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by K.R. PARTHASARATHY

§ 1. Introduction This is an expositary account of work done in collaboration with L. Accardi [1] in the area of quantum probability. We begin with the following naive philosophy. In the language of classical probability a martingale is a sequence of random variables which "remains constant in conditional mean" and for a gambler, corresponds to a game which is on average fair. While investigating laws of nature one adheres to the belief that nature plays a fair game. Putting these two intuitive ideas together one may ask whether some interesting physical law can be deduced from the principles of martingale theory. The aim of the present lecture is to indicate a small step in this direction by showing how the socalled canonical commutation and anticommutation relations of quantum fields in one dimension follow from martingale hypotheses.

Our formulation of the problem is very much inspired by P.Lévy's famous characterisation of the standard Brownian motion as a martingale X_t, $t \geq 0$ with continuous trajectories and satisfying the conditions $X_o = 0$, $\mathbb{E}((X(t)-X(s))^2 | \mathcal{F}_s) = t-s$ for $t > s$, $\mathcal{F}_s$ denoting the σ-algebra of the process upto time s. (See Theorem 11.8 in [5]).

§ 2. Lévy fields In the theory of quantum probability real valued random variables on a Borel space are replaced by selfadjoint operators in a complex separable Hilbert space $\mathcal{H}$. A stochastic process or, more generally, a generalised random function is replaced by a field of observables over a space of test functions. The essential difference lies in the fact that multiplication between real valued random variables is a commutative operation whereas multiplication of two observables is not always defined. However, it is to be noted that all bounded observables (concerning a given quantum system) constitute a real linear subspace in the noncommutative algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. For a brief introduction to quantum probability we refer the reader to [4].

To formulate the notion of a martingale and describe our problem we start with a family $X = \{X(f), f \in L_2(\mathbb{R}_+)\}$ of selfadjoint operators on $\mathcal{H}$ such that there exists a unit vector ϕ belonging to the domain of every operator of the form $X(f_1)X(f_2)\ldots X(f_n)$, $f_j \in L_2(\mathbb{R}_+)$, $1 \le j \le n$, $n = 1,2,\ldots$ where $L_2(\mathbb{R}_+)$ denotes the space of all complex valued square integrable functions on $\mathbb{R}_+ = [0,\infty)$. Let $\mathcal{M}$ denote the linear manifold in $\mathcal{H}$ generated by $\{\phi, X(f_1)X(f_2)\ldots X(f_n)\phi, f_j \in L_2(\mathbb{R}_+), 1 \le j \le n, n = 1,2,\ldots\}$. For any set $E \subset \mathbb{R}_+$ we denote by χ_E the indicator function of E and write

$$f_{[a,b]} = f\chi_{[a,b]},\ f_{a]} = f\chi_{[o,a]},\ f_{[b} = f\chi_{[b,\infty)}$$

with the conventions $f_{o]} = f_{[\infty} = 0$, $f_{\infty]} = f$ for any $f \in L_2(\mathbb{R}_+)$. Let $\mathcal{M}_{t]}$ be the linear manifold in $\mathcal{H}$ generated by $\{\phi, X(f^{(1)}_{t]})X(f^{(2)}_{t]})\ldots X(f^{(n)}_{t]})\phi, f^{(j)} \in L_2(\mathbb{R}_+), 1 \le j \le n, n = 1,2,\ldots\}$. We say that the family X is a <u>Lévy field</u> with <u>cyclic vector</u> ϕ and <u>covariance kernel</u> K if the following conditions are fulfilled :

(i) $\mathcal{M}$ is dense in $\mathcal{H}$;

(ii) $X(0) = 0$ and the correspondence $(f_1,f_2,\ldots,f_n) \to X(f_1)X(f_2)\ldots X(f_n)\phi$ is real multilinear in the variables $f_1, f_2,\ldots,f_n$ for each n;

(iii) for fixed $f^{(j)}$, $1 \le j \le n$ in $L_2(\mathbb{R}_+)$ the map $t \to X(f^{(1)}_{t]})X(f^{(2)}_{t]})\ldots X(f^{(n)}_{t]})\phi$ is continuous in the closed interval $[o,\infty]$;

(iv) for any $u,v \in \mathcal{M}_{t]}$, $f \in L_2(\mathbb{R}_+)$ $\langle u, X(f_{[t})v\rangle = 0$;

(v) there exists a complex valued strongly continuous functional $K(.,.)$ on $L_2(\mathbb{R}_+) \times L_2(\mathbb{R}_+)$ such that $\langle u, X(f_{[t})X(g_{[t})v\rangle = K(f_{[t}, g_{[t}) \langle u,v\rangle$ for all $u,v \in \mathcal{M}_{t]}$, $t \ge 0$, $f,g \in L_2(\mathbb{R}_+)$.

Condition (i) expresses the cyclicity of the vector ϕ whereas (ii) means that we are dealing with a field of observables over the test function space $L_2(\mathbb{R}_+)$

visualised as a real vector space. (iii) is a purely technical requirement. (iv) is martingale property of the field X expressed in the language of quantum probability where expectation values are defined in terms of inner products. Condition (v) is inspired by P.Lévy's condition on the conditional variance of increments in the characterisation of standard Brownian motion. Indeed, if H is the Hilbert space of all square integrable functions on the probability space of standard Brownian motion and X(f) is the selfadjoint operator of multiplication by $\int_0^\infty \mathrm{Re}\ f\,dw$, w denoting the sample path, then $X = \{X(f), f \in L_2(\mathbb{R}_+)\}$ is a Lévy field with cyclic vector 1 and covariance kernel $K(f,g) = \int_0^\infty (\mathrm{Re}\ f)(\mathrm{Re}\ g)dt$. In this case all the operators X(f) commute with each other.

As a noncommutative example of a Lévy field we mention the case when H is the antisymmetric Fock space over $L_2(\mathbb{R}_+)$ and $X(f) = a(f) + a^\dagger(f)$ where a(f) and $a^\dagger(f)$ are the fermion annihilation and creation fields in H associated with $f \in L_2(\mathbb{R}_+)$. Then X(f) is a bounded selfadjoint operator and

$$X(f)X(g) + X(g)X(f) = 2\ \mathrm{Re} \langle f,g\rangle.$$

X has covariance kernel $K(f,g) = \langle f,g\rangle$ and the fermion vacuum as its cyclic vector.

