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by Sylvestre GALLOT

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0. Introduction

In 1919, in the course of his fundamental work [PL], Paul Lévy proved an isoperimetric inequality for domains in a convex hypersurface of $\mathbf{R}^{n+1}$, which can be formulated as follows :

Let Ω be any domain in a convex hypersurface M of $\mathbf{R}^{n+1}$, let Ω^* be the corresponding symmetric domain on S^n (i.e. $\dfrac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(M)} = \dfrac{\mathrm{Vol}(\Omega^*)}{\mathrm{Vol}(S^n)}$ and Ω^* is a geodesic ball in S^n), then $\dfrac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(M)} \geqslant \dfrac{\mathrm{Vol}(\partial\Omega^*)}{\mathrm{Vol}(S^n)}$

A few years ago, M. Gromov ([GV 3]) popularised this aspect of P. Lévy's thoughts by extending the result and the method to all Riemannian manifolds whose Ricci curvature is larger than that of the standard sphere.

Since then, it has been observed ([GT 1], [GT 2]) that some inequality is still obtained if only some (eventually negative) uniform lower bound δ on Ricci curvature is given, together with a bound D on the diameter of the manifold.

This last result has been sharpened ([B–B–G]) and formulated exactly in the same way as Paul Lévy's one (except for the fact that the sphere S^n must be replaced by another one whose radius can be computed a priori in terms of δ and D).

The purpose of this paper is to show that the above L^∞-assumption on the negative part of Ricci curvature is not necessary and that a bound on some integral norm of the negative part of Ricci curvature is in fact sufficient :

THEOREM 3. — *Let α and D be any positive constants and p be any element of $]n+\infty]$. In any Riemannian manifold (M^n, g) whose diameter is bounded by D and whose Ricci curvature satisfies*

$$\frac{1}{\mathrm{Vol}\, M}\int_M \left(\frac{r_-}{\alpha^2} - 1\right)_+^{p/2} dv_g \leqslant \frac{1}{2}(e^{B(p)\alpha D} - 1)^{-1}$$

[where $r(x) = \inf\limits_{X \in T_xM\setminus\{0\}} \left[\dfrac{\mathrm{Ric}(X,X)}{(n-1)g(X,X)}\right]$ and $r_-(x) = \sup(0, -r(x))]$ every domain Ω satisfies

$$\frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}\, M} \geqslant \gamma(\alpha, D).\mathrm{Min}\left[\frac{\mathrm{Vol}\,\Omega}{\mathrm{Vol}\, M}, \frac{\mathrm{Vol}\, M\setminus\Omega)}{\mathrm{Vol}\, M}\right]^{1-\frac{1}{p}} ,$$

(where $B(\rho)$ and $\gamma(\alpha, D)$ are computed in theorems 2 and 3).

We explain the significance of these successive generalizations for Riemannian geometry. One trend of Riemannian geometry since the 50's has been to investigate how the local properties of a metric (e.g., curvature bounds) interact with the global topology of a manifold, or influence certain global quantities (*e.g.*, eigenvalues λ_i of the Laplacian). M. Gromov's theorem implies that, on the class of manifolds he considers, each eigenvalue of the Laplacian has a uniform lower bound. Our result has similar consequences (see theorems 6, 7, 8, 10, 11). In particular, *under the same assumptions "diameter bounded by* D *and "Ricci curvature satisfying* $\frac{1}{\mathrm{Vol}\, M}\int_M (\frac{r_-}{\alpha^2}-1)_+^{p/2} dv_g \leqslant \frac{1}{2}(e^{B(p)\alpha D}-1)^{-1}$*", we can compute constants* $A(p,\alpha,D), B(p,\alpha,D), Z(p,\alpha,D)$ *such that*

$$\lambda_i \geqslant A(p,\alpha,D) i^{2/p}$$

$$\mathrm{Vol}(M,g).k_M(t,x,y) \leqslant B(p,\alpha,D).t^{-p/2}$$

$$b_1(M^n) \leqslant n.Z(p,\alpha,D)$$

$$b_i(M^n) \leqslant \binom{n}{i} \left[Z(p,\alpha,D).\Big(\frac{1}{\mathrm{Vol}\, M}\int |\sigma|^{p/2}\Big)\right]^{\frac{2n}{p-n}} ,$$

where k_M *is the heat kernel of* (M,g) *with initial datum* δ_x *and where* σ *is the sectional curvature of* (M,g) .

Why are we interested in integral norms of the curvature? There are topological invariants of manifolds which can be easily estimated in terms of the average of curvature of any metric on the manifold. For example, on any surface M^2 , the Gauss-Bonnet formula gives $\chi(M) = \int_M \sigma(g).dv_g$ for any metric g on M^2 . The Chern-Weil formulas for characteristic classes in higher dimensions have a similar form : the characteristic number appears as the integral of some polynomial of degree $n/2$ in the curvature. This indicates that the best one should expect is estimates of topological data in terms of the average of the $\frac{n}{2} - th$ power of curvature.

Also, our theorems 3, 6, 7, 8, 10, 11 and 13 widen the possibility (already present in previous results) of dealing with singular objects. For instance, a polyhedral surface can obviously be smoothed while keeping the integral of curvature bounded (spectral properties of such singular objects has already been studied in [C–M–S] and by J. Cheeger in several works). Since, there is a limitation, and the assumption in theorems 3, 6, 10, 11 are nearly optimal (see counter-examples in the appendix). In Riemannian surgery (constructing new Riemannian manifolds by gluing known pieces together – this often requires torturing the metrics near the boundaries) it is hard to keep uniform control of curvature, but $L^{p/2}$-norms of curvature behave somewhat better. Thus it is a general rule that few invariants can be estimated in terms of these integral norms (see the appendix). This is why we have to assume that the $L^{p/2}$ norm of the part of Ricci curvature which is not bounded from below is small enough for at least one $p \in]n,+\infty[$. The appendix shows that no estimate can be obtained in terms of diameter and $\int_M r_-^{p/2}$ and that p cannot be replaced by n in the assumptions (except for $n=2$). Let us explain the role played by Ricci curvature :

NOTATIONS. — From the curvature tensor $R(\bullet,\bullet,\bullet,\bullet)$ of a Riemannian manifold, one defines the *sectional curvature* $\sigma(P)$ (at the point x and in the direction of a 2-dimensional subspace P of the tangent space T_xM) by choosing 2 orthogonal vectors $\{X,Y\}$ in P and writing

$$\sigma(P) = R(X,Y,X,Y) .$$

The *Ricci-curvature tensor* Ric *or* Ricci is the quadratic tensor defined, for any $X \in T_xM$, by

$$\mathrm{Ric}_x(X,X) = \sum_{i=1}^{n} R(X,e_i,X,e_i) = \sum_{i=1}^{n} \sigma(X,e_i)\cdot \|X\|^2 ,$$

where $\{e_i\}$ is an orthonormal basis of T_xM such that $e_1 = X/\|X\|$. The *scalar curvature* is the function on M defined by $\mathrm{scal}(x) = \mathrm{Trace}(\mathrm{Ric}_x)$.

Let us recall that $r(x) = \inf_{X\in T_xM\setminus\{0\}} \left[\frac{\mathrm{Ric}(X,X)}{(n-1)g(X,X)}\right]$ and $r_-(x) = \sup(0,-r(x))$.

A general feeling among Riemannian geometers is that the assumption *"scalar curvature bounded"* tells almost nothing on the manifold, whereas *"sectional curvature bounded"* is a rather strong condition. That's why a series of results ([GV 1,2,3,4], [GT 1,2,3], [L–Y 1], [B–G] and [B–B–G]) has led geometers to a vast program, known as *"Principe de la domination universelle de la courbure de Ricci"* (M. Berger) : bounding topological and geometric invariants on the set of all Riemannian manifolds (whatever their topology and their metric are) whose Ricci curvature is bounded from below and diameter from above, though this set contains an infinite number of possible topologies (see 3.A.a for more explanations about this *"principle"* applied to spectral invariants). Theorem 5 shows that the main step in this program is to establish an ad hoc isoperimetric inequality. The one we give in Theorem 3 allows us to extend the *"principe de la domination universelle de la courbure de Ricci"* to the case where the part of Ricci curvature which is not bounded from below has small $L^{p/2}$-norm for at least one $p \in]n,+\infty]$.

The proof of theorem 3 follows Paul Lévy's method. We improve the step where the volume of a tubular neighbourhood of a hypersurface is estimated in terms of its mean curvature (theorem 2) and show that the mean curvature of hypersurfaces which fulfil the equality-case in the isoperimetric inequality can be a priori computed in terms of the isoperimetric constant (lemma 4). As a by-product, we obtain an estimate on the volume of balls in Riemannian manifolds which has been conjectured by M. Gromov (this is kind of the opposite of an isoperimetric inequality) :

THEOREM 1. — *On any complete Riemannian manifold (M,g) with non finite volume, for any $x \in M$ and any $p \in]n,+\infty]$, when R goes to infinity,*

$$\frac{\mathrm{Vol}\,\partial B(x,R)}{\mathrm{Vol}\,B(x,R)} \leqslant C'(n,p)\left[\frac{\int_{B(x,R)} r_-^{p/2}\,dv_g}{\mathrm{Vol}\,B(x,R)}\right]^{1/p} + \mathrm{O}(\mathrm{Vol}\,B(x,R)^{-1/p}) .$$

Combined with a method due to M. Gromov ([GV 2]), this result gives bounds for the topological norms of homology classes and, in particular, for the simplicial volume. Following [GV 2], let us call $\|c\|$ *the l_1-norm of any real chain c; i.e.* if $\sum_{i\in I} c_i\cdot s_i$ is the decomposition of c in terms of elementary simplices s_i, then $\|c\| = \sum_{i\in I}|c_i|$. For every homology-class γ, let us define $\|\gamma\| = \inf\{\|c\| : c \text{ such that } [c]=\gamma\}$. The simplicial volume is, by definition, equal to $\|[M]\|$, where $[M]$ is the fundamental class of M, we then obtain the

THEOREM 8. — *Let (M^n,g) be any Riemannian manifold. For any $p \in]n,+\infty[$ and any closed i-dimensional chain c,*

$$\|[c]\| \leqslant i!C'(n,p)^i\,\mathrm{Vol}_i(c)\cdot \sup_{x\in\widetilde{M}}\left[\liminf_{R\to\infty}\left(\mathrm{Vol}\,\widetilde{B}(x,R)^{-1}\int_{\widetilde{B}(x,R)} r_-^{p/2}\cdot dv_{\tilde{g}}\right)\right]^{i/p}$$

where the $\widetilde{B}(x,R)$ are balls of radius R in the Riemannian universal covering $(\widetilde{M},\widetilde{g})$ of (M,g) and where C' is defined in Theorem 1. In particular, the simplicial volume is bounded by $\mathrm{Vol}(M,g)$ *multiplied by some $L^{p/2}$ average of the negative part of Ricci-curvature.*

Notice that, in dimension 2, it gives a bound of the simplicial volume in terms of a L^1-norm of r_- , which is not far from the Gauss-Bonnet formula. Another by product is the proposition 7, which gives an upper estimate of the infimum of the spectrum of a non compact manifold in terms of the average of the $\frac{p}{2}-th$ power of Ricci curvature, improving a result of H. Donnelly ([DY]).

We have discussed about the possibility of weakening the assumption *"Ricci curvature bounded from below"* (in the *"principe de la domination universelle ..."*) in an integral one, is it possible to weaken the assumption *"diameter bounded"*? We shall see in a further paper ([GT 5]) that it is possible to replace it by the assumption : *"the portion of the whole volume of the manifold which lies in a bounded tubular neighbourhood of a median hypersurface is not too small"*, using the concept of concentration introduced by Paul Lévy in [PL] and developed by M. Gromov and V.D. Milman (see [MN]). We have already proved (see [GT 2] and [GT 4]) that the assumption *"diameter bounded"* may be replaced, in some cases, by the weaker assumption *"Cheeger's isoperimetric constant bounded from below"*.

