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1. Introduction

The object of this note is to give a representation result for bounded solutions of certain evolution equations with local nonlinearlity, which loosely speaking are of the type:

$$(1.1) \qquad \partial_t u = L^* u - u^{k+1} \ , \quad k \ \text{integer} \geq 1 \ ,$$

where L is the generator of a Markov process (X_t) (in fact (0.1) is to be interpreted in an integral form, see (3.4)).

The method we present here is in a certain sense linked with the method of Wild trees in the form presented in [7], [8], and some of the constructions in the proof of uniqueness of the Boltzmann process in [4]. We deal here with a local nonlinearity and there are two steps in our construction.

Let us first explain the motivation for the present paper. In a previous work [5], we studied a system of N Brownian spheres of radius $N^{-1/(d-2)}$, in dimension $d \geq 3$, and radius $\exp\{-N\}$ in dimension 2, which are destroyed as soon as a collision occurs between any two of them. It was shown in [5] that the limiting density of presence of each living particle was given by the solution of the nonlinear equation

$$(1.2) \qquad \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u - cu^2 \ ;$$

$u\big|_{t=0} = u_0$, initial probability density of each particle (c here is a constant), see also [2]. It was also shown in [5] that when N becomes large for any fixed k the law of $(B^1,\tau_1) \ldots (B^k,\tau_k)$ (τ_ℓ death time of the ℓth particle) become asymptotically independent and one has an explicit description of the limit law (B,τ) in terms of the solution of (1.2) (the conditional law of τ given B is "exponential" of variable intensity $cu(s,B_s)$); such a result is called propagation of chaos. One key ingredient of the proof was to show a decay of the probability that Brownian particle 2 belongs to the tree of ancestors of particle 1 at time t (here one forgets about destructions). That is, that Brownian particle 2 has a collision with another particle which later has a collision with a particle ... with finally a particle which collides with Brownian particle 1 before time t. Collision means of course coming to a distance less than $N^{-1/(d-2)}$. In other words, 2 has a chain reaction leading to 1 before time t. This technique suggested that one should investigate this collision tree between independent particles in this "constant capacity" or rather "constant mean free path" limit for its own sake, try to describe its asymptotic behavior then build the interaction (the destruction of the particles) directly in this limit object and then recover (1.2) in this scheme.

In the present note we first construct the collision tree (which describes chain reactions leading to one particle) and then we build the interaction on this tree.

Let me first explain the construction of the tree in the case where $k = 1$ and $L = \frac{1}{2}\Delta$ in (1.1), for the sake of simplicity. One starts with a Brownian particle (the ancestor) B_s running until time t, with initial density u_0. Then one constructs trajectories having a collision with the ancestor as follows: One picks conditionally on B_s a Poisson distribution of points on $[0,t]$ with intensity $V(s,B_s)\,ds$. Here $V(s,x)$ is the solution of the heat equation with initial u_0. Now conditionally on (B_s) and on these times $0 < t_1 < \dots < t_\ell < t$ one constructs independent Brownian bridges, $W^1,\dots,W^n$: W_ℓ, $\ell \le n$ has the law of a Brownian motion on time $[0,t_\ell]$ with initial density u_0, conditioned to be equal to B_{t_ℓ} at time t_ℓ. These constitute the first generation. Now one performs the same thing on each of the W_ℓ, $1 \le \ell \le n$, as we have just described on B. And one obtains the second generation, and one goes on like this.

The second step is now to construct the interaction (telling whether a trajectory in the tree is already destroyed by the time it meets its direct ancestor, that is for instance time t_ℓ for W_ℓ). This is performed as follows: one easily sees that the collision tree is almost surely finite provided our intensities satisfy $V(s,x) \le k < \infty$. Thus one considers the bottom branches in the tree, that is those which do not have descendents. These are said to be alive. Now recursively one can tell for each particle, whether it is alive or dead at the time it meets its direct ancestor, by stating that if one of its direct descendents is alive, then the particle is dead. For instance in the case of B, our basic ancestor, it is dead if one of the $W_1,\dots,W_n$ is alive. The result is now that the density of presence of B when it is alive satisfies (1.2) (here we normalize c to be one). The fact that we are dealing with Brownian motion is irrelevant for this construction and for a "general" process evolution with generator L, we obtain now (1.1) with $k = 1$. (We have explained here the case $k = 1$, which is easier to describe).

