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MONODROMY AND THE KOWALEVSKAYA TOP.

J. P. Francoise

We consider algebraically completely integrable Hamiltonian systems which are separable [A-M], [Mo], [Moe] and [Mu]. For these systems, we prove that the symplectic form can be reduced to a simple expression involving Abelian forms. We use then Arnol'd's method [A] to define the Actions. The determination of the Actions turns out to be equivalent to a monodromy computation. The Actions are not given, in general, by simple functions of the first integrals. But we can write the corresponding Picard-Fuchs equations. We consider in detail the Kowalevskaya Top and we write down the 4-th order differential equation which is involved in this case.

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1. Algebraically Completely Integrable Hamiltonian System

We see here a completely integrable Hamiltonian System $(\underline{H},\omega)$ as an algebraic mapping $\underline{H} = (H_1, \dots, H_m):V^{2m} \to \mathbb{C}^m$ which is submersive on a non-empty Zariski open set $V^* = V^{2m} \setminus S$, where V^{2m} is a symplectic algebraic variety, and such that the fibers of $\underline{H}$ are Lagrangian for the symplectic form ω.

Definition. A completely integrable Hamiltonian system is algebraically separable if

i) there is a family of hyperelliptic curves

$$C_{\underline{c}} = \{(z,w) \in \mathbb{C}^2 / z^2 = \Phi_{\underline{c}}(w)\}$$

such that the fiber $\underline{H}^{-1}(\underline{c})$ is the affine part of $\mathrm{Jac}(C_{\underline{c}})$ and constants $v_i \in \mathbb{C}^m$ such that

$$\sum_{k=1}^{m} \frac{w_k^{j-1}\{H_i, w_k\}}{\sqrt{\Phi(w_k)}} = v_i \delta_{ij} \tag{1.1}$$

{ } is the Poisson bracket for ω and $v_i \neq 0$ for all $i = 1, \ldots, m$.

Let us consider a Hamiltonian system $(\underline{H}, \omega)$ which is a complexification of a real mapping $\underline{H}: V_R^{2m} \to R$. If the fibers $\underline{H}^{-1}(\underline{c})$, $\underline{c} \in R^m$ are compact, the connected components of the general fiber are real tori (Arnol'd-Liouville).

The system is said to be algebraically completely integrable when the fibers are affine part of Abelian varieties [A-M]. Most of the interesting completely integrable Hamiltonian systems have this property. For instance, the three cases of integrability of the motion of a rigid body about a fixed point and their extensions [R], [R-M], the Toda Lattice and its extensions by Kostant [K] the examples of J. Moser [Mo] ect. Furthermore for all these examples, the Abelian varieties are Jacobians of Riemann surfaces $C_{\underline{c}}$, $\mathrm{Jac}(C_{\underline{c}}) = H^0(C, \Omega_C^1)^*/H_1(C, Z)$.

Let us recall that if $z^2 = \Phi(w)$ is an equation for a hyperelliptic curve C, Φ being a polynomial of degree $2g$ or $2g+1$, the Abelian forms of the first kind $\left(\frac{w^j dw}{\sqrt{\Phi(w)}},\ j = 0, \ldots, g-1\right)$ generate a basis of $H^0(C, \Omega_C^1)$.

So the equation (1.1) means that the Hamiltonian flows of the functions H_i are linearized on the Jacobian and that they have a constant velocity v_i relatively to the basis of the Abelian forms of the first kind.

The velocities v_i are usually independent of $\underline{H}$ in the classical examples. The algebraic polarization of the complex tori is not given by the expected one (Projective embedding of the fibers $\underline{H}^{-1}(\underline{c})$ by homogenization) but is provided by the existence of Laurent developments for the solutions of the $\underline{H}$ [A.M.]. In this sense, following Torelli's theorem the smooth curves $C_{\underline{c}}$ are uniquely determined by the couple $(\underline{H},\omega)$.

We consider now an example.

2. <u>The Integration of Kowalevskaya Top.</u>

Euler's equations governing the motion of a rigid body about a fixed point are given by the following

$$
\begin{aligned}
&A\dot{p} + (C - B)qr = mg(y_0\gamma_3 - z_0\gamma_2)\\
&B\dot{q} + (A - C)pr = mg(z_0\gamma_1 - x_0\gamma_1)\\
&C\dot{r} + (B - A)pq = mg(x_0\gamma_2 - y_0\gamma_1)\\
&\dot{\gamma}_1 = r\gamma_2 - q\gamma_3\\
&\dot{\gamma}_2 = p\gamma_3 - r\gamma_1\\
&\dot{\gamma}_3 = q\gamma_1 - p\gamma_2
\end{aligned}
\tag{2.1}
$$

It is natural to restrict the vector field that they define to the algebraic variety $V^4_R \subset R^6$ given by the equations

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1 = 0$$

$$2(p\gamma_1 + q\gamma_2) + r\gamma_3 - 2\ell = 0,$$

$\ell \in R$ is fixed.