We say that two Lévy fields X_i in Hilbert space H_i with cyclic vector ϕ_i and same covariance kernel K on $L_2(\mathbb{R}_+) \times L_2(\mathbb{R}_+)$, i = 1,2, are said to be <u>equivalent</u> if there exists a unitary operator $U : H_1 \to H_2$ such that $U\phi_1 = \phi_2$, $UX_1(f_1)X_1(f_2)\dots X_1(f_n)\phi_1 = X_2(f_1)X_2(f_2)\dots X_2(f_n)\phi_2$ for all $f_i \in L_2(\mathbb{R}_+), 1 \le i \le n$, n = 1,2,... . One of the interesting problems of quantum martingale theory is the classification of Lévy fields upto equivalence.

§ 3. <u>Structure of the covariance kernel of a Levy field</u> Suppose X is a Lévy field over $L_2(\mathbb{R}_+)$ in H with cyclic vector ϕ and covariance kernel K. Let $H_{t]} = \bar{M}_{t]}$, the closure of the linear manifold $M_{t]}$ introduced in Section 2, along with the conventions $M_{o]} = H_{o]} = 0$, $M_{\infty]} = M$, $H_{\infty]} = H$. For any operator T let D(T) denote its domain.

<u>Lemma 3.1</u> For any $t \ge 0$, $f,g \in L_2(\mathbb{R}_+)$, $u,v \in H_{t]}$ the following holds : (i) $H_{t]} \subset D(X(f_{[t}))$; (ii) $\langle u, X(f_{[t})v\rangle = 0$; (iii) $\langle u, X(f_{[t})X(g_{[t})v\rangle =$

$\langle u,v\rangle\, K(f_{[t}, g_{[t})$.

Proof This is straightforward from the fact that selfadjoint operators are closed and for any two vectors u_1, u_2 in $M_{t]}$

$$||X(f_{[t})(u_1-u_2)||^2 = ||u_1-u_2||^2\, K(f_{[t}, f_{[t}). \qquad \square$$

Theorem 3.2 The covariance kernel K admits the integral representation

$$K(f,g) = \int_0^\infty \sigma(f,g,t)dt, \quad f,g \in L_2(\mathbb{R}_+)$$

where

$$\sigma(f,g,.) = (f_1,f_2)\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

$f = f_1 + if_2$, $g = g_1 + ig_2$ are the decompositions into real and imaginary parts and $\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ is a bounded, complex, 2×2 nonnegative definite matrix valued function on $\mathbb{R}_+$.

Proof By properties (ii), (iv) and (v) of a Lévy field we observe that

$$K(f, g_{t]}) = K(f_{t]}, g) = K(f_{t]}, g_{t]}) \text{ for all } t \geq 0. \tag{3.1}$$

Consider $L_2(\mathbb{R}_+)$ as the real Hilbert space $\mathfrak{h} = \overset{\circ}{L}_2(\mathbb{R}_+) \oplus \overset{\circ}{L}_2(\mathbb{R}_+)$ where o indicates that only real valued square integrable functions are included. Thus $f \in L_2(\mathbb{R}_+)$ is identified with (f_1, f_2). Since Re $K(f,g)$ and Im $K(f,g)$ are bounded bilinear forms on $\mathfrak{h}$ there exist operators A, B in $\mathfrak{h}$ such that

$$\mathrm{Re}\, K(f,g) = \langle f,Ag\rangle_o, \quad \mathrm{Im}\, K(f,g) = \langle f, Bg\rangle_o$$

where $\langle f,g\rangle_o = \int_0^\infty (f_1g_1 + f_2g_2)\, dt$ is the inner product in $\mathfrak{h}$. Equation (3.1) implies that the operators A and B commute with all multiplication operators $(f_1, f_2) \to (\phi f_1, \phi f_2)$ where ϕ is any real bounded Borel function. Since $K(f,g) = \overline{K(g,f)}$ it follows that A and B are multiplications by real 2×2 symmetric and skew symmetric matrix valued functions. This proves that $K(f,g)$ has the form stated in the theorem for some Hermitian matrix valued function $((\sigma_{ij}))$. Since $K(f,f) \geq 0$ for all f it follows that $((\sigma_{ij}(t)))$ is positive semidefinite almost everywhere. The boundedness of the operators A,B implies that $((\sigma_{ij}(t)))$ is essentially bounded. $\square$

Remark Let $C_1 = \{t : \text{rank}\ ((\sigma_{ij}(t))) = 1\}$, $C_2 = \{t : \text{rank}\ ((\sigma_{ij}(t))) = 2\}$. Then $K = K_1 + K_2$ where

$$K_1(f,g) = \int_{C_1} \sigma(f,g,t)dt$$

$$K_2(f,g) = \int_{C_2} \sigma(f,g,t)dt$$

where $\sigma(f,g,t)$ is as in Theorem 3.2.

§ 4. Stochastic integration with respect to a Lévy field For a Lévy field X over $L_2(\mathbb{R}_+)$ in $\mathcal{H}$ with cyclic vector ϕ and covariance kernel K given by Theorem 3.2 we write

$$X_f(a,b) = X(f_{[a,b]}) \quad \text{for all} \quad 0 \leq a < b < \infty,\ f \in L_2(\mathbb{R}_+).$$