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1. Volume of tubes

Let (M^n,g) be a complete Riemannian manifold whose dimension is n . Let H be a compact smoothly imbedded submanifold in M^n .

DEFINITION. — The R-tubular neighbourhood $T(H,R)$ of H is the set of all points in M whose distance to H is not greater than R .

We call *"mean curvature at the point $x \in H$"* the number $\eta(x) = \dfrac{1}{n-1}\mathrm{Trace}(II_x)$, where II_x is the second fundamental form of H at the point x .

THEOREM 1. — *Let (M^n,g) be a complete, non compact Riemannian manifold with non finite volume. For any compact submanifold H in M^n and any $p \in]n,+\infty]$, one has*

(i) if $\int_M r_-^{p/2}\cdot dv_g < +\infty$, then

$$\lim_{R\to+\infty}\frac{(\mathrm{Vol}\,T(H,R))^{1/p}}{R} = \lim_{R\to\infty}\frac{\mathrm{Vol}\,\partial T(H,R)}{(\mathrm{Vol}\,T(H,R))^{1-\frac{1}{p}}} = 0\ .$$

(ii) More generally, for any $s \in]-\infty, \frac{1}{p}[$,

$$\limsup_{R\to+\infty} \frac{1}{R} L_s(\operatorname{Vol} T(H,R)) \leqslant \limsup_{R\to+\infty} \frac{\operatorname{Vol} \partial T(H,R))}{[\operatorname{Vol} T(H,R)]^{1-s}}$$

$$\leqslant C'(n,p) \cdot \limsup_{R\to+\infty} \left(\frac{\int_{T(H,R)} r_-^{p/2} dv_g}{(\operatorname{Vol} T(H,R))^{1-ps}} \right)^{1/p} ,$$

where $L_s(x) = \dfrac{x^s}{s}$ when $s \neq 0$ and $L_0(x) = \log(x)$, and where

$$C'(n,p) = 2^{1/p}(n-1)^{\frac{p-1}{p}} \left(\frac{p(p-2)}{(p-1)(p-n)} \right)^{\frac{p-2}{2p}}$$

Remarks. — (i) is a sharp comparison with the euclidean case; (ii) is sharp for the hyperbolic space of dimension n (let s be equal to zero and p go to infinity), but also (up to a multiplicative constant) when $r_-(x) \simeq \dfrac{-C}{d(H,x)^2}$ when x goes to infinity. If $\dim M = 2$, (ii) is still true for $p = n$ and $C'(2,2) = \sqrt{2}$.

This theorem also gives a control on the growth of balls $B(x,R)$ (just consider the case $H = \{x\}$). An application is given in section 4.

In the general case (M^n compact or not, with finite or non finite volume), we are considering only hypersurfaces H in M^n (these hypersurfaces are compact or not, but with finite $(n-1)$-dimensional volume). The following theorem will be true for a more general class of hypersurfaces than the regular ones, *i.e.* for all hypersurfaces H satisfying the

REGULARITY PROPERTY. — For almost every $x \in M \setminus H$, any minimizing geodesic from x to H attains H at a regular point.

THEOREM 2. — *Let (M^n, g) be any complete Riemannian manifold and H an hypersurface satisfying the above regularity property. For every $p \in]n, +\infty]$ and any $R, \alpha \in]0, +\infty[$,*

$$\operatorname{Vol} T(H,R) \leqslant (e^{B(p).\alpha R} - 1)$$

$$\left[\frac{2}{(B(p)\alpha)} \operatorname{Vol}(H) + \frac{(n-1)^{p-1}}{(B(p)\alpha)^p} \int_H |\eta|^{p-1} dv_{g_H} + \int_{T(H,R)} \left(\frac{r_-}{\alpha^2} - 1 \right)_+^{p/2} \cdot dv_g \right],$$

where $\left(\dfrac{r_-}{\alpha^2} - 1\right)_+ = \sup\left(\dfrac{r_-}{\alpha^2} - 1, 0\right)$ bounds the part of Ricci curvature which goes below the constant $-(n-1)\alpha^2$, and where

$$B(p) = \left(\frac{2(p-1)}{p} \right)^{1/2} (n-1)^{1-\frac{1}{p}} \left(\frac{p-2}{p-n} \right)^{\frac{1}{2}-\frac{1}{p}} .$$

Proofs of theorems 1 and 2. — Let Ω be any domain in (M,g) whose boundary satisfies the above regularity property. Let $u_+ = \sup(u,0)$ and $\Omega_R = \{x : d(x,\Omega) < R\}$. For any positive R, α, ε and any $p \in]n, +\infty]$, we shall prove the two following inequalities,

(1) $$\operatorname{Vol}(\partial\Omega_R)^{p/(p-1)} - \operatorname{Vol}(\partial\Omega)^{p/(p-1)} \leqslant p(n-1)(p-1)^{-1}(\operatorname{Vol}\Omega_R - \operatorname{Vol}\Omega)$$

$$\left[\int_{\partial\Omega} \eta_+(x)^{p-1} dx + 2^{-((p-2)/2)}(n-1)^{-(p-1)} B(p)^p \int_{\Omega_R \setminus \Omega} r_-^{p/2} dv_g \right]^{1/(p-1)}$$

$$\text{(2)}\quad \mathrm{Vol}(\Omega_{R+\varepsilon}) \leqslant e^{B(p)\alpha\varepsilon}\,\mathrm{Vol}(\Omega_R) + \left(e^{B(p)\alpha\varepsilon}-1\right)\Big[-\mathrm{Vol}\,\Omega + \left(B(p)\alpha\right)^{-1}\mathrm{Vol}(\partial\Omega) +$$
$$(B(p)\alpha)^{-p}(n-1)^{p-1}\int_{\partial\Omega}\eta_+(x)^{p-1}dx + \int_{\Omega_{R+\varepsilon}\setminus\Omega}((r_-/\alpha^2)-1)_+^{p/2}dv_g\Big]$$

Proof of (1) and (2). — Let $\partial\widetilde{\Omega}$ be the set of regular points in the boundary of Ω . Let us call "*normal coordinates system*" the application Φ from $]-\infty,+\infty[\times\partial\widetilde{\Omega}$ onto M defined by $\Phi(t,x)=\exp_x(t\cdot N_x)$, where N_x is outward unit normal vector at the point x of $\partial\widetilde{\Omega}$. From Hopf-Rinow's theorem, Φ is surjective and is a diffeomorphism from some open subset V in $]-\infty,+\infty[\times\partial\widetilde{\Omega}$ onto an open subset $\overset{\circ}{M}$ in M whose complement is of measure zero. More precisely, $V=\{(t,x): t_-(x)<t<t_+(x)\}$, where $]t_-(x),t_+(x)[$ is the greatest interval on which the geodesic $t\mapsto\Phi(t,x)$ minimizes the distance from $\Phi(t,x)$ to $\partial\widetilde{\Omega}$. Let us define $J(t,x)$ by

$$\Phi^* dv_g = J(t,x)^{n-1}dt\ dx.$$

By Heintze-Karcher's theorem [H-K] (for another proof by analytic methods see [GT 4]), one has, for $t\in]t_-(x),t_+(x)[$,

$$\frac{\partial^2}{\partial t^2}J(t,x)+(n-1)^{-1}\cdot\mathrm{Ric}[\Phi_*(\partial/\partial t),\Phi_*(\partial/\partial t)]\cdot J(t,x)\leqslant 0\ .$$

Let us note $(\ \cdot\ )'$ instead of $\frac{\partial}{\partial t}(\ \cdot\ )$. By definition of r_- and a direct calculus, for any positive δ and in any point $\Phi(t,x)$ such that $(t,x)\in V$, we obtain

$$\left(\frac{J'}{J^\delta}\right)'+\delta\cdot J'^2/J^{1+\delta}=J''/J^\delta\leqslant r_-\cdot J^{1-\delta}\ .$$

As

$$(p/2)(p/(p-2))^{p-2)/2}x\leqslant \sup(1+x,0)^{p/2}\ ,$$

it comes

$$(p/2)(p.\delta/(p-2))^{(p-2)/2}(J'/J^\delta)'\cdot|J'/J^\delta|^{p-2}\leqslant r_-^{p/2}\cdot J^{(p-1)(1-\delta)}\ .$$

Let us take $\delta=(p-n)/(p-1)$. We integrate the above inequality from 0 to t and use the information on initial conditions, i.e. $J(0,x)=1$ and $J'(0,x)=\eta(x)$. We then obtain

$$\text{(3)}\qquad \left(\frac{J'}{J^\delta}\right)^{p-1}(t,x)\leqslant \eta_+(x)^{p-1}+2^{-((p-2)/2)}(n-1)^{1-p}B(p)^p\int_0^t r_-^{p/2}\cdot J(s,x)^{n-1}ds.$$

The important fact is that the last integral has a geometric meaning because J^{n-1} *is the density of the Riemannian measure.* Let us define

$$J_+(t,x)=\begin{cases} J(t,x) & \text{if } t\in]t_-(x),t_+(x)[\\ 0 & \text{elsewhere}\end{cases}$$

$$L(R)=\int_{\partial\Omega}J_+(R,x)^{n-1}dx\ ,$$

$$A(R)=\mathrm{Vol}(\Omega_R\setminus\Omega)=\int_{\partial\Omega}\int_0^R J_+(t,x)^{n-1}dt\ dx\ .$$

As J_+ is always nonnegative, it comes immediately that

$$\limsup_{h\to 0}[(J_+(R+h,x)-J_+(R,x))/h]\leqslant \sup(J'(R,x),0)\ .$$

Let us note $L'(R) = \limsup_{h\to 0}[(L(R+h) - L(R))/h]$. A direct computation, using equation (3), integration on $\partial\Omega$ and Hölder inequality, leads to

$$(4) \qquad L' \leqslant L^{(p-2)/(p-1)}\left[(n-1)^{p-1}\int_{\partial\Omega}\eta_+(x)^{p-1}dx + 2^{-(p-2)/2}B(p)^p\int_{\Omega_R\setminus\Omega} r_-^{p/2}dv_g\right]^{1/(p-1)}$$

we then apply the following

LEMMA. — *Let A be any solution of the inequations*

$$A'' \leqslant A'^{(p-2)/(p-1)}(a^p + b^p\cdot A)^{1/(p-1)} \ , \ A(0) = 0 \ , \ A' \geqslant 0 \ ,$$

where a and b are two positives constants. For any nonnegative R ,

(*) *If $b = 0$, then*

$$\left(\frac{p}{p-1}\right)a^{p/(p-1)}\cdot A(R) \leqslant \left[(p-1)^{-1}a^{p/(p-1)}\cdot R + A'(0)^{1/(p-1)}\right]^p - A'(0)^{p/(p-1)}$$

$$A'(R)^{p/(p-1)} \leqslant \left(\frac{p}{(p-1)}\right)a^{p/(p-1)}\cdot A(R) + A'(0)^{p/(p-1)} \ .$$

(**) *If $b > 0$, then for any positive ε*

$$A(R+\varepsilon) \leqslant e^{b\varepsilon}\cdot A(R) + (a^p b^{-p} + A'(0)b^{-1})(e^{b\varepsilon} - 1) \ .$$

This lemma can be proved by multiplying both sides of the inequality by $A'^{1/(p-1)}$ and integrating.

