Astérisque

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Astérisque, tome 157-158 (1988), p. 147-171

<http://www.numdam.org/item?id=AST_1988__157-158__147_0>

Société Mathématique de France
Astérisque 157-158 (1988)

Representation for functionals of superprocesses by multiple stochastic integrals, with applications to self-intersection local times[1]

by Eugene B. DYNKIN

ABSTRACT. The representation of functionals of a Gaussian process by the multiple Wiener–Ito integrals plays an important role in stochastic calculus. We establish a similar representation for a certain class of non–Gaussian measure–valued Markov processes. A process X of this class can be associated with every Markov process ξ and we call X a superprocess over ξ. The existence of local times and self–intersection local times for X depends on the behaviour of the transition density of ξ as $t\to 0$.

1. INTRODUCTION

1.1. Let ξ_t, $t\in\Delta$ be a Markov process in a measurable space $(E,\mathcal{B})$ with the transition function $p(s,x;t,dy)$ and let $\mathcal{M}$ be a set of measures on $(E,\mathcal{B})$. We say that an $\mathcal{M}$–valued Markov process X_t is ***a superprocess over*** ξ_t if, for all $r<t\in\Delta$, $\mu\in\mathcal{M}$ and $B\in\mathcal{B}$,

$$E_{r,\mu} X_t(B)=\int\mu(dx)p(r,x;t,B). \tag{1.1}$$

This implies

$$E_{r,\mu}<f,X_t>=<T_t^r f,\mu> \tag{1.2}$$

where

$$T_t^r f(x)=\int p(r,x;t,dy)f(y)$$

[1]Partially supported by National Science Foundation Grant DMS–8505020.

and $<f,\mu>$ means $\int f d\mu$. (The domain of integration is not indicated under the integral sign if this is the entire domain of the corresponding measure.)

1.2. In this paper we deal with a special class of superprocesses introduced and studied by S.Watanabe [16] and D.Dawson [1], [2] (see [3] for more references).

We start from a Markov process $\xi_t, t\in\Delta=[0,u]$ on $(E,\mathcal{B})$ assuming that its ***transition function*** $p(s,x;t,B)$ ***is*** $\mathcal{B}(\Delta)\times\mathcal{B}\times\mathcal{B}(\Delta)-$ ***measurable for every*** $B\in\mathcal{B}$ ($\mathcal{B}(\Delta)$ is the Borel σ–algebra in Δ). We define $\mathcal{M}$ as ***the space of all finite measures on*** $(E,\mathcal{B})$. We consider a system of particles which move independently according to the law of the process ξ_t. Each particle has the mass β. There are n identically distributed particles at time 0. At time α each particle dies leaving, with equal probabilities, 0 or 2 offspring, and the offspring develope independently in the same way.

By passing to the limit as $n\to\infty, \alpha,\beta\to 0$ and $n\beta\to 1$, $\beta/(2\alpha)\to 1$, we get a superprocess X_t over ξ_t for which

$$E_{r,\mu} e^{<f,X_t>} = e^{<\varphi_r,\mu>}. \tag{1.3}$$

Here f is an arbitrary negative measurable function and φ satisfies the integral equation

$$\varphi_r = \int_r^t T_s^r(\varphi_s^{\,2})ds + T_t^r f \tag{1.4}$$

on the interval $[0,t]$.

The existence and uniqueness of the solution of (1.4) and of the corresponding superprocess X have been proved in [7]. [Under the assumption that p is a stationary transition function and that the related semi–group is Feller and continuous this has been proved first in [16], see also [11]].

We put

$$T_s^r = 0 \text{ for } r>s,\ T_s^s f = f$$

and we rewrite (1.4) in the form

$$\varphi = \varphi * \varphi + h \tag{1.5}$$

where $\varphi_s = 0$ for $s>t$ and

$$(\varphi * \psi)_r = \int_\Delta T_s^r(\varphi_s \psi_s) ds \tag{1.6}$$

and

(1.7) $$h_r(x)=T_t^r f(x)$$

(the value of t is fixed).

If h is a bounded function then, for all sufficiently small α, the equation

$$\varphi=\varphi*\varphi+\alpha h$$

has a unique solution and this solution is an analytic function of α [see [2] or [7]].

1.3. Our investigation is based on an explicit expression of the moments of the random field $<f,X_t>$ in terms of the transition function p. The main step is done in the following:

THEOREM 1.1. *Let* $r< \min\{t_1,\dots,t_n\}\in\Delta$. *For arbitrary positive measurable functions* $f_1,\dots,f_n$,

(1.8) $$E_{r,\mu}<f_1,X_{t_1}>\dots<f_n,X_{t_n}> = \sum_{\Lambda_1,\dots,\Lambda_k} \prod_{i=1}^{k} \int_E W_{\Lambda_i}(r,x)\mu(dx),$$

the sum is taken over all partitions of $\{1,2,\dots,n\}$ ***into disjoint non–empty subsets*** $\Lambda_1,\dots,\Lambda_k$ $(k=1,2,\dots n)$, ***and***

(1.9) $$W_\Lambda=\prod_{i\in\Lambda}^{*} h^i$$

with

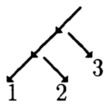
(1.10) $$h_r^i(x)=T_{t_i}^r f_i(x).$$

The symbol $\prod^*$ means the sum of $*$–products over all orders of factors and all orders of operations. For instance,

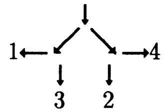
$$W_{\{1,2\}}=h^1*h^2+h^2*h^1,$$
$$W_{\{1,2,3\}}=(h^1*h^2)*h^3+h^1*(h^2*h^3)$$

+ ten more terms obtained by permutations of 1,2,3.

1.4. There exists an obvious 1–1 correspondence between $*$–monomials and directed binary trees with marked exits. For instance, the monomial $(h^1*h^2)*h^3$ corresponds to the tree



and monomial $(h^1 * h^3) * (h^2 * h^4)$ corresponds to the tree

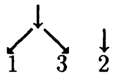


(cf. Wild (1951)).

The right side in (1.8) can be represented as the sum of terms

$$\langle H_{\mathcal{T}_1}, \mu\rangle \ldots \langle H_{\mathcal{T}_k}, \mu\rangle \tag{1.11}$$

where $H_{\mathcal{T}_i}$ is the $*$–product of $h^j, j \in \Lambda_i$ corresponding to a binary tree $\mathcal{T}_i$ marked my the elements of Λ_i. We associate with the term (1.11) a graph D whose connected components are marked trees $\mathcal{T}_1, \ldots, \mathcal{T}_k$. For example, the diagram $\underset{1}{\downarrow}\ \underset{2}{\downarrow}\ \underset{3}{\downarrow}$ corresponds to the term $\langle h^1, \mu\rangle\ \langle h^2, \mu\rangle\ \langle h^3, \mu\rangle$ and the diagram



corresponds to $\langle h^1 * h^3, \mu\rangle\ \langle h^2, \mu\rangle$.

1.5. In general, a diagram D is a directed graph with a set A of arrows and a set V of vertices (or sites). Writing $a: v \to v'$ indicates that v is the beginning and v' is the end of an arrow a.

For every vertex v, we denote by $a_+(v)$ the number of arrows which end at v and by $a_-(v)$ the number of arrows which begin at v. We consider only diagrams whose connected components are binary trees that is for every $v \in V$ there exist only three possibilities: i) $a_+(v)=0$, $a_-(v)=1$; (ii) $a_+(v)=1$, $a_-(v)=0$; (iii) $a_+(v)=1$, $a_-(v)=2$. We denote the coresponding subsets of V by V_-, V_+ and V_0, and we call elements of V_- ***entrances*** and elements of V_+ ***exits***. Put $a \in A_+$ if the end of a is an exit, and $a \in A_0$ if this is not the case.

Let $\mathbb{D}_n$ be the set of all diagrams with exits marked by 1,2,...,n. We label each site of $D \in \mathbb{D}_n$ by two variables – one with values in $\mathbb{R}_+$ and the other with values in E. Namely, (t_i, z_i) is the label of the exit marked by i, (r, x_v) is the label of an entrance v, and (s_v, y_v) is the label of a site $v \in V_0$.