We say that a map $\xi : \mathbb{R}_+ \to \mathcal{H}$ is adapted to X if $\xi(t) \in \mathcal{H}_{t]}$ for every t. Let $\mathcal{A}(X)$ denote the linear space of all strongly continuous maps from $\mathbb{R}_+$ into $\mathcal{H}$ adapted to X. To any finite partition $\mathcal{P}$ of an interval [a,b] into $a = t_o < t_1 < \ldots < t_n < t_{n+1} = b$, $\xi \in \mathcal{A}(X)$ and $f \in L_2(\mathbb{R}_+)$ we use Lemma 3.1 and associate the Riemann sum

$$S(\mathcal{P},f,\xi) = \sum_{j=0}^{n} X_f(t_j, t_{j+1})\xi(t_j).$$

Then by properties (i) - (v) of a Lévy field and Theorem 3.2 we obtain

$$\langle\phi, S(\mathcal{P},f,\xi)\rangle = 0,$$

$$\langle S(\mathcal{P},f,\xi), S(\mathcal{P},g,\eta)\rangle = \sum_{j=0}^{n} \langle\xi(t_j),\eta(t_j)\rangle \int_{t_j}^{t_{j+1}} \sigma(f,g,s)ds.$$

If $\mathcal{P},\mathcal{P}'$ are finite partitions of [a,b], [a',b'] respectively and $b \leq a'$ then

$$\langle S(\mathcal{P},f,\xi), S(\mathcal{P}',g,\eta)\rangle = 0.$$

These relations imply that as $\delta(\mathcal{P}) = \max_j (t_{j+1} - t_j)$ tends to 0 the Riemann sum $S(\mathcal{P},f,\xi)$ converges to a unique limit in $\mathcal{H}$. We denote this unique limit by $\int_a^b X_f(dt)\xi(t)$ and call it the integral of ξ with respect to X_f. It is routine to check that the following properties hold:

(i) $\langle \phi, \int_0^t X_f(ds)\xi(s)\rangle = 0,$

(ii) $\langle\int_a^b X_f(ds)\xi(s), \int_c^d X_g(ds)\eta(s)\rangle = 0$ if $0 \leq a < b \leq c < d,$

(iii) $\langle\int_a^b X_f(ds)\xi(s), \int_a^b X_g(ds)\eta(s)\rangle = \int_a^b \langle\xi(s),\eta(s)\rangle\, \sigma(f,g,s)ds$,

(iv) $\int_a^b X_f(ds)\xi(s) + \int_b^c X_f(ds)\xi(s) = \int_a^c X_f(ds)\xi(s)$ if $a < b < c,$

(v) the map $t \to \int_0^t X_f(ds)\xi(s)$ belongs to $A(X)$,

(vi) $\int_a^b X_f(ds)\xi(s)$ is linear in ξ and real linear in f.

<u>Lemma 4.1</u> Let $\{f_n\}$ be any sequence in $L_2(\mathbb{R}_+)$. Define inductively the adapted processes $\{\xi(f_1,f_2,\ldots,f_n,t)\}$ by

$$\xi(f_1, t) = X_{f_1}(0,t)\phi,$$
$$\xi(f_1,\ldots,f_n,t) = \int_0^t X_{f_1}(ds)\xi(f_2,f_3,\ldots,f_n,s), \quad n = 1,2,\ldots$$

Then the following holds:

(i) $\langle\phi,\xi(f_1,f_2,\ldots,f_n,t)\rangle = 0$ for all $t \geq 0$, $n = 1,2,\ldots$

(ii) $\langle\xi(f_1,\ldots,f_m,s), \xi(g_1,\ldots,g_n,t)\rangle = 0$ if $m \neq n$,

(iii) $\langle\xi(f_1,\ldots,f_n,s), \xi(g_1,\ldots,g_n,t)\rangle \int\limits_{0<t_n<t_{n-1}<\ldots<t_1<t\wedge s} \prod_{i=1}^n \sigma(f_i,g_i,t_i)dt_1dt_2\ldots dt_n$

where $t \wedge s$ is the minimum of t and s and $\sigma(f,g,t)$ is defined by Theorem 3.2.

<u>Proof</u> It is clear from property (iii) of a Lévy field that $\xi(f_1,t)$ is a strongly continuous adapted process and hence by induction each $\xi(f_1,f_2,\ldots,f_n,\cdot)$ is defined as an element of $A(X)$. Now (i) is a consequence of property (i) of integrals. By properties (ii) and (iii) of integrals

$$\langle\xi(f_1,f_2,\ldots,f_m,s),\xi(g_1,g_2,\ldots,g_n,t)\rangle = \int_0^{t\wedge s} \sigma(f_1,g_1,t_1)\langle\xi(f_2,\ldots,f_m,t_1), \xi(g_2,\ldots,g_n,t_1)\rangle dt_1.$$

Parts (ii) and (iii) of the lemma now follow by induction and part (i). $\square$

<u>Remark 1</u>. It is clear from the lemma that the repeated integrals $\xi(f_1,\ldots,f_n,t)$ with respect to the Lévy field X behave like n^{th} order chaos in the theory of

Brownian motion or like n-particle vectors in a boson Fock space. However, it is not clear whether the set of vectors $\{\phi, \xi(f_1, f_2, \ldots, f_n, t), f_j \in L_2(\mathbb{R}_+), 1 \leq j \leq n, n = 1,2,\ldots\}$ is total in $H_{t]}$ for each t. What additional conditions on a Lévy field would ensure this completeness property of a "Lévy chaos" ?

Remark 2 Define for any $f \in L_2(\mathbb{R}_+)$,

$$\psi_f(t) = \phi + \xi(f,t) + \xi(f,f,t) + \cdots + \xi(\underbrace{f,f,\ldots,f}_{n\text{-fold}},t) + \cdots$$

The convergence of this series is ensured by (ii) and (iii) in the lemma. In fact the convergence is even uniform in t. Furthermore

$$\langle \psi_f(t), \psi_g(s) \rangle = \exp K(f_{t]}, g_{s]}) \quad \text{for} \quad s,t \geq 0, \; f,g \in L_2(\mathbb{R}_+).$$

and there exists a unique vector $\psi(f) \in H$ such that

$$\lim_{t \to \infty} ||\psi_f(t) - \psi(f)|| = 0 .$$

It is clear that

$$\langle \psi(f), \psi(g) \rangle = \exp K(f,g).$$

We may call $\psi(f)$ the intrinsic exponential or coherent vector of the Lévy field X associated with f. It reminds us of the random variable $\exp(\int_0^\infty f\,dw - \frac{1}{2}\int_0^\infty f^2 dt)$ in Wiener space where w denotes the path of the standard Brownian motion and $f \in L_2(\mathbb{R}_+)$.

