d^D\delta^D + \delta^D d^D$. The following "$L^2$-Weitzenbock formula" for Δ^D due to Matsushima and Murakami [13] (see also [5,Chap.II] and [20,Chap.VII] for a proof) plays a fundamental role:

$$(3.4) \qquad \Delta^D = \Delta_1 + \Delta_2$$

where $\Delta_1 = d_1\delta_1 + \delta_1 d_1$, $d_1\delta_2 + d_2\delta_1 + \delta_1 d_2 + \delta_2 d_1 \equiv 0$ (this is a non trivial fact) and $\Delta_2 = d_2\delta_2 + \delta_2 d_2$ is given by :

$$(3.5) \qquad \Delta_2 \alpha(v_1,\ldots,v_p) = \sum_{k=1}^{n} \rho(e_k)^2.\alpha(v_1,\ldots,v_p) + \sum_{k=1}^{n} \sum_{i=1}^{p} (-1)^{i+1} \rho([v_i,e_k]).\alpha(e_k,\ldots,\hat{v}_i,\ldots,v_p)$$

To prove a vanishing theorem for a co-compact Γ it is therefore sufficient to show that the algebraic Laplacian Δ_2 is positive definite. We are mainly interested in the vanishing of the first and second cohomology groups with values in the adjoint representation of G in its Lie algebra $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{M}$. Since Δ_2 is obviously K-invariant we can study the cases of $\mathfrak{K}$-valued forms and $\mathfrak{M}$-valued forms separately. For a $\mathfrak{K}$-valued 1-form $\alpha : \mathfrak{M} \to \mathfrak{K} \subset \mathfrak{M} \wedge \mathfrak{M}$ with components $\alpha_{ij}{}^{k} = -\alpha_{ik}{}^{j}$, $d_2\alpha = 0$ means $\alpha_{ij}{}^{k} = \alpha_{ji}{}^{k}$. This implies $\alpha = 0$, since the first prolongation of $\mathfrak{O}(n)$ is trivial.(See example(ii) in §2).

For a $\mathfrak{K}$-valued 2-form $r : \mathfrak{M} \wedge \mathfrak{M} \to \mathfrak{K} \subset \mathfrak{M} \wedge \mathfrak{M}$, $d_2 r = 0$ means that r satisfies the algebraic Bianchi identity, and hence is a curvature like tensor. $\delta_2 r = 0$ means that the "Ricci" part of r vanishes. Now, by the rigidity theorem of Simons [21] mentioned in example (iii) of §2, this implies $r = 0$, provided rank(G/K) > 1 .

In order to study $\mathfrak{M}$-valued forms we introduce the operator Q acting on symmetric 2-tensors via:

$$(3.6) \qquad Q(s)_{ij} = \bar{R}_{i}{}^{pq}{}_{j} s_{pq}$$

where $\bar{R}$ is the curvature tensor of the symmetric space $\bar{M}$.

Q is a self-adjoint endomorphism with eigenvalues estimated by : (see [15,p.429])

$$(3.7) \qquad B(\mu,\mu) \geq Q \geq -1$$

where B is the Killing form of the compact dual $\mathfrak{U}$ of $\mathfrak{G}$ and μ is the highest weight of the adjoint representation.

An $\mathfrak{M}$-valued 1-form $\alpha : \mathfrak{M} \to \mathfrak{M}$ can be decomposed as $\alpha = a + s$, where a is skew-symmetric and s is symmetric. A computation now shows that :

$$(3.8) \qquad \Delta_2 a = a - \tfrac{1}{2}\hat{\bar{R}}(a)$$

where $\hat{\hat{R}}(a)$ is the curvature operator of G/K acting on a 2-form a. Hence Δ_2 is positive definite on the skew-symmetric component, since $\hat{\hat{R}}$ of $\bar{M}$ is <u>non-positive</u>.

For the symmetric component s we find that (see [15]):

(3.9) $\quad \Delta_2 s = s - 2Q(s)$

This proves that Δ_2 is positive on $\mathfrak{M}$-valued 1-forms <u>if</u> $B(\mu,\mu) < \frac{1}{2}$. A simple check of the tables for simple Lie algebras in [6] now shows that this is the case except for $\mathfrak{G} = \mathfrak{sl}(2;\mathbb{R})$.