We agree that $p(s,x;t,B)=0$ for $s>t$. For an arrow $a: v \to v'$ we put $p_a = p(s,w;s',dw')$ where (s,w) is the label of v and (s',w') is the label of v'. Using this notation we can restate Theorem 1.1

in a new form:

Theorem 1.1'. *For* $r<\min\{t_1,\dots,t_n\}\in\Delta$ *and all positive measurable* $f_1,\dots,f_n$,

$$E_{r,\mu}\langle f_1,X_{t_1}\rangle\dots\langle f_n,X_{t_n}\rangle=\sum_{D\in\mathbb{D}_n} c_D \tag{1.12}$$

where

$$c_D=\int\prod_{v\in V_-}\mu(dx_v)\prod_{a\in A}p_a\prod_{v\in V_0}ds_v\prod_{i=1}^{n}f_i(z_i). \tag{1.13}$$

Example. The diagram D corresponding to $\langle h^1*h^3,\mu\rangle\ \langle h^2,\mu\rangle$ can be labelled as follows

$$\begin{tikzcd}
& (r,x_1) \arrow[d] & & (r,x_2) \arrow[d] \\
& (s_1,y_1) \arrow[dl] \arrow[dr] & & (t_2,z_2) \\
(t_1,z_1) & & (t_3,z_3) &
\end{tikzcd}$$

(in contrast to the marking of the exits, the enumeration of V_- and V_0 is of no importance), and we have

$$c_D=\langle h^1*h^3,\mu\rangle\ \langle h^2,\mu\rangle=\int\mu(dx_1)\mu(dx_2)f_1(z_1)f_2(z_2)f_3(z_3)ds_1$$
$$\times p(r,x_1;s_1,dy_1)p(s_1,y_1;t_1,dz_1)p(s_1,y_1;t_3,dz_3)p(r,x_2;t_2,dz_2).$$

1.6. Let $\mathcal{W}$ be ***the space of all bounded measurable functions on*** $\Delta\times E$ ***with the topology induced by the bounded convergence.*** The operation $\varphi*\psi$ is a continuous mapping from $\mathcal{W}\times\mathcal{W}$ to $\mathcal{W}$. We denote by $\mathcal{K}$ ***the set of all functions of the form*** **(1.7)** ***with bounded*** f and we introduce the following assumption:

1.6.A. If $\mathcal{C}$ is a closed linear subspace of $\mathcal{W}$ and if $\mathcal{C}\supset\mathcal{K}$, then $\mathcal{C}=\mathcal{W}$.

We show in the Appendix that condition 1.6.A is satisfied if p is the transition function of a right process. In particular, 1.6.A holds for all classical diffusions.

It follows from Theorem 1.1 that

$$\langle f_1,X_{t_1}\rangle\dots\langle f_n,X_{t_n}\rangle \tag{1.14}$$

belongs to $L^2(P_\mu)$ for every $\mu\in\mathcal{M}$ and all bounded $f_1,\dots,f_n$. We fix a measure $\mu\in\mathcal{M}$ and we denote by L^2_n ***the minimal closed subspace of*** $L^2(P_\mu)$ ***which contains all the products*** (1.14).

Put

$$(\varphi,\psi)_n=\int\varphi(t_1,z_1;\ldots;t_n,z_n)\psi(t_{n+1},z_{n+1};\ldots;t_{2n},z_{2n})\,\gamma_{2n}(dt_1,dz_1;\ldots;dt_{2n},dz_{2n}) \tag{1.15}$$

and denote by $\mathcal{X}_n^0$ ***the set of functions*** φ ***for which*** $(|\varphi|,|\varphi|)_n<\infty$. Measures γ_{2n} will be specified in such a way that $(\varphi,\varphi)_n\geq 0$ for all $\varphi\in\mathcal{X}_n^0$. For every $\varphi\in\mathcal{X}_n^0$ we define a multiple stochastic integral

$$I_n(\varphi)=\int\varphi(t_1,z_1;\ldots;t_n,z_n)dZ_{t_1,z_1}\ldots dZ_{t_n,z_n} \tag{1.16}$$

with the property

$$E_\mu I_n(\varphi)I_n(\psi)=(\varphi,\psi)_n. \tag{1.17}$$

Hence I_n is an isometry from the pre–Hilbert space $\mathcal{X}_n^0$ to L_n^2. It has a unique continuation to an isometry from the completion $\mathcal{X}_n$ of $\mathcal{X}_n^0$ ***onto*** L_n^2. One can say that every functional of degree n has a unique representation (1.16) with a $\varphi\in\mathcal{X}_n$.

1.7. The case of n=1 is of special importance. First, we define $I_1(\varphi)$ for $\varphi\in\mathcal{X}$ by putting

$$I_1(\varphi)=\int\varphi(s,x)dZ_{s,x}=\langle f,X_t\rangle \tag{1.18}$$

for

$$\varphi(s,x)=T_t^s f(x). \tag{1.19}$$

In other words, we set

$$\int T_t^s f(x)dZ_{s,x}=\langle f,X_t\rangle \tag{1.20}$$

for every $t\in\Delta$ and every bounded measurable f.

By (1.12),

$$E_\mu I_1(\varphi_1)I_1(\varphi_2)=\int\varphi_1(t_1,z_1)\varphi_2(t_2,z_2)d\gamma_2 \tag{1.21}$$

with

$$\gamma_2(A_1\times B_1\times A_2\times B_2)=1_{A_1}(0)\mu(B_1)1_{A_2}(0)\mu(B_2) +2\int ds\,\mu(dx)p(0,x;s,dy)1_{A_1}(s)1_{B_1}(y)1_{A_2}(s)1_{B_2}(y). \tag{1.22}$$

Put $\varphi\in\mathcal{X}_1^0(t)$ of $\varphi\in\mathcal{X}_1^0$ and $\varphi(s,x)=0$ for all $s\in(t,u]$, $x\in E$. We call elements φ and ψ of $\mathcal{X}_1^0$ ***equivalent*** if $(\varphi-\psi,\varphi-\psi)_1=0$.

THEOREM 1.2. ***Classes of equivalent elements of*** $\mathcal{X}_1^0$ ***form a Hilbert space*** $\mathcal{X}_1$. ***Under condition***

1.6.A *there exists a unique isometry* I_1 *from* $\mathcal{X}_1$ *onto* L_1^2 *subject to condition* (1.18).

A random variable $Y \in L_1^2$ *is* $\mathcal{F}_t$*—measurable if and only if*

(1.23) $$Y=\int \varphi(s,x)dZ_{s,x}$$

for some $\varphi \in \mathcal{X}_1^0(t)$. *We have*

(1.24) $$E_\mu\{\int \varphi(s,x)dZ_{s,x}|\mathcal{F}_t\}=\int \varphi(s,x)1_{s\leq t}dZ_{s,x}$$

and

(1.25) $$E_\mu\int \varphi(s,x)dZ_{s,x}=\int 1_{s=0}\varphi(s,x)dZ_{s,x}=\int \varphi(0,x)\mu(dx).$$

For every $\varphi \in \mathcal{X}_1^0$,

(1.26) $$M_t^\varphi=\int \varphi(s,x)1_{s\leq t}dZ_{s,x},\ t\in\Delta$$

is a martingale, and formula (1.26) *describes all* L_1^2*—valued martingales.*

It is proved in [7] that, under broad assumptions, all martingales M_t^φ are continuous and have the quadratic variation

(1.27) $$<M,M>_t=2\int_0^t <\varphi(s,.)^2,X_s>ds.$$

(cf. [14]). In terminology of Metivier [12] and Walsh [15], $Z_{s,x}$ is ***a martingale measure.***

1.8. For an arbitrary n, we put

(1.28) $$\gamma_n=\sum_{D\in\mathbb{D}_n}\gamma_D$$

with

(1.29) $$\gamma_D(A_1\times B_1\times\ldots\times A_n\times B_n)$$

$$=\int\prod_{v\in V_-}\mu(dx_v)\prod_{a\in A_0}p_a\prod_{v\in V_0}ds_v\prod_{i=1}^n 1_{A_i}(s_{v_i})1_{B_i}(y_{v_i}).$$

Here v_i is the beginning of the arrow a_i leading to the exit with the mark i.

