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New invariants in the theory of knots

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New Invariants in the Theory of Knots

by Louis H. Kauffman

I. INTRODUCTION

In these notes I will concentrate on a diagrammatic approach to invariants of knots. We will talk about connections with graph theory, Hecke algebras and other topics. In the process we shall construct the Jones polynomial and its associated algebra. We'll also discuss generalizations of Jones polynomial due to myself and others. [Jones introduced his polynomial in 1984. Almost immediately, Hoste, Ocneanu, Millett, Lickorish, Freyd, Yetter, Przytycki, and Traczyk. had a significant generalization. Shortly, yet another invariant was crafted by Brandt, Lickorish, Millett and Ho. I generalized this one, and in the process found new approaches to the original Jones polynomial.] We'll also explain how proofs of some old conjectures about alternating knots emerge from this work (due to myself, Murasugi and Thistlethwaite). Many people have helped in this resurgence of the theory of knots. These notes are dedicated to all of them.

Let's begin by thinking about how one might go about making a theory of knots and links in three dimensional space. The typical example of a knot is illustrated in Figure 1.

A (realistic) B (schematic)

FIGURE 1

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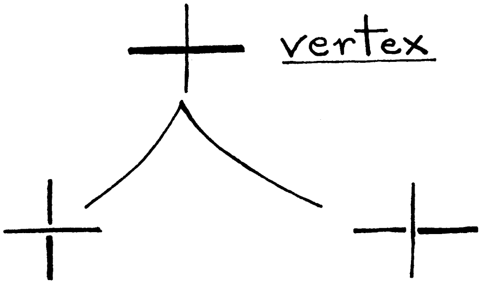
Actually, Figure 1 has two illustrations of a trefoil knot T. In the first (A), the trefoil is depicted realistically as a physical tube with thickness and shading in the three-dimensional space. This picture reminds you that the trefoil might be made of rope or rubber - and that such a model would exhibit thickness, tension, friction and other physical properties. In the second illustration (B), there is a schematic representation consisting in three continuous planar segments meeting at crossings. The crossings have local forms as shown in Figure 2.



two forms of crossing associated to a vertex

FIGURE 2

A schematic diagram of this type is a sufficient pattern to allow reconstruction of the knot or link from rope or string. It also encodes key topological properties, and allows the construction of a diagrammatic theory.

Thus, I shall regard a knot or link as extra structure (via crossing choices) on a (locally) 4-valent planar graph. Each vertex of such a graph has the form seen in Figure 2, and we shall call such a graph a universe. Thus in Figure 3 you see the trefoil and its universe.

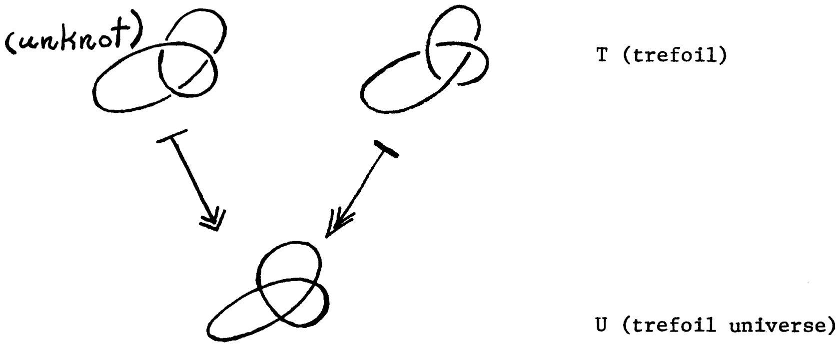
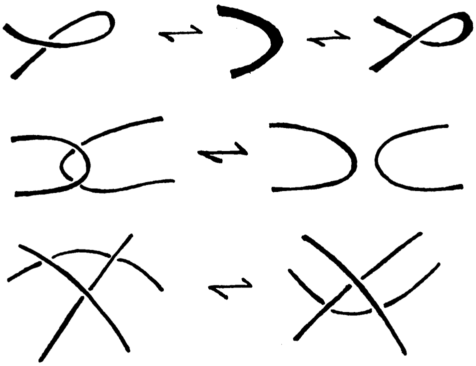


FIGURE 3

As Figure 3 indicates, you may regard the universe as the shadow, under projection to the plane, of the overlying knot or link. In general a universe of n vertices can be the projection of 2^n corresponding knots/links. Many of these will be unknotted or unlinked (see Figure 3 again).

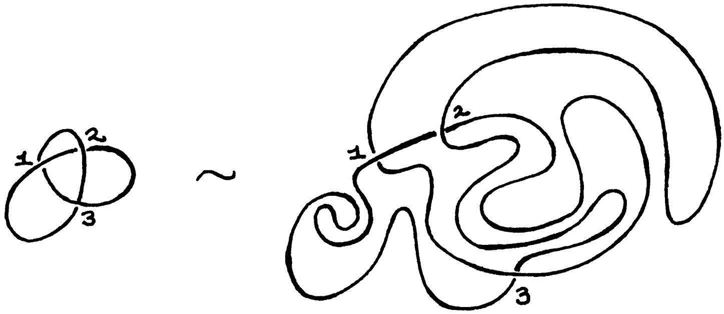
But we now need a definition of equivalence so that the words unknotted and unlinked make sense. This equivalence is generated by three fundamental types of diagram moves (the Reidemeister moves). See Figure 4. I have designated the Reidemeister moves as type I (add or remove a curl) type II (remove or add two consecutive under (over)crossings) and type III (triangle move). Reidemeister proved in the 1920's that these three moves (in conjunction with planar topological equivalences of the underlying universes) are sufficient to generate spatial isotopy. In other words, Reidemeister proved that two knots (links) in space can be deformed into each other (ambient isotopy) if and only if their diagrams can be transformed into one another by planar isotopy and the three moves (see [R]).

By a planar isotopy I just mean a motion of the diagram in the plane that preserves the graphical structure of the underlying universe. See Figure 5.



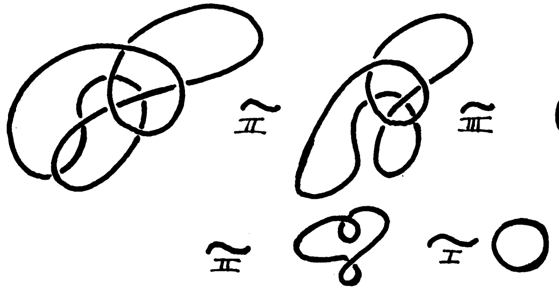
Reidemeister Moves

FIGURE 4



planar isotopy

FIGURE 5



ambient isotopy

FIGURE 6

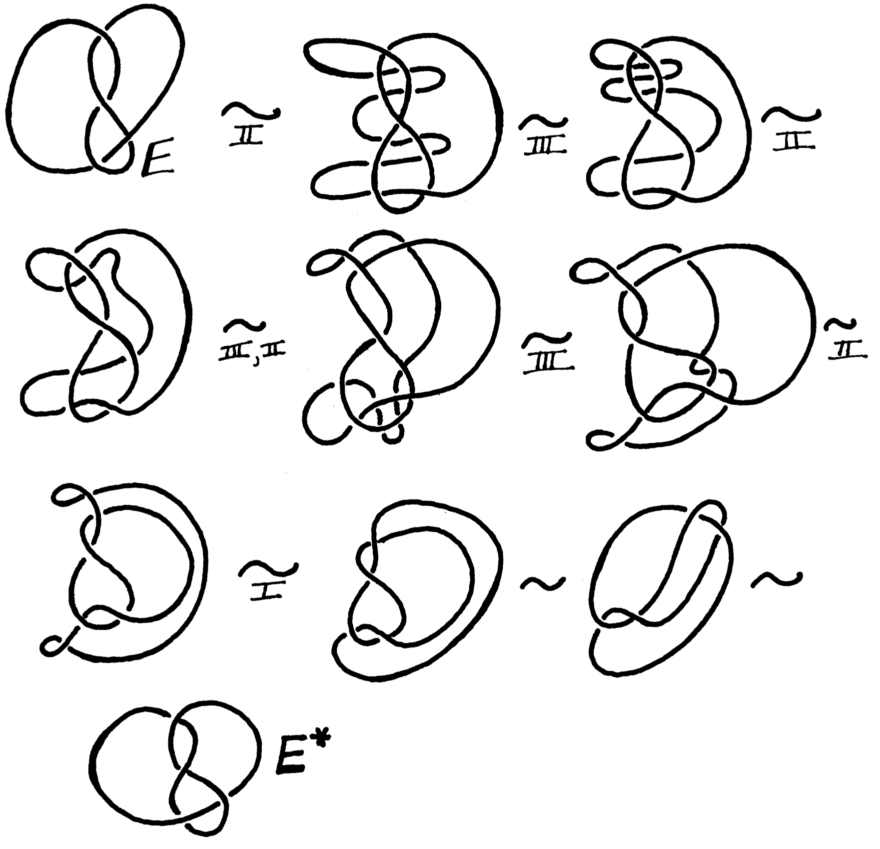
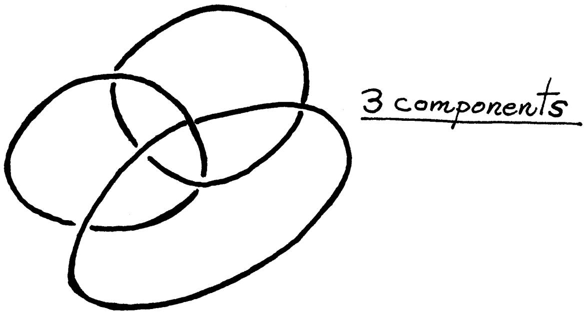


FIGURE 7

In Figure 6 an ambient isotopy to an unknotted circle is shown. Fogure 7 illustrates an ambient isotropy between the figure eight knot E and its mirror image E^*. A knot (link) is said to be an ambient isotropic (equivalent) to another if there is a sequence of Reidemeister moves and planar equivalences between them. We write ~ for equivalence. Thus $E \sim E^*$. (Note that the last two steps in the Figure 7 deformation are planar equivalences.)

A word about the Reidemeister moves is in order here. Reidemeister himself proved that the moves were sufficient to generate combinatorial isotopy of the corresponding embeddings of knots and links in three-dimensional space. His notion of combinatorial isotopy involved deformations of piecewise linearly embedded links, generated by elementary combinatorial isotopies. An elementary combinatorial isotopy was the result of replacing a segment on the link by the other two sides of a triangle linearly embedded in three-space so that its interior is disjoint from the curve (or the inverse of this operation). At the time that Reidemeister did his work, more general notions of ambient isotopy had not yet been formulated. The paper [GR] was the first to prove that combinatorial isotopy and ambient isotopy (in the sense of a parametrized piecewise linear deformation through embeddings) are equivalent. For a modern account of the equivalence, see [Z]. We have here used the term ambient isotopy as synonymous with the equivalence relation generated by the Reidemeister moves. The sense in which the theory to follow is quite elementary is the sense in which knot theory is seen to be generated from the formal diagrammatic system of the Reidemeister moves. Nevertheless, [Z] contains a good account of the equivalence of the combinatorial approach with isotopy in three-dimensional space.

The next thing one wants in a theory of knots are methods for distinguishing inequivalent knots and links. For example, we know that there is no equivalence between  and  : for the number of components remains invariant under ~. Note that you can determine the number of components even from a complicated diagram by choosing a point on some arc of the diagram and then taking a walk along the diagram - crossing crossings when you come to them. Each component is a complete cycle obtained in this way. (See Figure 8).

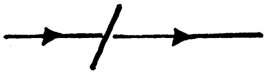


Counting Components

FIGURE 8

Note indeed that it is a consequence of the Reidemeister moves that the number of components is unchanged by equivalence. Thus the component count is our first invariant of knots and links. By itself it is, however not very strong.

In particular,  and  each have two components but are, in fact, inequivalent. The next invariant is the <u>linking number</u>. It gives a measure of how two curves wrap around each other. To define it we need notions of <u>orientation</u> and <u>sign</u>.

A link is said to be <u>oriented</u> if each of its components is assigned a direction indicated by arrow(s) on its arcs. The arrows are consistently arranged in the form  . Oriented crossings are given <u>signs</u> of ± 1 as shown in Figure 9.

$\varepsilon = +1$ $\varepsilon = -1$

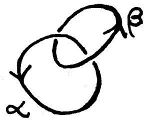
<u>Crossing Signs</u>

<u>FIGURE 9</u>

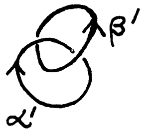
Given a link of two components α and β, let $\alpha \sqcap \beta$ denote the set of crossings of the component α with the component β. (Thus $\alpha \sqcap \beta$ does not include self-crossings of α or of β.) Then the linking number of α and β is defined by the formula:

$$\ell k(\alpha,\beta) = \frac{1}{2} \sum_{p \in \alpha \sqcap \beta} \varepsilon(p).$$

In other words, the linking number is one-half of the sum of crossing signs of one curve with another.

<u>Example</u> <u>1</u>. 

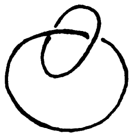
$\ell k(\alpha,\beta) = \frac{1}{2}(1+1) = 1$

<u>Example</u> <u>2</u>. 

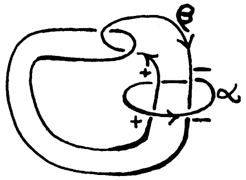
$\ell k(\alpha',\beta') = \frac{1}{2}(-1-1) = -1.$

Once an orientation has been assigned to a link of two components, it is immediate from the Reidemeister moves that the linking number is an invariant. For type I moves do not contribute to the linking number, while type II moves add or remove both a +1 <u>and</u> a −1. And type III moves do not alter the summation.

Thus the two examples above suffice to prove that the simplest link



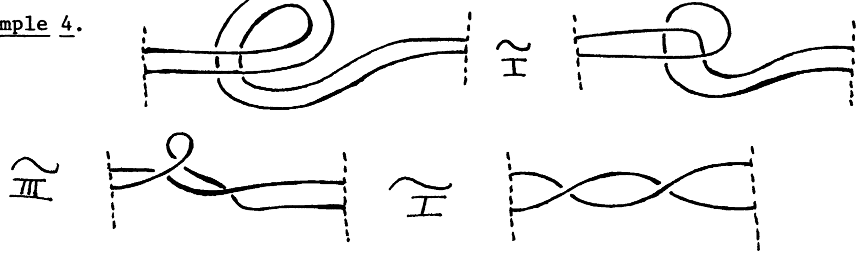
is indeed linked. For whatever orientation we assign to it, this link has a non-zero linking number.

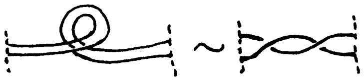
<u>Example</u> <u>3</u>. 

$\ell k(\alpha,\beta) = \frac{1}{2}(1+1-1-1) = 0.$

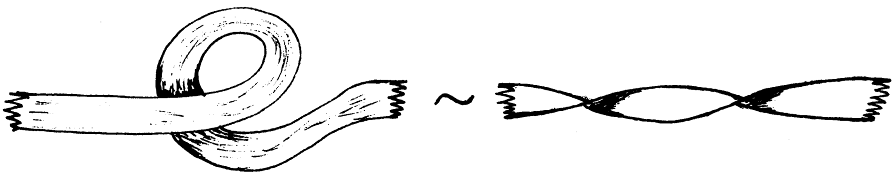
This is the Whitehead link (named after the topologist J.H.C. Whitehead). Even though it has linking number zero, it is linked.