§ 5. Lévy fields and commutation relations Let ε be a fixed constant equal to ± 1. Suppose the Levy field X satisfies the additional condition

$$\{X(f_{t]})X(f_{[t}) + \varepsilon X(f_{[t})X(f_{t]})\}u = 0$$

for all $u \in M$, $f \in L_2(\mathbb{R}_+)$, $t \geq 0$ in the notations of Section 2. We then say that X is a Lévy boson or fermion field according as $\varepsilon = -1$ or $+1$.

In the present stage of the development of our subject we are not in a position to classify all the Lévy boson or fermion fields with a given covariance kernel. It must be emphasised that in the classical form of Lévy's characterisation of Brownian motion the continuity of trajectories of the martingale under consideration plays a crucial part. In the quantum probabilistic context we do not have a clear picture of trajectories for a Lévy field. According to a well known result of Kolmogorov the continuity of trajectories of a classical stochastic process $\{x(t,\omega),\ t \geq 0\}$ can be ensured by a sufficient condition on the moments of the form $\mathbb{E}\,|x(t,.) - x(s,.)|^r \leq c|t-s|^{1+\delta}$ for some $r > 0, \delta > 0,\ c > 0$ and all $0 \leq s < t < \infty$. If we choose $r = 4$ such a condition admits a simple translation in the language of operators. We shall now make this statement precise in the case of a Lévy field. For any $0 \leq s < t < \infty$, $f,g \in L_2(\mathbb{R}_+)$ let

$$\theta(f,g,s,t) = \sup_{\substack{u \in M_{s]} \\ ||u||=1}} ||\{X_f(s,t)X_g(s,t) - K(f_{[s,t]}, g_{[s,t]})\}u||^2. \tag{5.1}$$

For any partition P of any finite interval $[0,t]$ by $0 = t_0 < t_1 < t_2 < \ldots < t_n < t = t_{n+1}$ let $\delta(P) = \max_j (t_{j+1} - t_j)$ and

$$V(f,g,P) = \sum_{j=0}^{n} \theta(f,g,t_j,t_{j+1}).$$

We say that a Lévy field with covariance kernel K satisfies the Kolmogorov condition (of the fourth order) if

$$\lim_{\delta(P) \to 0} V(f,g,P) = 0$$

for every $t > 0$. It is to be noted that the expression under the sup on the right hand side of (5.1) admits the interpretation of a fourth order moment involving the observables of the Lévy field in the language of quantum probability.

Remark Suppose there exist two families of nonnegative Radon measures $\{\mu_{f,g},\ f,g \in L_2(\mathbb{R}_+)\}$, $\{\nu_{f,g},\ f,g \in L_2(\mathbb{R}_+)\}$ in $\mathbb{R}_+$ such that $\nu_{f,g}$ is absolutely continuous and

$$||\{X_f(s,t)X_g(s,t) - K(f_{[s,t]}, g_{[s,t]})\}u||^2 \leq ||u||^2\, \mu_{f,g}((s,t))\nu_{f,g}((s,t)) \tag{5.2}$$

for all $u \in M_{s]}$, $0 \leq s < t < \infty$, $f,g \in L_2(\mathbb{R}_+)$. Then X obeys the Kolmogorov condition.

When X(f) is multiplication by $\int_0^\infty \mathrm{Re}\, f\, dw$ in $L_2(P)$ where P is the probability measure of standard brownian motion and w denotes the brownian path, condition (5.2) holds with

$$\mu_{f,g}(E) = 2\int_E [\mathrm{Re}\, f(t)]^2\, dt,\quad \nu_{f,g}(E) = \int_E [\mathrm{Re}\, g(t)]^2\, dt.$$

Our next result may be called an Ito's formula for Lévy boson or fermion fields satisfying Kolmogorov's condition.

<u>Theorem 5.1</u> Let X be a Lévy boson or fermion field over $L_2(\mathbb{R}_+)$ in $\mathcal{H}$ with cyclic vector ϕ and covariance kernel K and satisfying Kolmogorov's condition. Let $\xi, \eta \in \mathcal{A}(X)$ be such that $\xi(t) \in \mathcal{M}_{t]}$ for each t and the processes $\{X_f(0,t)\xi(t)\}$ $\{X_f(o,t)\eta(t)\}$ are defined as elements of $\mathcal{A}(X)$. Suppose

$$\zeta(t) = \int_o^t X_g(ds)\xi(s) + \int_o^t \eta(s)ds .$$

Then $\zeta(t) \in D(X_f(o,t))$ and

$$X_f(o,t)\zeta(t) = \int_o^t X_f(ds)\zeta(s) - \varepsilon \int_o^t X_g(ds)X_f(o,s)\xi(s)$$
$$+ \int_o^t \{\sigma(f,g,s)\xi(s) + X_f(o,s)\eta(s)\}ds$$

where $\varepsilon = -1$ or $+1$ according as X is Lévy boson or fermion field and $\sigma(f,g,.)$ is defined by Theorem 3.2.

<u>Proof</u> We shall prove the theorem when $\eta = 0$. The more general case is proved in exactly the same manner. By the definition of Lévy boson and fermion fields we have for any finite partition $\mathcal{P}$ of $[o,t]$ by $0 = t_o < t_1 <\ldots< t_n < t_{n+1} = t$

$$X_f(o,t) \sum_{j=0}^{n} X_g(t_j, t_{j+1})\xi(t_j)$$
$$= \sum_{j=0}^{n} X_f(o,t_j)X_g(t_j, t_{j+1})\xi(t_j)$$
$$+ \sum_{j=0}^{n} \{X_f(t_j,t_{j+1})X_g(t_j,t_{j+1}) - \int_{t_j}^{t_{j+1}} \sigma(f,g,s)ds\}\xi(t_j)$$
$$+ \sum_{j=0}^{n} \{\int_{t_j}^{t_{j+1}} \sigma(f,g,s)ds\}\xi(t_j)$$
$$+ \sum_{n\geq k>j\geq 0} X_f(t_k, t_{k+1})X_g(t_j, t_{j+1})\xi(t_j)$$
$$= S_1 + S_2 + S_3 + S_4 \tag{5.3}$$

where $S_i = S_i(\mathcal{P})$ denotes the i^{th} sum on the right hand side for $i = 1,2,3,4$. By the boson-fermion property

$$\lim_{\delta(\mathcal{P})\to 0} S_1(\mathcal{P}) = -\varepsilon \int_o^t X_g(ds)X_f(o,s)\xi(s).$$