Example. For the diagram D at the end of Subsection **1.4** (with r=0),

$$\gamma_D(A_1\times B_1\times A_2\times B_2\times A_3\times B_3)=\int\mu(dx_1)ds_1p(0,x_1;s_1,dy_1)$$
$$\times 1_{A_1}(s_1)1_{B_1}(y_1)1_{A_3}(s_1)1_{B_3}(y_1)1_{A_2}(0)\mu(B_2).$$

Let

$$\ell_p(\varphi)=\int\prod_{i=1}^{2p}\varphi(t_i,z_i)\gamma_{2p}(dt,dz).$$

Lemma 1.1. *For all* $\varphi_1,\ldots,\varphi_n\in\mathscr{W}$,

$$E_\mu\prod_{i=1}^{n}I_1(\varphi_i)=\int\prod_{i=1}^{n}\varphi_i(t_i,z_i)\ \gamma_n(dt,dz). \tag{1.30}$$

Moreover (1.30) *holds for unbounded* φ_i *if* $\ell_p(|\varphi_i|)<\infty$ *for* $i=1,\ldots,n$ *and some* $p\geq n/2$.

Theorem 1.3. *Under condition* 1.6.A, *there exists a unique mapping* I_n *from* $\mathscr{X}_n^0$ *to* L_n^2 *such that*

$$I_n(\varphi_1\times\ldots\times\varphi_n)=I_1(\varphi_1)\ldots I_1(\varphi_n) \tag{1.31}$$

and (1.17) *is true for all* $\varphi,\psi\in\mathscr{X}_n^0$. *The image* $I_n(\mathscr{X}_n^0)$ *is everywhere dense in* L_n^2.

1.9. Now we assume that:

1.9.A. There exists a measure m (a reference measure) such that p(s,x;t,.) is absolutely continuous with respect to m for all s,t and x.

It is shown in [9] that the density p(s,x;t,y) can be chosen to be jointly measurable in s,x,t,y and to satisfy the relation

$$\int p(s,x;t,y)\ dy\ p(t,y;v,z)=\ p(s,x;v,z) \tag{1.32}$$

for all $x,z\in E$, $s<t<v\in\Delta$ (for sake of brevity, we write dy for m(dy)).

Define the delta functions δ_z, $z\in E$ and δ^n, n=2,3,... as the linear functionals

$$\int\delta_z(x)f(x)dx=f(z) \tag{1.33}$$

$$\int\delta^n(x_1,\ldots,x_n)f(x_1,\ldots,x_n)dx_1\ldots dx_n=\int f(x,\ldots x)dx \tag{1.34}$$

Heuristically, the local time at point z is given by the formula

$$L_z(B)=\int_B<\delta_z,X_t>dt,\quad B\in\mathcal{B}(\Delta) \tag{1.35}$$

and the self–intersection local time of order n is given by the formula

$$L^n(B)=\int_B<\delta^n,X_{t_1}\times\ldots X_{t_n}>dt_1\ldots dt_n,\quad B\in\mathcal{B}(\Delta^n). \tag{1.36}$$

It follows from the construction of the multiple stochastic integral that, for not too bad functions f,

(1.37)
$$\int_B \langle f, X_{t_1}\times\ldots\times X_{t_n}\rangle\, dt_1\ldots dt_n$$
$$=\int F(s_1,x_1;\ldots;s_n,x_n)dZ_{s_1,x_1}\ldots dZ_{s_n,x_n}$$

where

(1.38)
$$F(s_1,x_1;\ldots;s_n,x_n)$$
$$=\int 1_B(t_1,\ldots,t_n)f(y_1,\ldots,y_n)\prod_{i=1}^{n}p(s_i,x_i;t_i,y_i)dt_i dy_i.$$

By extrapolating, heuristically, this expression to the delta functions, we get

(1.39)
$$L_z(B)=\int K_{z,B}(s,x)dZ_{s,x},$$
$$L^n(B)=\int K^n_B(s_1,x_1;\ldots;s_n,x_n)dZ_{s_1,x_1}\ldots dZ_{s_n,x_n}$$

where

(1.40)
$$K_{z,B}(s,x)=\int_B p(s,x;t,z)dt,$$
$$K^n_B(s_1,x_1;\ldots;s_n,x_n)=\int_B dt_1\ldots dt_n\int_E dz\prod_{i=1}^{n}p(s_i,x_i;t_i,z).$$

The stochastic integrals in the right sides of (1.40) make sense if $K_{z,B}\in\mathcal{X}^0_1$ and $K^n_B\in\mathcal{X}^0_n$. Using Theorem 1.1' we give conditions for this in terms of the transition density $p(s,x;t,y)$.

Let

(1.41)
$$G(s,y;z)=\int_\Delta p(s,y;t,z)dt,$$
$$H(z,\zeta)=\int dsdy\; G(s,y;z)G(s,y;\zeta).$$

Theorem 1.4. *Suppose that* 1.6.A, 1.9.A *and the following conditions* 1.9.B,C *are satisfied:*

1.9.B. *The measure* μ *has a bounded density relative to* m, *i.e.* $\mu(dx)\le c\,dx$ *for some constant* c.

1.9.C. *There exists a* $C<\infty$ *such that* $\int dy\; p(s,y;s',y')\le C$ *for all* $s,s'\in\Delta$, $y'\in E$.

If

(1.42)
$$H(z,z)<\infty,$$

then $K_{z,B}\in\mathcal{X}^0_1$ *and therefore there exists local time* L_z.

Theorem 1.5. *Suppose that conditions* 1.6.A, 1.9.A,B *are satisfied and, in addition, that:*

1.9.D. *For every* $\beta>0$, *there exists a constant* $C<\infty$ *such that* $p(s,x;t,y)\le C$ *for all* $x,y\in E$ and $s,t\in\Delta$ such that $t-s>\beta$.

1.9.E. *There exists* $\beta>0$, *such that* $B\subset\{|t_i-t_j|\ge\beta\}$ *for all* $i\ne j$.

If

$$\sup_{s,y}\int G(s,y;z)G(s,y;\zeta)H(z,\zeta)^{n-1}dz\,d\zeta<\infty, \tag{1.43}$$

then $K_B^n\in\mathcal{X}_n^0$, *and there exists the self–intersection local time* L^n *of order* n.

Remark. Random variables $L_z(B)$ and $L^n(B)$ are defined only up to equivalence. The technique used in theory of additive functionals (see, e.g., [8] and [5]) allows to choose a version of these random variables such that $L_z(.)$ is a measure on Δ and $L^n(.)$ is a measure on Δ^n (the latter "explodes" on diagonals $D_{ij}=\{t:t_i=t_j\}, i\ne j$ but it is σ–finite on the complement of their union).

1.10. Consider an elliptic differential operator of the second order

$$\sum_{i,j=1}^{d} a^{ij}(s,x)D_iD_jf+\sum_{i=1}^{d} b^i(s,x)D_if-c(s,x)f,\ s\in\Delta=[0,u],\ x\in\mathbb{R}^d. \tag{1.44}$$

Under broad assumptions on the coefficients (see,e.g.,[4], Appendix, Theorem 0.4) the corresponding parabolic differential equation has a fundamental solution $p(s,x;t,y)$, and this solution is the transition density (relative to the Lebesgue measure) of a continuous Markov process which we call ***a classical diffusion in*** $\Delta\times\mathbb{R}^d$. Moreover, there exist constants M and $\alpha>0$ such that

$$p(s,x;t,y)\le M\, q_{t-s}^d(\alpha r) \text{ for all } s<t\in\Delta,\ x,y\in E \tag{1.45}$$

where $r=|y-x|$ and

$$q_t^d(r)=(2\pi t)^{-d/2}e^{-r^2/2t} \tag{1.46}$$

(of course, $q_{t-s}^d(|y-x|)$ is the Brownian transition density).

Put

$$Q_d^s(r)=\int_0^s q_t^d(r)dt. \tag{1.47}$$

THEOREM 1.6. *Local times* L_z *exist for the classical superdiffusion in* $\Delta\times\mathbb{R}^d$ *if* $d\le3$.