We have thus proved the classical vanishing theorem for $H^1(\Gamma;ad)$ due to A.Weil [23], which is the infinitesimal version of the rigidity theorem of Mostow[19] about compact locally symmetric spaces of dimension ≥ 3.

By an argument very similar to the above we proved in [15] that there are no harmonic $\mathfrak{M}$-valued 2-forms if $\dim(M) > 6$. Combined with Simons theorem on $\mathfrak{K}$-valued 2-forms this leads to the following result:

<u>*Proposition* 3.10</u>

<u>Let</u> G/K <u>be an irreducible symmetric space of non-compact type of dimension</u> > 6 <u>and rank</u> > 1. <u>Then</u> Δ_2 <u>is positive definite on 2-forms with values in the adjoint representation</u>. <u>In particular,</u> $H^2(\Gamma;ad) = 0$ <u>for any co-compact discrete subgroup</u> $\Gamma \subset G$.

<u>*Remark*</u>: The above result is also true in the case of the dual space of the Cayley projective plane, which is a rank 1 symmetric space with G = a real form of F_4 with maximal compact subgroup Spin(9). Here we cannot use Simons result, but a direct computation (see[16]) shows that there are no curvature like tensors which are Ricci flat with values in $\mathfrak{O}(9) \subset \mathfrak{F}_4$. (Compare also[1] and [7])

§4 CURVATURE DEFORMATIONS

Let $A:TP \rightarrow \mathfrak{G}$ be a connection form on a G-principal bundle P over a compact manifold M. P is not required to be G-structure. Let $E = P \times_{ad} \mathfrak{G}$ be the Lie algebra bundle associated to the adjoint representation. The curvature F of A is then a 2-form with values in E and the difference of any 2 connections is an E-valued 1-form. The connection A induces a covariant differentiation D on E, and hence also an exterior differential d^D on E-valued forms. The Bianchi identity can be written as:

$$(4.1) \qquad d^D F = 0$$

In general $d^D \circ d^D \neq 0$, but is given by the formula:

$$(4.2) \qquad d^D \circ d^D = F \wedge$$

However the fundamental sequence of operators:

$$(4.3) \qquad \Omega^0(M;E) \xrightarrow{d^D} \Omega^1(M;E) \xrightarrow{d^D} \Omega^2(M;E) \rightarrow \cdots$$

where $\Omega^p(M;E)$ denotes the E-valued p-forms on M, although not a complex, is still elliptic in the sense that the sequence of symbols is exact. If we now introduce positive definite metrics, denoted by $<\ ,\ >$ on M and also for the Lie algebra $\mathfrak{G}$, then we can define the adjoint δ^D of d^D by:

$$(4.4) \qquad \delta^D(v_2,\ldots,v_p) = -\sum_{k=1}^{n} (D_{e_k})(e_k,v_2,\ldots,v_p)$$

where $\{e_k\}$ is an orthonormal base on M. Since we want to deform connections in directions transversal to the gauge transformations of the bundle, which are infinitesimally given by $\Omega^0(M;E)$, we introduce the gauge condition:

$$(4.5) \qquad \delta^D \dot{A} = 0$$

for our deformations $\dot{A} \in \Omega^1(M;E)$.

Besides the Bianchi identity, the curvature also satisfies:

$$(4.6) \qquad \delta^D \circ \delta^D F = 0$$

This follows easily by taking the adjoint of (4.2) and applying it to F itself.

Therefore the most natural direction to deform a connection is $-\delta^D F$, and we consider the evolution equation:

$$(4.7) \qquad \dot{A} = \frac{d}{dt}A = -\delta^D F$$

This can be regarded as the gradient flow for the Yang-Mills functional $\int |F|^2$. The change in curvature is $\frac{\partial}{\partial t}F = \dot{F} = d^D \dot{A}$, and therefore by the Bianchi identity, we obtain the following heat equation for the evolution of the curvature:

$$(4.8) \qquad \frac{\partial}{\partial t}F + \Delta^D F = 0$$

The equation (4.7) for A is not strictly parabolic because we are dealing here with a geometric problem with a large group of symmetries giving rise to various integrability conditions. However, we can still show that (4.7) can be integrated for a short positive time on any compact manifold.