We now obtain (1) by applying the part (*) of the lemma to the equation (4). We obtain (2) by applying the part (**) of the lemma to the equation (4) modified by the splitting of the integral of Ricci curvature associated to the inequality :

$$r_-^{p/2} \leqslant 2^{\frac{p-2}{2}}\left[\alpha^p + (r_- - \alpha^2)_+^{p/2}\right] \ .$$

End of the proof of theorem 1. — Let Ω_ε be the ε-tubular neighbourhood of H , where ε is smaller than the injectivity radius of the exponential map, defined on the subbundle of TM over H which is normal to H . Then Ω_ε has regular boundary. We end the proof by noticing that $\limsup_{R\to+\infty}\frac{f(R)}{R} \leqslant \limsup_{R\to+\infty} f'(R)$ and applying equality (1) to the domain $\Omega = \Omega_\varepsilon$, which gives

$$\frac{\mathrm{Vol}[\partial T(H,R)]}{[\mathrm{Vol}\,T(H,R)]^{1-s}} \leqslant C'(n,p)\frac{\left(\int_{T(H,R)} r_-^{p/2}dv_g\right)^{1/p}}{\mathrm{Vol}\,T(H,R)^{\frac{1}{p}-s}} + \mathrm{O}\left[\mathrm{Vol}\,T(H,R)^{s-\frac{1}{p}}\right] \qquad \blacksquare$$

End of the proof of the theorem 2. — Apply (2) to the domain $\Omega = \cap_{\varepsilon>0}\overline{\Omega_\varepsilon}$, whose boundary is made of two copies of H . In the particular case where H is the boundary of some domain Ω' , we obtain the same result by applying (2) both to Ω' and $M\setminus\Omega'$. ■

2. An isoperimetric inequality

THEOREM 3. — *Let α and D be any positive constants. Let p be any element of $]n,+\infty]$. In any Riemannian manifold (M^n,g) whose diameter is bounded by D and whose Ricci curvature satisfies*

$$\frac{1}{\operatorname{Vol} M}\int_M\left(\frac{r_-}{\alpha^2}-1\right)_+^{p/2}dv_g\leqslant\frac{1}{2}(e^{B(p)\alpha D}-1)^{-1},$$

every domain Ω (with regular boundary) satisfies

$$\frac{\operatorname{Vol}(\partial\Omega)}{\operatorname{Vol}(M)}\geqslant\gamma(\alpha,D)\cdot\inf\left[\frac{\operatorname{Vol}\Omega}{\operatorname{Vol} M},\frac{\operatorname{Vol} M\setminus\Omega}{\operatorname{Vol} M}\right]^{1-\frac{1}{p}}.$$

where $\gamma(\alpha,D)=B(p)\cdot\alpha\cdot\inf\left[2^{-\frac{1}{p-1}},\frac{1}{4}(e^{B(p)\alpha D}-1)^{-1}\right]$ and where $B(p)$ is given in theorem 2.

Remark. — The assumptions of this theorem cannot be improved (*cf.* the examples in the Appendix). When $p=+\infty$, these assumptions reduce to the usual ones : "*diameter bounded and Ricci-curvature bounded from below*". Notice that, if $p<+\infty$, singularities of the metric are allowed, provided that the $L^{p/2}$ norm of their curvature is small enough. When $n=2$, the value $p=2$ is allowed and then $B(p)=1$.

LEMMA 4. — *Let us define the isoperimetric constant* $\operatorname{Is}(p)$ *as the infimum (for all domains Ω in M^n satisfying* $\operatorname{Vol}(\Omega)\leqslant\operatorname{Vol}(M)/2$) *of the quantity* $\operatorname{Vol}(\partial\Omega)\cdot\operatorname{Vol}(\Omega)^{(1/p)-1}\cdot\operatorname{Vol}(M)^{-1/p}$. *For any $p\in]n,+\infty[$, there exists a minimal current Ω in M such that*

$$\operatorname{Vol}(\partial\Omega)\cdot\operatorname{Vol}(\Omega)^{(1/p)-1}\cdot\operatorname{Vol}(M)^{-1/p}=\operatorname{Is}(p).$$

This current has the following properties :

- *(i) For almost every point x in M , any geodesic of minimal length from x to $\partial\Omega$ reaches $\partial\Omega$ at a regular point x' . Moreover, there exists a neighbourhood U of x' in M such that $U\cap\partial\Omega$ is smooth.*
- *(ii) Let us call $\widetilde{\partial\Omega}$ the set of all regular points of $\partial\Omega$. The mean curvature η of $\widetilde{\partial\Omega}$ is constant and satisfies $|\eta|\leqslant(p-1)[p(n-1)]^{-1}\operatorname{Vol}(\partial\Omega)/\operatorname{Vol}(\Omega)$. Moreover, if $\operatorname{Vol}(\Omega)\neq\operatorname{Vol}(M)/2$, then $\eta=(p-1)[p(n-1)]^{-1}\operatorname{Vol}(\partial\Omega)/\operatorname{Vol}(\Omega)$.*

Proof. — For the sake of simplicity, let us suppose that $\operatorname{Vol}(M,g)=1$ (this condition can always be obtained by multiplying the metric by a constant factor, this trivial change does not modify the problem). For any $s\in]0,1[$, let us consider the set W_s of all domains Ω with regular boundary in M such that $\operatorname{Vol}(\Omega)=s$ and the minimum $h_M(s)$ of the functional $\Omega\mapsto\operatorname{Vol}(\partial\Omega)$ restricted of W_s . It is proved in [AN] that this minimum is reached for some open submanifold Ω whose boundary is a rectifiable current which is sufficiently regular for our purpose; more precisely : if the tangent cone to $\partial\Omega$ at some point $x'\in\partial\Omega$ is contained in a half-space, then there exists a neighbourhood U of x' in M such that $\partial\Omega\cap U$ is a smooth submanifold of U (see also [GV 3], [MI] and [BU]). Let us consider any point x in M such that the minimizing geodesic from x to $\partial\Omega$ has no conjugate point (this property is true for almost every x in M). Let us call x' the point where this geodesic reaches $\partial\Omega$ and S the image by the map $\exp_x$ of the sphere of radius $d(x,x')$ of T_xM . As there are no conjugate points on the geodesic, there exists a neighbourhood U of x' in M such that the connected component of $S\cap U$ which contains x' is a submanifold of U and divides U in two half spaces. As $\partial\Omega\cap U$ is contained in the half-space opposite to the geodesic and is tangential to S in x', then the above regularity property applies. This first part of the proof, due to M. Gromov ([GV 3]), proves property (i) provided that we are able to prove the existence of a domain Ω for which $\mathrm{Is}(p)$ is reached as a minimum of the corresponding functional (see Lemma 4). In order to prove this, let us consider a sequence of domains Ω_k such that $F(\Omega_k)$ converges to $\mathrm{Is}(p)$ (where $F(\Omega) = \mathrm{Vol}(\partial\Omega)\cdot \mathrm{Vol}(\Omega)^{(1/p)-1}$). Compactness of $[0,1/2]$ guarantees that some subsequence Ω_i is such that $\mathrm{Vol}(\Omega_i)$ goes to some $s\in[0,1/2]$. From the asymptotic isoperimetric inequality of [B-M], $F(\Omega_i)$ goes to infinity if $\mathrm{Vol}(\Omega_i)$ goes to 0 , we may then suppose that $s\in]0,1/2]$. By Rauch's comparison theorem, there exists some positive constant ε such that any ball $B(x,t)$ in M whose radius is less than ε satisfies

$$(3/4)t^n\cdot \mathrm{Vol}(B^n)\leqslant \mathrm{Vol}[B(x,t)]\leqslant (5/4)t^n\cdot\mathrm{Vol}(B^n)\ ,$$
$$(3/4)t^{n-1}\cdot \mathrm{Vol}(S^{n-1})\leqslant \mathrm{Vol}[\partial B(x,t)]\leqslant (5/4)t^{n-1}\cdot\mathrm{Vol}(S^{n-1})\ .$$

From the first inequality, using a mean value argument, one proves the existence of a particular choice of x_i and ε_i for each i such that the new sequence of domains $\Omega_i' = \Omega_i\backslash[B(x_i,\varepsilon_i)\cap\Omega_i]$ [resp. $\Omega_i'=\Omega_i\cup B(x_i,\varepsilon_i)$] is of constant volume s and such that

$$\mathrm{Vol}[B(x_i,\varepsilon_i)\cap\Omega_i]\geqslant (3s/5)\,\mathrm{Vol}\,B(x_i,\varepsilon_i)$$
$$[\text{resp. } \mathrm{Vol}[B(x_i,\varepsilon_i)\cap(M\backslash\Omega_i)]\geqslant (3/5)(1-s)\,\mathrm{Vol}\,B(x_i,\varepsilon_i)]\ .$$

As i goes to infinity, $\mathrm{Vol}(\Omega_i)-s$ goes to zero. The two last inequalities then imply that $\mathrm{Vol}[B(x_i,\varepsilon_i)]$ also goes to zero and so does $\mathrm{Vol}[\partial B(x_i,\varepsilon_i)]$ by Rauch's comparison theorem. This implies that $F(\Omega_i')$ goes to $\mathrm{Is}(p)$ and so that $\mathrm{Is}(p)=h_M(s)\cdot s^{(1/p)-1}$. Almgren's theorem then proves the existence of a rectifiable current which fulfils $\mathrm{Is}(p)$ and has property (i). Let us still call Ω this domain verifying $\mathrm{Vol}(\partial\Omega)\cdot\mathrm{Vol}(\Omega)^{(1/p)-1}=\mathrm{Is}(p)$, we then prove (ii) by a variational argument. Let v be any function on $\partial\Omega$ with support in the regular part of $\partial\Omega$ and let us consider the variation $H_t(x)=H(t,x)=\Phi(x,t\cdot v(x))$, where Φ is the normal coordinates system from $\widetilde{\partial\Omega}\times\mathbf{R}$ onto M . Let us define $\partial\Omega_t=H_t(\partial\Omega)$ and call Ω_t the connected component of $M\backslash\Omega_t$ which comes from Ω in the variation. Notice that, as the variation let the singularities fixed, it doesn't change anything in the topology of $\partial\Omega$. The first variation calculus gives (at the point $t=0$)

$$\frac{d}{dt}(\mathrm{Vol}\,\Omega_t)=\int_{\partial\Omega} v\,dv_g\quad ,\quad \frac{d}{dt}(\mathrm{Vol}\,\partial\Omega_t)=(n-1)\int_{\partial\Omega}\eta\cdot v\,dv_g\ .$$

Applying these formula to any variation H such that $\mathrm{Vol}(\Omega_t)=\mathrm{Vol}(\Omega)$, we prove that η is constant. Applying them one more time when H is such that the integral of v on $\partial\Omega$ is non trivial and noticing that the fonction $t\mapsto\sup[F(\Omega_t),F(M\backslash\Omega_t)]$ attains its minimum in $t=0$, we get (ii).

End of the proof of the Theorem 3. — Property (i) of Lemma 4 allows us to apply the Theorem 2 to the minimizing domain of the Lemma 4. Replacing η by the estimate of Lemma 4, (ii), we get

$$\mathrm{Vol}(M)\leqslant (e^{B(p)\alpha D}-1)\Big[2(B(p)\alpha)^{-1}\,\mathrm{Vol}(\partial\Omega)+\Big(\frac{p-1}{p}\Big)^{p-1}\Big(\frac{\mathrm{Is}(p)}{B(p)\alpha}\Big)^p\,\mathrm{Vol}(M)$$
$$+\int_M[(r_-/\alpha^2)-1]_+^{p/2}dv_g\Big]\ .$$

A direct calculus then gives $\mathrm{Is}(p)\geqslant\gamma(\alpha,D)$ and proves the isoperimetric inequality of Theorem 3.

3. Bounds on eigenvalues of the Laplace-Beltrami operator

A. The compact case. When (M,g) is a compact Riemannian manifold without boundary, it's a classical result that the spectrum is a discrete sequence

$$0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_i\leqslant\cdots$$

going to infinity, whose asymptotic behaviour is $\lambda_i\sim 4\pi^2\Big(\dfrac{i}{\mathrm{Vol}\,B^n\cdot\mathrm{Vol}(M,g)}\Big)^{2/n}$.

a) What geometric assumptions are to be considered ?. — What we are looking for is the following : given a constant C , what are the biggest possible sets of differentiable structures M and metrics g such that $\lambda_i(M,g)\geqslant C\cdot i^{2/p}$ for ***each*** i . It is classical (examples of Calabi and Cheeger) that an estimate as independant of the geometry as the asymptotic one cannot be expected for small i and that the sets of Riemannian manifolds to consider must at least satisfy (in addition to the obvious condition *"dimension bounded"*) assumptions of two kinds :

(i) A local one, which bounds the ratio between the Riemannian measure and the euclidean one.