THEOREM 1.7. *Self–intersection local times* L^n *of order* n *exist for the classical superdiffusion* *in* $[0,u]\times\mathbb{R}^d$ *if*

$$\int_{\mathbb{R}^d} [Q^{2u}_{d-2}(|x|)]^n dx = \text{const.}\times\int_0^\infty Q^{2u}_{d-2}(r)^n r^{d-1} dr<\infty. \tag{1.48}$$

COROLLARY. *Self–intersection local times* L^n *exist for the classical superdiffusion in* $\Delta\times\mathbb{R}^d$:

(a) *for all* n *if* $d\le 4$;

(b) *for* $n\le 4$ *if* $d=5$;

(c) *for* $n\le 2$ *if* $d=6$ *or* 7.

Theorem 1.6 for the super–Brownian motion has been proved, first, by Iscoe [11].

Perkins has proved that the pairs (d,n) listed in the Corollary are exactly those pair for which the super–Brownian motion in $\mathbb{R}^d$ has, with positive probability, more than countable set of "n–multiple points" (z is an n–multiple point for X_t if z belongs to the support of X_{t_i} for n distinct times $t_1,\dots,t_n$). Presenting this result in his talk at Cornell in fall, 1986, Perkins conjectured the statement on self–intersection local times formulated in the Corollary.

1.11. **Acknowledgements.** The author is deeply indebted to D.Dawson, I.Iscoe and E.Perkins for stimulating discussions.

2. MOMENT FUNCTIONS

2.1. In this section we prove Theorem 1.1. Our starting point is formula (1.3). The first step is the evaluation of

$$E_{r,\mu} \exp \{\alpha_1<f_1,X_{t_1}>+\dots+\alpha_n<f_n,X_{t_n}>\}$$

where $r<t_1<\dots<t_n\in\Delta$, $f_1,\dots,f_n$ are positive measurable functions on E and $\alpha_1,\dots,\alpha_n$ are negative numbers.

LEMMA 2.1. *For every measure* μ *and for every* $i=1,2,\dots,n$,

$$E_{r,\mu} \exp \sum_{j=i}^{n} \alpha_j<f_j,X_{t_j}> =\exp\int F_i(r,x;\alpha_n^i)\mu(dx) \tag{2.1}$$

where $\alpha_n^i=\{\alpha_i,\alpha_{i+1},\dots,\alpha_n\}$ *and*

$$F_i(r,x;\alpha_n^i)=\log E_{r,\delta_x} \exp \sum_{j=i}^{n} \alpha_j \langle f_j, X_{t_j}\rangle \quad \text{for } r \le t_i, \tag{2.2}$$
$$=0 \quad \text{for } r>t_i.$$

(*Here* $\delta_x(B)=1_B(x)$ *is the unit measure concentrated at* x.)

The functions F_i *are connected by the following relations*

$$F_i(r,x;\alpha_n^i)-\int_{\Delta\times E} ds\, p(r,x;s,dy)F_i(s,y;\alpha_n^i)^2 \tag{2.3}$$
$$=\int_E p(r,x;t_i,dy)[\alpha_i f_i(y)+F_{i+1}(t_i,y;\alpha_n^{i+1})]$$

with $F_{n+1}=0$.

Proof. For i=n, formulae (2.1) and (2.3) follow from (1.3). Suppose that they are true for i+1 and prove that they are valid for i. Indeed, for $r<t_i$,

$$E_{r,\delta_x} \exp \sum_{j=i}^{n} \alpha_j \langle f_j, X_{t_j}\rangle$$
$$=E_{r,\delta_x}[\exp \alpha_i \langle f_i, X_{t_i}\rangle\, P_{t_i,X_{t_i}} \exp \sum_{j=i+1}^{\infty} \alpha_j \langle f_j, X_{t_j}\rangle\,]$$
$$=E_{r,\delta_x} \exp\, [\alpha_i \langle f_i+F_{i+1}(t_i,\cdot;\alpha_n^{i+1}), X_{t_i}\rangle],$$

and (1.3) inplies (2.1) and (2.3).

2.2. It follows from the remark at the end of Section 1.2 that $F_i(r,x;\alpha_n^i)$ defined by (2.3) are analytic functions of α_n^i in a neighborhood of the origin. The next step is to establish that

$$F_i(r,x;\alpha_n^i)=\sum_{\Lambda\subset\{i,\dots,n\}} \alpha_\Lambda W_\Lambda(r,x) \mod\{\alpha_i^2,\dots,\alpha_n^2\}. \tag{2.4}$$

Here Λ runs over non–empty subsets of {i,...,n},

$$\alpha_\Lambda=\prod_{i\in\Lambda}\alpha_i,$$

$W_\Lambda(r,x)$ is given by formulae (1.9),(1.10) and writing F=G $\mod\{\alpha_i^2,\dots,\alpha_n^2\}$ means that each term in the power series F–G is divisible by α_j^2 for some j=i,i+1,...,n.

Let

$$\partial/\partial\alpha_\Lambda=\prod_{i\in\Lambda}\partial/\partial\alpha_i.$$

Since $F_i(r,x,0)=0$, by Taylor's formula,

(2.5) $$F_i(r,x;\alpha_n^i)=\sum \alpha_\Lambda W_\Lambda^i(r,x) \quad \mathrm{mod}\{\alpha_i^2,\dots,\alpha_n^2\}$$

where Λ runs over all non–empty subsets of $\{i,\dots,n\}$,

(2.6) $$W_\Lambda^i(r,x)=\partial F_i(r,x;\alpha_n^i)/\partial\alpha_\Lambda \text{ evaluated at } \alpha_n^i=0.$$

To prove (2.4), it is sufficient to show that

(2.7) $$W_\Lambda^i(r,x)=W_\Lambda(r,x) \text{ for all } \Lambda\subset\{i,\dots,n\}.$$

By (2.3),(1.10) and (2.5),

$$W_i^i(r,x)=h_i(r,x)$$

and

$$W_j^i(r,x)=\int p(r,x;t_i,dy)W_j^{i+1}(t_i,y) \text{ for } j>i.$$

Hence (2.7) holds if $|\Lambda|=1$. If $|\Lambda|>1$, then by (2.3)

(2.8) $$W_\Lambda^i(r,x)=\sum \int_{\mathbb{R}_+\times E} ds\, p(r,x;s,dy)W_{\Lambda_1}^i(s,y)W_{\Lambda_2}^i(s,y)$$

with the sum running over all (ordered) partitions of Λ into disjoint non–empty subsets Λ_1 and Λ_2. Thus (2.7) holds for Λ if it holds for all $\tilde\Lambda$ with $|\tilde\Lambda|<|\Lambda|$.

2.3. Formula (1.8) follows from (2.4) since the left side is equal to the coefficient at $\alpha_1\dots\alpha_n$ in

$$E_{r,\mu}\exp\sum_{j=1}^n \alpha_j<\varphi_j,X_{t_j}> = \exp\{\sum_\Lambda\alpha_\Lambda\int W_\Lambda(r,x)\mu(dx)+R_\alpha\}$$

where $R_\alpha=0 \bmod \{\alpha_1^2,\dots,\alpha_n^2\}$.

3. STOCHASTIC INTEGRALS

3.1. For $n=1$, the inner product (1.15) with γ_2 defined by (1.22) can be rewritten in the following form

(3.1) $$(\varphi,\psi)_1=\int\varphi(t_1,z_1)\psi(t_2,z_2)d\gamma_2$$
$$=\int\varphi(0,z)\mu(dz)\int\psi(0,z)\mu(dz)+\int\varphi(s,y)\psi(s,y)\Lambda(ds,dy)$$

where Λ is a measure on $\Delta\times E$ given by the formula

$$\Lambda(C)=2\int ds\mu(dx)p(0,x;s,dy)1_C(s,y). \tag{3.2}$$

A function φ belongs to $\mathcal{H}_1^0$ if and only if $\varphi\in L^2(\Lambda)$ and $\varphi(0,x)$ is μ–integrable. The space of μ–integrable functions f on E with the inner product $(f,g)=\langle f,\mu\rangle\langle g,\mu\rangle$ becomes a one–dimensional Euclidean space if we identify functions f,g such that $\int f d\mu=\int g d\mu$. Note that $\varphi,\psi\in\mathcal{H}_1^0$ are equivalent if and only if $\varphi=\psi$ Λ–a.e. and $\int\varphi(0,x)\mu(dx)=\int\psi(0,x)\mu(dx)$. Therefore classes of equivalent elements of $\mathcal{H}_1^0$ form a Hilbert space $\mathcal{H}_1$.