Example 4.



Thus  , keeping the endpoints fixed.

This is a familiar phenomenon that you can illustrate with a belt (the two arcs forming the edges of the belt.)

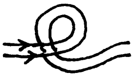


If the edges of the belt are oriented in the same direction then we can see what the linking number contribution will be from either of these forms.

$\frac{1}{2}(-1-1) = -1$

$\frac{1}{2}(-1-1) = -1$

(self-crossings are not counted)

In the curl form  it is worth noting that the linking contribution is the same as the contribution of the self-crossing  .

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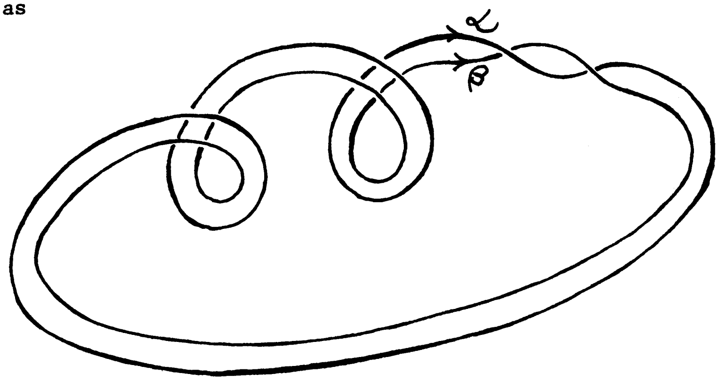
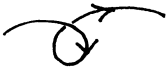
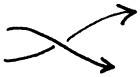
We can use these observations to find the linking number of a more complex link such as

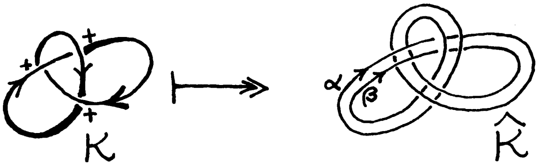


 each curl contributes +1 to the linking number

 occurs <u>twice</u> giving $\frac{1}{2}(1+1) = 1$

$\therefore \quad \ell k(\alpha,\beta) = 2 + 1 = 3$.

In fact, we can always find the linking number of a link that is built from a knot diagram by adding a parallel strand:

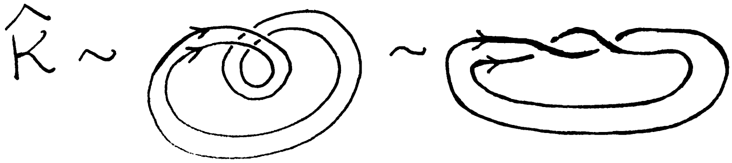


$$\ell k(\alpha,\beta) = w(K) = 3.$$

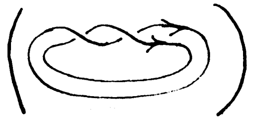
The resulting link has linking number equal to the sum of the crossing signs of K. We have denoted this sum by $w(K)$. It is called the writhe of K (or the twist number of K). The writhe $w(K)$ is not an invariant of K since it changes by ± 1 under the type I move. But the writhe is an invariant of the associated link $\hat{K}$ obtained by drawing parallel strands as above.

Example 5.

$$\ell k(\hat{K}) = w(K) = -1$$

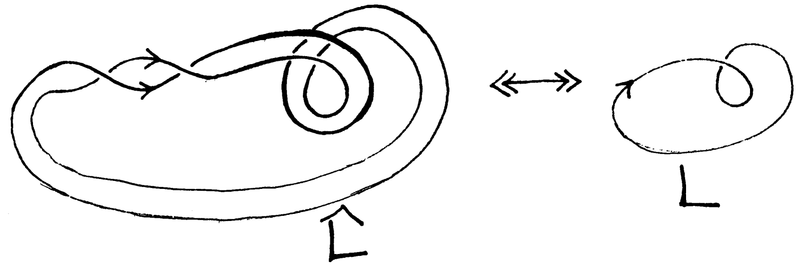


In the context of the associated parallel link $\hat{K}$, it is appropriate to call $w(K)$ the writhe of $\hat{K}$ and to reserve the word twist and a number $T(\hat{K})$ for the twisting of the strands. Thus we shall call  one full positive twist. And we'll write

T  = +1.

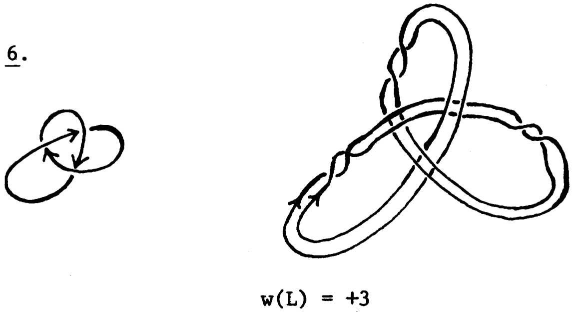
Then for links $\hat{L}$ composed of parallel twisted strands we have the formula $\ell k(\hat{L}) = w(L) + T(\hat{L})$. (See [Wh].)

The linking number for parallel twisted strands is the sum of the writhing and twisting. *Thus*



$$\left.\begin{array}{l} w(L) = +1 \\ T(\hat{L}) = +1 \end{array}\right\} \Rightarrow \ell k(\hat{L}) = 1 + 1 = 2.$$

Example 6.



$w(L) = +3$

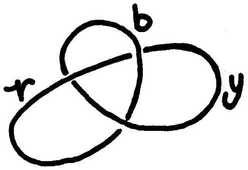
$T(\hat{L}) = -3$

$\ell k(\hat{L}) = 3 - 3 = 0.$

This is another example of a link with zero linking number that is nevertheless linked.

Remark: The formula [Wh] $\ell k(\hat{L}) = w(L) + T(\hat{L})$ can be regarded as a kind of "conservation law" for links of closed parallel (twisted) strands. Neither $w(L)$ nor $T(\hat{L})$ are individually topological invariants. But since the sum of writing and twisting is a topological invariant, this sum must remain a constant. You can observe this conservation by playing with a rubber band. And it has been used to help understand the geometry of closed double-stranded DNA. ([BCW],[F])

There is much more so say about linking numbers - many other points of view. See [Knots and Links by Dale Rolfsen] or my books [Formal Knot Theory; On Knots] for other points of view. We now pass on to the next problem: to show that the trefoil  is indeed knotted. The most elementary proof of this fact that is known to me runs as follows: Color the arcs of the trefoil diagram red (r), blue (b) and yellow (y).



Say that a knot diagram is tri-colored if every arc is colored r, b or y and at any given crossing either all three colors appear or only one color appears. Of course to be tri-colored there must be arcs of each color in the diagram. Then prove (exercise!) that for knots (one component) tri-coloration is preserved under the Reidemeister moves.

Since the trefoil is tri-colored and the unknot is not tri-colored, this method articulates a topological property of the trefoil and shows that it can not be unknotted. In the next section we will show that the trefoil is chiral. That is, we will prove that it is not equivalent to its mirror image. In the old days (before 1984) this was something that required a lot of mathematical background. Now we can prove it using only diagrams and a few definitions and calculations. That new invariants can be both simple and powerful makes the subject of knots very exciting.

In the next section we construct the bracket invariant and show how it gives rise to the Jones polynomial and to chirality for the trefoil. Section 3 uses the bracket to get at subtle facts about alternating knots and links. Section 4 gives more discussion of the bracket and its relation to braids and the algebra of Jones' original representation. Section 5 discusses 2-variable generalized polynomials and the historical background of Alexander and Conway polynomials. Section 6 shows how the bracket (hence the Jones polynomial) is directly related to the Potts model in statistical physics. Sections 7 and 8 explain and generalize a relation with the Tutte polynomial in graph theory that was discovered by Thistlewaite [T1]. Section 9 discusses the knot theory of graphs embedded in three-dimensional space. Section 10 has speculations and problems.

It should be pointed out that there are knot diagrams that are equivalent to the unknot, but only through sequences of Reidemeister moves that increase the number of crossings in the diagram before (eventually) bringing the crossing number to zero. For example, consider the diagram in Figure 10.

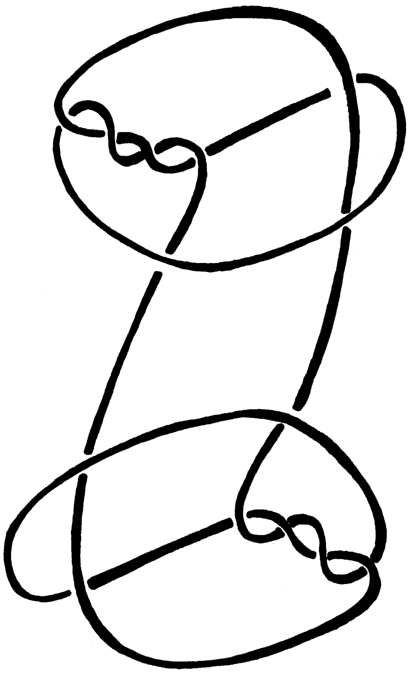


FIGURE 10

This diagram certianly represents (is equivalent to) the unknot. However, it admits no Reidemeister move that decreases <u>or</u> leaves constant the number of crossings. Therefore, this unknot diagram must be made more complicated before it becomes simpler. It is this phenomenon that makes any theory of invariance under Reidemeister moves non-trivial. (A challenge - prove that this phenomenon requires at least a diagram with ten crossings. That is, show that any unknot diagram with less than 10 crossings can be undone by simplifying Reidemeister moves.)

II. THE BRACKET POLYNOMIAL.

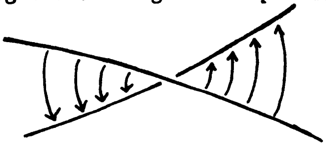
I begin by defining a 3-variable polynomial on unoriented link diagrams. Given an unoriented link diagram K, $[K] \in Z[A,B,d]$ will denote the corresponding polynomial in commuting variables A, B and d. The bracket polynomial satisfies the <u>axioms</u>:

Bracket Axioms

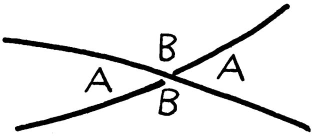
1. [><] = A[≍] + B[ )( ]
   [><] = B[≍] + A[ )( ]

2. [O K] = d[K]
   [O] = d

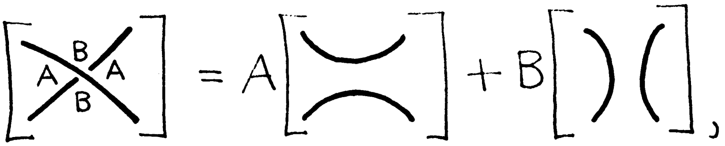
Some explanation of these rules is in order. First note that <u>an unoriented crossing discriminates two out of the four regions incident at its vertex</u>. This can be done conventionally by rotating the over-crossing line counterclockwise and choosing the two regions swept out. Thus



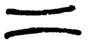
By using this convention, we can label the regions A and B respectively:

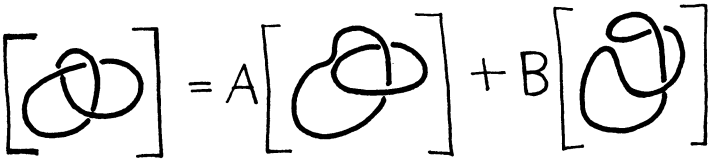


The formula in (1) then reads



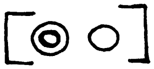
and we see that A corresponds to a splice that "opens the A-channel" while B corresponds to a splice that opens the B-channel. By this convention, the second equation in (1) is correct, and a repetition of the first.

The crossings in these equations stand for larger diagrams that contain them. Thus  ,  ,  are assumed to be parts of otherwise identical diagrams. In this sense the expansion formula (1) stands for infinitely many particular formulas such as



The second equation asserts that an extra disjoint circle placed anywhere in the diagram multiplies the value of bracket by d. In particular

$$[\text{any } N \text{ disjoint simple closed curves}] = d^N.$$

Thus  $= d^3$. Clearly, these axioms lead to a recursive calculation of [K] by continued expansion to evaluations of collections of simple closed curves. To see that [K] is well-defined, it suffices to re-formulate it as a sum over states S of the universe U underlying K.

Let U be the universe for K. A state S of U is a choice of splitting for each vertex of U. I denote such a choice by a marker at the vertex, thus (see Figure 11).

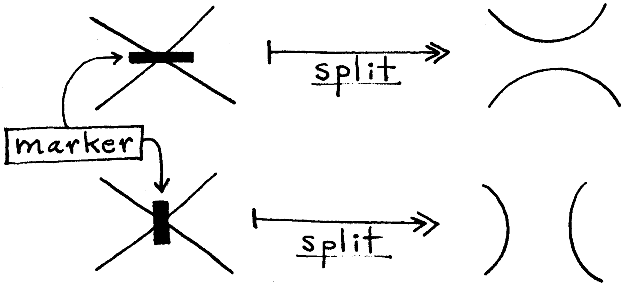
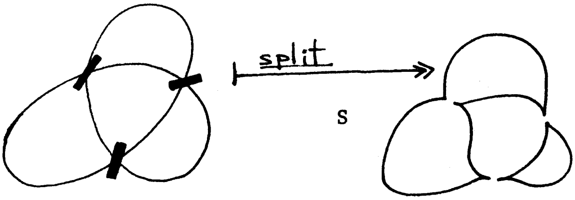


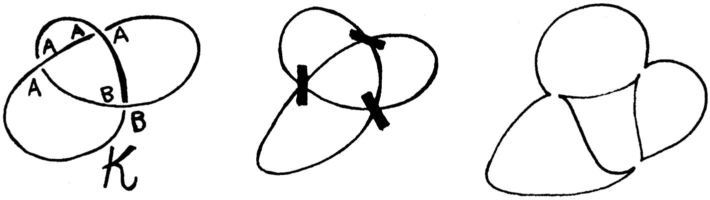
Figure 10



A State of the trefoil universe.
Figure 11

Figure 11 shows a state of the trefoil universe and its corresponding splitting.

Given a state S, let $|S|$ denote the number of components in its splitting. Let $i_K(S)$ denote the number of A-channels opened in S and $j_K(S)$ denote the number of B-channels in S. For example,

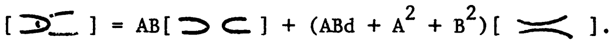


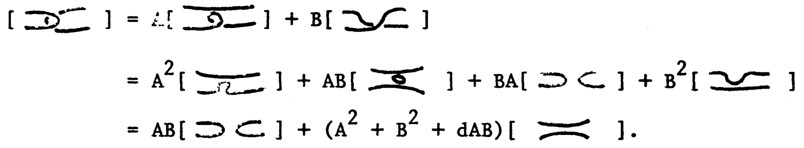
$i_K(S) = 2 \quad , \quad j_K(S) = 1.$

<u>Lemma 2.1</u>. $[K] = \sum_S A^{i_K(S)} B^{j_K(S)} d^{|S|}$. This formula for the value of the bracket follows directly from the axioms (by expanding using (1) and (2)). It gives a unique value for the bracket on diagrams (no Reidemeister moves yet) and can be taken as the <u>definition</u> of $[K]$ for a strictly logical development.

<u>We now ask</u>: Under what restrictions on A, B and d will $[K]$ become a topological invariant of knots and links?