(ii) A global one, which says that big homothetic changes of the metric are not allowed.

About (i), we shall replace here the usual assumption *"Ricci curvature bounded from below"* by the weaker condition : *"the part of Ricci curvature which goes below the fixed bound has not too big $L^{p/2}$ norm"* (cf. theorem 6).

About (ii), this assumption must be chosen in order to contain *"collapsing"* sequences of metrics g_k in the sense of [C–G] (*i.e.* metrics g_k whose injectivity radius and volume go to zero when k goes to infinity and whose diameter and sectional curvature are bounded), because it is known that the $\lambda_i(g_k)$ remain in fixed intervals (see [CH], [L–Y 1], [GV 3], [GT 1-2-3-4]) when k go to infinity. This excludes the assumption *"Volume bounded"*, because inefficient, and any assumption on the injectivity radius, because too strong (so standard arguments using local maps and analysis in $\mathbf{R}^n$ are inefficient here).

The assumption (ii) we choose here is the usual one : *"diameter bounded"*, but it would be possible to replace it by weaker ones (see [GT 4] section 6 and [GT 5]).

b) Isoperimetric inequalities are equivalent to inequalities on eigenvalues or Sobolev constants. — Let h be any C^1 increasing function from $]0,\frac12]$ to $]0,+\infty[$ such that, *for some* $r\in]-\infty,1[$, $\lim\limits_{s\to 0}\Big(\dfrac{h(s)}{s^{1-r}}\Big)$ *exists and lies in* $]0,+\infty[$. Let us still call h the function from $]0,1[$ to $]0,+\infty[$ given by $h(s)=h(1-s)$. Considering ***the set*** $\mathcal{M}_h$ ***of Riemannian manifolds*** (M,g) ***all of whose domains*** Ω ***satisfy***

$$\frac{\mathrm{Vol}\,\partial\Omega}{\mathrm{Vol}\,M}\geqslant h\Big(\frac{\mathrm{Vol}\,\Omega}{\mathrm{Vol}\,M}\Big)\ ,$$

we are interested in proving that some spectral geometric invariants are uniformly bounded on $\mathcal{M}_h$ and that their extrema on $\mathcal{M}_h$ can be computed. The invariants we consider here are the eigenvalues $\lambda_i(M,g)$ of the Laplace-Beltrami operator on the whole of M , the first eigenvalue $\lambda_1^D(\Omega,g)$ of the Laplace-Beltrami operator on any subdomain $\Omega\subset M$ (for Dirichlet boundary conditions), the heat kernel $k_{(M,g)}(t,x,y)$ of (M,g) with initial data δ_x and the Sobolev constants

$$\lambda_{q,m}(M,g)=\inf_{f\in C^\infty(M)}\left[\frac{\|\nabla f\|_{L^m}}{\inf_{a\in\mathbf{R}}\|f-a\|_{L^q}\cdot\mathrm{Vol}(M,g)^{\frac1m-\frac1q}}\right]$$

which occur in the embeddings $H_1^m \hookrightarrow L^q$.

It has been well known since [BI] that the datum of $\lambda_{\frac{n}{n-1},1}$ is equivalent to the datum of the euclidean-like isoperimetric constant

$$\inf_{\Omega \subset M} \left[\frac{\operatorname{Vol} \partial\Omega}{\inf(\operatorname{Vol}\Omega, \operatorname{Vol} M \setminus \Omega)^{1-\frac{1}{n}} \cdot \operatorname{Vol}(M)^{1/n}} \right] .$$

The following theorem generalizes this equivalence to other invariants and isoperimetric inequalities.

THEOREM 5. — *Let $\Lambda_1(h), \Lambda_i(h), \Lambda_{q,m}(h)$ be the infima of $\lambda_1(M,g), \lambda_i(M,g)$, $\lambda_{q,m}(M,g)$ when (M,g) runs in $\mathcal{M}_h$. Let*

$$k_t(h) = \sup\{\sup_{x,y} \operatorname{Vol}(M,g) \cdot k_{(M,g)}(t,x,y) : (M,g) \in \mathcal{M}_h\} ,$$

then

(i) $\Lambda_1(h)$ is non zero iff $r \geqslant 0$. In this case,

$$\Lambda_1(h) = \inf \left\{ \frac{\int_0^{1/2} |u'(s)|^2 \cdot h(s)^2 ds}{\int_0^{1/2} u(s)^2 ds} \ : \ u \in C_0^\infty\Big(]0, \frac{1}{2}[\Big) \right\} .$$

(ii) $\Lambda_i(h)$ goes to infinity with i iff $r > 0$. In this case,

$$\Lambda_i(h) \geqslant C(h) \cdot i^{2r} .$$

(iii) For every $(M,g) \in \mathcal{M}_h$ and any domain $\Omega \subset M$,

$$\lambda_1^D(\Omega, g) \geqslant \inf \left\{ \frac{\int_0^{\frac{\operatorname{Vol}\Omega}{\operatorname{Vol}M}} |u'(s)|^2 \cdot h(s)^2 ds}{\int_0^{\frac{\operatorname{Vol}\Omega}{\operatorname{Vol}M}} u(s)^2 ds} \ : \ u \in C_0^\infty\Big(]0, \frac{\operatorname{Vol}\Omega}{\operatorname{Vol}M}[\Big) \right\} .$$

The right-hand side of this inequality is non trivial iff $r \geqslant 0$.

(iv) $k_t(h)$ is finite iff $r > 0$. In this case

$$k_t(h) = \int_0^1 \Big[\frac{\partial u}{\partial s}(t,s)\Big]^2 ds ,$$

where u is the solution of the equation $\dfrac{\partial u}{\partial t} - h^2(x)\dfrac{\partial^2 u}{\partial^2 s} = 0$, which satisfies boundary conditions $u(t,0) = 0$, $u(t,1) = 1$ and initial datum $u(0,\bullet) = \chi_{]0,1]}$.

(v) $\Lambda_{q,m}(h)$ is non zero iff $r \geqslant \frac{1}{m} - \frac{1}{q}$. In this case,

$$\Lambda_{q,m}(h) = \inf \left\{ \frac{\left(\int_0^1 |u'(s)|^m h(s)^m ds\right)^{1/m}}{\inf_{a \in \mathbf{R}} \left(\int_0^1 |u(s) - a|^q ds\right)^{1/q}} \ : \ u \in C_0^\infty(]0,1[) \right\} .$$

Remarks. — A complete proof of this theorem is given in [GT 4]. Previously, (i), (ii), (iii) and (v) where proved in [GT 2] and [GT 3] (in the particular case $h(s) = s^{1-\frac{1}{n}}$) and the inequality $k_t(h) \leqslant \int_0^1 \left[\frac{\partial u}{\partial s}\right]^2 ds$ was established in [B–G] [another proof, by comparison between operators, is due to G. Besson (see the appendix of [BD])].

From the datum of h , it is possible to build metrics of revolution on cylinders for which each invariant considered in (i), (iii), (iv) and (v) is equal to the corresponding bound on $\mathcal{M}_h$

(see [B–G]). It is then possible to make small changes on these metrics in order that they lie in $\mathcal{M}_h$ and approach the bound; this proves that the bound is in fact an extremum(see [GT 4]).

c) Bounding the spectrum and the Sobolev constants in terms of integral norms of Ricci-curvature. — On any Riemannian manifold (M, g) let

$$Is(p) = \inf_{\substack{\Omega \subset M \\ \partial\Omega \text{ regular}}} \left(\frac{\operatorname{Vol} \partial\Omega}{\min(\operatorname{Vol}\Omega, \operatorname{Vol} M \setminus \Omega)^{1-\frac{1}{p}} \cdot \operatorname{Vol} M^{1/p}} \right) .$$

As any Riemannian manifold belongs to $\mathcal{M}_h$ where $h(s) = Is(p) \cdot \inf(s, 1-s)^{1-\frac{1}{p}}$, we may apply the Theorem 5. Changing the variable s for the new variable $\rho(s) = \int_0^s \frac{1}{h(x)} dx$ in the bounds given by the theorem 5, it appears that the computation of these bounds is possible in the same way as done for euclidean balls. So the bounds given by theorem 5 (i), (ii), (iii) and (iv) are computable in terms of Bessel functions (see [B–G] section 5 and the appendix of [GT 4] for more details). The Bliss's lemma ([AU], prop. 2.18, p.42) gives the exact computation of the right-hand side of the equality of theorem 5 (v). Using the lower bound of $Is(p)$ given by the theorem 3, we then obtain

THEOREM 6. — *Let α and D be any positive constants. Let p be any element of $]n, +\infty]$. On any Riemannian manifold (M^n, g) whose diameter is bounded by D and whose Ricci curvature satisfies*

$$\frac{1}{\operatorname{Vol} M} \int_M \left(\frac{r_-}{\alpha^2} - 1\right)_+^{p/2} dv_g \leqslant \frac{1}{2}(e^{B(p)\alpha D} - 1)^{-1} ,$$

one has

(i) For every $i \in \mathbf{N}$,

$$\lambda_i(M, g) \geqslant \left(\frac{C_i(p)}{p}\right)^2 \cdot 2^{2/p} \cdot Is(p)^2 \cdot i^{2/p} \geqslant \left(\frac{C_i(p)}{p}\right)^2 \cdot 2^{2/p} \cdot \gamma(\alpha, D)^2 i^{2/p} .$$

(ii) For any domain $\Omega \subset M$ whose volume is less than $\frac{\operatorname{Vol} M}{2}$

$$\lambda_1(\Omega, g) \geqslant \left(\frac{C_1(p)}{p}\right)^2 \cdot Is(p)^2 \left(\frac{\operatorname{Vol} M}{\operatorname{Vol}\Omega}\right)^{2/p} \geqslant \left(\frac{C_1(p)}{p}\right)^2 \gamma(\alpha, D)^2 \cdot \left(\frac{\operatorname{Vol} M}{\operatorname{Vol}\Omega}\right)^{2/p}$$

(iii) For any $(x, y, t) \in M \times M \times \mathbf{R}^+$, then

$$\operatorname{Vol}(M, g) \cdot k_{(M,g)}(t, x, y) \leqslant k^*\left(\frac{2^{2/p}}{p^2} \cdot Is(p)^2 \cdot t\right) \leqslant k^*\left(\frac{2^{2/p}}{p^2} \gamma(\alpha, D)^2 \cdot t\right)$$

where $k^*(t) \leqslant C(p) \cdot t^{-p/2}$.