Lemma 3.1. *$\mathcal{K}$ is everywhere dense in $\mathcal{H}_1^0$.*

Proof. Let $\mathcal{C}$ be a closed subspace of $\mathcal{H}_1^0$ and let $\mathcal{C}\supset\mathcal{K}$. Since the bounded convergence implies the convergence in $\mathcal{H}_1^0$, 1.6.A implies that $\mathcal{C}\supset\mathcal{W}$. Since $\mathcal{W}$ is everywhere dense in $\mathcal{H}_1^0$, $\mathcal{C}=\mathcal{H}_1^0$.

3.2. Proof of Theorem 1.2. The first statement of the theorem has been already proved. The second statement follows immediately from Lemma 3.1 and the fact that $I_1(\mathcal{K})$ contains all functionals $\langle f,X_t\rangle$.

Note that $T_t^sT_v^t=1_{s\le t\le v}T_v^s$ and, by (1.20),

$$\int 1_{s\le t\le v}T_v^s f dZ_{s,x}=\int T_t^sT_v^t f dZ_{s,x}=\langle T_v^t f,X_t\rangle.$$

On the other hand,

$$\int T_v^s f dZ_{s,x}=\langle f,X_v\rangle.$$

Let $t<v\in\Delta$. By Markov property and (1.2),

$$E_\mu\{\langle f,X_v\rangle\,|\,\mathcal{F}_t\}=E_{t,X_t}\langle f,X_v\rangle=\langle T_v^t f,X_t\rangle.$$

Hence (1.24) holds for functions $\varphi\in\mathcal{K}$. By Lemma 3.1 it holds for all $\varphi\in\mathcal{H}_1^0$. This implies that (1.23) describes all $\mathcal{F}_t$–measurable functions in L_1^2 and also the statement on L_1^2–valued martingales .

By setting t=0 in (1.18) and (1.19), we get $\int 1_{s=0}f(x)dZ_{s,x}=\langle f,X_0\rangle=\langle f,\mu\rangle$. Therefore

$$\int 1_{s=0}\varphi(s,x)dZ_{s,x}=\int\varphi(0,x)\mu(dx). \tag{3.6}$$

Formula (1.25) follows from (1.24) and (3.6) since $E_\mu Y=E_\mu\{Y\,|\,\mathcal{F}_0\}$.

3.3. Proof of Lemma 1.1. By Theorem 1.1' formula (1.30) holds for $\varphi\in\mathcal{K}$. This implies, in particular, that $E_\mu I_1(\varphi)^{2p}<\infty$ for all $\varphi\in\mathcal{K}$ and every positive integer p. Lemma 3.1 implies that (1.30) holds for all $\varphi_i\in\mathcal{W}$.

To prove the second part of Lemma 1.1, we start from a function φ such that $\ell_p(|\varphi|)<\infty$ and we consider a sequence of elements of $\mathcal{K}$

$$\varphi_m(s,x)=\varphi(s,x) \text{ if } |\varphi(s,x)|\le m, \quad =0 \text{ otherwise.} \tag{3.7}$$

By the dominated convergence theorem,

$$E_\mu\big[I_1(\varphi_m)-I_1(\varphi_k)\big]^{2p}=E_\mu\big[I_1(\varphi_m-\varphi_k)^{2p}\big]=\ell_p(\varphi_m-\varphi_k)\to 0 \quad \text{as } m,k\to 0.$$

Hence $I_1(\varphi_m)\to Y$ in $L^{2p}(P_\mu)$. We conclude that $I_1(\varphi_m)\to Y$ in $L^2(P_\mu)$ and therefore φ_m converges in $\mathcal{X}_1^0$ to a φ such that $I_1(\varphi)=Y$. Thus $I_1(\varphi)\in L^{2p}(P_\mu)$. By the dominated convergence theorem,

$$E_\mu I_1(\varphi)^{2p}=\lim E_\mu I_1(\varphi_m)^{2p}=\lim \ell_p(\varphi_m)=\ell_p(\varphi). \tag{3.8}$$

By Hölder's inequality we get that

$$E_\mu \prod_{i=1}^{n} |I_1(\varphi_i)|<\infty \tag{3.9}$$

if $E_\mu I_1(\varphi_i)^{2p}=\ell_p(\varphi_i)\le\ell_p(|\varphi_i|)<\infty$ for $i=1,\dots,n$ and some $p\ge n/2$.

By applying (3.8) to $\varphi=\alpha_1\varphi_1+\dots+\alpha_{2p}\varphi_{2p}$ and by comparing the coefficients at $\alpha_1\dots\alpha_{2p}$ we obtain

$$E_\mu \prod_{i=1}^{2p} I_1(\varphi_i)=\int \prod_{i=1}^{2p} \varphi_i d\gamma_{2p}. \tag{3.10}$$

We get (1.30) from (3.10) by setting $\varphi_{n+1}=\dots=\varphi_{2p}=\kappa$ where $\kappa(s,x)=1_0(s)$ and taking into account that $I_1(\kappa)=<1,\mu>$ and

$$\gamma_k(A_1\times B_1\times\dots\times A_k\times B_k)=<1,\mu>\ \gamma_{k-1}(A_1\times B_1\times\dots\times A_{k-1}\times B_{k-1})$$

for $A_k=\{0\}$, $B_k=E$.

3.4. Proof of Theorem 1.3. Denote by $\mathcal{W}^n$ the set of all monomials $\varphi_1\times\dots\times\varphi_n$ with $\varphi_1,\dots,\varphi_n\in\mathcal{W}$. It follows from Lemma 1.1 that (1.17) holds for functions $\varphi,\psi\in\mathcal{W}^n$ if we define $I_n(\varphi)$ for $\varphi\in\mathcal{W}^n$ by formula (1.31). Since $I_n(\mathcal{W}^n)$ contains functions $<f_1,X_{t_1}>\dots<f_n,X_{t_n}>$ which generate L_n^2, Theorem 1.3 will be proved if we show that the closure $\mathcal{C}\ni\{\ \mathcal{W}^n$ in $\mathcal{X}_n^0$ coincides with $\mathcal{X}_n^0$. Since $\mathcal{W}^n$ is closed under multiplication, $\mathcal{C}$ contains all bounded measurable functions on $(\Delta\times E)^n$. It remains to note that, if $\phi\in\mathcal{X}_n^0$, then

$$\phi_m=\phi \text{ if } |\phi|\le m,$$
$$=0 \text{ otherwise}$$

tends to ϕ in $\mathcal{X}_n^0$ as $m\to\infty$.

4. LOCAL TIMES AND SELF–INTERSECTION LOCAL TIMES

4.1. Proof of Theorem 1.4. By (1.39),(3.1) and (3.2), $K_{z,B}\in\mathcal{X}_1^0$ if and only if

(4.1) $$a_1=\int\mu(dx)K_{z,B}(0,x)<\infty$$

and

(4.2) $$a_2=\int\mu(dx)ds\ p(0,x;s,y)dy\ K_{z,B}(s,y)^2<\infty.$$

By (1.40) and (1.41),

(4.3) $$K_{z,B}(s,y)\le G(s,y;z)$$

and, by 1.9.B,C,

(4.4) $$a_1\le cu,\quad a_2\le cH(z,z)$$

which implies Theorem 1.4.