The structure of the collision tree was guessed with the help of the work performed in [6]. There we deal with N independent diffusions, and prove a propagation of chaos result for the triplet made of the ith trajectory, its first collision time T_i, with any of the other X^j, $j \neq i$, and the trajectory of the particle it encounters at time T_i. (In this situation one has to redefine what collision means). The limit law which arises for the triplet is as follows: conditionally on X, T is an "exponential" of variable intensity $V(s,X_s)$, where $V(s,x)$ is the solution of the heat equation $\frac{\partial V}{\partial t} = L^* V$ associated with X. Conditionally on (X,T) the law of the colliding particle is the law of our basic diffusion conditioned on being equal to X_T at time T.

This result describes the limiting aspect of the first branch in the collision tree. And the limit obtained in indeed coherent with the law of (B, t_1, W_1) in the limit tree constructed in the present note (with the notations used above).
The present note in some sense shows that one may try to derive the limit behavior of the interacting annihilating particles by reading their interaction on what should be the right candidate for the limiting aspect of the collision tree for the independent noninteracting particles.

2. The basic Tree

We consider a conservative Markov process $(D(\mathbb{R}_+,E), (X_t)_{t\geq 0}, P^x, x\in E)$ with right continuous trajectories having left limits, on a Polish state space E, and measurable transition density $p_t(x,y)$, with respect to a σ-finite measure m on E, satisfying the Chapman-Kolmogorov equation:

(2.1) $\quad p_{t+s}(x,y) = \int_E p_t(x,z)\, p_s(z,y)\, m(dz)\ , \qquad t,s > 0,\quad x,y \in E$, and

(2.2) $\quad \int p_t(x,y)\, m(dx) \leq C_T < \infty\ , \qquad \text{for}\quad t \leq T\ .$

We suppose we are given an initial law $u_0(dx) = V(x)\; m(dx)$, $V\in L^\infty(dm)$, and we impose

(2.3) $\quad V(t,y) = \int p_t(x,y)\; V(x)\; m(dx) > 0\ , \qquad \text{for } t > 0\ .$

We denote by P the law of our Markov process with initial law u_0, and by P^t the image of this law on $D([0,t],E)$, for $t > 0$. We assume that we can define the probabilities $P^{t,x}$ on $D([0,t],E)$ for $t > 0$, $x\in E$ (defining the conditional law of P^t given $X_t = x$), for which $P^{t,x}(X_t = x) = 1$,

(2.4) $\quad P^t = \int P^{t,x}\; u_t(dx)\ , \qquad \text{for}\quad u_t(dx) = V(t,x)\; m(dx)$, and

(2.5) $\quad \text{for } 0\leq s < t,\quad A\in \sigma(X_u,\ u\leq s),\ P^{t,x}[A] = V(t,x)^{-1}\; E^t[p_{t-s}(X_s,x)]_A]$.

Remark 2.1: Applying [1], p. 73-79, to the $P^{t,x}$-supermartingale $p_{t-s}(X_s,x)^{-1}$ (this expression is defined to be zero where $p_{t-s}(X_s,x) = 0$), it is not difficult to see that if $p_t(\cdot,x)$ converges uniformly to zero on the complementary of each neighborhood of x, when t goes to 0, then one can construct the laws $P^{t,x}$ in such a way that $X_{t^-} = X_t = x$ (see also [6], section 3). □

We denote by D the space $\cup_{t>0} D([0,t],E)$ that we identify with $(0,\infty)\times D([0,1],E)$, through the map $(t,\psi)\to\psi(u/t)$, $0\le u\le t$, and denote by $\tau: D\to(0,\infty)$, the first coordinate. We now construct the random tree which represents the limiting aspect of the various "chain reactions" leading to one specific particle, when one studies the motion of many small independent particles as in [6].

Recall that following [3], a tree is a subset π of $U=\cup_{n\ge 0}\mathbb{N}^{*n}$, which contains the sequence $\emptyset$, such that u belongs to π whenever uj, $j\in\mathbb{N}^*$ belongs to π, and such that for any $u\in\pi$, there exists $\nu_u\in\mathbb{N}$, such that $uj\in\pi$, whenever $1\le j\le\nu_u$. We let Ω denote the set of trees with marks in D: $\omega=(\pi,(\phi^u, u\in\pi))$, and for $u\in\pi$, $T_u\omega$ will be the translation of the marked tree ω at u. We let G_n be the n^{th} generation of the tree, that is to say, the finite subset $\pi\cap\mathbb{N}^{*n}$, and F_n the σ-field generated by G_k, $0\le k\le n$, and ϕ^u, $u\in G_k$, $0\le k\le n$.