(iv) If $\frac{1}{p} = \frac{1}{m} - \frac{1}{q}$ (critical case), then

$$K(p, m) \cdot \lambda_{q,m}(M, g) \geqslant Is(p) \geqslant \gamma(\alpha, D) ,$$

where the constants can be computed as follows :

$$K(p, m) = p^{1-\frac{1}{p}-\frac{1}{m}} \left(\frac{m-1}{p-m}\right)^{1-\frac{1}{m}} \left(\frac{m}{m-1}\right)^{1/p} \left[\frac{\Gamma(\frac{p}{m}) \cdot \Gamma(p(1-\frac{1}{m}))}{2\Gamma(p)}\right]^{-\frac{1}{p}}$$

$C_1(p) = 1^{\text{st}}$ zero of the Bessel function $J_{\frac{p-2}{2}}$, where

$$J_l(x) = \Gamma(l+1) \cdot \sum_{k=0}^{\infty} (-1)^k \frac{(t/2)^{2k}}{k! \Gamma(k+l+1)} .$$

$$k^*(t) = 1 + \sum_{i=1}^{\infty} \alpha_k \cdot e^{-\nu_k^2 t} + \beta_k e^{-j_k^2 t} ,$$

where j_k (resp. ν_k) is the k^{th} zero of $J_{\frac{p-2}{2}}$ (resp. $J'_{\frac{p-2}{2}}$) in $]0, +\infty[$, and where

$$\alpha_k = \frac{1}{2^{p-3} \cdot p \cdot \Gamma(\frac{p}{2})^2} \frac{j_k^{n-2}}{J_{p/2}^2(j_k)} ,$$

$$\beta_k = \frac{1}{2^{p-3} \cdot p \cdot \Gamma(\frac{p}{2})^2} \frac{\nu_k^{n-2}}{J_{\frac{p-2}{2}}^2(\nu_k)} .$$

$C_i(p) =$ solution x of $k^*\Big(\dfrac{1}{x^2 \cdot i^{2/p}}\Big) = \dfrac{i}{e}$.

Remarks. — (i) can be deduced directly from (iii) because

$$\frac{i}{e} \leqslant \sum_{k \leqslant i} e^{-\lambda_k/\lambda_i} \leqslant k^*\Big(\frac{2^{2/p}}{p^2} \frac{Is(p)^2}{\lambda_i}\Big) .$$

The assumptions of the theorem 6 cannot be improved (see the Appendix).

B. Bounding the first eigenvalue from above in the non compact case. When the volume of (M, g) is non finite, constant functions no longer lie in $H_1(M, g)$, so the infimum of the spectrum [that we still call $\lambda_0(M, g)$] is sometimes non zero and is not always on eigenvalue. In [DY], H. Donnelly proved that $\lambda_0(M, g) \leqslant \dfrac{(n-1)^2}{4} \|r_-\|_{L^\infty}$. We give there a version of this result in terms of $\|r_-\|_{L^{p/2}}$:

PROPOSITION 7. — *For any complete Riemannian manifold (M^n, g) with non finite volume and any $p \in]n, +\infty[$,*

$$\begin{aligned}\lambda_0(M, g) &\leqslant (1/4) \inf_x \Big[\limsup_{R\to\infty} \Big(\frac{\mathrm{Vol}(\partial B(x, R))}{\mathrm{Vol}(B(x, R))}\Big)\Big]^2 \\ &\leqslant (1/4) C'(n, p)^2 \inf_x \limsup_{R\to\infty} \Big[\mathrm{Vol}(B(x, R))^{-1} \int_{B(x,R)} r_-^{p/2} dv_g\Big]^{2/p}\end{aligned}$$

where C' is the constant defined in Theorem 1. In particular, to obtain the inequality $\lambda_0(M, g) \leqslant C'(n, p)^2 \alpha^2/4$, it is sufficient that $\int_M (r_- - \alpha^2)_+ dv_g$ is finite.

Remarks. — This proposition implies in particular that every manifold with infinite volume, whose Ricci-curvature is bounded from below by $-(n-1)\alpha^2$ outside a compact subset, satisfies $\lambda_0(M, g) \leqslant \frac{(n-1)^2 \alpha}{4}$. On the other hand, it is sharp for hyperbolic spaces because, when p goes to infinity, $C'(n, p)$ goes to $n - 1$.

Proof. — Let x be any point in M . Let us define $L(R) = \mathrm{Vol}(\partial B(x, R))$ and $A(R) = \mathrm{Vol}(B(x, R))$. For any function u from $\mathbf{R}^+$ to $\mathbf{R}$ such that $I(u) = \int_0^\infty \big[u'(t)^2 + u(t)^2\big] L(t) dt$ is finite, the minimax principle, applied to the function $f(y) = u[d(x, y)]$, leads to

$$\lambda_0(M, g) \leqslant \Big[\int_0^\infty u'(t)^2 L(t) dt \Big/ \int_0^\infty u(t)^2 L(t) dt\Big] .$$

Let $c = \limsup_{R\to\infty}\left[\mathrm{Vol}(\partial B(x,R))/\mathrm{Vol}(B(x,R))\right]$. For any positive ε , there exists some number R_0 such that, for any $R \geqslant R_0$, $L(R) \leqslant A(R)\,(c+\varepsilon)$. By integration, it gives

$$A(R) \leqslant A(R_0)e^{(c+\varepsilon)(R-R_0)}$$
$$L(R) \leqslant (c+\varepsilon)A(R_0)e^{(c+\varepsilon)(R-R_0)}$$

So, if $u(t) = e^{-(c+2\varepsilon)t/2}$, $I(u)$ is finite. A direct calculus leads to

$$\lambda_0(M,g) \leqslant (c+2\varepsilon)^2/4 ,$$

we prove the first inequality by making ε go to zero.

The second inequality comes from Theorem 1,(ii), where $s=0$. ∎

4. Bounds on Gromov's norms in homology

In [GV 2] (pp. 34–36), M. Gromov proved that the norms $\|[c]\|$ of the homology-classes (see definition in the introduction) can be bounded by means of the growth of big balls in the Riemannian universal covering $(\widetilde{M},\tilde{g})$ of the considered complete Riemannian manifold (M,g) . He proved, for any i-dimensional closed chain c , that

$$\|[c]\| \leqslant i!\,\sup_x\left[\liminf_{R\to+\infty}\frac{\mathrm{Vol}\,\partial B(x,R)}{\mathrm{Vol}\,B(x,R)}\right]^i\cdot \mathrm{Vol}_i(c) ,$$

where the right-hand side of the inequality may be improved by multiplication by $\left(\dfrac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n+1)/2)}\right)^n$ when $[c]=[M]$. In the proof of theorem 1, we established, as an application of (2), that

$$\frac{\mathrm{Vol}[\partial T(H,R)]}{\mathrm{Vol}\,T(H,R)} \leqslant C'(n,p)\left(\frac{\int_{T(H,R)} r_-^{p/2}\,dv_g}{\mathrm{Vol}\,T(H,R)}\right)^{\frac1p} + \mathrm{O}\left[\mathrm{Vol}\,T(H,R)^{-\frac1p}\right] .$$

Replacing H by $\{x\}$ and noticing that $\|[c]\|$ is always zero when $\mathrm{Vol}(\widetilde{M},\tilde{g}) < +\infty$ (because the bounded cohomology is trivial when the fundamental group is finite), we get the

THEOREM 8. — *Let (M^n,g) be any Riemannian manifold. For any $p \in]n,+\infty[$ and any closed i-dimensional chain c ,*

$$\|[c]\| \leqslant i!C'(n,p)^i\,\mathrm{Vol}_i(c)\cdot\sup_{x\in\widetilde{M}}\left[\liminf_{R\to\infty}\left(\mathrm{Vol}\,\widetilde{B}(x,R)^{-1}\int_{\widetilde{B}(x,R)} r_-^{p/2}\cdot dv_{\tilde{g}}\right)\right]^{i/p}$$

$$\|[M]\| \leqslant n!C'(n,p)^n\left(\frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n+1)/2)}\right)^n\mathrm{Vol}(M)\sup_{x\in\widetilde{M}}\left[\liminf_{R\to\infty}\left(\mathrm{Vol}\,\widetilde{B}(x,R)^{-1}\int_{\widetilde{B}(x,R)} r_-^{p/2}dv_{\tilde{g}}\right)\right]^{\frac np}$$

where the $\widetilde{B}(x,R)$ are balls of radius R in the Riemannian universal covering $(\widetilde{M},\widetilde{g})$ of (M,g) and where C' is defined in Theorem 1.

Remarks. —

– There existed a previous estimate by M. Gromov in which the right-hand side of the inequality depends on $\|r_-\|_{L^\infty}$ and $\mathrm{Vol}(M,g)$. He conjectured as possible to replace this L^∞-norm by a $L^{n/2}$-norm. The above inequality is a step in that direction.

– A second step would be to obtain the right-hand side of the above inequality in terms of $\left(\mathrm{Vol}(M)^{-1}\int_M r_-^{p/2}\cdot dv_g\right)$. We first have to see in what cases this quantity is the limit, when R goes to infinity, of $\left(\mathrm{Vol}\,\widetilde{B}(x,R)^{-1}\int_{\widetilde{B}(x,R)} r_-^{p/2}\cdot dv_{\tilde{g}}\right)$. M. Gromov recently pointed to me that this is true (from [BN]) when the sectional curvature of (M,g) is supposed to be negative. It would induce that, in this case, the simplicial volume is bounded in terms of $\left(\mathrm{Vol}(M)^{-1}\int_M \mathrm{Scal}(g)^{p/2}dv_g\right)$.

– The theorem 8 implies that the volume of any immersed submanifold of M is bounded from below by some (eventually trivial) topological invariant of the immersion.

5. Topological or geometric invariants of harmonic type bounded by curvature integrals

Let (M,g) be any compact Riemannian manifold. We consider Riemannian vector-bundles $E\to M$ equiped with a metric $\langle\cdot,\cdot\rangle$ on each fiber and a connection D compatible with the metric. Let $|s(x)|$ or $|s|$ denote the norm of a section s at some point x and $\|s\|_p=\left(\int_M |s|^p dv_g\right)^{1/p}$ its L^p-norm. The space of sections s such that $|s|^2$ (resp. $|Ds|^2+|s|^2$) is integrable is noted $L^2(M,E)$ (resp. $W_1(M,E)$). The rough laplacian D^*D is the symmetric operator associated to the quadratic form $Q(s)=\int_M |Ds|^2\cdot dv_g$ on $W_1(M,E)$.

A geometric (or topological) invariant $\delta(M,g)$ (or $\delta(M)$) is called harmonic (resp. *subharmonic) if their exists some Riemannian fiber bundle $(E,\langle\cdot,\cdot\rangle,D)$ and some section $\mathcal{R}$ of the symmetrized tensor product $E^*\odot E^*$ such that* $\delta(M,g)=\dim[\mathrm{Ker}(D^*D+\mathcal{R})]$ (resp. $\delta(M,g)\leqslant \mathrm{index}(Q_{\mathcal{R}})$), *where $Q_{\mathcal{R}}$ is, in all this section, the quadratic form defined on $W_1(M,E)$ by* $Q_{\mathcal{R}}(s)=\int_M[|Ds|^2+\mathcal{R}(s,s)]\cdot dv_g$, and where its index is the maximal dimension of any vector-subspace on which $Q_{\mathcal{R}}\leqslant 0$.

Examples. — *Betti numbers* [$b_i=\dim(H^i(M,\mathbf{R}))$] *are harmonic topological invariants. The index of a Dirac operator, the dimension of the moduli-space of Einstein metrics, the number of eigenvalues lying in* $[0,\lambda]$ *for the Hodge-de Rham laplacian, etc. are subharmonic invariants.*

The proof of this assertion comes from Hodge's theory and from Weitzenbock's formulae. About this class of invariants, one can aim different levels of results :

a) Vanishing theorems. — The (trivial) theorem here is : if $\mathcal{R}$ is everywhere positive definite , so is $Q_{\mathcal{R}}$ and the corresponding invariant is trivial. What is not trivial here is to find the curvature-hypothesis which implies that $\mathcal{R}>0$. However, only pointwise algebraic computations are needed there (see for instance [G-M 1]).

b) Pinching theorems. — What occurs when the diameter is bounded and when the curvature is allowed to go a little below zero? A reference-result in this field is the

THEOREM 9 (M. Gromov, see [B-K]). — *There exists a positive function ε on $\mathbf{N}\backslash\{0,1\}\times\mathbf{R}^+$ with the following property : a manifold M is diffeomorphic to a compact quotient of a nilpotent Lie group iff it admits a metric g satisfying*

$$\|\sigma(g)\|_{L^\infty}<\varepsilon(\dim M,\mathrm{diam}(g))\ .$$

where $\sigma(g)$ is the sectional curvature of g .