Obviously

$$\lim_{\delta(\mathcal{P})\to 0} S_3(\mathcal{P}) = \int_0^t \sigma(f,g,s)\xi(s)ds.$$

When $j \neq k$ we have from the covariance condition (v) of Lévy fields, Theorem 3.2 and property (iii) in Lemma 3.1

$$<\{X_f(t_j,t_{j+1})X_g(t_j,t_{j+1}) - \int_{t_j}^{t_{j+1}} \sigma(f,g,s)ds\}\xi(t_j),$$

$$\{X_f(t_k,t_{k+1})X_g(t_k,t_{k+1}) - \int_{t_k}^{t_{k+1}} \sigma(f,g,s)ds\}\xi(t_k)> = 0 .$$

Thus

$$||S_2(\mathcal{P})||^2 = \sum_{j=0}^{n} ||\{X_f(t_j,t_{j+1})X_g(t_j,t_{j+1}) - \int_{t_j}^{t_{j+1}} \sigma(f,g,s)ds\}\xi(t_j)||^2$$

$$\leq \sup_{0\leq s\leq t} ||\xi(s)||^2 \, V(f,g,\mathcal{P})$$

where $V(f,g,\mathcal{P}) \to 0$ as $\delta(\mathcal{P}) \to 0$ in view of Kolmogorov's condition. We can express $S_4(\mathcal{P})$ as

$$S_4(\mathcal{P}) = \sum_k X_f(t_k,t_{k+1}) \int_0^{t_k} X_g(ds)\xi(s)$$

$$+ \sum_k X_f(t_k,t_{k+1})\{\sum_{j\leq k-1} [X_g(t_j,t_{j+1})\xi(t_j) - \int_{t_j}^{t_{j+1}} X_g(ds)\xi(s)]\}.$$

As $\delta(\mathcal{P}) \to 0$ the first sum converges to $\int_0^t X_f(ds)\zeta(s)$. Writing

$$\omega_\xi(\delta) = \sup_{\substack{0\leq a\leq b\leq t \\ (b-s)<\delta}} ||\xi(a) - \xi(b)||^2$$

and using the properties of stochastic integrals we get

$$||\sum_k X_f(t_k, t_{k+1})\{\sum_{j\leq k-1} [X_g(t_j,t_{j+1})\xi(t_j) - \int_{t_j}^{t_{j+1}} X_g(ds)\xi(s)]\}||^2$$

$$= \sum_k \int_{t_k}^{t_{k+1}} \sigma(f,f,s)|| \sum_{j\leq k-1} \int_{t_j}^{t_{j+1}} X_g(ds)(\xi(t_j)-\xi(s))||^2 \, ds$$

$$= \sum_{k>j} \int_{t_k}^{t_{k+1}} \sigma(f,f,s)ds \int_{t_j}^{t_{j+1}} \sigma(g,g,s')ds'\omega_\xi(\delta(\mathcal{P}))$$

$$\leq \omega_\xi(\delta(\mathcal{P})) \int_0^t \sigma(f,f,s)ds \int_0^t \sigma(g,g,s)ds .$$

The strong continuity of ξ implies that $\omega_\xi(\delta(\mathcal{P})) \to 0$. Thus the left hand side of (5.3) converges to the required limit and at the same time $\sum_{j=0}^{n} X_g(t_j,t_{j+1})\xi(t_j)$

converges to $\zeta(t)$ as $\delta(P) \to 0$. Since $X_f(o,t)$ is closed it follows that $\zeta(t)$ belongs to $D(X_f(o,t))$ and

$$X_f(0,t)\zeta(t) = \int_0^t X_f(ds)\zeta(s) - \varepsilon \int_0^t X_g(ds)X_f(o,s)\xi(s)$$
$$+ \int_0^t \sigma(f,g,s)\xi(s)ds. \qquad \Box$$

For processes $\xi_1, \xi_2, \ldots, \xi_n, \zeta$ in $A(X)$ let

$$\xi(t) = \xi(0) + \sum_{j=1}^{n} \int_0^t X_{f_j}(ds)\eta_j(s) + \int_0^t \zeta(s)ds$$

where $\xi(0) \in M_{o]}$. Then we write

$$d\xi = \sum_j X_{f_j}(dt)\eta_j(t) + \zeta(t)dt.$$

With this differential notation we have the following result.

<u>Theorem 5.2</u> Let $f_j \in L_2(\mathbb{R}_+)$, $j = 1,2,\ldots$,

$$\xi(f_1,f_2,\ldots,f_n,t) = X_{f_1}(o,t)X_{f_2}(o,t) \ldots X_{f_n}(o,t)\phi \tag{5.4}$$

where the left hand side is interpreted as the constant vector ϕ if $n = 0$. Suppose X is a Lévy boson or fermion field satisfying the Kolmogorov condition. Then

$$d\xi(f_1,\ldots,f_n,t) = \sum_{i=1}^{n} (-\varepsilon)^{i-1} X_{f_i}(dt)\xi(f_1,\ldots,\hat{f}_i,\ldots,f_n, t)$$
$$+ \sum_{1\le i<j\le n} (-\varepsilon)^{i+j-1}\sigma(f_i,f_j,t)\ (f_1,\ldots,\hat{f}_i,\ldots,\hat{f}_j,\ldots,f_n, t)dt$$

where over a letter implies omission and $\varepsilon = -1$ or 1 according as X is bosonic or fermionic.