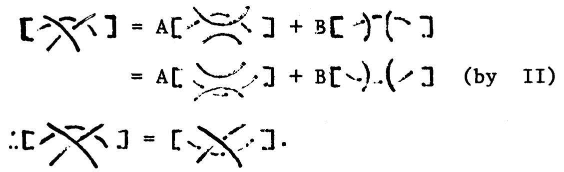
This question is easy to answer via the next lemma. (See [K5].)

<u>Lemma 2.2</u>. 

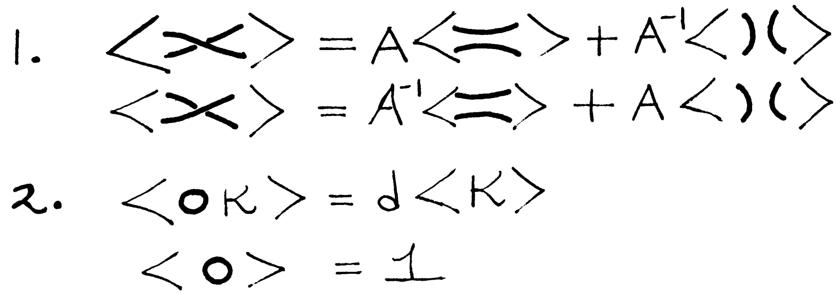
<u>Proof</u>: 

Thus with $AB = 1$ and $d = -A^2 - B^2$ we obtain invariance under the second Reidemeister move.

<u>Lemma 2.3</u>. If  then [ ] is also invariant under the type III move.

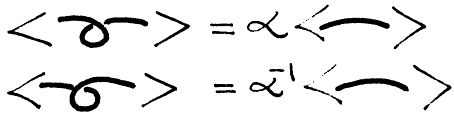
<u>Proof</u>: 

So for the rest of this section we'll take $B = A^{-1}$, $d = -A^2 - A^{-2}$ and also write $\langle K\rangle = d^{-1}[K]$ so that $\langle 0\rangle = 1$. We then have



This special bracket is invariant under moves II and III. It behaves as follows under the type I move:

<u>Lemma</u> <u>2.4</u>. Let $\alpha = -A^3$. Then



We can do two things at this point. We can understand that $\langle K\rangle$ is a special kind of invariant <u>and</u> it is possible to create an ambient isotopy invariant from $\langle K\rangle$ for K oriented. First we call the equivalence generated by moves II and III <u>regular isotopy</u>. Thus $\langle K\rangle$ is a regular isotopy invariant.

Recall from section 1 that the twist number $w(k)$ for an oriented link K is also a regular isotopy invariant. (Recall that $w(K)$ is the sum of all crossing signs). Thus <u>if</u> K <u>is oriented</u> we define $f_K = \alpha^{-w(K)}\langle K\rangle$ where $\langle\ \rangle$ forgets the particular orientation. Then f_K <u>is an ambient isotopy invariant for oriented knots and links K</u>. (This is proved by noting that f_K is a regular isotopy invariant, and that f_K is, by construction, invariant under Reidemeister move I.

For mirror images we have

Lemma 2.5. Let K^* denote the mirror image of K obtained by reversing all the crossings. Then $\langle K^*\rangle(A) = \langle K\rangle(A^{-1})$ and $f_{K^*}(A) = f_K(A^{-1})$.

We omit the (easy) proof.

Now it turns out that f_K is a version of Vaughan Jones' original polynomial. To see this we need a definition of the Jones polynomial. Later we will have a deeper look at this. For now it suffices to say that Jones' polynomial $V_K(t)$ is determined by the axioms: (see [J]], [J2], [J3].)

1) $t^{-1}V_{\times} - tV_{\bar{\times}} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{\rightrightarrows}$

2) $V_{\bigcirc} = 1$

3) $V_K(t)$ is an invariant of ambient isotopy.

Lemma 2.6. $f_K(t^{-1/4}) = V_K(t)$.

Proof:

$$\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$$

$$\langle \bar{\times} \rangle = A^{-1}\langle \asymp \rangle + A\langle)(\rangle$$

$$\therefore\ A^{+1}\langle \times \rangle - A^{-1}\langle \bar{\times} \rangle = (A^2 - A^{-2})\langle \asymp \rangle$$

$$A\alpha\langle \times \rangle\alpha^{-w(\times)} - A^{-1}\alpha^{-1}\langle \bar{\times} \rangle\alpha^{-w(\bar{\times})} = (A^2 - A^{-2})\langle \rightrightarrows \rangle\alpha^{-w(\rightrightarrows)}$$

$$A\alpha f_{\times} - A^{-1}\alpha^{-1}f_{\bar{\times}} = (A^2 - A^{-2})f_{\rightrightarrows} \qquad (f_K = \alpha^{-w(K)}\langle K\rangle)$$

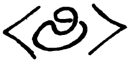
$$-A^4 f_{\times} + A^{-4}f_{\bar{\times}} = (A^2 - A^{-2})f_{\rightrightarrows}$$

Let $A = t^{1/4}$. Then

$$t^{-1}f_{\times} - tf_{\bar{\times}} = (\sqrt{t} - \frac{1}{\sqrt{t}})f_{\rightrightarrows}$$

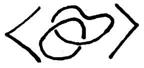
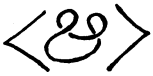
QED

Having constructed the bracket, here are some sample computations:

1.  $= A$  $+ A^{-1}$ 

$$= A(\alpha) + A^{-1}(\alpha^{-1})$$

$$= -A^4 - A^{-4}.$$

2.  $= A$  $+ A^{-1}$ 

T

$$= A(-A^4 - A^{-4}) + A^{-1}(-A^{-3})^2$$

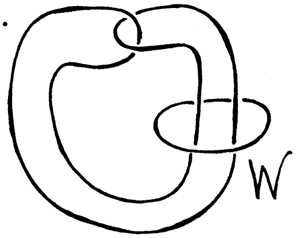
$$= -A^5 - A^{-3} + A^{-7}$$

$$f_T = \alpha^{-3}\langle T\rangle = -A^{-9}\langle T\rangle = A^{-4} + A^{-12} - A^{-16}$$

Thus $f_T(A) \neq f_T(A^{-1})$ and hence the trefoil knot is chiral. There is no ambient isotopy of the trefoil to its mirror image. This is the simplest known proof of the chirality of the trefoil knot. Note that all the machinery was developed from scratch, and it is all elementary.

By Lemma 2.6 we have the Jones polynomial for the trefoil as well:

$$V_T(t) = t + t^3 - t^4.$$

3. 

Exercise: Calculate <W> and show that it is a non-trivial knot.

III. ALTERNATING KNOTS AND LINKS.

The bracket polynomial can be used to get at some subtle properties of alternating knots and links. This comes about because we can determine a specific formula for the terms of highest and lowest degree in A for such links.

Recall that a link is said to be alternating if it has an alternating diagram. This is a diagram where the crossings alternate under-over-under-over- ... as one travels along the link (crossing at the crossings). Now view Figure 12.

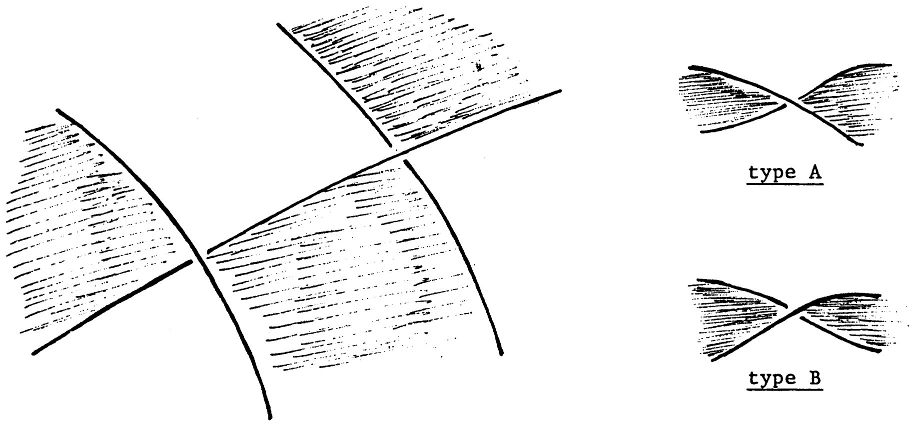


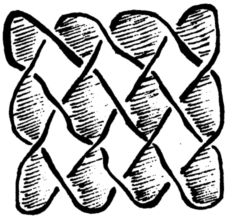
Figure 12

It should be clear from this figure that in a checker board shading of the diagram for an alternating link, the shaded regions at each crossing are all of the same type (A or B where this is the same discrimination that we used to define the bracket.). It is assumed that the underlying universe for the diagram is connected.

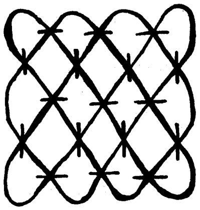
We will use this observation to guess the highest degree term in $\langle K \rangle$, and then prove that our guess is correct. Recall that the bracket is given by a summation

$$\langle K \rangle = \sum_S A^{i_K(S)} A^{-j_K(S)} d^{|S| - 1}.$$

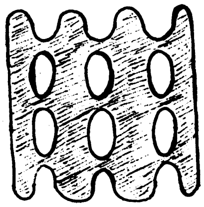
Our guess is that the highest degree is contributed by that state S where all the markers open A-channels. Such a state will contribute a term of the form $A^v d^{|S| - 1}$ where v is the number of vertices (crossings) in the diagram K. And our checker board observation shows that in the case of alternating links this A-channel state S has W components ($|S| = W$) where W is the number of white (unshaded) regions, and all the A-channels are colored black. View Figure 13.



K



S



$V = 17$

$W = 7$, $B = 12$

$R = 7 + 12 = 19 = v + 2$

$|S| = 7 = W$

S (split)

The A-channel state.

Figure 13

We see that $|S| = W$ exactly because we have split all the shaded crossings, connecting the shaded part into one big shaded region, whose boundary components are boundaries of the white regions.

Thus S constributes the term

$$A^V d^{W-1} = A^V(-A^2 - A^{-2})^{W-1}.$$

Hence we assert

Theorem 3.1. Let K be a reduced alternating diagram. Then the highest degree term in $\langle K \rangle$ has degree given by the formula

$$\max \deg\langle K \rangle = V + 2(W-1)$$

where V is the number of vertices in the diagram, W is the number of unshaded regions (shading corresponding to type A crossings). This term has coefficient equal to ± 1 in $\langle K \rangle$. The term of minimal degree is also monic and has degree

$$\min \deg\langle K \rangle = -V - 2(B-1)$$

where B is the number of shaded regions.

Comment. A diagram is reduced if it has no crossing that is an isthmus. A crossing is said to be an isthmus if any two of the four local regions at the crossing are parts of the same region in the whole diagram. See Figure 14.

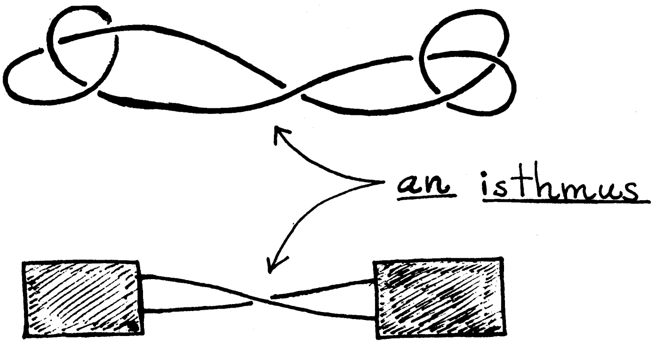


Figure 14

Proof of Theorem 3.1. Let S be the A-channel state. Note that any other state S' can be obtained from S by flipping some subset of S's markers. For any state S', let $\langle K|S'\rangle = A^{i_K(S')} B^{j_K(S')} d^{|S'| - 1}$ denote the contribution of this state to the bracket summation. ($B = A^{-1}$, $d = -A^2 - A^{-2}$). Thus $\langle K\rangle = \sum_{S'} \langle K|S'\rangle$.

Now observe the following facts:

(i) If S' is obtained from S" by flipping an A-channel marker to a B-channel marker, then $\max\deg\langle K|S'\rangle \le \max\deg\langle K|S''\rangle$. The inequality is strict exactly when S' has fewer components that S". That is when $|S'| = |S''| - 1$.

(ii) If S' is obtained from the A-channel state by one flip, then $|S'| = |S| - 1$.

Assertion (i) is obvious, for if $\langle K|S''\rangle = A^x d^{|S''| - 1}$, then $\langle K|S'\rangle = A^{x-2} d^{|S'| - 1}$. Since S' is obtained from S" by one flip, we know that $|S'| = |S''| \pm 1$. If

$|S'| = |S''| + 1$, then

$$\langle K|S'\rangle = A^{x-2}d^{|S''|}$$

hence $\underline{\max \deg\langle K|S'\rangle = \max \deg\langle K|S''\rangle}$. If $|S'| = |S''|$, then

$$\langle K|S'\rangle = A^{x-2}d^{|S'|-1-1}$$

and (using $d = -A^2 - A^{-2}$),

$$\underline{\max \deg\langle K|S'\rangle = \max \deg\langle K|S''\rangle - 4}.$$

This verifies assertion (i).

Assertion (ii) is a consequence of our hypothesis of no isthmus. For suppose that $|S''| = |S| + 1$. Begin tracing along one of the components of S' at the changed marker. Note that due to our construction of the state S, this tracing (when drawn parallel to the component in the white regions) will encircle all or part of the original knot diagram. If $|S'| = |S| + 1$ then the two cusps (> <) at the site of the changed marker will lie on separate components of S. Thus we will end up encircling a part of the diagram showing that this site was an isthmus. This is a contradiction. Hence $|S'| = |S| - 1$ and $\max \deg\langle K|S'\rangle = \max \deg\langle K|S\rangle - 4$.

It follows from (i) and (ii) that $\max \deg\langle K|S''\rangle < \max \deg\langle K|S\rangle$ for all states S''. Thus

$$\begin{aligned} \max \deg\langle K\rangle &= \max \deg\langle K|S\rangle \\ &= V + 2(W-1). \end{aligned}$$

This completes the proof of the theorem.

We are now in a position to deduce the following (see [K5], [M2], [T1].)

Theorem (Kauffman-Murasugi-Thistlethwaite). The number of crossings in a reduced alternating projection of a link L is a topological invariant of L.

Proof: Let span(L) denote the difference between the maximal and minimal degrees of $\langle L\rangle$. Since $f_L = \alpha^{-w(L)}\langle L\rangle$ is an ambient isotopy invariant of L, we conclude that span(L) is also an ambient isotopy invariant. By 3.1

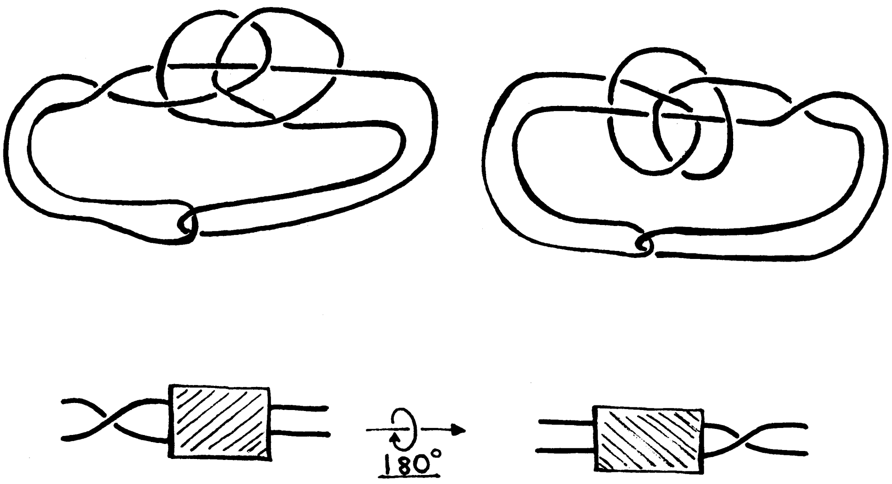
$$\max \deg\langle L\rangle = V + 2(W-1)$$
$$\min \deg\langle L\rangle = -V - 2(B-1)$$

where V is the number of crossings in the diagram, W the number of white regions, B the number of black regions. (In a shading where all A-type crossings are shaded.) Thus

$$\begin{aligned}\text{span}(L) &= V + 2(W-1) - (-V - 2(B-1)) \\ &= 2V + 2(W + B - 2).\end{aligned}$$

But $W + B = R$, the total number of regions in the diagram, and $R = V + 2$. Hence $\text{span}(L) = 4V$. This completes the proof.