What occurs when we replace the pinching of sectional curvature by a pinching of Ricci-curvature ? and the L^∞ condition by a $L^{p/2}$ condition? [To see that this hypothesis would be an important weakening of the above one, just note that it is satisfied by every Ricci-flat manifold (and there are many of them by Aubin-Yau's proof of Calabi's conjecture)] . It is the aim of the following theorem

THEOREM 10. — *Let ε be the function defined by*

$$\varepsilon(p) = (\operatorname{Log} 2)^p \cdot \sup[2^{5p} A(p)^p, 2B(p)^p]^{-1}$$

where $B(p)$ is defined in Theorem 2 and $A(p)$ in Proposition 13. If a manifold M^n admits a metric g such that, for at least one $p \in]n, +\infty[$,

$$\operatorname{diam}(g)^p [\operatorname{Vol}(g)^{-1} \int_M r_-^{p/2} dv_g] < \varepsilon(p)$$

then $b_1(M) \leqslant b_1(\mathbf{T}^n)$.

Moreover, if such a metric also satisfies

$$\operatorname{diam}(g)^n [\operatorname{Vol}(g)^{-1} \int_M |\sigma|^{n/2} \cdot dv_g] < \varepsilon(p)$$

then $b_i(M) \leqslant b_i(\mathbf{T}^n)$ for every i .

Remark. — The assumptions of this theorem cannot be improved (see the Appendix).

This theorem is a corollary of Theorem 11 and will be proved later.

c) Finiteness theorems. — We then allow the curvature to have the two signs and only suppose that its negative part is controlled in some $L^{p/2}$ sense. It is obvious that we can get nothing with only local considerations, we really need arguments of global geometry. We shall prove the

THEOREM 11. — *There exists a function $\mathcal{Z}$ (resp. $\mathcal{X}$) defined on $(\mathbf{R}^+)^3$ (resp. on $(\mathbf{R}^+)^4$) with the following property : for any compact Riemannian manifold (M^n, g) whose diameter is bounded by some number D , for any $p \in]n, +\infty[$ and any α satisfying*

$$\operatorname{Vol}(M)^{-1} \cdot \int_M [(r_-/\alpha^2) - 1]_+^{p/2} \cdot dv_g \leqslant (1/2)(e^{B(p)\alpha D} - 1)^{-1}$$

(where $B(p)$ is defined in Theorem 2) then

$$b_1(M) \leqslant n \cdot \mathcal{Z}(p, D, \alpha)$$

$$b_i(M) \leqslant \binom{n}{i} \cdot \mathcal{X}(p, D, \alpha, \operatorname{Vol}(M)^{-1} \cdot \int_M |\sigma|^{p/2}) \text{ for any } i$$

$\widehat{A}(M) \leqslant 2^{(n/2)-1} \mathcal{Z}(p, D, \alpha)$ where $\widehat{A}(M)$ is the index of the Dirac operator ,

More generally, for any subharmonic invariant $\delta(M, g)$, let V be any positive function such that $\mathcal{R}(\cdot,\cdot) \geqslant -V(x) \cdot \langle \cdot, \cdot \rangle$ on any fiber E_x [where $\mathcal{R}$ is, as above, the zero-order term of the elliptic operator corresponding to $\delta(M, g)$], then

$$\delta(M, g) \leqslant \dim(E_x) \cdot \mathcal{X}\big(p, D, \alpha, \operatorname{Vol}(M)^{-1} \int_M V^{p/2} dv_g\big) .$$

Remarks. —

1. — For $b_1(M)$ for instance previous bounds had been given in terms of $\|r_-\|_{L^\infty}$ and diameter in [GV 1] and [GT 2]. These results are contained in Theorem 11.

2. — Examples in the appendix show that the diameter must occur in the estimate and that it is impossible to bound $b_1(M)$ in term of $\left[\mathrm{Vol}(M)^{-1}\cdot\int_M r_-^{n/2}\cdot dv_g\right]$ and the diameter. In fact, the assumptions of the theorem 11 are almost optimal (see the Appendix).

Before proving Theorem 11, let us establish the two following propositions. For any elliptic self-adjoint operator L , let us call $\lambda_0(L)$ (resp. $N_L(\lambda)$) the infimum of its spectrum (resp. the number of values of its spectrum which are inferior or equal to λ). We then have the following comparison theorem between a Schrödinger operator on a fiber bundle and its analogous on the basis :

PROPOSITION 12 ([G-M 2], Proposition 5 and Remark 9). — *Let $\mathcal{R}_-(x) = \min_{v\in E_x}[v^{-2}\mathcal{R}(v,v)]$. If l is the dimension of the fiber E_x , we always have :*

$$\mathrm{Index}(D^*D+\mathcal{R}) = \mathrm{Index}(Q_{\mathcal{R}}) \leqslant (l+1)N_{\Delta+\mathcal{R}_-}[(1-8(l+1)^2)\lambda_0(\Delta+\mathcal{R}_-)]-1\ .$$

Sketch of the proof. — The function $\Phi : s \to |s|$ sends any subspace $\mathcal{E}$ of $L^2(M,E)$ on a cone in $L^2(M,\mathbf{R})$ (here $|s|$ denotes the function $x \mapsto |s(x)|$). Let $\mathcal{H}$ be any finite dimensional subspace of $L^2(M,\mathbf{R})$ and P be the canonical projection from $L^2(M,\mathbf{R})$ onto its projective space, it is proved in [G-M 2] that, if $P\circ\Phi(\mathcal{E})$ lies in the ε -neighbourhood of $P(\mathcal{H})$ (for $\varepsilon = [8(l+1)^2]^{-1/2}$), then $\dim(\mathcal{E}) < \dim(\mathcal{H})\cdot(l+1)$.

Now, suppose that $Q_{\mathcal{R}} \leqslant 0$ in restriction to $\mathcal{E}$ and define $\mathcal{H}$ as the vector-space spanned by the eigenfunctions of $(\Delta+\mathcal{R}_-)$ whose eigenvalues are not greater than $[1-8(l+1)^2]\lambda_0(\Delta+\mathcal{R}_-)$. If $s\in\mathcal{E}$, the component of the function $|s|$ which is orthogonal to $\mathcal{H}$ is, by the min-max principle, small enough and $P\circ\Phi(\mathcal{E})$ lies in a ε-neighbourhood of $P(\mathcal{H})$ for some $\varepsilon \leqslant [8(l+1)^2]^{-1/2}$. The result follows.

PROPOSITION 13. — *Let (M^n,g) be any complete Riemannian manifold. For any potential-function V, define $V_- = \sup(-V,0)$. For any $p\in]n,+\infty[$, let us put*

$$A(p) = 2^{1-(1/p)} + p^{(p-2)/(2p)}(p-2)^{-1/2}[2\Gamma(p)\cdot\Gamma(p/2)^{-2}]^{1/p}\ .$$

Then, for every positive ε ,

(i)

$$\lambda_0(\Delta+V) \geqslant -\left(\frac{2^{1/2}A(n+\varepsilon)}{\mathrm{Is}(n+\varepsilon)}\right)^{2(n+\varepsilon)/\varepsilon}\left(\mathrm{Vol}(M)^{-1}\int_M V_-^{(n+2\varepsilon)/2}dv_g\right)^{2/\varepsilon}$$
$$-2\left(\mathrm{Vol}(M)^{-1}\int_M V_-^{(n+\varepsilon)/2}\right)^{2/(n+\varepsilon)},$$

(ii) For any $i\in\mathbf{N}\backslash\{0\}$ and for any $a\in]0,1[$,

$$\lambda_i(\Delta+V) \geqslant \lambda_i(\Delta)(1-a) - \left(\frac{A(n+\varepsilon)}{a^{1/2}\mathrm{Is}(n+\varepsilon)}\right)^{2(n+\varepsilon)/\varepsilon}\left(\mathrm{Vol}(M)^{-1}\int_M V_-^{(n+2\varepsilon)/2}dv_g\right)^{2/\varepsilon}.$$

(iii) If, for some $a\in]0,1/2]$, $\mathrm{Vol}(M)^{-1}\int_M V_-^{n/2}dv_g < [a^{1/2}\cdot A(n)^{-1}\cdot\mathrm{Is}(n)]^n$ then

$$\lambda_0(\Delta+V) \geqslant -2\left(\mathrm{Vol}(M)^{-1}\int_M V_-^{n/2}dv_g\right)^{2/n}$$

$$\lambda_i(\Delta+V) \geqslant (1-a)\lambda_i(\Delta) \text{ for any } i\in\mathbf{N}\backslash\{0,1\}\ .$$

Remarks. —

– A similar result was proved by the author from [GT 3] [just replace, in the iterations of Sobolev inequalities used in [GT 3], the inequality $\int_M V_- \cdot f^2 \leqslant \|V_-\|_{L^\infty} \int_M f^2$ by $\int_M V_- \cdot f^2 \leqslant \|V_-\|_{L^{p/2}} \left(\int_M f^{2p/(p-2)}\right)^{(p-2)/p}$] but this proof only worked when $\dim(M) \geqslant 3$. Recently, P. Bérard and G. Besson ([B–B]) gave another proof (which also works when $\dim M \geqslant 3$) which improves the $L^{p/2}$ estimate of the potential to a critical $L^{n/2}$ one. However, the geometric and topological estimates one could deduce from these two methods still depended on $\|r_-\|_{L^\infty}$, for lack of an estimate of the isoperimetric constant $Is(p)$ in term of $\|r_-\|_{L^{p/2}}$, which is given here by the theorem 3. Examples of the Appendix prove that the difficulty of bounding the isoperimetric constant is the critical point which prevents us from estimating invariants in terms of a $L^{n/2}$-norm of the curvature.

Proof of Proposition 13. — Let E be any subspace of $W_1(M, \mathbf{R})$ such that $\int_M |df|^2 + \int_M V \cdot f^2 \leqslant \lambda \int_M f^2$ for every $f \in E$. For every α and for every $p \geqslant n$, by Hölder's inequality, we have

$$\text{(5)} \qquad \int_M |df|^2 \leqslant (\lambda + \alpha) \int_M f^2 + \left(\int_M (V_- - \alpha)_+^{p/2}\right)^{2/p} \left(\int_M |f - \overline{f}|^{2p/(p-2)}\right)^{(p-2)/p}$$
$$+ 2\overline{f} \int_M (V_- - \alpha)_+ (f - \overline{f}) + \overline{f}^2 \int_M (V_- - \alpha)_+ ,$$

where $\overline{f} = \mathrm{Vol}(M)^{-1} \int_M f$ is supposed to be nonnegative (elsewhere just change f in $-f$).

From Theorem 6, (iv) (that we modify as in the proof of [GT 2], Corollaire 2.6) we also have

$$\text{(6)} \qquad \|f - \overline{f}\|^2_{L^{2p/(p-2)}} \leqslant \mathrm{Vol}(M)^{-2/p} [A(p)/\mathrm{Is}(p)]^2 \|df\|^2_{L^2} ,$$

Inequalities (5) and (6) give

$$\text{(7)} \qquad \int_M |df|^2 \leqslant (\lambda + \alpha) \int_M f^2 + \left(\mathrm{Vol}(M)^{-1} \int_M (V_- - \alpha)_+^{p/2}\right)^{2/p}$$
$$\left(\frac{A(p)}{\mathrm{Is}(p)} \|df\|_{L^2} + \mathrm{Vol}(M)^{1/2} \cdot \overline{f}\right)^2 .$$

Let us take, for any $q > p$,

$$\alpha = [a^{-1/2} A(p)/\mathrm{Is}(p)]^{2p/(q-p)} [\mathrm{Vol}(M)^{-1} \int_M V_-^{q/2}]^{2/(q-p)} .$$

As

$$\int_M (V_- - \alpha)_+^{p/2} \leqslant (\mathrm{Vol}\{V_- \geqslant \alpha\})^{1-(p/q)} \left(\int_M (V_- - \alpha)_+^{q/2}\right)^{p/q} ,$$

$$\alpha^{q/2} \left(\mathrm{Vol}\{V_- \geqslant \alpha\}\right) \leqslant \int_M V_-^{q/2} ,$$

we immediately deduce

$$\text{(8)} \qquad \left[\mathrm{Vol}(M)^{-1} \int_M (V_- - \alpha)_+^{p/2}\right]^{2/p} \leqslant a \cdot \mathrm{Is}(p)^2 / A(p)^2$$

Replacing in (7) and making $a = 1/2$, we obtain $\lambda \geqslant -\alpha - 2\left[\mathrm{Vol}(M)^{-1} \int_M V_-^{p/2}\right]^{2/p}$ which gives (i), and (iii) by making $\varepsilon \to 0_+$.