4.2. Proof of Theorem 1.5. By (1.15),(1.28) and (1.29), $K_B^n\in\mathcal{X}_N^0$ if and only if, for every $D\in\mathbb{D}_{2n}$,

(4.5) $$c(D)=\int_{B\times B} dt_1\ldots dt_{2n}\int q_D(t_1,z;\ldots;t_n,z;t_{n+1},\zeta;\ldots;t_{2n},\zeta)dzd\zeta$$

is finite. Here

(4.6) $$q_D(t_1,z_1;\ldots;t_{2n},z_{2n})$$
$$=\int\prod_{v\in V_-}\mu(dx_v)\prod_{v\in V_0}ds_v dy_v\prod_{a\in A_0}p_a\prod_{i=1}^{2n}p(s_{v_i},y_{v_i};t_i,z_i)$$

and $p_a=p(s,w;s',w')$ for an arrow a with the beginning labelled by (s,w) and the end labelled by (s',w'). (In contrast to (1.29), p_a is a transition densitu, not a transition function.) The exits marked by $1,\ldots,n$ are called the ***z–exits*** and those marked by $n+1,\ldots,2n$ are called the ***ζ–exits***. Our goal is to show that, under conditions of Theorem 1.5, $q_D<\infty$ for all $D\in\mathbb{D}_{2n}$.

Fix a diagram $D^0\in\mathbb{D}_{2n}$ and denote by $\mathbb{D}^0$ the set of all diagrams obtained from D^0 by cutting some arrows. (Possibly, no arrow is cut, so $D^0\in\mathbb{D}^0$.) We say that a vertex $v\in D$ is ***accessible*** from $v'\in D$ if there exists a path $\pi: i_1 \to i_2 \to \ldots \to i_m$ with vertices $i_1, i_2, \ldots, i_m$ such that $i_1=v'$, $i_m=v$ and the arrows $i_1 \to i_2, \ldots, i_{m-1} \to i_m$ are not cut. A vertex v is accessible from V_- if it is accessible frome some $v'\in V_-$. Denote by $\mathbb{D}^*$ the set of all $D\in\mathbb{D}^o$ with the property: at least one z–exit and at least one ζ–exit are accessible from V_-. put $v\in V_0'$ if $v \in V_0$ and if all three arrows to which v belongs are cut. For every $D\in\mathbb{D}^*$ we define c(D) by (4.5)–(4.6) with $p_a=p(s,w;s',w')$ replaced by $1_{s<s'}$ for all cut arrows and with dy_a dropped for all $v\in V_0'$.

Example. Let

$$D^o: \qquad \begin{array}{ccccccc} (0,x_1) \to & (s_1,y_1) & \to & (s_2,y_2) & \to (t_4,z_4) & \qquad & (0,x_2) \\ & \downarrow & & \downarrow & & & \downarrow \\ & (t_3,z_3) & & (t_2,z_2) & & & (t_1,z_1) \end{array}$$

and let D be obtained from D^o by cutting three arrows touching label (s_2,y_2). Then $D\in\mathbb{D}^*$ and

$$c(D)= \int_{B\times B} dt_1 \ldots dt_4 \int \mu(dx_1)\mu(dx_2)p(0,x_1;s_1,dy_1)ds_1 ds_2 p(s_1,y_1;t_3,\zeta)$$

$$\times\, 1_{s_1<s_2<t_4, s_2<t_2} p(0,x_2;t_1,z).$$

We say that a family $\mathbb{D}'\subset\mathbb{D}^*$ ***dominates*** a diagram $\tilde{D}\in\mathbb{D}^*$ if every $D\in\mathbb{D}'$ is obtained from $\tilde{D}$ by cutting a non–empty set of arrows and if

$$c(\tilde{D}) \le \text{const.} \times \sum_{D\in\mathbb{D}'} c(D).$$

A diagram D of $\mathbb{D}^*$ is called ***maximal*** if it is not dominated by any family $\mathbb{D}'\subset\mathbb{D}^*$. Theorem 1.5 will be proved if we demonstrate that $c(D)<\infty$ for all maximal D.

Fix a maximal element D of $\mathbb{D}^*$.

Proposition 4.1. *If* $v\in V_0$ ***belongs to two cut arrows, then*** $v\in V_0'$.

Proof. Let a be the third arrow which contains v and let (s,y) be its label. Suppose that a is not cut. Its cutting produces from D another diagram $D'\in\mathbb{D}^*$. We claim that D' dominates D. Indeed, the variable y enters only one factor in (4.6). By integrating with respect to dy and by using condition 1.9.C and the inequality $\int p(s',y';s,y)dy\le 1$, we note that $c(D)\le \text{const.} c(D')$

Proposition 4.2. ***Only one*** z–***exit and only one*** ζ–***exit are accessible from*** V_-.

Proof. Suppose that two z–exits v and v' are accessible from V_-, that $\pi: i_1 \to \ldots \to i_m$ and $\pi': i_1' \to \ldots \to i_{m'}'$ are the corresponding paths and $a_1, \ldots, a_N$ are all arrows in these paths enumerated in an arbitrary order. We shall arrive at a contradiction by proving that D is dominated by the family $D_1,\dots,D_N$ where D_k is obtained from D by cutting a_k.

Let s_ℓ and s'_ℓ be the time variables in the labels of i_ℓ and i'_ℓ. Note that $s_1=s'_1=0$ and $s_m=t_j$, $s'_{m'}=t_{j'}$ where j,j' are the marks of the exits v,v'. Therefore

$$t_j-t_{j'}=(s_2-s_1)+\dots+(s_m-s_{m-1})-(s'_2-s'_1)-\dots-(s'_{m'}-s'_{m'-1}). \tag{4.7}$$

The differences in parentheses are in a 1–1 correspondence with arrows a_k.

By 1.9.E, $|t_j-t_{j'}|\ge\beta$ for all $t=(t_1,\dots,t_n)\in B$. Therefore, for every $t\in B$, at least one of the differences in (4.7) is larger than or equal to $\alpha=\beta/N$. Put $t\in B_k$ if this is true for the difference corresponding to a_k. Since $\{B_k\}$ cover B, we get an upper bound for c(D) by replacing the integrand q_D in (4.5) by $\sum_k 1_{t\in B_k} q_D$. It remains to note that, by 1.9.D, $q_D\le\text{const.}\,q_{D_k}$ for $t\in B_k$.

Proposition 4.3. ***For every*** $v\in V$ ***there exists at most one*** z***–exit and at most one*** ζ***–exit accessible from*** v.

Proof is analogous to that of Proposition 4.2.

For every vertex $v\in D$ there exists a unique maximal path $\pi: i_1\to i_2\to\dots\to i_m$ such that $i_m=v$ and all arrows $i_\ell\to i_{\ell+1}$ are not cut. Denote by π_k the maximal path to the exit marked by k and by v_k its initial vertex. It follows from Propositions 4.1,2,3 that:

(a) Every non–cut arrow belongs to one of paths $\pi_1,\dots,\pi_{2n}$;

(b) $v_1,\dots,v_n$ (corresponding to the z–exits) are distinct and only one of them v_ℓ belongs to V_-;

(c) $v_{n+1},\dots,v_{2n}$ (corresponding to the z–exits) are distinct and only one of them $v_{\ell'}$ belongs to V_-;

(d) For every $k=1,\dots,n$ except $k=\ell$, there exists one and only one $k'\in\{n+1,\dots,2n\}$ such that $v_k=v_{k'}$.

Therefore we have the following picture:

(4.8)
$$
\begin{tikzcd}
v_\ell \arrow[d, dotted, "\pi_\ell"] & v_{\ell'} \arrow[d, dotted, "\pi_{\ell'}"] & & v_k=v_{k'}\ (k\neq\ell) \arrow[dl, dotted, "\pi_k"'] \arrow[dr, dotted, "\pi_{k'}"] & \\
z & \zeta & z & & \zeta
\end{tikzcd}
$$

if $v_\ell \neq v_{\ell'}$ (exits are labelled by z and ζ) or

(4.9)
$$
\begin{tikzcd}
& v_\ell=v_{\ell'} \arrow[d, dotted, "\rho"] & & & v_k=v_{k'}\ (k\neq\ell) \arrow[dl, dotted, "\pi_k"'] \arrow[dr, dotted, "\pi_{k'}"] & \\
& v \arrow[dl, dotted, "\sigma"'] \arrow[dr, dotted, "\sigma'"] & & z & & \zeta \\
z & & \zeta & & &
\end{tikzcd}
$$

if $v_\ell \neq v_{\ell'}$. Here ρ is the common part of the paths π_ℓ and $\pi_{\ell'}$; v is the end of ρ, and σ,σ' are the parts of π_ℓ and $\pi_{\ell'}$ starting from v.