The trajectories ϕ^{uj}, $1\le j\le\nu_u$, will represent the particles which hit the trajectory ϕ^u at time $\tau(\phi^{uj})$ (the probabilities on the tree will be such that $\tau(\phi^u)>\tau(\phi^{uj})$, a.s.). Using Kolmogorov's extension theorem, in a very similar fashion to [3], for each $\psi\in D$, we build a probability R_ψ on Ω, characterized by:

(2.5) $R_\psi[\phi^\emptyset=\psi]=1$,

(2.6) for $u\in U$, and f_u positive measurable functions on Ω,

$$E_\psi\Big[\prod_{u\in G_n} f_u\circ T_u / F_n\Big]=\prod_{u\in G_n} E_{\phi^u}[f_u]\ ,$$

(2.7) Under R_ψ, with $\tau(\psi)=t$, we have a.s.

$$\nu=kp,\ p\ge 0,\ 0<\tau(\phi^k)<\dots<\tau(\phi^{kp}),\ \tau(\phi^{k\ell+1})$$
$$=\tau(\phi^{k\ell+2})=\dots=\tau(\phi^{k(\ell+1)})\ ,\quad \text{for } 0\le\ell<p\ ,$$

and

$\sum_{0\le k\ell<\nu}\delta_{(\phi^{k\ell+1},\dots,\phi^{k(\ell+1)})}$ is a Poisson measure on D^k, with intensity $Q_\psi=\int_0^t V(s,\psi_s)^k\,(P^{s,\psi_s})^{\otimes k}\,ds$.

<u>Remark 2.2.</u> Notice that in the case k = 1, that is the case of binary collisions, if we set $\tau(\phi^1)=\infty$, when $\nu(\ )=0$, we see that when $\tau(\psi)=1$,

$$R_\psi[\tau(\phi^\ell)>t]=\exp-\int_0^{t\wedge 1} V(s,\psi_s)\,ds\ ,$$

and conditionally on $\tau(\phi^1)=s<1$, ϕ^1 is P^{s,ψ_s} distributed. This corresponds to the limit distribution for the first incoming particle found in [6], provided we let ψ be P^1 distributed. □

We are now going to prove:

<u>Lemma 2.3.</u> (2.8) R_ψ - a.s. the marked tree ω is finite,

(2.9) Under R_ψ, $M(\omega) = \sum_{0\leq k\ell<\nu} \delta_{(T_{k\ell+1}(\omega),\dots,T_{k(\ell+1)}(\omega))}$ is a Poisson point measure on Ω^k with intensity

$\mathfrak{Q}_\psi = \int_0^t V(s,\psi_s)^k \, (R^{s,\psi_s})^{\otimes k} \, ds$, if we let

$R^{t,x} = \int R_\psi \, P^{t,x}(d\psi)$, for $t > 0$, $x \in E$.

Proof. 1) Set for $0 \leq p < n$, and $c = k\|V\|^k_{L^\infty([0,t]\times E)}$:

$$\begin{aligned} v_p &= E_\psi[\sum_{u\in G_{n-p}} 1/p! \ (c\tau(\phi^u))^p] \\ &= E_\psi[E_\psi[\sum_{u\in G_{n-p-1},\ uj\in G_{n-p}} 1/p! \ (c\tau(\phi^{uj}))^p / F_{n-p-1}]] \\ &= E_\psi[\sum_{u\in G_{n-p-1}} E_{\phi^u}[\sum_{1\leq j\leq\nu} 1/p! \ (c\tau(\phi^{uj}))^p]] \\ &\leq E_\psi[\sum_{u\in G_{n-p-1}} \int_0^{\tau(\phi^u)} 1/p! \ (ct)^p \ c \ dt] \\ &= E_\psi[\sum_{u\in G_{n-p-1}} 1/(p+1)! \ (c\tau(\phi^u))^{p+1}] \ = \ v_{p+1} . \end{aligned}$$

We see that $v_0 \leq v_n$, so $E_\psi[\mathrm{card}(G_n)] \leq 1/n! \ (ct)^n$, and $E_\psi[\sum_n \mathrm{card}(G_n)] \leq e^{ct}$, which yields (2.8).