We get by restriction on V^4 a Hamiltonian system for the symplectic form:

$$\omega = \frac{2}{\gamma_3} dp \wedge d\gamma_2 - \frac{2}{\gamma_3} dp \wedge d\gamma_1 - \frac{r}{\gamma_3^2} d\gamma_1 \wedge d\gamma_2 \tag{2.2}$$

and the Hamiltonian is

$$H = p^2 + q^2 + \frac{r^2}{2} - c\gamma_1, \qquad c = \frac{mgx_0}{C} \tag{2.3}$$

for the Kowalevskaya Top which corresponds to the values

$$A = B = 2C, \quad y_0 = z_0 = 0$$

of the parameters.

In that case, we have an extra integral K:

$$K = [(p + iq)^2 + c(\gamma_1 + i\gamma_2)][(p - iq)^2 + c(\gamma_1 - i\gamma_2)]. \tag{2.4}$$

We define by $\underline{H} = (K,H): V^4_{\mathbb{C}} \to \mathbb{C}$ a completely integrable Hamiltonian system.

If we follow S. Kowalevskaya's computation [Ko], [Go], we choose $x_1 = p + iq$, $x_2 = p - iq$, γ_1, γ_2 as a system of coordinates on V^4. We introduce the polynomials

$$R(x) = -x^4 + 2Hx^2 + 4c\ell x$$

$$R(x_1,x_2) = -x_1^2x_2^2 + 2Hx_1x_2 + 2c\ell(x_1 + x_2) + c^2 - K$$

$$R_1(x_1,x_2) = -2Hx_1^2x_2^2 - (c^2 - K)(x_1 + x_2)^2 - 4c\ell x_1x_2(x_1 + x_2)$$
$$+ 2H(c^2 - K) - 4c^2\ell^2.$$

We use then

$$w_1 = \frac{R(x_1x_2) - \sqrt{R(x_1)R(x_2)}}{(x_1-x_2)^2}$$

(2.5)

$$w_2 = \frac{R(x_1x_2) + \sqrt{R(x_1)R(x_2)}}{(x_1-x_2)^2}$$

and the polynomials

$$\phi(w) = (w + H)(w^2 + c^2 - K) - 2c^2\ell^2$$

(2.6)

$$\Phi(w) = -2(w^2 - K)\phi(w).$$

Now an algebraic computation shows that the equations (2.1) are equivalent to

$$\dot{w}_1 = -\{H,w_1\} = \sqrt{\Phi(w_1)}/w_1 - w_2$$

(2.7)

$$\dot{w}_2 = -\{H, w_2\} = \sqrt{\Phi(w_2)}/w_1 - w_2$$

and so, we have

$$\frac{dw_1}{\sqrt{\Phi(w_1)}} + \frac{dw_2}{\sqrt{\Phi(w_2)}} = 0$$

(2.8)

$$\frac{w_1 dw_1}{\sqrt{\Phi(w_1)}} + \frac{w_2 dw_2}{\sqrt{\Phi(w_2)}} = -dt.$$

Let us introduce the hyperelliptic Riemann surface C defined by $z^2 = \Phi(w)$ in $\mathbb{C}P_2 = \{(z,w)\}$. It is a compactification of a double cover of the plane minus three cuts and so it is a Hyperelliptic curve of genus 2. Let $\mathrm{Jac}(C) = H^0(C,\Omega_C^1)^*/H_1(C,Z)$ be the Jacobian variety of C. The forms $\left(\frac{dw}{\sqrt{\Phi(w)}}, \frac{wdw}{\sqrt{\Phi(w)}}\right)$ provide a basis of $H^0(C,\Omega_C^1)$. We can associate to (w_1,w_2) an element $(z_1,w_1) - (z_2,w_2)$ of the Picard group $\mathrm{Pic}_0(C)$ where $z_1^2 = \Phi(w_1)$ and $z_2^2 = \Phi(w_2)$. So the equations (2.8) describe a linear motion on $\mathrm{Jac}(C)$ and the real tori given by Arnol'd-Liouville are real part of Abelian varieties on which the motion is linear.

If we introduce

$$Q(w,\underline{H},x_1,x_2) = (x_1 - x_2)^2 w^2 - 2R(x_1,x_2)w - R_1(x_1,x_2)$$

we deduce from (2.5) that

$$Q(w_1,\underline{H},x_1,x_2) = Q(w_2,\underline{H},x_1,x_2) = 0.$$

So we have an identity

$$\frac{\partial Q}{\partial w_i} dw_i + \frac{\partial Q}{\partial x_1} dx_1 + \frac{\partial Q}{\partial x_2} dx_2 + \frac{\partial Q}{\partial H} dH + \frac{\partial Q}{\partial K} dK \Big|_{w=w_i} = 0$$

for $i = 1, 2$.

We can deduce from this identity that

$$\{K, w_1\} = 2w_2\sqrt{\Phi(w_1)}/w_1 - w_2 \tag{2.9}$$

$$\{K, w_2\} = 2w_1\sqrt{\Phi(w_2)}/w_2 - w_1 .$$

So the condition (1.1) holds for the Kowalevskaya top with $v_1 = +2$ and $v_2 = 1$.