Discussion. This result is one of a number of classical conjectures about alternating knots and links that go back to the original compilations of knot tables by Tait and Little at the end of the last century. They also conjectured that a reduced alternating projection is minimal in the sense that it has the least number of crossings of any projection of that link. This is also true, as we shall see. Beyond this however, is the Tait flyping conjecture. This states that any two reduced alternating projections of the same (up to ambient isotopy) link can be obtained from another by flyping. A flype is a move on a tangle (with two inputs and two outputs) obtained by rotating the tangle by 180°. See Figure 15. Among other things, the flyping conjecture implies that the twist number, w(K), of a reduced alternating projection is an ambient isotopy invariant of K. At this writing, the full conjecture remains open. Morwen Thistlethwaite has proved that w(K) is an ambient isotopy invariant for reduced alternating diagrams. His proof uses my extension of the Brandt-Lickorish-Millett-Ho polynomial to two variables. (See section 5 of these notes for a discussion of this polynomial.)

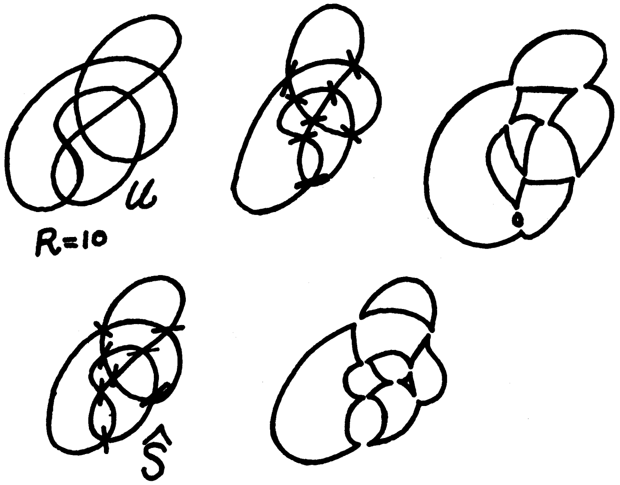


Flyping
Figure 15

The next lemma [K5] gives a quick proof of the general inequality span(K) $\leq$ 4V (first proved independently by Murasugi and Thistlethwaite).

<u>Lemma</u> <u>3.3</u>. Let S be any state of a universe U. Then $|S| + |\hat{S}| \le R$ where R is the number of regions in U and $\hat{S}$ is the <u>dual state for</u> S obtained by reversing all the markers of S.

I omit the proof of this lemma. See Figure 16 for an illustration



$|S| = 2$

$|\hat{S}| = 4$

$|S| + |\hat{S}| = 6 < 10$

<u>Figure 16</u>

We then use Lemma 3.3 to prove

<u>Proposition</u> <u>3.4</u>. For any diagram reduced K, $\text{span}(K) \le 4V$ where V is the number of crossings in K.

<u>Proof</u>: Let S be that state for K such that every crossing is split in the A-direction. Then the same argument as in the proof of 3.1 shows that

$$\max \deg\langle K\rangle \le V + 2(|S| - 1)$$

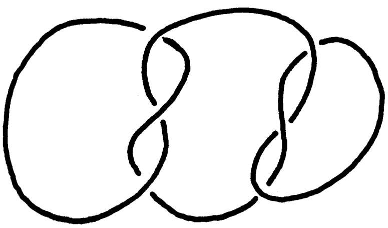
and

$$\min \deg\langle K\rangle \ge -V - 2(|\hat{S}| - 1).$$

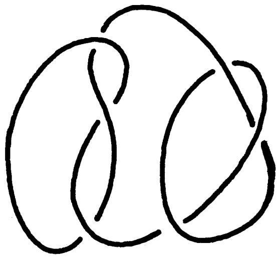
Therefore $\operatorname{span}(K) \le 2V + 2(|S| + |\hat{S}| - 2) \le 4V$.

A diagram is said to be <u>prime</u> if it can not be cut in two points belonging to different arcs of the diagram by any simple closed curve in the plane. Thus

is the form of a typical non-prime diagram. A non-prime alternating knot can have a non-alternating diagrammatic representative with the same number of crossings. For example, here are two ambient isotopic diagrams of the six-crossing square knot:



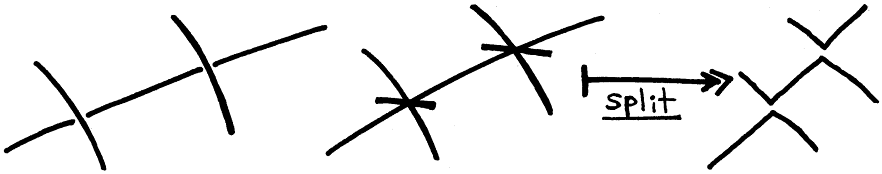
<u>non-alternating</u>



<u>alternating</u>

Nevertheless, a refinement of Lemma 3.3 by Wu [Wu] shows that $\operatorname{span}(K) < 4V$ when K is any reduced, prime, non-alternating diagram. This gives us a quick proof of this inequality, also due to Murasugi and Thistlethwaite. Mr. Wu's very nice observation is that in a state S where every crossing is split in

A-direction for a non-alternating diagram there must appear splits in the pattern



corresponding to two consecutive over or under crossings. Call this a pair of parallel markers. Mr. Wu notes that if S has at least one pair of parallel markers, then $|S| + |\hat{S}| \leq V$. Repeating the argument of 3.4, we obtain the strong inequality span(K) < 4V when K is reduced, prime, and non-alternating. Thus we know that <u>a reduced alternating projection has a minimal number of crossings among all diagrams for the link</u>.

This is a remarkable application of these techniques. It is the first result of this kind in knot theory, and has a number of ramifications. For example, D.W. Sumners has used it to show that the number of knots grows at least exponentially as a function of minimal crossing number. See also [Ki].

Mirror Images

We now turn to the consequences of Theorem 3.1 for chirality of alternating links. Let K be a reduced alternating projection as in 3.1. Then max deg<K> = V + 2(W-1) and min deg<K> = -V - 2(B-1). Thus (using $f_K = \alpha^{-w(K)}\langle K\rangle$) we have

$$\max \deg f_K = -3w(K) + V + 2(W-1)$$
$$\min \deg f_K = -3w(K) - V - 2(B-1).$$

If K is ambient isotopic to its mirror image K^*, then $f_{K^*}(A) = f_K(A^{-1})$ implies $f_K(A) = f_K(A^{-1})$. Hence $-\min \deg f_K = \max \deg f_K$, thus

$$3w(K) + V + 2(B-1) = -3w(K) + V + 2(W-1).$$

Therefore

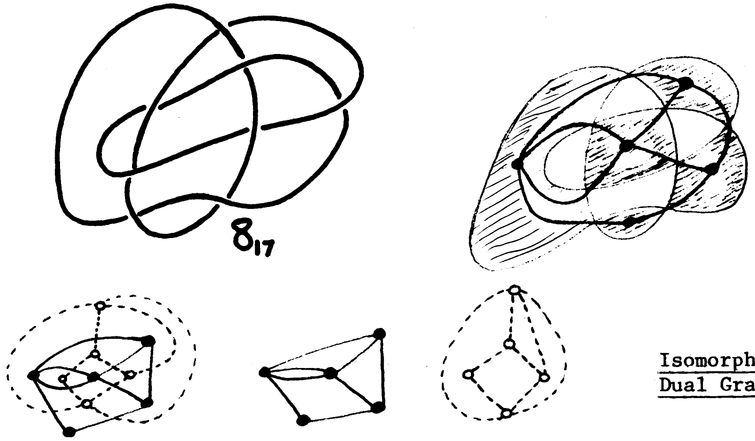
$$6w(K) = 2(W-B), \text{ or}$$
$$3w(K) = W - B.$$

Thus we have a necessary condition for an alternating link to be achiral (equivalent to its mirror image). You can check that it follows from this equation that if the absolute value of the twist number, $|w(K)|$, is greater than or equal to one third the number of crossings, then the link is chiral. This is a step in the direction of the

Theorem [T2]. K reduced, alternating, $w(K) \neq 0$ implies K is chiral. In fact, w(K) is a topological invariant for K reduced and alternating.

Murasugi [M2] also proved the invariance of w(K). His method is to note the sum s(K) of the maximal and minimal degrees is an invariant of the ambient isotopy class of K. For K reduced alternating, we have (from the above) that $s(K) = -6w(K) + 2(W-B)$. He then uses another technique (the signature of knots and links) to show that $(W-B)-w(K)$ is an ambient isotopy invariant for the reduced alternating diagram K. Hence $w(K)$ must also be invariant. In section 5 we'll give another proof of the invariance of $w(K)$.

There exist many reduced prime alternating achiral knots with twist number zero. I <u>conjecture</u> that each such not only satisfies $W = B$, but that the graph associated to the white regions is isomorphic to the graph associated to the black regions. See Figure 17 for the example of the knot 8_{17}.



Isomorphic Graph and Dual Graph

FIGURE 17

IV. BRAIDS AND DIAGRAMS.

Let's now consider the specialization of the bracket to the case of braids. The <u>n-strand braid group</u> B_n is generated by elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ (and their inverses) subject to the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i-j| > 2$$

The meaning of these generators and relations should become clear from Figure 18. A braid is a collection of unknotted strands, proceeding downward from n points (top row) to n points (bottom row). The strands wind around one another throughout the descent. Given a braid $b \in B_n$, its <u>closure</u> $\bar{b}$ is the knot or link obtained by attaching the n points in the top row to their counterparts on the bottom row. (Again see Figure 18). By definition, the value of the bracket on a braid is its value on the closure of the braid:

$$\langle b \rangle = \langle \bar{b} \rangle .$$

Braids b_1, b_2 are multiplied by attaching the n points on the bottom row of the first to the n points on the top row of the second. (Again see Figure 18.)

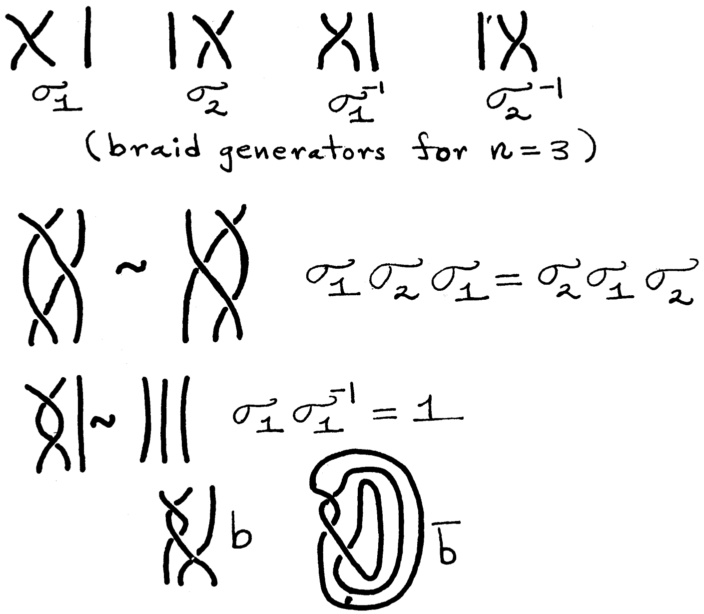


Figure 18

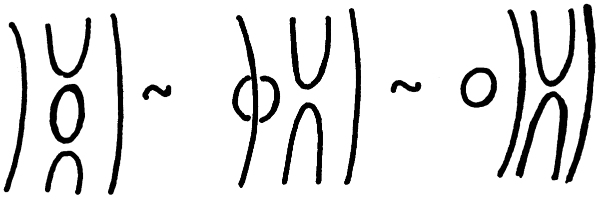
Now consider the states of a braid-universe. It should be apparent from Figure 19 that these can be constructed as diagrammatic products of the elementary diagrams $h_1, \ldots, h_{n-1}$ with relations

$$h_i^2 = dh_i$$

$$h_i h_{i\pm1} h_i = h_i$$

$$h_i h_j = h_j h_i \quad \text{for} \quad |i-j| > 2.$$

Here we take d to represent the closed loop obtained by plugging h_i into itself. As diagrams, the states can be multiplied just as we multiply braids. Since the result of such multiplications can produce extra closed loops, we need to impose a mixed topological and combinatorial equivalence relation to capture the resulting structure. Since, for computing the bracket it is irrelevant where a closed component is (we only count them), I define two diagrams to be equivalent if one can be obtained from another by regular isotopy relative to the endpoints, with free regular isotopy for closed loops. Thus



illustrates the equivalence behind the identity $h_2^2 = dh_2$. A mixture of braid generators and h_i's produces a more intricate structure.

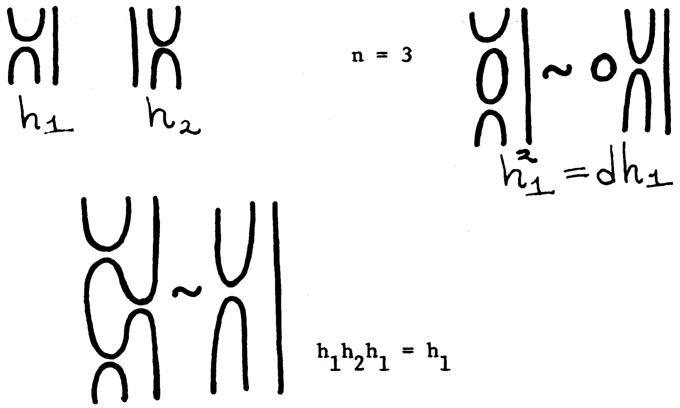
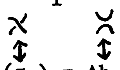


Figure 19

Because the braid-states have a multiplicative structure, we see that the bracket expansion <  > = A<  > + A^{-1}<  > can be construed <u>for braids</u> as a mapping $\rho: B_n \to D_n$ where D_n is the free additive algebra over $Z[A,A^{-1}]$ with multiplicative generators h_i and relations (*) above. That is, we define



$$\rho(\sigma_i) = Ah_i + A^{-1}$$

$$\rho(\sigma_i^{-1}) = A^{-1}h_i + A$$

and take $d = -A^2 - A^{-2}$ in D_n. Then the formalism we have used to prove that <K> is an invariant of ambient isotopy also proves (via the braiding relations) that ρ is a representation of the n-strand braid group to the algebra D_n. Furthermore there is a function $tr: D_n \to Z[A,A^{-1}]$ that we may interpret as the linear extension of tr(h) where h is a product of h_i's. And $tr(h) = d^{|h| - 1}$ where $|h|$ is the number of disjoint circles in the state corresponding to h. Then $tr \circ \rho(b) = \langle b \rangle$ and <u>this gives a diagrammatic interpretation to the original construction of the Jones polynomial via representations</u>.