Proposition 4.9

The evolution equation $\dot{A} = -\delta^D F$ for connections on a compact Riemannian manifold has a unique smooth solution for some time interval $[0,t_0]$ with $t_0 > 0$ for any smooth initial connection $A(0)$.

Proof: We apply Theorem 5.1 of R.S.Hamilton [11] to the equation and check the conditions therein. Condition (A) follows from (4.6). Condition (B) follows from the fact that the symbol sequence of (4.3) is independent of the connection and is exact. In fact, we could follow Hamilton further and prove a more precise statement:

Proposition 4.10

The equation $\dot{A} = -\delta^D F$ can be integrated for a maximal time interval $[0,T]$, with $0 < T < \infty$, and if $T < \infty$, then $\lim_{t \to T} max|F(t)| = \infty$.

Proof: We follow Hamilton's proof of Theorem 4.1 in [11]. The Bochner-Weitzenböck formula for the operator Δ^D is:

$$(4.11) \qquad \Delta^D F = \bar{\Delta} F + Q_2(F) + \hat{F}(F)$$

where:

(i) $\bar{\Delta} F = -\operatorname{tr} \nabla^2$ is the rough Laplacian,

(ii) $Q_2(F)$ is an algebraic term linear in F and the curvature of M and

(iii) $\hat{F}(F)$ is a term quadratic in F. ($\hat{F}(F)_{ij} = -\sum_{k=1}^{n} [F_{ik}, F_{kj}]$)

If $F^{(k)}$ denotes the k^{th} covariant derivative of F, then by repeated differentiation of (4.11) we obtain the equation:

$$(4.12) \qquad \left(\frac{\partial}{\partial t} + \Delta \right) F^{(k)} = \sum_{i+j=k} F^{(i)} * F^{(j)}$$

where $F^{(i)} * F^{(j)}$ is a certain linear combination of the i^{th} and j^{th} derivatives of F, with coefficients depending on the metric and the curvature (including its derivatives) of the base manifold M.

Using now the general interpolation inequalities in §12 of [11] and integrating over M we get:

$$(4.13) \qquad \frac{\partial}{\partial t} \|F^{(k)}\|_2^{\,2} + 2\|F^{(k+1)}\|_2^{\,2} \leq C\|F\|_\infty \|F^{(k)}\|_2^{\,2}$$

where $\| \ \|_2$ and $\| \ \|_\infty$ denote the L^2, resp. the C^0-norm.

Hence if $\|F(t)\|_\infty \leq C$ for all $t \in [0,T)$, then we have uniform a-priori bounds on all the L^2-norms of the derivatives of F. The interpolation inequalities together with the Sobolev inequalities now give $\|F^{(k)}\|_\infty \leq C(k)$ for all k, and this proves the proposition.

Now let λ_2 denote the minimal eigenvalue over M of the operator Q_2 appearing in (4.11). Q_2 acting on a 2-form is given by:

$$(4.14) \qquad Q_2(\beta)(u,v) = \beta(\mathrm{Ric}(u),v) + \beta(\mathrm{Ric}(v),u) - \beta(\hat{R}(u,v))$$

where $\hat{R}$ is the curvature operator on 2-forms.

If we normalize the metric on $\mathfrak{G}$ so that $2|[X,Y]| \leq |X|.|Y|$ then the last term appearing in (4.11) can be estimated by:

$$(4.15) \qquad |\langle \hat{F}(F), F \rangle| \leq |F|^3$$

As an example of using the heat flow method to deform curvature we can now prove the following:

Proposition 4.16

If A is a connection on a G-principal bundle over a compact Riemannian manifold M whose curvature F satisfies: $\|F\|_\infty < \lambda_2$, with λ_2 and the norm defined above, then the flow (4.7) can be integrated for all time and $A(t)$ converges to a smooth flat connection $\bar{A} = A(\infty)$ as $t \to \infty$.