Now, let E' be the subspace of all functions f in E satisfying $\int_M f\, dv_g = 0$. By the min-max principle, there exists one choice of E such that $\dim(E') \geqslant N_{\Delta+V}(\lambda) - 1$. Applying (7) and (8) to every $f \in E'$, we have

$$(1-a)\int_M df^2 \leqslant (\lambda+\alpha)\int_M f^2 \ ,$$

and we conclude by using the min-max principle. ■

End of the proof of the Theorem 11. — Applying Propositions 12 and 13, a direct calculus leads to

$$\operatorname{Index}(D^*D+\mathcal{R}) \leqslant -1 + (l+1)\cdot N_\Delta\Big(32(l+1)^2\Big[\big(\operatorname{Vol}(M)^{-1}\int_M |\mathcal{R}_-|^{\frac{n+\varepsilon}{2}}\big)^{\frac{2}{n+\varepsilon}}$$
$$+\, 2^{\frac{n}{\varepsilon}}(A(n+\varepsilon)/\operatorname{Is}(n+\varepsilon))^{\frac{2(n+\varepsilon)}{\varepsilon}}\cdot\big(\operatorname{Vol}(M)^{-1}\int_M |\mathcal{R}_-|^{(n+2\varepsilon)/2}\big)^{2/\varepsilon}\Big]\Big) \ .$$

As bounding from below the eigenvalues of Δ is equivalent to give upper bounds for N_Δ , we may apply the Theorem 6 (i) which gives (for $p = n+\varepsilon$)

$$\operatorname{Index}(D^*D+\mathcal{R}) \leqslant l\cdot\mathcal{X}\Big(p, D, \alpha, \operatorname{Vol}(M)^{-1}\int_M |\mathcal{R}_-|^{(n+2\varepsilon)/2}dv_g\Big) \ .$$

Theorem 11 comes by expliciting the relation between $\mathcal{R}$ and the curvature in each application. For instance :

$$b_1(M) = \dim[\operatorname{Ker}(D^*D + \operatorname{Ricci})]$$
$$b_i(M) = \dim[\operatorname{Ker}(D^*D + \mathcal{R})] \ ,$$

where $\mathcal{R}$ can be bounded from below by the curvature-operator (see [G-M 1]), which can be bounded in each point by $|\sigma|$ (see [KR]).

$$\widehat{A}(M) \leqslant \dim[\operatorname{Ker}(D^*D + \operatorname{Scal}/4)] \ ,$$

where the potential is here the function "scalar curvature of (M, g)".

End of the proof of the Theorem 10. — Use the same argument, but notice that $\lambda_1(\Delta) \geqslant \operatorname{Is}(p)^2/A(p)^2$ by Theorem 6 and make $\alpha = \operatorname{Log}(2)/[D\cdot B(p)]$ in Theorem 3. Then apply Proposition 13, (iii).

d) Manifolds with boundary (without convexity assumption). — In this case two new problems arise :

The quadratic form whose index must be bounded is

$$Q_{\mathcal{R}}(s) = \int_M \left[|Ds|^2 + \mathcal{R}(s,s)\right] dv_g + \int_{\partial M} K(x)\cdot|s(x)|^2 dx \ ,$$

where K is the curvature of the boundary. In the non-convex case, we have to get a precise geometric bound of $\big(\int_{\partial M}|s|^2\big)/\big(\int_M |s|^2\big)$.

– In the right hand side of Proposition 13, the spectrum of Δ must be replaced by the spectrum with Neumann boundary condition. As isoperimetric inequalities are not avaible in this case, the Theorem 6 (i) must be proved by another argument.

See [G-M 2] for one kind of answer to this problem. In the case where the boundary is convex, previous results where given by [L–Y 3] and [MR] (this last one extends to the case where the boundary is minimal).

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Appendix. *Constructing Riemannian manifolds by surgery without keeping control of spectrum and topology, while controlling $L^{p/2}$-norms of Ricci-curvature and diameter. Discussion about the sharpness of the results.*

Let r_α be the part of Ricci-curvature which goes below the a priori fixed constant $-\alpha^2$. More precisely : $r_\alpha(x) = \sup(-\alpha^2 - r(x), 0)$. Let us recall the general scheme of Theorems 3, 6, 10 and 11 :

For every manifold (M, g) which satisfies

(i) $\frac{1}{\mathrm{Vol}(M,g)} \int_M |r_\alpha|^{p/2} \cdot dv_g \leqslant \varepsilon(\alpha, D)$ *for at least one* $p \in]n, +\infty]$,

(ii) $\mathrm{diam}(M, g) \leqslant D$,

then the invariant involved (isoperimetric constant $Is_g(p)$ in theorem 3, eigenvalues $\lambda_i(M, g)$ of the Laplace-Beltrami operator, etc. in theorem 6, Betti numbers $b_i(M)$ etc. in theorems 10 and 11) are bounded (from below or above) by a non trivial constant $C(p, \alpha, D)$ (to be replaced by $\chi(p, \alpha, D, \frac{1}{\mathrm{Vol}\, M} \int_M |\sigma|^{p/2} dv_g)$ when Betti numbers of order $i \geqslant 2$ are concerned).

The exact value of $\varepsilon(\alpha, D)$ is computed in each theorem involved.

Are these results sharp? As the assumption (i) seems rather complicated, is it possible to improve or simplify it? The answer is negative. More precisely :

– Both in the isoperimetric and spectral estimates (theorems 3, 6) and in the topological estimates (theorems 10, 11),

- it is impossible to replace the $L^{p/2}$ assumption in (i) by a $L^{n/2}$ one (see A.2). However, the question is still open for the simplicial volume (theorem 8),
- it is impossible to replace the assumption "$\frac{1}{\mathrm{Vol}\, M} \int_M |r_\alpha|^{p/2} \cdot dv_g \leqslant \varepsilon(\alpha, D)$" in (i) by "$\frac{1}{\mathrm{Vol}\, M} \int_M |r_\alpha|^{p/2} dv_g$ bounded" (i.e. $\varepsilon(\alpha, D)$ must be small enough see A.3 and A.5),
- it is impossible to replace the assumption (ii) "diameter bounded" by "volume bounded, even if $||\sigma||_{L^\infty}$ is bounded (see A.6).

– In the isoperimetric and spectral estimates, it is moreover impossible to replace

$$\text{“}\frac{1}{\mathrm{Vol}(M,g)} \int_M |r_\alpha|^{p/2} \leqslant \varepsilon(\alpha, D)\text{” by “}\int_M |r_\alpha|^{p/2} \leqslant \varepsilon(\alpha, D)\text{”}$$

in the assumption (i) (see A.4).

We shall get counter-examples by gluing together thin cylinders $C(\alpha, \varepsilon, \eta) = (C, g_{\alpha,\varepsilon,\eta})$ with large boundaries, whose negative part of Ricci curvature has small $L^{p/2}$-norm. Let

$$C = [-1, 1] \times S^{n-1} \quad \text{and} \quad g_{\alpha,\varepsilon,\eta} = (dt)^2 + b^2(t) \cdot g_{S^{n-1}} ,$$

where $b(t) = \eta(t^2 + \varepsilon^2)^{\alpha/2}$. Let us call $\sigma_{\min}(x)$ the infimum, for every 2-dimensional subspace P of $T_x M$, of the sectional curvature of P .

A.1. *For every* $(\alpha, \varepsilon, \eta) \in]\frac{p-1}{n-1}, +\infty[\times]0, 1[\times]0, +\infty[$, *one has*

$$\int_{C(\alpha,\varepsilon,\eta)} |r_-|^{p/2} \leqslant \int_{C(\alpha,\varepsilon,\eta)} |\sigma_{\min}|^{p/2}$$

$$\leqslant B \cdot \eta^{n-1} \left[\frac{\left(\alpha(\alpha-1) + (\alpha^2 - \frac{1}{\eta^2})_+ + \alpha^2\varepsilon^2\right)^{p/2}}{\alpha - (\frac{p-1}{n-1})} + \frac{\varepsilon^{(n-1)\alpha-(p-1)}}{2p - 1 - \alpha(n-1)} \right]$$

$$\operatorname{Vol} C(\alpha,\varepsilon,\eta) \geqslant \frac{2\eta^{n-1}}{\alpha(n-1)+1}$$

Proof. — For every X, Y tangent to $\{t\} \times S^{n-1} \subset C$, from the equation of Jacobi fields and Gauss-Codazzi formula, we get

$$\sigma(\frac{\partial}{\partial t}, X) = \frac{-b''}{b}(t) = -\frac{\alpha(\alpha-1)}{t^2+\varepsilon^2} - \frac{\alpha(2-\alpha)\varepsilon^2}{(t^2+\varepsilon^2)^2} ,$$

$$\sigma(X,Y) = \frac{1}{b^2(t)} - (\frac{b'}{b})^2(t) \geqslant \frac{1}{(t^2+\varepsilon^2)}\Big(\frac{1}{(1+\varepsilon^2)^{\alpha-1}\eta^2} - \alpha^2\Big) .$$

As the Riemannian measure is

$$dv_{g_{\alpha,\varepsilon,\eta}} = \eta^{n-1}(t^2+\varepsilon^2)^{\frac{(n-1)\alpha}{2}} ,$$

a direct computation then gives **A.1** ∎

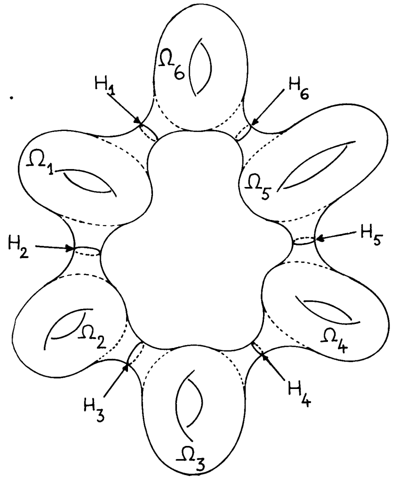
A.2. *For every positive ε, D , and in every dimension $n \geqslant 3$, there exists a sequence of Riemannian manifolds (M_k, g_k) which satisfies*

(i) $\operatorname{diam}(M_k, g_k) \leqslant D$, $\int_{M_k} |r_-|^{n/2} dv_{g_k} \leqslant \int_{M_k} |\sigma_{\min}|^{n/2} \cdot dv_{g_k} < \varepsilon$ *and* $\frac{1}{\operatorname{Vol}(M_k,g_k)} \int_{M_k} |\sigma_{\min}|^{n/2} dv_{g_k} < \varepsilon$,

(ii) $\lim_{k\to\infty} Is_{g_k}(p) = 0$,

(iii) *For every i* , $\lim_{k\to\infty} \lambda_i(M_k, g_k) = 0$,

(iv) $\lim_{k\to\infty} b_i(M_k) = +\infty$ for every $i \in \{1, \dots, n-1\}$.