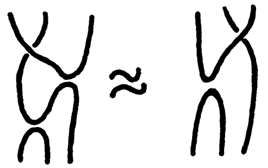
This approach has been generalized (see [KC], [K7], [L]) but the algebra of the h_i's remains the most transparent structure in this context. And while it may seem transparent, it is in fact rather opaque! We do not yet know whether there is a non-trivial knot with trivial Jones polynomial.

<u>The Mixed Algebra</u>

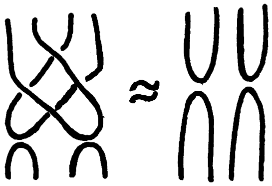
From our context, it is very natural to consider a mixture of products of braid generators σ_i and the state-elements h_i. At this writing, an abstraction of this algebra has been used by Birman and Wenzel [BW], and one very beautiful systematization of it by Yetter [Y]. For our purposes we shall write such diagrams up to regular isotopy. Thus we do not have relations $\sigma_i h_i = h_i$ since (for example) $\sigma_1 h_1$ has diagram

and it requires a type I move to obtain the cancellation.

And since diagram multiplication does yield an extra loop d when squaring h_i, we hope to retain the relation $h_i^2 = dh_i$. Let M_n denote this (multiplicative) extension of braids via the h_i's. Obviously, we want a better formal definition of M_n, but first consider some examples:

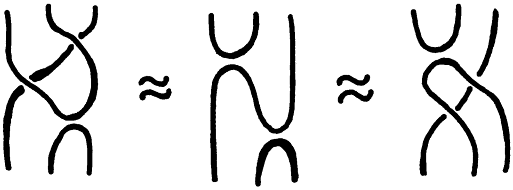
1°.  $\sigma_1 h_2 h_1 = \sigma_2^{-1} h_1.$

This is a fundamental type of mixed relation. Note how the pairing of maxima/minima to produce the h_1 on the left-hand-side comes from different arcs than on the right-hand-side!

2°. 

$\sigma_2\sigma_3\sigma_1\sigma_2h_1h_3 = h_1h_3$.

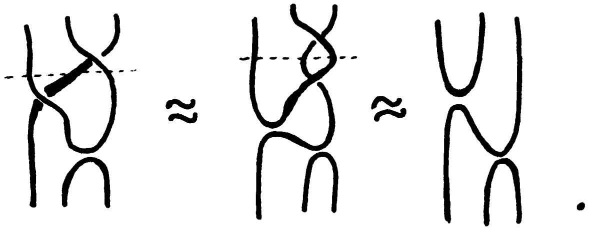
We are allowing regular isotopy of the strands relative to the endpoints and to the (vertical) sides of the box in which the tangle sits.

3°. 

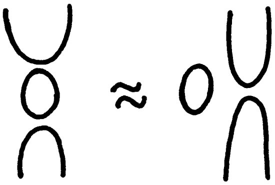
$\sigma_2\sigma_1h_2 = h_1h_2 = h_1\sigma_2\sigma_1$

It may begin to look like there is a myriad of possible relations in M_n. This is true, but the kind of relation illustrated in example 1°. plus the usual braiding and h_i relations is sufficient to generate the others. (For details see [Y] and compare with [K6] and [K7].)

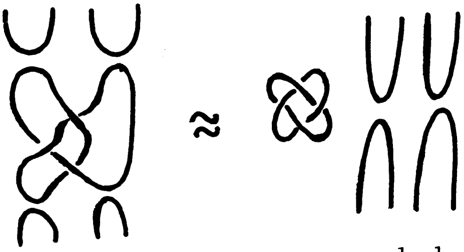
For example:



Thus one can actually give generators and relations for M_n, just as for the classical braid group. But in order to do so a decision must be made about handling appearances of closed loops. We take



bringing closed loop forms outside the rectangle, then it is natural to move whole knots and links in and out as in:



Thus in this formulation, we can write $h_1h_3\sigma_2\sigma^{-1}\sigma^{-1}\sigma_2h_1h_3 = \wedge h_1h_3$ where $\wedge$ is the Hopf link. By allowing multiplication by disjoint union with knots and links, we go beyond a simple set of generators and relations. However the formalism is useful in some contents. For example, using [K] = d<K> we have $[K \sqcup K'] = [K][K']$ so that the square bracket preserves this outer multiplicative structure.

Finally, the outer form of multiplication then fits in with a generalized tensor formalism (see [P]) with two types of multiplication corresponding to ordinary tensor product and to different forms in index contraction such as

matrix multipilication. This if  $\leftrightarrow A_{\alpha}^{\beta}$ and  $\leftrightarrow B_{\alpha'}^{\beta'}$
then  $\leftrightarrow A_{\alpha}^{\beta} B_{\alpha'}^{\alpha}$ (meaning the sum over all occurrences of α by the summation convention). Connection by a single connecting line corresponds to ordinary matrix multiplication. Connection using multiple connecting lines can correspond to multiplication in a tensor product.

We can regard  and  as diagrams for matrices $M_{\alpha\beta}$ and $M^{\alpha'\beta'}$ respectively. Then  corresponds to the tensor product $h = M_{\alpha\beta}M^{\alpha'\beta'}$, and $h^2 = M_{\alpha\beta}M^{\alpha'\beta'}M^{\alpha\beta}M_{\alpha''\beta''} = (M_{\alpha\beta}M^{\alpha\beta})h = \Delta h$. This formalism corresponds directly to the diagram for h^2 with  corresponding to the scalar $\Delta = M_{\alpha\beta}M^{\alpha\beta}$.

In this way our diagram algebra can be interpreted as the underlying structure for specific matrix representations of the multiplicative structure of the h_i's. See [K7] for a complete exposition of this.

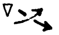
In this last comment we have informally presented two points of view about the extended braid-like multiplicative structures that appear so naturally from the braid-states, By restricting to internal multiplication (matching upper and lower strands) one obtains significant generalizations of the Artin braid group. Adding outer multiplication by closed forms (knots and links) creates close correspondences with representations and abstract tensor products.

V. GENERALIZED POLYNOMIALS.

There are, at present, two two-variable generalized polynomial invariants for knots and links, each a generalization of the Jones polynomial. These are the Homfly polynomial and the Kauffman polynomial. In these notes I will denote the Homfly polynomial by $P_K(\alpha,Z)$ and the Kauffman polynomial by $F_K(\alpha,Z)$. In this section we will just briefly touch on the formalisms of these polynomials. And I shall begin by recalling the Conway polynomial, and telling a bit of the tale leading from Alexander to Conway to generalized polynomials.

In the beginning [A] was Alexander and his invention/discovery of the Alexander polynomial $\Delta_K(t)$. Alexander probably discovered this polynomial by thinking about covering spaces, but his paper was strictly combinatorial, using linear algebra, determinants and the Reidemeister moves. He showed that if two oriented knots or links K, K' are ambient isotopic then $\Delta_K(t) \doteq \Delta_{K'}(t)$ where $\doteq$ means equal up to a multiple of $\pm t^n$ for some integer n. The polynomial was seen to be quite good at distinguishing knots and links, although it did not distinguish a knot or link from its mirror image.

The Alexander polynomial has been an extraordinary and useful tool in knot theory since its inception. Attempts to model and reformulate it led to much new work and different points of view. One of the most notable of these approaches is R.H. Fox's [CF], discovery of the free differential calculus, a technique for extricating the Alexander polynomial from any presentation of the fundamental group of its complement. Then in 1970 John Horton Conway published a remarkable paper [Con] in which he showed that the Alexander polynomial could be sharpened to an invariant with a simple recursive definition. The Conway polynomial, $\nabla_K(z)$, is determined by the conditions:

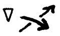
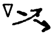
1. ∇  $-\ \nabla$  $= z\nabla$ 

2. ∇  $= 1$

3. $\nabla_K(z) = \nabla_{K'}(z)$ whenever K and K' are ambient isotopic.

Conway explained that his polynomial was related to the Alexander polynomial by the formula $\Delta_K(t) \doteq \nabla_K(\sqrt{t} - 1/\sqrt{t})$. (Note $\doteq$ means equality up to a factor of $(\pm 1)t^n$ for some integer n.)

Eight years later, Conway became enthusiastic once again about this polynomial and he lectured about it in a number of places. This time people heard him and their interest led to some papers about the polynomial (see e.g., [K1], [G], [Co]). The focus was primarily on how to use this recursive scheme, and on understanding the relation to the Alexander polynomial. Some use was made of the extra information in the Conway polynomial. (It can distinguish many links with even number of components from their mirror images.) This author wrote a monograph [K2] on combinatorial and diagrammatic work related to $\nabla_K(z)$. In particular, I found a states model for $\nabla_K(z)$ with a state-summation that is a bit more intricate than our model for the bracket. This model allowed a new proof of the theorems of Murasagi and Crowell ([M1], [Cr]) on the genus of alternating knots, and a generalization of these results to a category I called alternative knots.

But curiously, no one tried to generalize Conway's recursive scheme itself. No one asked what would happen if the first formula were modified to (say) ∇  $+\ \nabla$  $= z\nabla$  ? And then in 1984 Vaughan Jones lectured on his new invariant, derived from a representation of the Artin braid group into a von Neumann algebra [J2]. And Vaughan proved (among other things) that his (Laurent) polynomial satisfied an identity

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V_{L_0}(t).$$

With this formula standing in juxtaposition to the Conway formula, a number of people lept at once to the generalization

$$\alpha^{-1}P_{L_+} - \alpha P_{L_-} = zP_{L_0},$$

giving a two-variable polynomial $P_K(\alpha,z)$ specializing both to the Conway $(\alpha = 1)$ and the Jones $(\alpha = t, z = \sqrt{t} - 1/\sqrt{t})$ polynomials. This is the Homfly polynomial [HOMFLY].

Some time passed, and then Brandt, Lickorish, Millet and (independently) Ho found ([BLM], [H]) yet another new invariant polynomial $Q_K(z)$ satisfying $Q_{L_+} + Q_{L_-} = z(Q_{L_0} + Q_{L_\infty})$ for unoriented links. This is a one-variable polynomial, distinct from the Homfly polynomial. It does not distinguish mirror images.

I then had the good fortune to recognize how to put another variable into the context of the Q-polynomial ([K4], [K6]). The idea is to work in the regular isotopy category (as we explained for the bracket) and let a polynomial L be defined via:

1) $L_{L_+} + L_{L_-} = z(L_{L_0} + L_{L_\infty})$

2) $L_{\text{(positive curl)}} = \alpha L$

$L_{\text{(negative curl)}} = \alpha^{-1}L$

3) $L_{\bigcirc} = 1$

4) $L_K = L_{K'}$ whenever K and K' are regularly isotopic.

Then L is normalized to form an invariant of ambient isotopy for oriented knots and links via the equation

$$F_K = \alpha^{-w(K)} L_K$$

where $w(K)$ is the twist number of the diagram K. The polynomial F_K turns out to be quite good at distinguishing knots and links from their mirror images. It appears to be a proper companion to the Homfly polynomial $P_K(\alpha,z)$. (The reader should note that our names of polynomials by letter and variable choice may differ from those given elsewhere in the literature. The translations are always straightforward.) Ocneanu and Jones discovered how to put a trace on the Hecke algebra generated by elements c_i satisfying braiding relations and the Conway-type relation

$$c_i - c_i^{-1} = z$$

to produce the Homfly polynomial in a fashion analogous to the representation for $V_K(t)$. Hugh Morton worked extensively with this algebra, producing very good programs for computing the Homfly polynomial for braids. Morton continues doing deep theoretical work related to these polynomials. (See [M]).

Birman and Wenzel [BW] have given a similar treatment for the Kauffman polynomial by using an algebra with relations $c_i + c_i^{-1} = z(1 + E_i)$ where c_i corresponds to a braid generator and E_i shows the formal properties of our h_i's. More will come of this. David Yetter [Y] has given a good general content for diagram-related algebras. See also [K5], [K6], and [K7].

Both of the two-variable generalized polynomials have the Jones polynomial as a special case. For F_K this is most easily seen via adding:

$$< \; $$$$ \; > + < \; $$$$ \; > = (A + A^{-1}) \, (< \; $$$$ \; > + < \; $$$$ \; >).$$

$$< \; $$$$ \; > = \alpha < \; $$$$ \; >$$

$$\alpha = -A^3.$$

$$< \; $$$$ \; > = \alpha^{-1} < \; $$$$ \; >$$

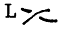
Thus $\langle K \rangle = L_K(-A^3, A + A^{-1})$ and $f_K = F_K(-A^3, A + A^{-1})$. Since $V_K(t) = f_K(t^{-1/4})$ we conclude that $V_K(t) = F_K(-t^{-3/4}, t^{-1/4} + t^{1/4})$. This was observed by Lickorish [L] by a different route.

The two two-variable polynomials can be extablished via direct inductive definition. It is an open question whether there exist models for the workings of these polynomials that connect them directly with geometry beyond the geometry of diagrams. I believe that such connections will come about, and that they will be of great importance for topology as a whole.

Remark. Thistlethwaite's simple proof [T2] of the invariance of the writhe for reduced alternating diagrams uses the Kauffman polynomial: He observes that for L_K with K reduced alternating, the highest term in z has coefficient $k(a+a^{-1})$, $k > 0$ and power z^{n-1} where n is the number of crossings in K. Thus $L_K = k(a + a^{-1})z^{n-1}$ + (other lower degrees in z) for K reduced alternating. Since $F_K = \alpha^{-w(K)}L_K$ is an ambient isotopy invariant, it follows at once that $w(K)$ is also an ambient isotopy invariant. The proof of Morwen's observation is a direct structural induction:

(i) If  is reduced alternating, then  and  are both alternating, and at least one is reduced.

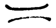
(ii) Use (i) and the recursion

$$L \; $$$$ \; + L \; $$$$ \; = z(L \; $$$$ \; + L \; $$$$ \;).$$

The separation of the α and z variables is the crucial ingredient in the proof.

VI. GRAPHS AND STATISTICAL PHYSICS.

Recall that in section III, I defined a generalized bracket polynomial for diagrams so that $[K] \in Z[A,B,d]$ and

1. [] = A[] + B[ )( ]

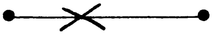
2. $[0\ K] = d[K]$

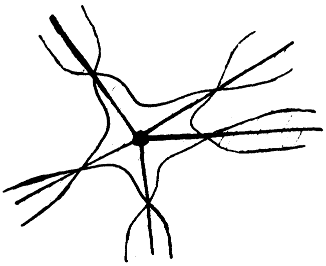
   $[0] = d.$

We then created an invariant of regular isotopy via $B = A^{-1}$, $d = -A^2 - A^{-2}$ and $\langle K\rangle = d^{-1}[K]$ (so that $\langle 0\rangle = 1$). The square bracket can be specialized in other ways. In particular, it is (for the right choice of A and B) the dichromatic (Whitney-Tutte) polynomial for a planar graph. This in turn can be seen to be a way of expressing the partition function for the Potts model (a generalization of the Ising model) in statistical physics [B].

To understand this connection it is important to realize the

Theorem 6.1. Universes are in one-to-one correspondence with planar graphs.