We associate with a path $\pi: i=i_1\to i_2\to\ldots\to i_m=j$ a function

$$Q_\pi(s_i,y_i;s_j,y_j)=\int\prod_{\alpha=2}^{m} p(s_{i_{\alpha-1}},y_{i_{\alpha-1}};s_{i_\alpha},y_{i_\alpha})\prod_{\alpha=2}^{m-1}dy_{i_\alpha}ds_{i_\alpha}.$$

Clearly,

$$c(D)\leq\int dz d\zeta F(z,\zeta)\prod_{k\neq\ell}Q_{\pi_k}(s_{v_k},y_{v_k};t_k,z)Q_{\pi_{k'}}(s_{v_k},y_{v_k};t_{k'},\zeta)ds_{v_k}dy_{v_k}dt_k dt_{k'} \tag{4.10}$$

where

$$F(z,\zeta)=\int\mu(dw)Q_{\pi_\ell}(0,w;t_\ell,z)\mu(d\tilde{w})Q_{\pi_{\ell'}}(0,\tilde{w};t_{\ell'},\zeta)dt_\ell dt_{\ell'}$$

in the case (4.8), or

$$F(z,\zeta)=\int\mu(dw)Q_\rho(0,w;s_v,y_v)Q_\sigma(s_v,y_v;t_\ell,z)Q_{\sigma'}(s_v,y_v;t_{\ell'},\zeta)ds_v dy_v dt_\ell dt_{\ell'}$$

in the case (4.9). By (1.32),

$$Q_\pi(s_i,y_i;s_j,y_j)=p(s_i,y_i;s_j,y_j)\int 1_{s_{i_1}\leq\ldots\leq s_{i_m}}ds_{i_2}\ldots ds_{i_{m-1}}\leq\frac{u^{m-2}}{(m-2)!}p(s_i,y_i;s_j,y_j).$$

By (4.10),

$$c(D)\leq\text{const.}\times\int F(z,\zeta)H(z,\zeta)^{n-1}dz d\zeta. \tag{4.11}$$

Note that

$$(4.12)\qquad F(z,\zeta)\le \text{const.}\times\int \mu(dw)G(0,w;z)\mu(d\tilde{w})G(0,\tilde{w};\zeta)$$

or

$$(4.12')\qquad F(z,\zeta)\le \text{const.}\times\int \mu(dw)p(0,w;s,y)G(s,y;z)G(s,y;\zeta)dsdy.$$

By (4.11),(4.12) and (4.12'), condition (1.43) implies that $c(D)<\infty$.

4.3. Proof of Theorems 1.6 and 1.7. The Chapman–Kolmogorov equation for the Brownian transition density implies

$$(4.13)\qquad \int Q_d^s(|y-z|)Q_d^s(|\zeta-y|)dy\le\int_0^{2s} tq_t^d(|\zeta-z|)dt=\frac{1}{2\pi}Q_{d-2}^{2s}(|z-\zeta|).$$

Since

$$(4.14)\qquad \int_s^u q_{u-s}^d ds=\int_0^{u-s} q_s^d ds\le Q_d^u,$$

we have from (1.41) and (1.45)

$$(4.15)\qquad G_d(s,y;z)\le Q_d^u(\alpha|y-z|)\ .$$

By (1.41) and (4.13),

$$(4.16)\qquad H_d(z,\zeta)\le \text{const.}Q_{d-2}^{2u}(\alpha|z-\zeta|).$$

Therefore

$$H_d(z,z)\le \text{const.}Q_{d-2}^{2u}(0)=\text{const.}\int_0^{2u} t^{-(d-2)/2}dt<\infty \text{ for } d\le 3,$$

and Theorem 1.6 follows from Theorem 1.4.

By (4.15) and (4.16),

$$(4.17)\qquad \begin{aligned}&\int G_d(s,y;z)\ G_d(s,y;\zeta)\ H_d(z,\zeta)^{n-1}\ dz\ d\zeta\\ &\le \text{const.}\int Q_d^u(\alpha|y-z|)Q_d^u(\alpha|y-\zeta|)Q_{d-2}^{2u}(\alpha|z-\zeta|)^{n-1}dz\ d\zeta.\end{aligned}$$

Changing variables by the formulae $z'=\zeta-z$, $\zeta'=\zeta-y$, we establish that the integral in the right side is equal to

$$(4.18)\qquad \int Q_d^u(\alpha|z-\zeta|)Q_d^u(\alpha|\zeta|)Q_{d-2}^{2u}(\alpha|z|)^{n-1}dz\ d\zeta.$$

By applying (4.13) to the integral relative to $d\zeta$, we get that (4.18) is dominated by

$$(4.19)\qquad \text{const.}\int Q_{d-2}^{2u}(\alpha|x|)^n dx=\text{const.}\int Q_{d-2}^{2u}(|x|)^n dx.$$

Thus Theorem 1.7 follows from Theorem 1.5.

4.4. **Proof of Corollary to Theorem 1.7.** For $k\le 1$,

$$Q_k^{2u}(r)\le \text{const}.e^{-r^2/4u}. \tag{4.20}$$

Therefore condition (1.48) holds for $d\le 3$ and all n.

Change of variables $s=r^2/2t$ in (1.47) yields

$$Q_d^{2u}(r)=\text{const}.r^{2-d}\, S_d(r^2/4u) \tag{4.21}$$

where

$$S_d(t)=\int_t^\infty s^{(d-4)/2}e^{-s}ds. \tag{4.22}$$

For $d\ge 3$, $S_d(t)\le S_d(0)<\infty$. By (4.21), $Q_{d-2}^{2u}(r)\le \text{const}.r^{4-d}$ if $d\ge 5$, and we see (1.48) holds for $n\le 4$ if $d=5$ and for $n\le 2$ if $d=6$ or 7.

Finally, $S_2(t)\le e^{-t}$ for $t>1$ and, by Lemma 2.1 in [6], $S_2(t)\le \text{const}.(|\log t|+1)$ for all t. Therefore (1.48) is satisfied for $d=4$ and all n.

5. CONCLUDING REMARKS

5.1. Time $s=0$ plays a special role in the definition of the martingale measure $Z_{s,x}$. On the contrary, all points of the interval Δ are in the same position for the martingale measure $Z^0_{s,x}$ defined by the formula

$$\int\varphi(s,x)dZ^0_{s,x}=\int\varphi(s,x)dZ_{s,x}-P_\mu\int\varphi(s,x)dZ_{s,x}=\int\varphi(s,x)dZ_{s,x}-\int\varphi(0,x)\mu(dx) \tag{5.1}$$

(cf.(1.25)). In [7] (written after the first draft of the present paper had been already finished) we introduce the stochastic integral with respect to $Z^0_{s,x}$ directly, starting from the formula

$$\int_r^t T_t^s f(x)dZ^0_{s,x}=\langle f,X_t\rangle - \langle T_t^r f,X_r\rangle \tag{5.2}$$

instead of (1.18)–(1.19) (this is closer to the original approach of Walsh and Metivier). The construction in Sections 1.6 and 1.8 can be used to define multiple stochastic integrals relative to $Z^0_{s,x}$. The only change is that the set $\mathbb{D}_n$ in (1.28) must be replaced by its subset $\tilde{\mathbb{D}}_n$ specified by the condition: $D\in\tilde{\mathbb{D}}_n$ if every connected component of D contains more than one arrow. In particular, the first term in (1.22) must be dropped.

5.2. Suppose that an integrand φ depends on a parameter α with values in a measurable space $(A,\mathcal{A})$. Assuming that $\varphi_\alpha(s,x)$ is jointly measurable in α,s,x and that $\varphi_\alpha\in\mathcal{H}_1^0$ for every $\alpha\in A$, we can choose an $\mathcal{A}$–measurable version of the integral $I_1(\varphi_\alpha)$. Moreover, if ν is a measure on $\mathcal{A}$ such that $\varphi=\int\varphi_\alpha\nu(d\alpha)\in\mathcal{H}_1^0$, then $I_1(\varphi)=\int I_1(\varphi_\alpha)\nu(d\alpha)$. Multiple integrals I_n have an analogous property.