2) Let F be a positive function on Ω^k,

$$\begin{aligned} E_\psi[\exp\{-\langle M,F\rangle\}] &= E_\psi[\prod_{0\leq k\ell<\nu} \exp\{-F(T_{k\ell+1}(\omega),\dots,T_{k(\ell+1)}(\omega))\}] \\ &= E_\psi[\prod_{0\leq k\ell<\nu} R_{\phi^{k\ell+1}} \otimes \dots \otimes R_{\phi^{k(\ell+1)}}[\exp\{-F\}]] \\ &= \exp\{ \int_0^t V(s,\psi_s)^k \int dP^{s,\psi_s}(\phi^1) \dots dP^{s,\psi_s}(\phi^k) \\ &\qquad \times\{R_{\phi^1} \otimes \dots \otimes R_{\phi^k}[\exp\{-F\}] - 1\}\} \, ds \\ &= \exp\{ \int_0^t V(s,\psi_s)^k \int (dR^{s,\psi_s})^{\otimes k}(\exp\{-F\} - 1) , \end{aligned}$$

which yields (2.9). □

3. The Interaction

We are now going to construct a supplementary mark Z^u, $u \in \pi$, on our basic tree, which will tell us whether ϕ^u is alive or not at time $\tau(\phi^u)$, in the sense that it has collided with a k-uple of trajectories $\phi^{u,k\ell+1},\dots,\phi^{u,k(\ell+1)}$, $0 \leq k\ell < \nu_u$, that were all alive at the time of collision

$\tau(\phi^{u,k\ell+1}) = \ldots = (\phi^{u,k(\ell+1)})$.

To this end, using (2.8), we define $D_u(\omega)$, for $u \in \omega$, to be the number of generations descending from u, that is $\max\{|v|, uv \in \omega\}$, which is R_ψ-a.s. finite. We set for $u \in \pi$: $Z^u = 1$, if $D_u = 0$, and by induction, if $D_u = p+1$, using the fact that for $1 \le j \le \nu_u$, $D_{uj} = D_u - 1$, we set

$$Z^u = (1 - \textstyle\sum_{0 \le k\ell < \nu_u} Z^{u,k\ell+1} \ldots Z^{u,k(\ell+1)})_+ ,$$

in other words, Z^u is zero as soon as a collision with a k-uple of particles with marks Z equal to 1 happens.

We denote by R^t (resp. $R^{t,x}$) the law on Ω defined by $R^t = \int R_\psi P^t(d\psi)$ (resp. $R^{t,x} = \int R_\psi P^{t,x}(d\psi)$). We are now going to study our candidate for the solution of the nonlinear evolution equation, namely the measurable function:

$$(3.1) \qquad u(t,x) = V(t,x)\, R^{t,x}[1(Z^\emptyset = 1)], \quad t > 0,$$
$$u(0,x) = V(x).$$

For convenience we will write ψ for $\phi^\emptyset$ and Z for $Z^\emptyset$, in what follows. As the next lemma shows, $u(t,x)$ appears as the density of presence of ψ when it survives:

<u>Lemma 3.1</u>. For f bounded measurable,

$$(3.2) \qquad \int u(t,x)\, f(x)\, dm(x) = E^{R^t}[f(\psi_t)\, 1(Z = 1)] .$$

<u>Proof</u>: $\int u(t,x)\, f(x)\, dm(x) = \int V(t,x)\, R^{t,x}[Z=1]\, f(x)\, dm(x)$

$$= \int V(t,x) \int P^{t,x}(d\psi)\, f(\psi_t)\, R_\psi[Z=1]\, dm(x)$$
$$= E^{R^t}[f(\psi_t)\, 1(Z=1)]. \qquad \square$$

<u>Remark 3.2</u>. It is easy to see that when $\tau(\psi) = T$, then for $0 < t \le T$, if one sets $\pi_t = \{u\ \ \pi, \tau(\phi^u) \le t$, or $u = \emptyset\}$, and $\omega_t = (\pi_t, (\phi^u, u \ne \emptyset$ in π_t, $\psi|_{[0,t]}$ for $u=\emptyset))$, then the law induced by ω_t on Ω, under R_ψ is precisely $R_{\psi|_{[0,t]}}$. As a consequence if one defines under R_ψ, Z_s, $0 \le s \le T$, by:

$$(3.3) \qquad Z_s = (1 - \textstyle\sum_{0 \le k\ell < \nu} Z^{k\ell+1} \ldots Z^{k(\ell+1)}\, 1(\tau(\phi^{k\ell+1}) \le s))_+ ,$$

one has

$$\int u(s,x)\, f(x)\, dm(x) = E^{R^T}[f(\psi_s)\, 1(Z_s = 1)], \quad \text{for } 0 \le s \le T . \qquad \square$$

We can now state our main result:

<u>Theorem 3.2</u>: $u(t,x)$, $0 \le t \le T$, $x \in E$, is the unique solution in $L^\infty([0,T]\times E, ds\, dm(x))$ of the equation:

$$(3.4) \qquad w(t,x) = V(t,x) - \int_0^t \int_E p_{t-s}(y,x)\, w^{k+1}(s,y)\, ds\, dm(y), \quad dt \otimes dm \text{ - a.s.}.$$

<u>Proof</u>: Let us check that $u(t,x)$ is a solution of (3.4). Using (2.2), (3.1), we

see that $u(\cdot,\cdot) \in L^\infty([0,T]\times E)$. On the other hand, we have:

$$(3.5) \qquad R_\psi[Z=1] = R_\psi[\{\textstyle\sum_{0\le k<\ell<\nu_\emptyset} \prod_{1\le p\le k} 1(Z^{k\ell+p}=1)=0\}]$$
$$= \exp\left\{-\int_0^t V(s,\psi_s)^k\, R^{s,\psi_s}[Z=1]^k\, ds\right\}$$
$$= \exp\{-\int_0^t u(s,\psi_s)^k\, ds\}\ .$$

From this we see that

$$u(t,x) = V(t,x)R^{t,x}[Z=1] = V(t,x)\,(1-\int_0^t ds\ P^{t,x}(d\psi)\,[u(s,\psi_s)^k \exp\{-\int_0^s u(r,\psi_r)^k dr\}])$$
$$= V(t,x)-\int_0^t ds\int dP^t(\psi)\ p_{t-s}(\psi_s,x)\ u(s,\psi_s)^k\ \exp\{-\int_0^s u(r,\psi_r)^k\, dr\}$$
$$= V(t,x)\ -\int_0^t ds\ \ dm(y)\ V(s,y)\ p_{t-s}(y,x)\ u(s,y)^k \int P^{s,y}(d\psi)$$
$$\times \exp\{\ -\int_0^s u(r,\psi_r)^k\, dr\}$$
$$= V(t,x)-\int_0^t ds\int dm(y)\ p_{t-s}(y,x)\ u(s,y)^k\ V(s,y)\ R^{s,y}[Z=1]\ \ ,\ \text{using (3.5)}.$$

The last quantity is precisely the right member of (3.4), and we have thus proved that $u(\cdot,\cdot)$ is a solution of (3.4). There remains to show the uniqueness statement.

Let w_1 and w_2 be two solutions in $L^\infty([0,T]\times E,\ ds\ dm(x))$ of (3.4), and set $d_t = \|w_1-w_2\|_{L^\infty([0,t]\times E)}$, then d_t is an increasing function, and it is easy to see that for a.e. t in $[0,T]$, $\|w_1(t,\cdot)-w_2(t,\cdot)\|_{L^\infty(dm)} \le d_t$.

We now see that for a.e. t, $0 \le t \le T$,

$$\|w_1(t,\cdot)-w_2(t,\cdot)\|_{L^\infty(dm)} \le \sup_x \int_0^t \int_E p_{t-s}(y,x)\ \|w_1^{k+1}(s,\cdot)$$
$$- w_2^{k+1}(s,\cdot)\|_{L^\infty(dm)}\quad ds\ dm(y),\qquad \text{using (2.2)},$$
$$\le \int_0^t ds\ \text{const.}\ d_s\ \le \int_0^{t'} ds\ \text{const.}\ d_s\ ,\quad \text{if } t' \ge t.$$

Since $d_t = \text{ess}\sup_{s\le t} \|w_1(s,\cdot)-w_2(s,\cdot)\|_{L^\infty(dm)}$, it satisfies: $d_t \le \int_0^t ds\ \text{const.}\, d_s$, which implies that $d_t = 0$, identically, and proves our uniqueness claim. □

<u>Remark 3.3</u>. As a by product of our representation we can see that taking ψ, with $\tau(\psi) = 1$. If S denotes the first time ψ has a collision with a k-uple of particles with marks Z all equal to 1 ($S = +\infty$, if such a collision does not occur), then:

$$R_\psi[S > t] = \exp\left\{-\int_0^{t\wedge 1} u(s,\psi_s)^k\ ds\right\},$$

and conditionally on $S = t < \infty$, the law of the k incoming particles is the k-fold product of

$$(V/u)(t,\psi_t)\ \exp\left\{-\int_0^t u(s,\phi_s)^k\ ds\right\} P^{t,\psi_t}(d\phi)\ . \qquad \square$$

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