Proof: To each universe, shade it so that the unbounded region is unshaded (i.e., 2-color the regions). Associate a graph to U, $\Gamma(U)$, so that the vertices of $\Gamma(U)$ correspond to the shaded regions of U and the edges correspond to crossings shared by shaded regions. Given a graph G, associate to it a universe V(G) by placing a crossing of the form  on each edge of G and connecting these crossings at each vertex as shown in Figure 20. It is easy to verify that $\Gamma(V(G)) = G$ and $V(\Gamma(U)) = U$. This completes the proof.



connection at a vertex

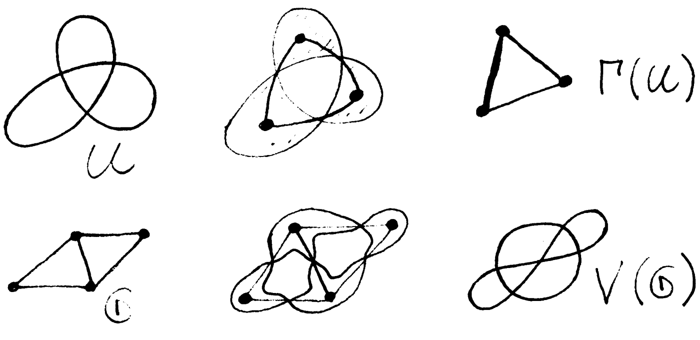


Figure 20

The dichromatic polynomial $Z_G(q,v) \in Z[q,v]$ is defined for graphs G by the recursive formulas:

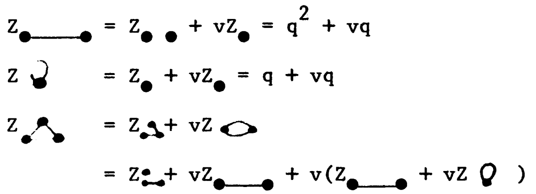
1) $Z_{\_\_\bullet\_\_\bullet\_\_} = Z_{\_\_\bullet\ \ \bullet\_\_} + vZ_{\_\_\bullet\_\_}$

2) $Z_{\bullet G} = qZ_G$

$Z_{\bullet} = q$

The first formula asserts that the value of Z on a graph G is equal to the sum of the value of Z on a graph G' obtained by deleting one edge from G plus v multiplied by the Z for G'', the graph obtained by collapsing this edge to a point. The second formula asserts that the addition of an extra vertex to a graph G multiplies the dichromatic polynomial by q, and that the value of Z for an isolated vertex is q.

Examples:

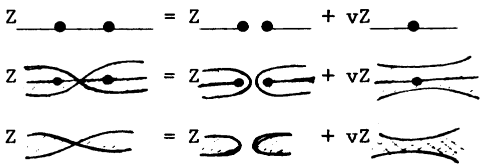


$= q(q^2 + vq) + v(q^2 + vq) + v(q^2 + vq) + v^2(q + vq)$

$= (q + 2v)(q^2 + vq) + v^2(q + vq).$

For $v = -1$, the dichromatic polynomial specializes to the chromatic polynomial. That is, $Z_G(q,-1) = K_G(q)$ is the number of ways to vertex-color the graph G with q colors so that no two adjacent vertices receive the same color. That this is so easily seen from the recursion formula since K counts all possibilities for these two vertices, while K corresponds to those cases where the two vertices receive the same color.

Now consider how the recursion formula (1) diagrams:

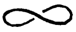


We see that deletion and contraction in the graphs become the two ways of splicing the crossing in the knot diagrams (universe). And the expansion for Z is formally a bracket expansion.

Some further translation is then required to actually re-write Z_G as a bracket. First let $K(G)$ be the alternating link diagram associated with $V(G)$ so that all shaded crossings are of type A. Then

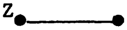
__Theorem 6.2__. $Z_G(q,v) = q^{N/2}[K(G)]$ where N is the number of vertices of G, and the bracket is expanded with $A = q^{-1/2}v$, $B = 1$, $d = q^{1/2}$ so that [] $= q^{-1/2}v$[] + [ ) ( ] and $[0] = q^{1/2}$.

__Example__. K() = 

$q^{N/2}$[] $= q(q^{-1/2}v$[] + [])

$= q(q^{-1/2}vq^{1/2} + (q^{1/2})^2)$

$= qv + q^2$

q[] = Z 

Thus we see that the square bracket is fundamentally related both to knot theory and to graph theory. This connection raises many questions. We would like to know whether qualitative information can be transferred between these two subjects. The square bracket gives a picture of a parameter space A,B,d and subvarieties along which [K] is topological or dichromatic. More work is needed here.

Just to complete this picture I shall explain how the partition function for the Potts model in statistical physics is a dichromatic polynomial (see [B], [K7]). The partition function has the form

$$Z_G = \sum_{\sigma} e^{-E(\sigma)}$$

where σ runs over all "states" of the lattice G (we will let G be a planar graph) and $E(\sigma)$ is the energy of the given state. In the Potts model the energy has the form

$$E(\sigma) = \frac{1}{kT} \sum_{<i,j>} \delta(\sigma_i, \sigma_j)$$

where $<i,j>$ denotes an edge of G with vertices i,j and σ_i and σ_j are the state's assignments to these vertices. We assume that each vertex can be freely assigned one of q values, and that a state σ is such an assignment. In this formula δ is the Kronecker delta

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

and T is the <u>temperature</u> of the system, while k is a constant (Boltzman's constant).

The partition function has many uses in this subject. For example, the probability of being in a given energy state E is

$$p(E) = e^{-E}/Z_G.$$

<u>Proposition 6.3</u>. For $E(\sigma) = \frac{1}{kT} \sum_{<i,j>} \delta(\sigma_i, \sigma_j)$ and q local states, let

$$v = e^{-\frac{1}{kT}} - 1.$$

Then <u>the partition function is the dichromatic polynomial in</u> q <u>and</u> v:

$$\sum_{\sigma} e^{-E(\sigma)} = Z_G(q,v).$$

<u>Proof</u>: $$\sum_{\sigma} e^{-E(\sigma)} = \sum_{\sigma} e^{-\frac{1}{kT}\sum_{<i,j>} \delta(\sigma_i,\sigma_j)}$$

$$= \sum_{\sigma} \prod_{<i,j>} \left(e^{-\frac{1}{kT}\delta(\sigma_i,\sigma_j)}\right)$$

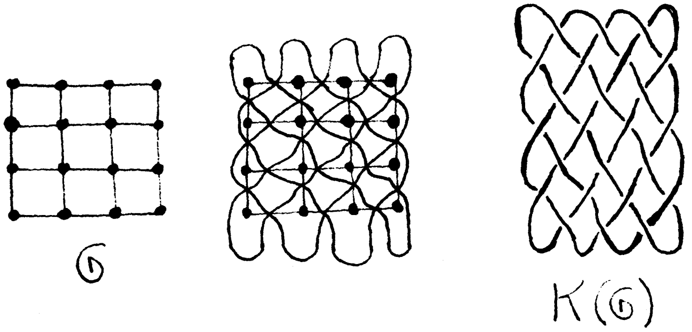
$$\sum_{\sigma} e^{-E(\sigma)} = \sum_{\sigma} \prod_{<i,j>} (1 + v\delta(\sigma_i,\sigma_j)).$$

It is easy to see that the right hand side of this equation satisfies the recursion relation for the dichromatic polynomial in q and v.

$$(Z_{—\bullet—\bullet—} = Z_{—\bullet\quad\bullet—} + vZ_{—\bullet—})$$

$$(Z_{\bullet\ G} = qZ_G)$$

Combining 6.2 and 6.3 we see that the partition function is a bracket expansion. This gives a theoretical explanation for the appearance of the algebra of the h_i's (see section 4) in the structure of the Potts model for the square lattice. (See [B].) This lattice corresponds to the plat closure (see Figure 21) of a particular braid. Any bracket evaluation for a braid is expressed in terms of this operator algebra. It remains to be seen whether the bracket formulation for the Potts model will shed light on its physics. The relationship between the Potts model and the Jones polynomial via its operator algebra was first observed by Vaughan Jones. Our formulation shows the direct connection through translating graphs and link diagrams.



Translating the Square Lattice

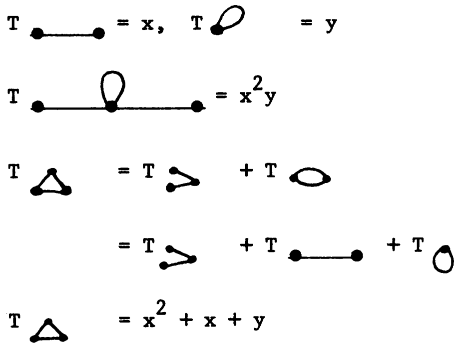
Figure 21

VII. THE BRACKET AND THE TUTTE POLYNOMIAL.

In the last section we showed how the bracket could be used to give the dichromatic polynomial for planar graphs. Here we shall reformulate the state expansion of the general bracket function, showing that it can be calculated solely from states with a single component. Our reformulation generalizes work of Morwen Thistlethwaite. He showed [Tl] how to do this for the bracket and the Jones polynomial. The ideas go back to a generalization of the dichromatic polynomial known as the Tutte polynomial. (See [Tu]). The Tutte polynomial is defined recursively as follows.

To each graph G is associated a polynomial $T_G(x,y) \in Z[x,y]$. If G is composed solely of isthmuses and loops then $T_G = x^i y^\ell$ where i is the number of isthmus and ℓ is the number of loops. The polynomial satisfies the recursion $T_G = T_{G'} + T_{G''}$ where G' and G'' are the graphs obtained by deleting and contracting (respectively) an edge that is neither a loop nor an isthmus.

Examples:

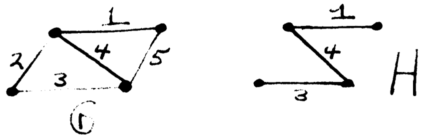


The dichromatic polynomial $Z_G(q,v)$ (see section 6) is related to the Tutte polynomial $T_G(x,y)$ by the formula $Z_G(q,v) = qv^{N-1}T_G(1 + qv^{-1}, 1 + v)$ where N is the number of vertices of G. For example, $qv(1 + q/v) = qv + q^2 = Z$  and $q(1 + v) = q + qv = Z$  show that the formula is correct for a single loop and isthmus. This formula shows that the dichromatic polynomial and the Tutte polynomial determine one another. Thus

$$T_G(x,y) = \frac{1}{(x-1)(y-1)^N} Z_G((x-1)(y-1),(y-1)).$$

Tutte proved a remarkable theorem showing that his polynomial could be computed from weightings assigned to the maximal trees of the graph G. This weighting is dependent upon an ordering of the edges of G, but $T_G(x,y)$ is independent of the particular ordering.

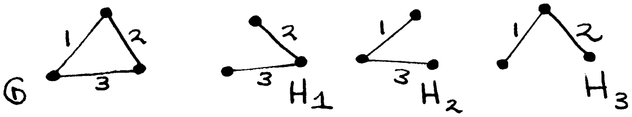
<u>Definition</u> <u>7.1</u>. Let G be a graph whose edges have been labelled $1,2,3,\ldots,n$. Let $H \subset G$ be a maximal tree in G. Let $i \in \{1,2,\ldots,n\}$ denote an edge of H. Let H_i denote H - (the i^{th} edge). Since H is a maximal tree, H_i has two components. One says that i is <u>internally active</u> if $i < j$ for every edge j in $G-H$ and endpoints in both components of H_i. Let $i \in G-H$ be an external edge. One says that i is <u>externally</u> <u>active</u> if $i < j$ for all edges j on the cycle in H extending from one end of i to the other.

Example: 

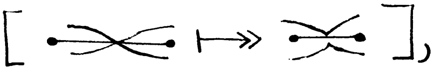
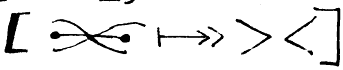
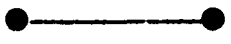
Here the edge labelled 1 is internally active. Edge number 2 is externally active.

Theorem(Tutte). Let $\mathcal{H}$ denote the collection of maximal trees in a graph G. Let $i(H)$ denote the number of internally active edges in G, and $e(H)$ the number of externally active edges in G (with respect to a given tree H). Then the Tutte polynomial is given by the formula

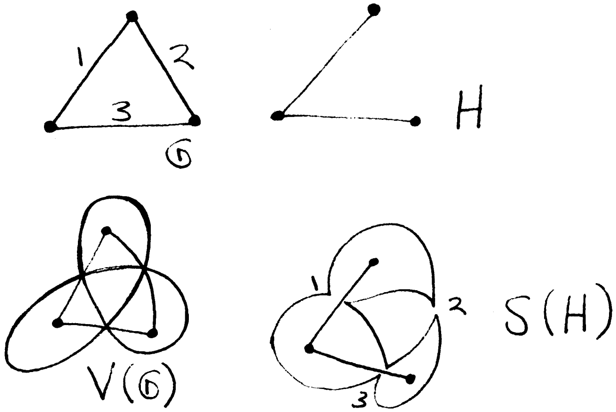
$$T_G(x,y) = \sum_{H \subset \mathcal{H}} x^{i(H)} y^{e(H)}.$$

Example: 

$$T_G = y + x + x^2.$$

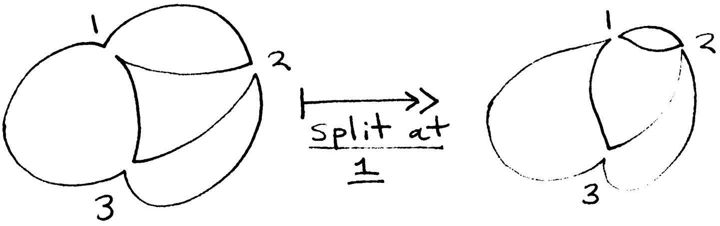
It may seem from the definitions of internal and external activity that they are somewhat different. Actually there is a symmetry of definition for planar graphs. It is through this symmetry that I like to see the relationship of the Tutte weightings with universes and with knot theory. To see this symmetry consider the universe V(G) (sometimes called the medial graph) associated with a planar graph G. Each maximal tree $H \subset G$ determines a state $S = S(H)$ of V(G) with one component. This state is obtained by splitting V(G) along the edges of H , and splitting all other crossings in the opposite fashion  (where  is not in H).

Example:



The edge labelling of G becomes a vertex labelling of V(G). Call the vertices (crossings) of S(H) internal or external according as they are split with cusps pointing to the inside or to the outside of the Jordan trail S(H). Thus in the example above, 1 and 3 are internal and 2 is external. Given a Jordan trail S (a universe with $|S| = 1$) with vertices (sites ><) labelled $1,2,\ldots,n$, call a site i active if $i < j$ for all sites j with cusps in the two components resulting from splitting S at i. (Compare [K2].)

Example:

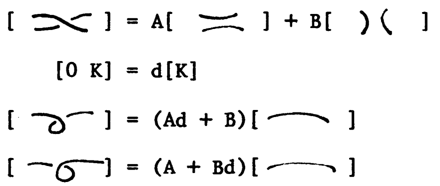


Since $1 < 2$, 1 is active.

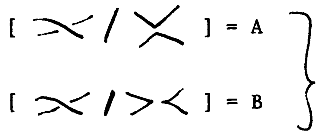
A site on a trail is <u>internally active</u> if it is internal and active. A site is externally active if it is external and active. By replacing the trees in G by Jordan trails on V(G), we obtain a symmetrical definition of $T_G(x,y)$.