Proof: From the formula (4.11) and the assumptions of the Proposition we get the following differential inequality for the positive function $u = |F|$ on M:

$$(4.17) \qquad (\frac{\partial}{\partial t} + \Delta)u \leq -\lambda_2 u + u^2$$

By the maximum principle we obtain the C^0-estimate:

$$(4.18) \qquad \|F(t)\|_\infty \leq \frac{\lambda_2.\epsilon.\exp(-\lambda_2 t)}{\epsilon.(1 - \exp(-\lambda_2 t)) + \lambda_2}$$

where $\epsilon = \|F(0)\|_\infty$.

So if $\epsilon < \lambda_2$ then $F(t) \to 0$ as $t \to \infty$.

The proof of the last proposition, which describes the underlying method used in our proof of pinching theorems for symmetric spaces of compact type [14], uses very strongly the fact that the Laplacian is coercive, allowing one to use the maximum principle.This is also the basic method used by Hamilton in [11]. This easy method is obviously not applicable to manifolds which are not positively curved, since we cannot make use of the simple Weitzenböck formula (4.11). In order to deform curvature on an almost symmetric space of non-compact type we have to use a Laplacian to which we can not apply the maximum principle directly. However, in case we know that the relevant Laplacian is L^2-positive definite, we get exponential decay of the L^2-norm of the curvature, at least for a short time before the C^0-norm explodes (possibly). Now since the curvature is evolving along a heat flow with only zero$^{\text{th}}$ order non-linearity one can estimate the C^0-norm in terms of the L^2-norm, provided that the initial curvature is already small in the C^0-norm.

Now let M be a compact manifold with an almost homogeneous structure P of type G/K, where G/K is an irreducible symmetric space of non-compact type. Let $A = A_1 + A_2$ be a Cartan connection for P with A_1, $\mathfrak{K}$-valued and A_2, $\mathfrak{M}$-valued. As was explained in §2, A_2 induces a gauge transformation $TM \to TM$, or equivalently a bundle automorphism $A_2 : Gl(M) \to Gl(M)$. By applying this gauge transformation (or rather its inverse) to the K-structure P and the Cartan connection A, we can assume that A_2 is in fact the canonical 1-form $\theta = \bar{A}_2$ on P. The Cartan curvature F splits now as $F = F_1 + F_2$, where the $\mathfrak{K}$-valued component F_1 is the difference of the curvature of the K-connection A_1 , whose covariant derivative we will denote by ∇ , to the curvature of the model space G/K, and the $\mathfrak{M}$-valued component F_2 is just the usual torsion of this connection. The covariant derivative induced by the full Cartan connection will be denoted by D. Since K is compact and acts orthogonally on $\mathfrak{M}$, the K-structure P defines a Riemannian metric on M, which is invariant under the connection ∇. Using the natural metrics given by the structure we want to integrate along the heat flow (4.7) starting with a given initial Cartan connection of small curvature. The evolution of the curvature is given by (4.8) and we therefore have to investigate the positivity of the Laplacian: $\Delta^D = d^D\delta^D + \delta^D d^D$ on 2-forms with values in the adjoint bundle $E = P \times_{ad} \mathfrak{G}$ over M. The Weitzenböck formula we need now is a generalization of the formula (3.4) to the case of a not necessarily integrable almost symmetric structure. A computation shows that:

$$\text{(4.19)} \qquad \Delta^D \beta = \Delta_1 \beta + \Delta_2 \beta + \hat{F}_2(\beta)$$

for an E-valued 2-form β where Δ_1 , Δ_2 are defined as in (3.4) and $\hat{F}_2(\beta)$, the correction term, depends only on the $\mathfrak{M}$-valued torsion part F_2 of the Cartan curvature. If we define $S: TM \times TM \to TM$ by:
$< S(u,v), w > = < F_2(w,u), v > + < F_2(w,v), u >$, then $\hat{F}_2(\beta)$ is given by:

$$(4.20)\quad \hat{F}_2(\beta)(u,v) = \sum_{k=1}^{n} \left\{ [S(u,e_k), \beta(e_k,v)] - [S(v,e_k), \beta(e_k,u)] - [S(e_k,e_k), \beta(u,v)] \right\}$$

where $\{e_k\}$ is an orthonormal base. If A_2 is not the canonical 1-form on P, the above formula is true after a gauge transformation with A_2^{-1} .