<u>Proof</u> For $n = 2$ put $f = f_1$, $g = f_2$, $\xi(t) \equiv \phi$, $\eta(t) \equiv 0$ in Theorem 5.1. Then

$$d\xi(f_1,f_2,t) = X_{f_1}(dt)\xi(f_2,t) - \varepsilon X_{f_2}(dt)\xi(f_1,t) + \sigma(f_1,f_2,t)dt .$$

Now a routine induction with the help of Theorem 5.1 yields the required result.

<u>Theorem 5.3</u> Let X be a Lévy boson or fermion field over $L_2(\mathbb{R}_+)$ in H with cyclic vector ϕ, covariance kernel K and satisfying the Kolmogorov condition. Then

$$\{X(f)X(g) + \varepsilon X(g)X(f)\}u = \{K(f,g) + \varepsilon K(g,f)\}u$$

for all $u \in M$, $f,g \in L_2(\mathbb{R}_+)$.

Proof Let $\xi(f_1,\ldots,f_n,t)$ be defined by (5.4) and

$$E(f_1,\ldots,f_n,t) = \langle \phi, \xi(f_1,\ldots,f_n,t)\rangle .$$

By Theorem 5.2 we have

$$\frac{d}{dt} E(f_1,\ldots,f_n,t) = \sum_{i<j} (-\varepsilon)^{i+j-1} \sigma(f_i,f_j,t)E(f_1,\ldots,\hat{f}_i,\ldots,\hat{f}_j,\ldots,f_n,t)$$

where σ is defined by Theorem 3.2. A routine algebra now shows that for $f_1,\ldots,f_m$, $f,g,g_1,\ldots,g_n$ in $L_2(\mathbb{R}_+)$ we have by induction on the pair (m,n)

$$\frac{d}{dt} \{E(f_1,\ldots,f_m,f,g,g_1,\ldots,g_n,t) + \varepsilon E(f_1,\ldots,f_m,g,f,g_1,\ldots,g_n,t)\}$$
$$= \frac{d}{dt} \{[K(f,g,t) + K(g,f,t)]E(f_1,\ldots,f_m,g_1,\ldots,g_n,t)\}.$$

Now the strong continuity property (iii) of the Lévy field yields the required result. $\square$

Corollary Let X, X' be Lévy boson (fermion) fields over $L_2(\mathbb{R}_+)$ in H, H' respectively with cyclic vectors ϕ,ϕ' and same covariance kernel K. Suppose, in addition, Kolmogorov's condition is fulfilled by both the fields. Then they are equivalent.

Proof Let

$$E^{\#}(f_1,f_2,\ldots,f_n,t) = \langle \phi^{\#}, X^{\#}_{f_1}(o,t)X^{\#}_{f_2}(o,t)\ldots X^{\#}_{f_n}(o,t)\phi^{\#}\rangle$$

where # indicates two identities with or without the prime '. Then from the proof of the theorem it is clear that the family of functions $E^{\#}$ obey the same set of differential equations

$$\frac{d}{dt} E^{\#}(f_1,\ldots,f_n,t) = \sum_{i<j} (-\varepsilon)^{i+j-1} \sigma(f_i,f_j,t)E^{\#}(f_1,\ldots,\hat{f}_i,\ldots,\hat{f}_j\ldots,f_n,t),$$

where

$$E^{\#}(f,t) = 0,$$

$$E(f_1,f_2,t) = E'(f_1,f_2,t) = K(\chi_{[o,t]}f_1, \chi_{[o,t]}f_2),$$

$$E^{\#}(f_1, f_2,\ldots,f_n,o) = 0.$$

Thus

$$E(f_1,\ldots,f_n,t) \equiv E'(f_1,\ldots,f_n,t).$$

Letting $t \to \infty$ one obtains

$$\langle\phi, X(f_1)\ldots X(f_n)\phi\rangle = \langle\phi', X'(f_1)\ldots X'(f_n)\phi'\rangle.$$

Now the properties of a Lévy field with a cyclic vector and covariance kernel imply that there exists a unitary isomorphism $U : H \to H'$ such that

$$UX(f_1)\ldots X(f_n)\phi = X'(f_1)\ldots X'(f_n)\phi'$$

for all f_j, $1 \leq j \leq n$, $n = 1,2,\ldots$. $\square$

§ 6. Construction of Lévy boson and fermion fields with a given covariance kernel

In view of the Corollary to Theorem 5.3 it suffices to construct one model of a Lévy boson or fermion field satisfying the Kolmogorov condition. Using the methods of quantum stochastic calculus [2], [3], [4], [6] we shall indicate briefly how to construct models of Lévy fields. Already Remark 2 in Section 4 shows that the background for a Lévy field is the notion of Fock space.

Let $\mathfrak{h}$ be any complex separable Hilbert space. The boson Fock space $\Gamma(\mathfrak{h})$ associated with $\mathfrak{h}$ is defined by

$$\Gamma(\mathfrak{h}) = \mathbb{C} \oplus \mathfrak{h} \oplus \ldots \oplus \mathfrak{h}^{\circledS n} \oplus \ldots$$

where $\circledS^{n}$ denotes n-fold symmetric tensor product. To any $u \in \mathfrak{h}$ the coherent or exponential vector $\psi(u)$ associated with u is defined by

$$\psi(u) = 1 \oplus u \oplus (2!)^{-\frac{1}{2}} u^{\otimes 2} \oplus \ldots \oplus (n!)^{-\frac{1}{2}} u^{\otimes n} \oplus \ldots$$

The set of all such coherent vectors is linearly independent and total in $\Gamma(\mathfrak{h})$. Furthermore

$$\langle\psi(u), \psi(v)\rangle = \exp \langle u,v\rangle.$$

This may be compared with the identity concerning inner products of intrinsic coherent vectors of Lévy fields in Remark 2 of Section 4. We adopt the convention that scalar products $\langle .,. \rangle$ are conjugate linear in the first and linear in the second variable. Let $\mathcal{E}$ denote the linear manifold generated by the set of all coherent vectors. To any $u \in \mathfrak{h}$ there exist closed operators $a(u)$, $a^{\dagger}(u)$ in $\Gamma(\mathfrak{h})$ satisfying

$$a(u)\,\psi(v) = \langle u,v\rangle\,\psi(v),$$
$$a^{\dagger}(u)\psi(v) = \frac{d}{d\varepsilon}\,\psi(v+\varepsilon u)\Big|_{\varepsilon=0} \quad \text{for all} \quad u,v \in \mathfrak{h}$$
$$\langle \xi,\, a(u)\eta\rangle = \langle a^{\dagger}(u)\xi,\eta\rangle \quad \text{for all} \quad \xi,\eta \in \mathcal{E}\,.$$