Incidentally, it is easy to see from this reformulation that $T_G(x,y) = T_{\bar{G}}(y,x)$ where $\bar{G}$ is the planar dual graph to the planar graph G. Each Jordan trail gives a pair of maximal trees, one for G and one for $\bar{G}$.

By now we are very close to the knot theory, and I can explain how to calculate the square bracket, [K] for link diagrams by using Tutte weightings. Recall that [K] has variables A,B,d and that

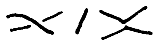
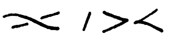


Let $\alpha = Ad + B$ and $\beta = A + Bd$. Now let K be a given diagram, and S be a state for K with one component ($|S| = 1$). Let $\mathcal{S}$ denote the collection of all states S with one component. Let the crossings of K be labelled 1,2,...,n. Each crossing of K will determine a local contribution at the corresponding site of S. If the site is <u>inactive</u> we retain the usual bracket contribution:



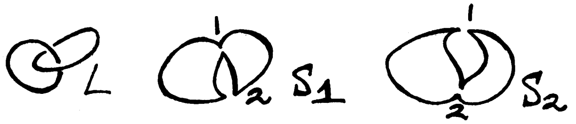
<u>inactive site</u>.

If the site is active then we take

$[$  $] = \beta$
$[$  $] = \alpha$
} active site.

Then $[K|S]$ is the product of all these local contributions, and we assert that $[K]$ is given by the formula $[K] = \sum_{S \in \mathcal{S}} [K|S]$.

Example 1.



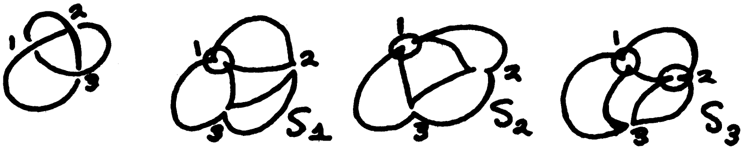
$[L] = \sum_{S} [L|S] = \beta B + \alpha A$ (1 active in S_1, 1 also active in S_2). Doing this the long way we have,

$[$  $] = A[$  $] + B[$  $]$

$= A(Ad + B) + B(A + Bd)$

$= A\alpha + B\beta.$

Example 2.

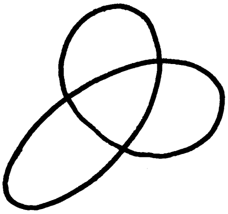


$[K] = \sum_{S} [K|S] = \beta BA + \alpha A^2 + \beta^2 B.$ (Circled sites are active.)

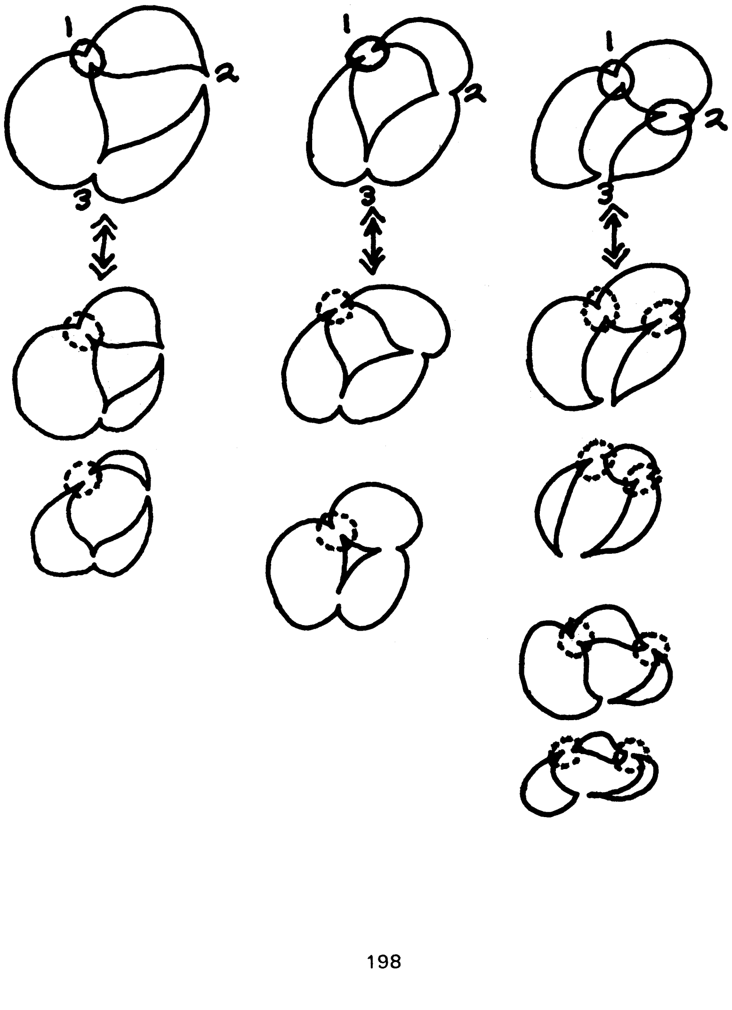
Note that for $\langle K \rangle$, $\alpha = -A^3$, $\beta = -A^3$, $B = A^{-1}$ so that

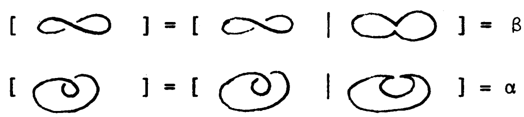
$$\langle K \rangle = -A^{-3} - A^5 + A^{-7},$$

our familiar value for the bracket of the trefoil. (See section 2.)



Exercise. Contemplate this example, and give a direct proof of the expansion of the bracket via connected states. (Hint: Show that each connected state gives rise to a collection of possibly disconnected states by re-splicing at some or all of the active sites.) This exercise can be expanded into a new approach to the theory of the Tutte polynomial. See [K9], [K10].

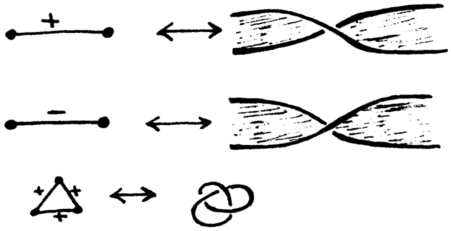


Example 3: 

These correspond graphically to basic contributions for isthmus and loop. Note however, that if we switch the crossing then the contributions flip (since we are cataloging the type of curl in the knot diagram).

Warning: It should be clear to both author and the reader by now that the square bracket function we are now using is normalized to 1 on the circle: [O] = 1. Let this cause no difficulty with regard to our earlier convention.

Example 4: This is really a reformulation. Note that diagrams of knots and links are in 1-1 correspondence with signed planar graphs where the signs are placed on the edges of the graph so that + corresponds to an A-channel, - corresponds to a B-channel.

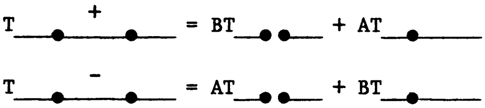


Our Tutte-reformulation of the generalized bracket then gives a generalized Tutte polynomial for signed graphs satisfying

1) If G has i_+ positive isthmus, i-negative isthmus, ℓ_+ positive loops, ℓ-negative loops, then

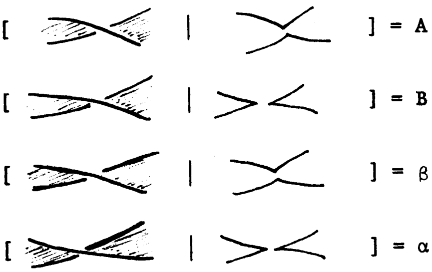
$$T_G = X^{i_+ + \ell_-} Y^{i_- + \ell_+}.$$

2) If the edge indicated below is not an isthmus or loop, then



Thus $T_G(A,B,x,y)$ is a graph theoretic version of the square bracket. ($X = A + Bd$, $Y = Ad + B$ recovers previous notation). It has a Tutte expansion in terms of spanning trees, and should be explored for its own sake. (N.B. $AX + B^2 = A^2 + BY$, and this condition is equivalent to stating that X has the form $A + Bd$, while Y has the form $Ad + B$. The polynomial is well-defined for A, B, X, Y satisfying this relation.)

Example 5: If K is an alternating diagram then all crossings have the same internal type. Thus the contributions take the form



inactive site

active site

From this it is easy to see some specifics about the topological bracket where $B = A^{-1}$, $\alpha = -A^3$, $\beta = -A^3$. Note that $\alpha = -A^4(A^{-1})$, $\beta = -A^{-4}(A)$. Thus for K alternating, we have

$$\langle K\rangle = A^{I-E}T_{G(K)}(-A^{-4}, -A^4)$$

where T is the (standard) Tutte polynomial and I denotes the number of internal sites on a trail, E the number of external sites on a trail. We note that I- E is a constant independent of the given choice of trail.

Now recall that the ambient isotopy invariant f_K is given by the formula

$$f_K = \alpha^{-w(K)}\langle K\rangle = (-A^{-3})^{w(K)}\langle K\rangle$$

and that the Jones polynomial $V_K(t)$ is given by the formula

$$V_K(t) = f_K(t^{-1/4}).$$

Thus for K alternating we have:

Theorem (Thistlethwaite). Let K be an alternating projection, G(K) the corresponding planar graph. Then the Jones polynomial $V_K(t)$ is equal to the Tutte polynomial $T_{G(K)}(-t, -t^{-1})$ up to a sign and factor a power of t. This is a remarkable observation from which it is now easy to deduce such facts as: the coefficients of the Jones polynomial of an alternating link alternate in sign (according to parity of degree). That is, the Jones polynomial of an alternating link is an alternating polynomial. The reader is referred to Thistlethwaite's paper for more details ([T1]).

VIII. FROM GRAPH THEORY TO KNOT THEORY.

It is interesting to speculate about alternate realities. How could the bracket polynomial, and hence the Jones polynomial have emerged from graph theory ?! One possible reconstruction is to suppose that graph theory had had in its possession our generalized Tutte polynomial for signed graphs.

Since knots and links can be encoded into signed graphs, it would then have been possible to look for a specialization of this Tutte polynomial that gives an invariant.

To taste the flavor of this reconstruction we must first examine the graph theoretic versions of the Reidemeister moves. View Figure 22 for this. Because of the translation to graphs via shaded regions, there are two versions of the type II move, and two versions of the type III move. Note that the two versions of the type II move are signed forms of deletion and contraction, while the type I move involves addition or removal of a branch or loop. These relationships could have been taken as a hint to try a Tutte polynomial for an invariant.

In any case, let's do this. We begin with the generalized Tutte polynomial for signed graphs, as explained in example 4 of the last section. This assigns a polynomial $T_G(A,B,x,y)$ to any signed graph G.

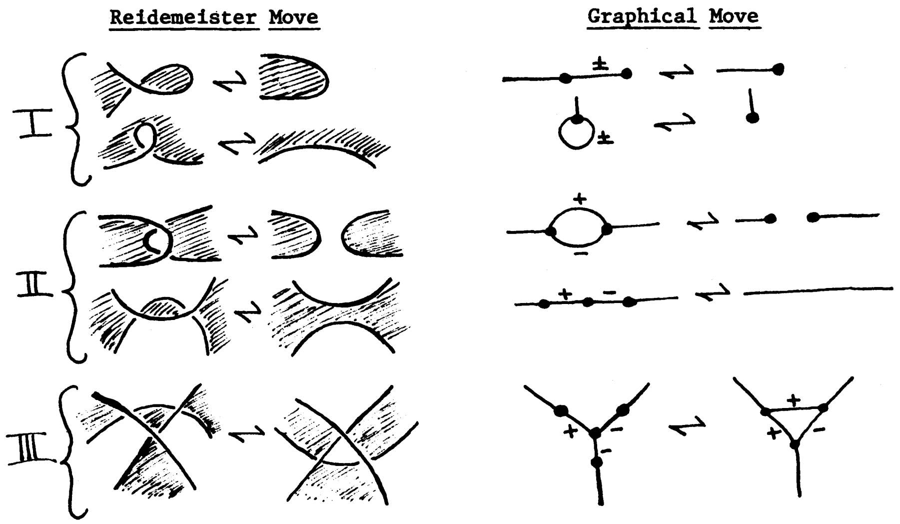
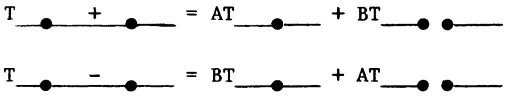


Figure 22

It is characterized by the rules: $(x = Ad + B,\ y = A + Bd)$.

1) If G has only isthmus and loops, then $T_G = x^{i_+ + \ell_-} y^{i_- + \ell_+}$ where i_+ is the number of positive isthmus, i_- is the number of negative isthmus, ℓ_+ is the number of positive loops, ℓ_- is the number of ngative loops.

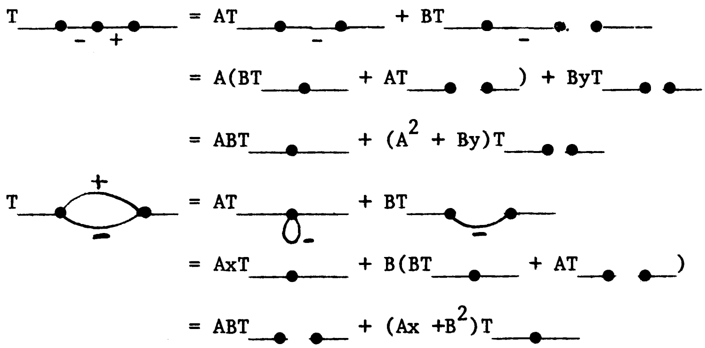
2) 

(the $\pm$ edge is not an isthmus.)

Let's investigate the behaviour of this polynomial under type II moves.

<u>Proposition 8.1</u>. In order for $T_G(A,B,x,y)$ be invariant under type II moves it is necessary and sufficient that

$$\begin{cases} B = A^{-1} \\ x = -A^{-3} \\ y = -A^{3}. \end{cases}$$

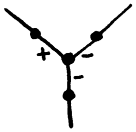
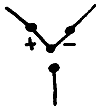
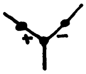
<u>Proof</u>: 

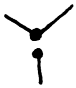
The rest of the proof follows from these identities.

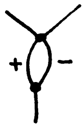
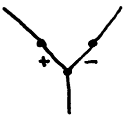
The rest of the story now proceeds just as for the bracket. With $B = A^{-1}$, $x = -A^{-3}$, $y = -A^{3}$ the polynomial T_G becomes an invariant of moves II <u>and</u> III for arbitrary graphs.

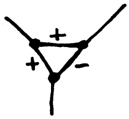
<u>Definition 8.2</u>. For any connected graph G, let $\Omega \in Z[A,A^{-1}]$ be the Laurent polynomial defined by $\Omega_G = T_G(A,A^{-1},-A^{-3},-A^{3})$. By Proposition 8.1, Ω_G is an invariant of graphical move II. Type III invariance is free:

<u>Proposition 8.2</u>. $\Omega_G = \Omega_{G'}$ if G and G' are related by a type III move.