For all symmetric spaces G/K with positive Δ_2 , as given by Prop.3.10 and the remark following it, we can now estimate the first eigenvalue of Δ^D from below by a uniform bound depending only on the dimension n and independent of the diameter and other geometric data of the manifold M, provided that we have a bound:

$$(*)\qquad \|F_2\|_\infty < 10\epsilon \qquad \text{with } \epsilon \text{ depending only on } n.$$

We begin therefore with initial curvature satisfying $\|F(0)\|_\infty < \epsilon$. Then there exists a universal time t_1, depending only on n, such that :

$$(**)\qquad \|F(t)\|_\infty < 10\epsilon \qquad \text{holds in the interval } [0,t_1].$$

This follows from the fact that in (4.11) the lowest order non-linear terms are universal polynomials quadratic in F. (Here we use the maximum principle). Therefore there exists a positive λ , depending only on n so that

$$(4.21)\qquad \|F(t)\|_2 \leq \|F(0)\|_2 \exp(-\lambda t) \qquad \text{for } t \in [0,t_1]$$

In order to control also the change in the connection during the flow we derive the evolution equation for $\dot{A}$:

$$\frac{d}{dt}\dot{A} = \frac{d}{dt}(-\delta^D F) = -\dot{\delta}^D F - \delta^D \dot{F} = -\dot{A} \vee F - \delta^D d^D \delta^D F$$
$$= -\dot{A} \vee F - \Delta^D \dot{A}$$

where $(\dot{A} \vee F)_i = \sum_{k=1}^{n} [\dot{A}_k, F_{ik}]$ and hence:

$$(4.22) \qquad (\frac{\partial}{\partial t} + \Delta^D)\dot{A} + \dot{A} \vee F = 0$$

Here we have made essential use of the basic identity (4.6), which is an integrability condition for the curvature.

Now $\Delta^D = \Delta_1 + \Delta_2 + \hat{F}_2$, and by the proof we gave in §2 of the vanishing theorem of A.Weil, we know that Δ_2 is positive on $\mathfrak{G}$-valued 1-forms for $n>2$. Therefore if (**) holds , with a smaller $\epsilon(n)$ if necessary, we obtain the following L^2-estimate for $\dot{A}$:

$$(4.23) \qquad \|\dot{A}(t)\|_2 \leq \|\dot{A}(0)\| \exp(-\lambda t) \qquad \text{for } t \in [0, t_1].$$

To bootstrap the L^2-estimates (4.21) and (4.23) into C^0-estimates we use the powerful iteration method of Moser [18].

First we make a running assumption that: (The square on the left hand side is deliberate, in order to leave some room for a contradiction.)

$$(***) \qquad \|F(t)\|_\infty{}^2 \leq 10\|F(0)\|_\infty$$

holds for a maximal time interval [0,T] , we can bound the coefficients of the non-linear terms in the evolution equations for F and A so that we have a parabolic differential inequality:

$$(4.24) \qquad (\frac{\partial}{\partial t} + \Delta)u \leq c(n)u$$

valid in the interval [0,T], where u could either be the function $|F|$ or $|\dot{A}|$. By applying now the Moser iteration technique to this inequality we get the estimate:

$$(4.25) \qquad \|u(T)\|_\infty^2 \leq 2^n c(n)^{n/2}\, T^{-n/2}\, V.(C_{iso})^{-1} \int_{T/2}^{1} V^{-1} \|u\|_2^2\, dt$$

where V is the volume and C_{iso} is the *isoperimetric constant* of the Riemannian manifold M defined by:

$$(4.26) \qquad C_{iso} = \inf\left\{(\mathrm{vol}\,\partial D)^n.(\mathrm{vol}\ D)^{1-n}\right\}$$

where the infimum is taken over all domains $D \subset M$ with smooth boundary and with $2.\mathrm{vol}D \leq \mathrm{vol}M$. This constant appears in Moser's estimate because its inverse is equal to the best constant for the Sobolev inequality:

$$\left(\|f\|_{\frac{n}{n-1}}\right)^n \leq C_{sob}\ \|df\|_1^n \quad , \text{for } f \text{ with } \int f = 0.$$