There exist selfadjoint operators $P(u)$, $Q(u)$ in $\Gamma(\mathfrak{h})$ such that $P(u) = -i(a(u) - a^{\dagger}(u))$, $Q(u) = a(u) + a^{\dagger}(u)$ on $\mathcal{E}$ for all $u \in \mathfrak{h}$. One has the following commutation relations in the domain $\mathcal{E}$:

$$[a(u),\, a(v)] = 0, \quad [a^{\dagger}(u),\, a^{\dagger}(v)] = 0, \quad [a(u),\, a^{\dagger}(v)] = \langle u,v\rangle.$$

$a(u)$ and $a^{\dagger}(u)$ are respectively called the annihilation and creation operators associated with u.

To any contraction operator T on $\mathfrak{h}$ there corresponds a contraction operator $\Gamma(T)$ called the second quantization of T satisfying the property

$$\Gamma(T)\psi(u) = \psi(Tu) \quad \text{for all} \quad u \in \mathfrak{h}\,.$$

The correspondence $T \to \Gamma(T)$ is a continuous $*$ homomorphism from the $*$ semigroup of contractions on $\mathfrak{h}$ to that on $\Gamma(\mathfrak{h})$. In particular, $\Gamma(T)$ is unitary whenever T is unitary.

Suppose now that K is the covariance kernel of a Lévy field over $L_2(\mathbb{R}_+)$ with a cyclic vector ϕ. In view of Theorem 3.2 and the subsequent Remark there exists a 2×2 bounded positive definite matrix valued function $((\sigma_{ij}(t)))$ such that

$$\begin{aligned} K &= K_1 + K_2\,, \\ K_j(f,g) &= \int_{C_j} (f_1,f_2)\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}\begin{pmatrix}g_1\\ g_2\end{pmatrix} dt\,, \quad j = 1,2, \\ C_j &= \{t : \text{Rank}\ ((\sigma_{k\ell}(t))) = j\}\,, \quad j = 1,2, \end{aligned} \tag{6.1}$$

where $f = f_1 + if_2$, $g = g_1 + ig_2$. Define

$$\rho_{\pm}(t) = p_{\pm}(t)^{\frac{1}{2}} \begin{pmatrix}\sigma_{11}(t) & \pm\ \text{Re}\ \sigma_{12}(t)\\ \\ \pm\ \text{Re}\ \sigma_{12}(t) & \sigma_{22}(t)\end{pmatrix}^{\frac{1}{2}}, \quad t \in C_2$$

where

$$p_{\pm}(t) = \frac{1}{2}\,\{1 \pm \frac{\text{Im}\ \sigma_{12}(t)}{(\sigma_{11}(t)\sigma_{22}(t) - [\text{Re}\ \sigma_{12}(t)]^2)^{\frac{1}{2}}}\}\,.$$

Let

$$((\sigma_{ij}(t))) = \begin{pmatrix} \rho_1(t) \\ \\ \rho_2(t) \end{pmatrix} (\bar{\rho}_1(t), \bar{\rho}_2(t)), \quad t \in C_1$$

where ρ_1, ρ_2 are suitable Borel functions on C_1. In the Hilbert space $L_2(C_2)$ define real linear maps $S_\pm$ by putting

$$\begin{pmatrix} \mathrm{Re}\,(S_\pm f)(t) \\ \\ \mathrm{Im}\,(S_\pm f)(t) \end{pmatrix} = \rho_\pm(t) \begin{pmatrix} \mathrm{Re}\, f(t) \\ \\ \mathrm{Im}\, f(t) \end{pmatrix}, \quad t \in C_2 .$$

With these notations we shall now construct explicitly a Lévy boson field satisfying the Kolmogorov condition and having covariance kernel K. To this end consider the Hilbert space

$$H = \Gamma(L_2(C_1)) \otimes \Gamma(L_2(C_2)) \otimes \Gamma(L_2(C_2)). \tag{6.2}$$

Let ϕ be the product of the coherent vectors $\psi(0)$ in each of the three Fock spaces appearing in H, so that

$$\phi = \psi(0) \otimes \psi(0) \otimes \psi(0).$$

Note that H may also be viewed as the Fock space $\Gamma(\mathfrak{h})$ where $\mathfrak{h} = L_2(C_1) \oplus L_2(C_2) \oplus L_2(C_2)$. Let $\mathcal{E}$ denote the linear manifold generated by coherent vectors in H. For any $f \in L_2(\mathbb{R}_+)$ let f_{C_1}, f_{C_2} denote the restrictions of f to C_1, C_2 respectively. From the remarks made at the beginning of this section and the theory of tensor products it follows that for any $f \in L_2(\mathbb{R}_+)$ one can associate a unique selfadjoint operator $X^-(f)$ in H such that on the domain

$$\begin{aligned} X^-(f) = {} & \{a(\bar{\rho}_1 \,\mathrm{Re}\, f_{C_1} + \bar{\rho}_2 \,\mathrm{Im}\, f_{C_1}) + a^\dagger(\bar{\rho}_1 \,\mathrm{Re}\, f_{C_1} + \bar{\rho}_2 \,\mathrm{Im}\, f_{C_1})\} \otimes 1 \otimes 1 \\ & + 1 \otimes \{a(S_+ f_{C_2}) + a^\dagger(S_+ f_{C_2})\} \otimes 1 \\ & + 1 \otimes 1 \otimes \{a(S_- \bar{f}_{C_2}) + a^\dagger(S_- \bar{f}_{C_2})\} \end{aligned}$$

where the right hand side is expressed in the factorisation (6.2).