<u>Proof</u>: Ω  $= A\Omega$  $+ B\Omega$ 

(type II) $= A\Omega$  $+ B\Omega$ 

(type II) $= A\Omega$  $+ B\Omega$ 

$= \Omega$ 

Remark: We want to be able to perform the type II move even when it disconnects the graph. Hence we need a formula for $T_{G \sqcup G'}$, where $\sqcup$ denotes disjoint union. Note that T  $= Ax + By$. Define $T_{G \sqcup G'} = (Ax + By)T_G T_{G'}$. It is then easy to see that Ω has the right invariance properties for this move. Note that for $B = A^{-1}$, $x = -A^{-3}$, $y = -A^3$ we have $Ax + By = -A^{-2} - A^2 = d$, the corresponding bracket factor.

This section is just intended as a sketch of the graph-theoretical formulation. Note that Ω_G is an invariant of type II and type III moves for arbitrary (not necessarily planar) signed graphs. This is likely to be a useful extension of knot theory to arbitrary neyworks involving those transformations.

What about the type I move? ΩG multiplies by x or y under a type I move. In the abstract graph theoretic setting we do not have a direct analog of the twist number. Thus it remains to be seen whether ΩG can be normalized to form an invariant of all three move types.

To finish the translation we state the now obvious

Theorem 8.3. Let G be a planar signed graph. Let $K(G)$ be the knot/link diagram corresponding to G. Then $\langle K(G) \rangle = \Omega_G$. The bracket polynomial for knots and links is a specialization of the generalized Tutte polynomial for signed graphs.

Finally, we return to the generalized Tutte polynomial $T_G(A,B,x,y)$ for signed graphs G, and note that it has a spanning tree expansion. Given an ordering of the edges of G and a maximal tree $H \subset G$, define contributions from the edges of G as follows:

internally active, +	x
externally active, −	x
internally active, −	y
externally active, +	y
internally inactive, +	A
externally inactive, −	A
internally inactive, −	B
externally inactive, +	B

Let $G(H)$ denote the product of the contributions of the edges of G relative to activities for H. Then

$$T_G = \sum_H G(H)$$

where this summation extends over all maximal trees in G.

Technical Caveat: It is necessary and sufficient for T_G to be well-defined that $Ax + B^2 = A^2 + By$. (Note that we are in the category of signed graphs.) This certainly holds for the topological case. In the general case we can rephrase this condition, as we did for the square bracket, by introducing a variable d and writing $x = A + Bd$, $y = Ad + B$. This shows that square bracket is directly generalized to arbitrary signed graphs by $T_G(A,B,x,y) = T_G(A,B,d)$.

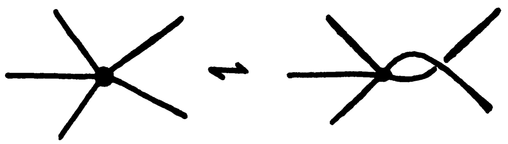
The sufficiency is seen in proving that this polynomial has a spanning tree expansion. (See [K9] and compare [T1].) For necessity expand a triangle graph 

in two different ways. You will find that $AX + B^2 = A^2 + BY$ is needed for agreement.

IX. THE KNOT THEORY OF IMBEDDED GRAPHS.

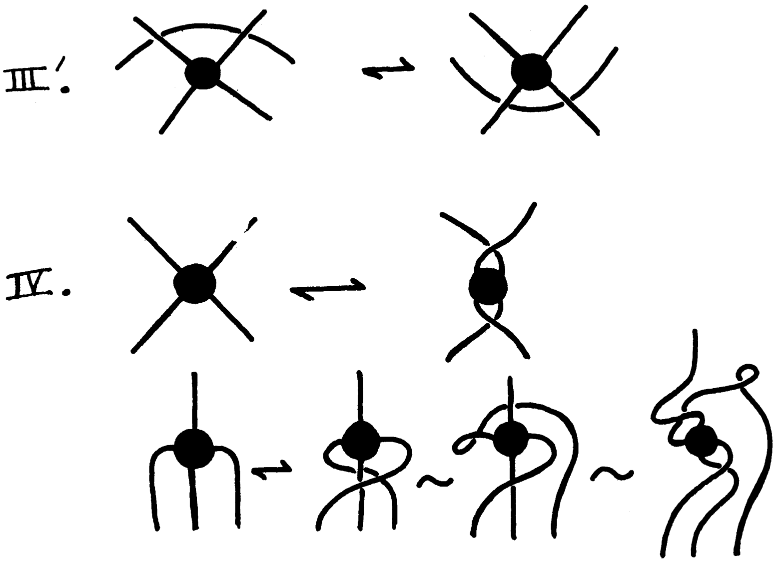
It will not do to mention graphs without pointing out the now active extension of knot theory considering embeddings of arbitrary graphs in Euclidean three space. Here one would like to answer the usual ambient isotopy questions of knot theory in this larger context. In particular, we want polynomial (or other simple) invariants of graphs in space. The most general notion of ambient isotopy for graphs in space allows topological vertices. Strands coming into a topological vertex behave independently. In diagrams, this means that moves are allowed (see Figure 23) that create arbitrary braiding at a vertex. At the vertex, any two adjacent strands can be given a twist. In general this notion of ambient isotopy is both fundamental and difficult. Nevertheless, progress is being made (see [S] and [W]) on the general classification.

One may also consider a rigid vertex. Here the vertex is thought of as a rigid object with topological strands attached at specific sites. Then I note [K8] that it is possible to define some useful invariants in this case. A similar approach was seen independently by Ken Millett. Here is a sketch of my viewpoint about the rigid vertices. We shall restrict ourselves to 4-valent vertices as shown in Figure 24.



A topological vertex move

Figure 23

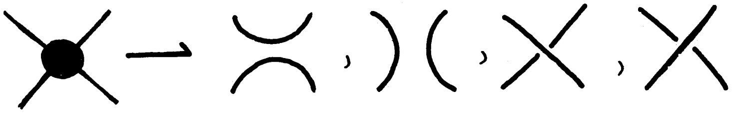


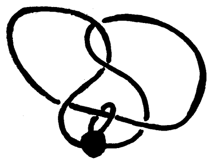
Rigid Vertex Moves

Figure 24

Figure 24 indicates the extra move-types that must be added to the list of Reidemeister moves in order to have a theory of rigid vertex equivalence. Note that the rigidity of the vertex forces double braiding when it is turned by 180°. I have denoted by III' the analog of the type III move. The second move under IV (with a three strand twist) can be accomplished up to ambient isotopy by the first type IV. (We will not consider regular isotopy of graphs here.) My method for obtaining invariants of RV4-graphs (4-valent rigid vertex graphs) is to associate to such a graph G a collection of knots and links $L(G)$ obtained as described below. This can be done for both oriented and non-oriented graphs. Here we consider only non-oriented graphs.

Definition 9.1. Let G be an RV4-graph. Let $L(G)$ be the collection of knots and links obtained from G by choosing one replacement of each of the following types at each vertex of G:



Example: G =  (J. Simon's graph.)

$L(G)$ = 

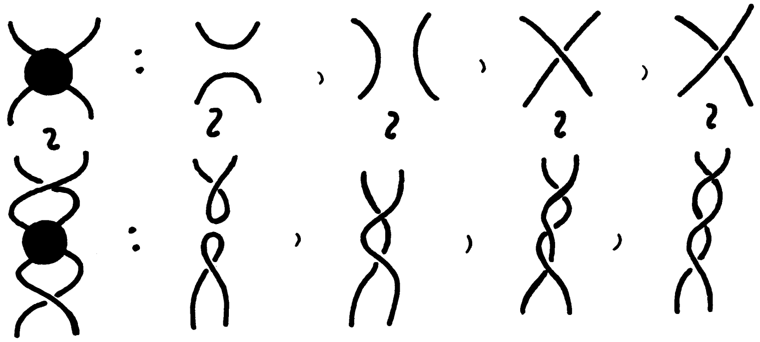
In general, if G has n rigid vertices, then $L(G)$ will contain 4^n diagrams, some trivial, some ambient isotopic.

Definition 9.2. Let X be a collection of knots and links. Two such collections will be said to be ambient isotopic (X ~ X') if every member of the first collection is ambient isotopic to some member of the second collection and vice versa.

The notation ~ will be used both for RV4-equivalence of graphs and for ambient isotopy of knots and links.

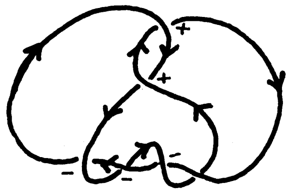
Theorem 9.3. Let G and G' be equivalent RV4-graphs in three dimensional space. Then their associated link collections are ambient isotopic - $L(G) \sim L(G')$.

Proof: Observe that the extra moves III' and IV (Figure 24) preserve the elements of $L(G)$ up to ambient isotopy. For example;

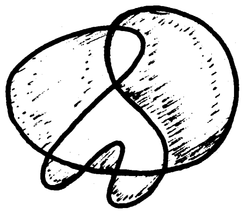


This completes the proof.

This is a very useful theorem for studying RV4-graphs. For example, we can immediately conclude that the graph G in the example above is not equivalent to its mirror image. For if this were so then the individual knots and links in $L(G)$ (being distinct) would have to each be achiral. We can then chekc that this is not the case by using our results about the Jones polynomial for alternating knots. Recall that we have shown that if K is achiral then $3w(K) = W - B$ (after Theorem 3.1) where $w(K)$ is the twist number of K and W and B are the numbers of white and black regions in a shading where the A-channels are shaded. Choosing $K_1 \in L(G)$ we find



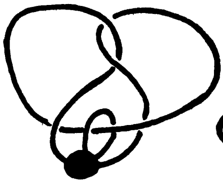
$$w(K_1) = 2 - 4 = -2$$
$$3w(K_1) = -6$$



$$W = 3$$
$$B = 5$$
$$W - B = -2$$

Since $-6 \neq -2$ we conclude the Simon's graph G is not equivalent (RV4) to its mirror image. (Wylbur Whitten has verified by other techniques that this graph is not topologically equivalent to its mirror image.) Note that we could have used the invariance of the writhe for reduced, prime alternating diagrams (sections 3 and 5) for a shorter verification. This one uses only facts completely developed in the present paper.

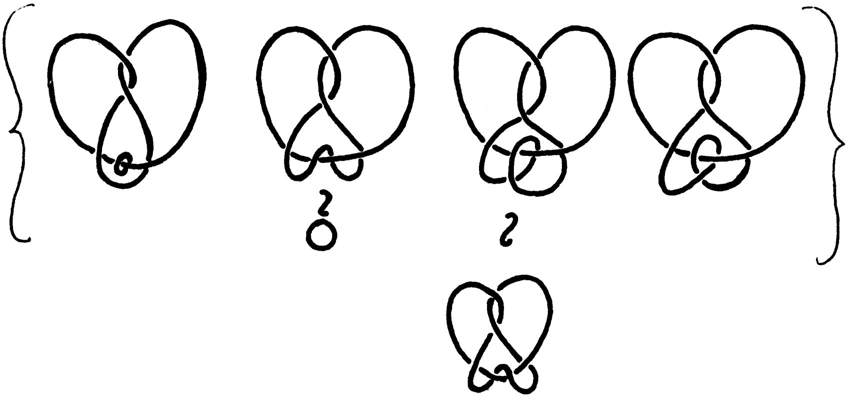
Example:



G'

Examination will reveal that this graph is obtained from the graph G of the previous example by reversing two crossings.

$L(G) =$



We see that to prove G not equivalent to G' it suffices to show that K_2' is not ambient isotopic to K_2. But we also know from section 3 that $\text{span}(K_2') < 4V$ where V is the number of crossings in this diagram, since K_2' is a non-alternating diagram. On the other hand, $\text{span}(K_2) = 4V$ $(V = 7)$ since K_2 is an alternating diagram. Therefore, without further calculation we know that K_2 and K_2' are different and hence that G and G' are not RV4-equivalent graphs.

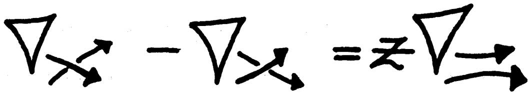
To further verify that G' is chiral requires a calculation of $\langle K_2' \rangle$. We omit this and asser that the calculation indeed shows that K_2' is chiral, and hence that G' is RV4-chiral.

These examples and our theorem relating equivalences of graphs with ambient isotopy for collections of knots and links show how there can be a good collaboration between problems of graph-embedding and new invariants of knots and links such as the Jones polynomial. Since graphs can be used to model the configurations of molecules and other naturally-occuring networks, it is to be expected that there will be many fruitful applications of these ideas.

It is interesting to note that in the case of RV4-graphs there is a mixed - mechanical/topological model that is nevertheless susceptible to a topological analysis. This, in itself, is a good sign for applications where there will always be a mixture of topology and other structures.

X. PATTERNS AND SPECULATIONS.

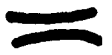
I always thought that the Conway identity



bore a striking resemblance to the exchange identities of quantum physics such as the Heisenberg form of the uncertainty principle $PQ - QP = \hbar i$. And that the crossing  and its reverse  were something like a complex number and its conjugate.

From the present vantage these speculations are not nonsense at all. They appear like hints about the remarkable connections that have subsequently appeared. The operator algebras that produced the Jones polynomial and generalizations were traditionally studied with quantum physics as the technical and inspirational source. The writing of the Conway identity into a Hecke algebra related to braids via $\sigma_i - \bar{\sigma}_i = z$ is a direct algebraic version of this sort of identity.

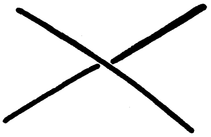
Another view of complex numbers lets us think of ordered pairs of real numbers [A,B] and [B,A] as conjugates. If $W = a + ib$ let $A = \frac{1}{2}(a+b)$, $B = \frac{1}{2}(a-b)$.) Then 1 corresponds to [1,1] and $\sqrt{-1}$ corresponds to [1,-1] so that $a + ib$, $= a[1,1] + b[1,-1] = [a+b, a-b]$, and $[A,B] - [B,A] = (A-B)[1,-1] = (A-B)\sqrt{-1}$. This has a formal resemblance to both Conway identity and Heisenberg formula. And in the bracket expansion

[] = A[] + B[]

[] = B[] + A[]

the reversal of a crossing appears as this form of conjugation.

Simple aspects of formalism lie at the root of these similarities. Beyond all the conceptual apparatus, the fundamental point is that an unoriented crossing



discriminates two regions our of four. All the rest, whether in braids or in diagram form builds on this distinction. Formulas like $PQ - QP = \hbar i$ rely on the left-right distinct along a line. And a formula such as

[] = A[] + B[] relies on the discrimination of characters along a line that are identical after a rotation.

Thus we are initially mapping one simple order into another. These are designer's comments: for the possibility of distinguishing handedness of topological objects in three-dimensional space goes back through diagrams, symbols and distinctions to the possibility of finding handedness in the plane.

And the simplest forms of left and right in the plane are  and . Thus the detection and discrimination of  and  (or  and ) in a formalism may be fundamental to its sensitivity to handedness. This is my personal explanation for favoring regular isotopy and detecting the curls in the diagrams.

There are more questions that you can shake a stick at. What deeper insight will unlock the really hidden secrets of these diagrams. Does the Jones polynomial, or its generalization detect knottedness? Is there a further relationship to physics based on these diagrams (a la Feynman diagrams or Penrose spin nets)? (See [K11] and [K12] for an unfolding of this remark.) How can one understand the mirror image problem for knots and links completely? What is the relationship between these new techniques and the classical methods using homotopy theory, covering spaces algebraic topology? What information about slice knits and knot concordance is in the new invariants? What is the next simple idea that will turn the subject upside down? How do you follow the hints?

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