THE RIEMANNIAN MANIFOLD (M_6, g_6)

Proof. — The manifold M_k is obtained, in a standard way, by gluing together k copies of the torus $\mathbf{T}^n$ (after excision of 2 balls from each torus) by means of cylinders, in order to form a closed chain (see the picture). We get a first metric $\tilde{g}_k$ on M_k by equiping each torus with a flat metric whose diameter and injectivity radius are fixed and respectively greater than 4 and 2. The cylinders are equiped with the metric $g_{\alpha_k,\varepsilon_k,\eta_k}$ defined in A.1, where $\alpha_k = 1 + k \cdot e^{-k}$, $\varepsilon_k = \exp(-e^k)$, $\eta_k = \frac{1}{\alpha_k}(1+\varepsilon_k^2)^{\frac{2-\alpha}{2}}$. If the radii R_k of the excised balls are chosen equal to $\dfrac{1+\varepsilon_k^2}{\alpha_k}$, then the metric $\tilde{g}_k$ thus obtained is C^1 and, by applying A.1,

$$\int_{M_k} |\sigma_{\min}(\tilde{g}_k)|^{n/2} dv_{\tilde{g}_k} \leqslant B'k\Big[(k.e^{-k})^{\frac{p-2}{2}} + e^{-(n-1)k}\Big] .$$

This estimate is unchanged when replacing $\tilde{g}_k$ by $g_k = \dfrac{1}{k^4}\tilde{g}_k$. As $\operatorname{diam}(M_k, g_k) = \mathrm{O}(\frac{1}{k})$ and $\operatorname{Vol}(M_k, g_k) \cdot k^{2n-1}$ is bounded from below, the assumptions (i) are proved. The assumption (ii) is obviously true, since each minimal hypersurface H_i in each cylinder satisfies $\operatorname{Vol}(H_i) = k^{2-2n} \cdot \mathrm{O}(\varepsilon_k^{(n-1)\alpha_k})$. Let H_i be the minimal hypersurface of the i^{th} cylinder ($i \in \mathbf{Z}/k.\mathbf{Z}$) and Ω_i be the domain bounded by H_i and H_{i+1} . The min-max principle applied to the functions

$$u_i = \min\Big[\frac{1}{\varepsilon_k} d(\bullet, H_i), 1, \frac{1}{\varepsilon_k} d(\bullet, H_{i+1})\Big] \cdot \chi_{\Omega_i}$$

gives

$$\lambda_{k-1}(M_k, g_k) \leqslant \frac{\int_{\Omega_i} |du_i|^2 . dv_{g_k}}{\int_{\Omega_i} u_i^2 \cdot dv_{g_k}} = k^4 \cdot \mathrm{O}(\varepsilon_k^{\alpha_k(n-1)-1}) .$$

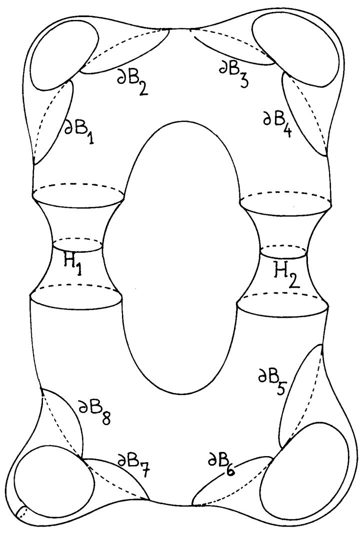
This proves (iii). We then prove (iv) by computing the cohomology groups from the Mayer-Vietoris' sequence. ∎

A.3. *For any $n \in \mathbf{N} \setminus \{0,1\}$ and $p \in]n, +\infty[$, for any positive D , there exists a sequence of n-dimensional Riemannian manifolds (M_k, g_k) which satisfies*

(i) $\operatorname{diam}(M_k, g_k) \leqslant D$ *and* $\frac{1}{\mathrm{Vol}(M_k, g_k)} \int_{M_k} |r_-|^{p/2} dv_{g_k} \leqslant C(n,p) D^{n-p}$

(ii) $\lim_{k\to\infty} b_1(M_k) = +\infty$

Proof. — Let $C_k = C(\alpha, \varepsilon_k, \eta_k)$ (with α fixed in $]\frac{p-1}{n-1}, +\infty[\setminus \{\frac{2p-1}{n-1}\}$, $\varepsilon_k = e^{-k}$ and $\eta_k = e^{\alpha k}$) be a cylinder as defined in A.1. We get a new cylinder C'_k by gluing, to each boundary of C_k , a cylinder $B = [0,2] \times S^{n-1}$ equiped with the metric $g_B = (dt)^2 + \eta_k^2 (1+\varepsilon_k^2)^\alpha \cdot g_{S^{n-1}}$. We obtain $(M_0, \overline{g}_k)$ by gluing together 2 copies of C'_k . We get $(M_k, \tilde{g}_k)$ from $(M_0, \overline{g}_k)$ by excising, from the thick parts B of M_0 , balls $B_1, \ldots, B_{2k}$ of radius $R = \frac{1}{\alpha}(1+\varepsilon_k^2)$ and by gluing a cylinder $C(\alpha, \varepsilon_k, \eta'_k)$ (where $\eta'_k = \frac{(1+\varepsilon_k^2)^{\frac{2-\alpha}{2}}}{\alpha}$) to each pair of boundaries $(\partial B_1, \partial B_2), \ldots, (\partial B_{2k-1}, \partial B_{2k})$. (see the picture). The values of R and η'_k are such that the singularities of $\tilde{g}_k$ at the gluings only increase the positive part of the curvature. By A.1, we get



THE RIEMANNIAN MANIFOLD (M_4, g_4)

$$\int_{M_k} |r_-(\tilde{g}_k)|^{p/2} \leqslant C(n,p,\alpha)\big[e^{(n-1)k\alpha} + k\big]$$

$$\mathrm{Vol}(M_k, \tilde{g}_k) \geqslant C'(n,p,\alpha)\big[e^{(n-1)k\alpha} + k\big] .$$

Let us call H_1 and H_2 the two minimal hypersurfaces of (M_0, g_0) . Every $x \in M_0$ satisfies $d(x, H_1) + d(x, H_2) = 6$, so

$$\operatorname{diam}(M_k, \tilde{g}_k) \leqslant \operatorname{diam}(M_0, g_0) + 2 \leqslant 8 + \operatorname{diam}(H_1) \leqslant 8 + \pi .$$

Changing $\tilde{g}_k$ in $g_k = \left(\frac{D}{8+\pi}\right)^2 \cdot \tilde{g}_k$, we get a metric which satisfies the assumptions (i). Using the Mayer-Vietoris' sequence, we get $b_1(M) = k+1$, which gives (ii).

A.4. *For any $n \in \mathbf{N} \setminus \{0,1\}$ and $p \in]n, +\infty[$, for any positive D , there exists a sequence of Riemannian metrics g_k on $M = S^{n-1} \times S^1$ which satisfies*

(i) $\operatorname{diam}(M, g_k) \leqslant D$ *and* $\lim_{k\to\infty} \int_{M_k} |r_-|^{p/2} dv_{g_k} = 0$.

(ii) $\lim_{k\to\infty} \mathrm{Is}_{g_k}(p) = 0$.

(iii) For each $i \in \mathbf{N}$, $\lim_{k\to\infty} \lambda_i(M, g_k) = 0$.

Proof. — In the construction of example A.2, just replace the k copies of the torus by k copies of the sphere $S^n(e^{-k})$ (whose radius is e^{-k}), α_k by some fixed $\alpha \in \left]\frac{p-1}{n-1}, \frac{2p-1}{n-1}\right[$, ε_k by e^{-k}, η_k by $\frac{(1+\varepsilon_k^2)^{\frac{2-\alpha}{2}}}{[\alpha^2+e^{2k}(1+\varepsilon_k^2)^2]^{1/2}}$ and R_k by $e^{-k}\cdot \text{Arc tg}\left[\frac{e^k}{\alpha}(1+\varepsilon_k^2)\right]$. The new values of the radii R_k of the excised balls and of η_k are computed such that he metric $\tilde{g}_k$, obtained by gluing the new cylinder $C(\alpha,\varepsilon_k,\eta_k)$ to $S^n(e^{-k})$, is smooth. By applying A.1, we get

$$\int_{M_k} |r_-|^{p/2} dv_{\tilde{g}_k} \leqslant C(n,p,\alpha)\cdot k\cdot \eta_k^{n-1}$$

$$\operatorname{diam}(M,\tilde{g}_k) < 2k$$

Changing $\tilde{g}_k$ for $g_k = \frac{1}{k^4}g_k$, we prove (i). The proof of (ii) and (iii) is the same as in A.2.

A.5. *For any $n \in \mathbf{N}\setminus\{0,1\}$ and $p \in]n,+\infty[$, for any positive D, there exists a sequence of Riemannian metrics g_k on $M = S^{n-1}\times S^1$ which satisfies*

(i) $\operatorname{diam}(g_k) \leqslant D$ *and* $\frac{1}{\operatorname{Vol}(M_k,g_k)}\int_{M_k}|r_-|^{p/2} \leqslant C(n,p)\cdot D^{n-p}$

(ii) $\lim_{k\to\infty} \operatorname{Is}_{g_k}(p) = 0$

(ii) For each $i\in\mathbf{N}$, $\lim_{k\to\infty}\lambda_i(M,g_k) = 0$.

Proof. — Let us consider the manifold $(M_0,\overline{g}_k)$ of the construction of the example A.3. The metric $\overline{g}_k$ may also be considered as a metric on $S^1\times S^{n-1}$ which only depends on the component in the factor S^1. Let $\mathcal{E}$ be the subspace of $L^2(S^{n-1},\text{can})$ spanned by the eigenfunctions $\varphi_0,\dots,\varphi_i$ corresponding to $\lambda_0,\dots,\lambda_i$. We define a canonical injective application ψ from $\mathcal{E}$ in $L^2(M_0,dv_{\overline{g}_k})$ by $\psi(u)(s,x) = u(x)$. The min-max principle and a direct computation give

$$\lambda_i(M_0,\overline{g}_k) \leqslant \sup_{u\in\mathcal{E}}\frac{||d(\psi(u))||_{L^2}}{||\psi(u)||_{L^2}} \leqslant \frac{\lambda_i(S^{n-1},\text{can})}{\eta_k^2}\,\frac{4+\int_0^1(t^2+\varepsilon_k^2)^{\alpha\left(\frac{n-3}{2}\right)}dt}{4+\int_0^1(t^2+\varepsilon_k^2)^{\alpha\left(\frac{n-1}{2}\right)}dt}$$

$$\lambda_i(M_0,\overline{g}_k) \leqslant \lambda_i(S^{n-1},\text{can})\cdot\varepsilon_k^{\alpha+1}$$

changing $\overline{g}_k$ for $g_k = \left(\frac{D}{8+\pi}\right)^2\cdot\overline{g}_k$, we prove (iii). The same arguments as in A.3 prove (i). The proof of (ii) is obvious. ∎

A.6. *For any $n\in\mathbf{N}\setminus\{0,1,2\}$, there exists a sequence of n-dimensional Riemannian manifolds (M_k,g_k) which satisfies*

(i) $-1\leqslant\sigma(g_k)\leqslant 0$ *and* $\operatorname{Vol}(M_k,g_k) = 1$

(ii) $\lim_{k\to\infty}\lambda_i(M_k,g_k) = 0$

(iii) $\lim_{k\to\infty} b_1(M_k) = +\infty$

Proof. — Let $M_k = \mathbf{T}^{n-2}\times H_k$, where H_k is a compact surface of genus k. Let g_0 be a given flat metric on $\mathbf{T}^{n-2}$ and $\tilde{g}_k$ be a metric on H_k whose curvature is equal to -1. Let $g_k = \frac{1}{4\pi(k-1)\cdot\operatorname{Vol}(g_0)}\cdot g_0\oplus\tilde{g}_k$. Then (M_k,g_k) obviously satisfies (i) and (iii). It is a classical result that $\tilde{g}_k$ may be chosen such that $\lim_{k\to\infty}\lambda_i(H_k,\tilde{g}_k) = 0$, which implies (ii). ∎