<u>Theorem 6.1</u> The family $X^- = \{X^-(f), f \in L_2(\mathbb{R}_+)\}$ defined as above by (6.3) is a Lévy boson field having cyclic vector ϕ, covariance kernel K determined by (6.1) and satisfying the Kolmogorov condition.

Proof This follows essentially from the basic properties of annihilation and creation operators in boson Fock spaces. (See [4]). We shall just verify that X^- has covariance kernel K. In the decomposition (6.2) we have

$$X^-(f)\phi = (\bar{\rho}_1 \operatorname{Re} f_{C_1} + \bar{\rho}_2 \operatorname{Im} f_{C_1}) \otimes \psi(0) \otimes \psi(0) + \psi(0) \otimes S_+ f_{C_2} \otimes \psi(0) + \psi(0) \otimes \psi(0) \otimes S_- \bar{f}_{C_2} .$$

In particular,

$$\begin{aligned} K(f,g) &= \langle X^-(f)\phi \ , \ X^-(g)\phi\rangle \\ &= \langle\bar{\rho}_1 \operatorname{Re} f_{C_1} + \bar{\rho}_2 \operatorname{Im} f_{C_1}, \bar{\rho}_1 \operatorname{Re} g_{C_1} + \bar{\rho}_2 \operatorname{Im} g_{C_1}\rangle \\ &\quad + \langle S_+ f_{C_2}, S_+ g_{C_2}\rangle + \langle S_- \bar{f}_{C_2}, S_- \bar{g}_{C_2}\rangle \\ &= \int_{C_1} (\operatorname{Re} f, \operatorname{Im} f) \begin{pmatrix} |\rho_1|^2 & \rho_1\bar{\rho}_2 \\ \bar{\rho}_1\rho_2 & |\rho_2|^2 \end{pmatrix} \begin{pmatrix} \operatorname{Re} & g \\ \operatorname{Im} & g \end{pmatrix} dt \\ &\quad + \langle S_+ f_{C_2}, S_+ g_{C_2}\rangle + \langle S_- \bar{f}_{C_2}, S_- \bar{g}_{C_2}\rangle . \end{aligned}$$

By definition the first term on the right hand side is $K_1(f,g)$. Noting that

$$\langle f,g\rangle = \int_0^\infty (\operatorname{Re} f, \operatorname{Im} f) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \operatorname{Re} g \\ \operatorname{Im} g \end{pmatrix} dt$$

we observe that

$$\begin{aligned} &\langle S_+ f_{C_2}, S_+ g_{C_2}\rangle + \langle S_- \bar{f}_{C_2}, S_- \bar{g}_{C_2}\rangle \\ &= \int_{C_2} (\operatorname{Re} f, \operatorname{Im} f) \left\{ {}^{\rho_+}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}^{\rho_+} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\rho_-}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}^{\rho_-}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \begin{pmatrix} \operatorname{Re} g \\ \operatorname{Im} g \end{pmatrix} dt \end{aligned}$$

A fairly simple calculation shows that the matrix within the brackets { } is $((\sigma_{ij}(t)))$ and hence the right hand side of the above equation is $K_2(f,g)$. $\square$

To construct a Lévy fermion field with the covariance kernel K given by (6.1) we adopt the methods of quantum stochastic integration used in [3], [6]. Consider the Hilbert space $\mathcal{H}$ in (6.2). In $L_2(C_1)$ and $L_2(C_2)$ consider the families of reflection operators $\{R_i(s), s \geq 0\}$, i = 1,2 respectively defined by

$$[R_i(s)f](t) = \begin{cases} -f(t) & \text{if} \quad t \in C_i, \quad t < s \\ f(t) & \text{if} \quad t \in C_i, \quad t \geq s . \end{cases}$$

Then $\Gamma(R_i(s)) = J_i(s)$ is a unitary operator valued process in $\Gamma(L_2(C_i))$, i = 1,2.

Define the operators

$$J(s) = J_1(s) \otimes J_2(s) \otimes J_2(s)$$

in H using the factorisation (6.2) and put

$$X^+(f) = \int_0^\infty J(s)\, X_f^-(ds)$$

where $\{X_f^-(t)\}$ is the adapted process defined by $X_f^-(t) = X^-(f_{t]})$. The square integrability of f and the boundedness of the functions $\sigma_{ij}(t)$ imply that the operators $X^+(f)$ are well defined on $\mathcal{E}$. Using quantum Ito's formula for quantum stochastic integrals with respect to the creation, annihilation and gauge processes as outlined in [2], [3], [4],and [6] one can prove that $X^+(f)$ extends to a bounded selfadjoint operator satisfying the anticommutation relations

$$X^+(f)\, X^+(g) + X^+(g)\, X^+(f) = 0$$

whenever f and g have disjoint supports. Since $J(s)$ fixes the vector $\phi = \psi(0) \otimes \psi(0) \otimes \psi(0)$ it follows that $X^+(f)\phi = X^-(f)\phi$ and hence $\langle X^+(f)\phi, X^+(g)\phi\rangle = K(f,g)$ for all $f,g \in L_2(\mathbb{R}_+)$. This yields the following theorem.

<u>Theorem 6.2</u> The family $X^+ = \{X^+(f),\ f \in L_2(\mathbb{R}_+)\}$ is a Lévy fermion field over $L_2(\mathbb{R}_+)$ in H satisfying the Kolmogorov condition with cyclic vector ϕ and covariance kernel K.

<u>Remark</u> Suppose that $\operatorname{Im} K(f,g) \equiv 0$. Then the selfadjoint operators $X^-(f)$, $f \in L_2(\mathbb{R}_+)$ in Theorem 6.1 commute. In this case X^- is a classical Gaussian field of random variables in the state ϕ.

If $\operatorname{Re} K(f,g) \equiv 0$ then the selfadjoint operators $X^+(f)$, $f \in L_2(\mathbb{R}_+)$ anticommute with each other and one obtains a Grassmann field.

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Summary : Starting from martingale hypotheses the canonical commutation and anticommutation relations for fields of observables in one dimension are derived.

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