Combining this with the L^2-estimates (4.21) and (4.23), and using $\|u\|_2^2 \leq V\ \|u\|_\infty^2$, we obtain the following exponential decay estimates for the C^0-norm of F and A:

$$(4.27) \qquad \|u(t)\|_\infty \leq C(n)(\widetilde{C}_{iso})^{-1}\ \|u(0)\|_\infty \exp(-\lambda t)$$

where $\widetilde{C}_{iso} = C_{iso} V^{-1}$.

Optimal estimates for the isoperimetric constant in terms of the lower bound for the Ricci curvature (scaled correctly w.r.t. diameter) have been obtained by P.Berard and S.Gallot [3] based on Gromov's isoperimetric inequality [10]. In particular Gallot [7] gives the following esimate:

$$\widetilde{C}_{iso} = V^{-1}\, C_{iso} \;\geq\; C(n,\alpha)\, d^{-n} \tag{4.28}$$

where d is the diameter and $C(n,\alpha)$ depends only on n and $\alpha = d^2 . \min_{|v|=1} Ric(v)$.

Substituting such an of estimate in (4.27) would give us a contradiction to the fact that (***) holds for some finite maximal time interval, provided the initial curvature $|F(0)|$ is less than ϵ, where ϵ now depends on the constants appearing in the above estimate. The assumption $|F(0)|$ small implies that the curvature and torsion of the Cartan connection are near that of the model space $\bar{M}$ so that we have universal bounds on the quantities on which all our constants depend.

Once C^0-estimates are established higher order estimates, as in the proof of Prop.4.9 above, follow in a rather standard fashion since the non-linearity of the parabolic equation for F occurs only in the zeroth order terms. This proves that the flow exists for all time and converges smoothly to a limit connection with zero curvature.

The exponential decay of the C^0-norm of A shows that the limit connection $A(\infty)$ is still a Cartan connection satisfying the non-degeneracy condition C3. By applying the smoothing Lemma of [2] as in [16], which is to say that we let some time elapse along the heat flow before we start doing the estimates, we can assume that $\dot{A}(0) = -\delta^D F(0)$ is of the same order of magnitude as $F(0)$.

Thus we have arrived at the following main result of this paper:

THEOREM.

Let G/K be an irreducible Riemannian symmetric space of non-compact type of rank > 1 or the dual of the Cayley projective plane . Then for any $n > 6$ and $d > 0$ there exists an $\epsilon(n,d) > 0$ depending only on n and d such that if M is an n-dimensional compact almost homogeneous manifold of type G/K with diameter $\leq d$ and if A is a Cartan connection for M whose curvature F satisfies $max|F| < \epsilon(n,d)$, then A can be deformed to a flat Cartan connection. In particular M is diffeomorphic to a compact quotient of G/K by a discrete group of isometries $\Gamma \subset G$.

Remark: The theorem is also true for the 3-dimensional hyperbolic space as model, since by Poincare duality $H^2(\bar{M};ad) \simeq H^1(\bar{M};ad)$ which vanishes by the theorem of A.Weil.

Problems:

(i) (suggested by M.Gromov) Can one get rid of the dependence of the pinching constant ϵ on the diameter? There is good evidence that the isoperimetric constant for the standard locally symmetric spaces of non-compact type of rank ≥ 2 do not depend on the diameter.

(ii) Can one prove a pinching theorem corresponding to the rigidity phenomena discovered by M.Berger and J.Simons? In other words can one prove the above pinching theorem assuming only that the torsion part of the Cartan connection F_2 is small ?

(iii) Can one find a proof of a version of Mostow's rigidity theorem using a Weitzenböck formula, imitating Y.T.Siu's method for the Kählerian case ?

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