5.3. Let $K_{z,B}$ be defined by (1.40). For every bounded measurable function ρ,

$$F=\int\rho(z)K_{z,B}dz=\int_B T_t^s\rho dt$$

is bounded and therefore belongs to $\mathcal{H}_1^0$. If $K_{z,B}\in\mathcal{H}_1^0$ for all z, then

$$\int\rho(z)L_z(B)dz=\int_B<\rho,X_t>dt. \tag{5.3}$$

Indeed, $I_1(F)=\int I_1(T_t^s\rho)dt$ is equal to the right side in (5.3) by (1.18)–(1.19).

5.4. Note that

$$L^n(B)=\lim_{k\to\infty}\int_B<\rho_k,X_{t_1}\times\ldots\times X_{t_n}>dt_1\ldots dt_n \tag{5.4}$$

in L_n^2 if $\rho_k\to\delta^n$ in the following sense. Put

$$R_{z_1,\ldots,z_n}(s_1,x_1;\ldots;s_n,x_n)=\int_B dt_1\ldots dt_n\int_E\prod_{i=1}^n p(s_i,x_i;t_i,z_i).$$

It is sufficient that

$$\int\rho_k(z_1,\ldots,z_n)R_{z_1,\ldots,z_n}dz_1\ldots dz_n\to\int R_{z,\ldots,z}dz$$

in $\mathcal{H}_n^0$. This follows immediately from (1.37) through (1.40).

5.5. Measures γ_n on $(\Delta\times E)^n$ defined by (1.28) are symmetric, that is

$$\int\varphi d\gamma_n=\int\varphi_\sigma d\gamma_n$$

where φ_σ is obtained from φ by a permutation σ of pairs $(t_1,z_1),\ldots,(t_n,z_n)$. Therefore $(\varphi_\sigma,\varphi_\sigma)_n=(\varphi_\sigma,\varphi)_n=(\varphi,\varphi)_n$ and $(\varphi-\varphi_\sigma,\varphi-\varphi_\sigma)_n=0$. We conclude that $I_n(\varphi)$ does not change if we replace φ by its symmetrization.

APPENDIX

0.1. We say that a Markov process $\xi_t, t\in\Delta=[0,u]$ in a measurable space $(E,\mathcal{B})$, with a transition function $p(s,x;t,B)$ is ***right*** if:

0.1.A. For every $r<t\in\Delta$ and every finite measure μ, $p(s,\xi_s;t,B)$ is right continuous on $[r,t)$ a.s.$P_{r,\mu}$.

0.1.B. The σ–algebra $\mathcal{B}(\Delta)\times\mathcal{B}$ is generated by functions $\varphi(s,x)$ such that $\varphi(s,\xi_s)$ is right–continuous for all paths.

Obviously, both conditions are satisfied for every classical diffusion.

0.2. As in subsection 1.6, $\mathcal{W}$ means the space of all bounded measurable functions on $\Delta\times E$. Put $\varphi\in\mathcal{W}_0$ if $\varphi\in\mathcal{W}$ and if $T_t^s\varphi_t\to\varphi_s$ (pointwise) as $t\downarrow s\in[0,u)$. If $f\in\mathcal{W}$, $\varphi\in\mathcal{W}_0$ and if,for every $s\in[0,u)$,

$$(T_t^s f_t - f_s)/(t-s)\to\varphi_s \text{ boundedly as } t\downarrow s, \tag{0.1}$$

then we put $f\in\mathcal{D}_A$ and $A_t f_t=-\varphi_t$.

We note that:

0.2.A. If $\varphi\in\mathcal{W}_0$, then

$$f_s(x)=\int_\Delta T_t^s\varphi_t dt \tag{0.2}$$

belongs to $\mathcal{D}_A$ and $A_s f_s=\varphi_s$.

0.2.B. If $f\in\mathcal{D}_A$, then

$$T_t^s A_t f_t = d^+ T_t^s f_t/dt \tag{0.3}$$

and

$$\int_\Delta T_t^s A_t f_t dt = \lim_{t\uparrow u} T_t^s f_t - f_s. \tag{0.4}$$

0.2.C. Let $\mathcal{C}$ be a closed subspace of $\mathcal{W}$ (relative to the bounded convergence) and let $F_t\in\mathcal{C}$ for every $t\in\Delta$. If $F_t(s,x)$ is uniformly bounded and right continuous in t for all s,x, then

$$\phi(s,x)=\int_\Delta F_t(s,x)dt \tag{0.5}$$

belongs to $\mathcal{C}$.

Proof of 0.2.A,B,C is similar to the proof of analogous statements in the time–homogeneous case (see [4], section 1.6).

0.3. Theorem 0.1. *Condition* 1.6.A *is satisfied if* p *is the transition function of a right* *process.*

Proof. Suppose that $\mathcal{C}$ is a closed subspace of $\mathcal{W}$ which contains $\mathcal{K}$. Let $\varphi\in\mathcal{W}_0$. By 0.2.A,B,

$$\int_\Delta T_t^s\varphi_t dt=\lim_{t\uparrow u} T_t^s f_t - f_s \tag{0.6}$$

where f is given by (0.2). Obviously, $F_t(s,x)=T_t^s\varphi_t(x)$ is right continuous in t and, by 0.2.C, the right side in (0.4) is an element of $\mathcal{C}$. Since $T_t^s f_t\in\mathcal{C}$, f belongs to $\mathcal{C}$ as well.

For a fixed t, $\mathcal{C}$ contains $T_t^s f_t$ and therefore it contains the function in the left side of (0.1). Consequently, $\mathcal{C}$ contains φ.

Functions φ described in 0.1.B belong to $\mathcal{W}_0$ and therefore they belong to $\mathcal{C}$. Since $\mathcal{C}$ contains a multiplicative system which generates $\mathcal{B}(\Delta)\times\mathcal{B}$, it contains $\mathcal{W}$.

REFERENCES

[1] D. A. Dawson, Stochastic evolution equations and related measure processes, J. Multivariant Analysis **3** (1975), 1–52.

[2] D. A. Dawson, The critical measure diffusion process, Z.Wahrsheinlichkeitstheorie und Verw.Gebiete **40** (1977), 125–145.

[3] D.A.Dawson, Measure–valued processes, qualitative behaviour and stochastic geometry, Preprint, 1986.

[4] E. B. Dynkin, Markov processes, Springer–Verlag, 1965.

[5] E. B. Dynkin, Multiple path integrals, Advan. Appl. Math. **7** (1986), 205–219.

[6] E. B. Dynkin, Self–intersection local times, occupation fields and stochastic integrals, Advan.Math. **65** (1987), 254–271.

[7] E. B. Dynkin, Superprocesses and their linear additive functionals, Submitted to Trans.Amer.Math.Soc.

[8] E. B. Dynkin, R. K. Getoor, Additive functionals and entrance laws, J.Funct. Anal.,**62** (1985), 221–265.

[9] E. B. Dynkin, S. E. Kuznetsov, Determining functions of Markov processes and corresponding dual regular classes, Soviet Math.–Doklady **15** (1974), 20–23.

[10] I. Iscoe, A weighted occupation time for a class of measure–valued branching processes, Probab.Th.Rel.Fields **71** (1986), 85–116.

[11] I. Iscoe, Ergodic theory and local occupation time for measure–valued critical branching Brownian motion, Stochastics, **18** (1986), 197–243.

[12] M. Metivier, Semimartingales, Walter de Gruyter, 1982.

[13] E. A. Perkins, A space–time property of a class of measure–valued branching diffusions, Preprint, 1986.

[14] S. Roelly–Coppoletta, A criterion of convergence of measure–valued processes: application to measure branching processes, Stochastics, **17** (1986), 43–65.

[15] J. Walsh, An introduction to stochastic partial differential equations, In École d'Été de Probabilités de Saint–Flour XIV – 1984, 266–439, Springer–Verlag, 1986.

[16] S. Watanabe, A limit theorem of branching processes and continuous state branching processes, J.Math.Kyoto Univ. **8** (1968), 141–167.

[17] E. Wild, On Boltzmann's equation in the kinetic theory of gases, Proc.Camb.Phil.Soc. **47** (1951), 602–609.

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