3. Preparation of the Symplectic Form

Proposition 3.1. *If $(\underline{H},\omega)$ is algebraically separable, then there are functions $\tilde{q}_j$ such that the forms $d\tilde{q}_j\Big|_{\underline{H}^{-1}(\underline{c})}$ are sums of Abelian integrals and such that*

$$\omega = \sum_{j=1}^{m} d\tilde{q}_j \wedge dH_j .$$

Proof. We start with the expression of the symplectic form ω in the coordinates $(\underline{H},\underline{w})$

$$\omega = \sum_{j,\ell} A_{j\ell} dH_j \wedge dH_\ell + B_{j\ell} dw_j \wedge dH_\ell + C_{j\ell} dw_j \wedge dw_\ell .$$

We have

$$(3.1) \qquad -dH_i = \sum_{j,\ell} B_{j\ell}\{H_i,w_j\}dH_\ell + 2C_{j\ell}\{H_i,w_j\}dw_\ell .$$

It is convenient at this point to introduce a matrix notation. Let F, B, W, V be the matrices whose general terms are:

$$(F)_{ij} = \{H_i,w_j\} \qquad (B)_{ij} = B_{ij}$$

$$(W)_{ij} = \frac{w_i^{j-1}}{\sqrt{\Phi(w_1)}} \qquad (V)_{ij} = v_i \delta_{ij} .$$

Then, the equation (1.1) gives

$$F \cdot W = V.$$

From (3.1), we deduce that

$$F \cdot B = -1$$

and since $\det(V) = \prod_{i=1}^{m} v_i \neq 0$, that

$$B = -WV^{-1}$$

or

$$(3.2) \qquad B_{j\ell} = -\frac{1}{V_\ell} w_j^{\ell-1} / \sqrt{\Phi(w_j)}.$$

Another consequence of (3.1) is

$$C_{j\ell}\{H_i, w_j\} = 0$$

and because $\det F \neq 0$, we have

$$(3.3) \qquad C_{j\ell} = 0 \quad \text{for all} \quad j,\ \ell.$$

We introduce now the pre-angles $\tilde{q}_j$ in the following way. The symplectic form can be written:

$$\omega = \sum_{j=1}^{m} \eta_j \wedge dH_j$$

Let $\tilde{\eta}$ be a one-form such that $\omega = d\tilde{\eta}$ defined on an appropriate universal cover. We have

$$\tilde{\eta} = \sum_i \alpha_i dH_i + \beta_i dw_i$$

$$\frac{\partial \beta_i}{\partial w_k} = \frac{\partial \beta_k}{\partial w_i}.$$

Let us introduce a function $\tilde{S}$ such that $\beta_i = \frac{\partial \tilde{S}}{\partial w_i}$ and write:

$$\tilde{\eta}' = \eta - d\tilde{S} = \sum(\alpha_i - \frac{\partial\tilde{S}}{\partial H_i})dH_i$$

$$\tilde{q}_i = \alpha_i - \frac{\partial\tilde{S}}{\partial H_i}$$

$$\omega = d\tilde{\eta}' = \sum d\tilde{q}_i \wedge dH_i .$$

4. Arnol'd's Definition of the Actions

A system of Action-angles for $(\underline{H},\omega)$ can be defined following Arnol'd [A] when $\underline{H}:V_R^{2m} \to R$ has compact fibers. In that case the connected components of $\underline{H}^{-1}(\underline{c})$ are tori and we define an Action-angle coordinates system, relatively to a basis $\gamma_j(\underline{c})$ of the homology of the real tori $\underline{H}^{-1}(\underline{c})$, as coordinates $(\underline{p},\underline{q})$ so that

i) $\omega = \sum_{j=1}^{m} dq_j \wedge dp_j$

ii) $\underline{H} = \underline{H}(\underline{p})$ (the first integrals do not depend on the angles)

iii) $\int_{\gamma_j(\underline{c})} dq_i = \delta_{ij}$.

Basic references for Action-angles are Arnol'd [A], Nekhoroshev [N]. A nice example of R. Cushman of non-existence of global Action-angles is analyzed in [Du]. See also [F.M.]. Action-angles are very useful, for instance, for the quantization of classical mechanical systems [G-S].

For algebraically separable Hamiltonian systems, we have previously prepared the symplectic form

$$\omega = \sum_{j=1}^{m} d\tilde{q}_j \wedge dH_j .$$

So the Actions are determined as functions of $\underline{H}$ by the periods

$$\psi_j(\underline{H}) = -\sum_{i,k} \int_{\gamma_j(\underline{H})} \frac{w_k^{i-1}\, dw_k}{v_k\sqrt{\Phi(w_k)}} .$$

The periods are given by Abelian integrals of the first kind. Thus, their computation is a problem of Algebraic Geometry once we know explicitly how the Hyperelliptic curves $\mathbf{C}_{\underline{c}}$ depend on $\underline{H}$.

5. Computation of the Angles

Proposition 5.1. *The angles are given by*

$$q_i = \sum_{j=1}^{m} T_{ij}(\underline{H})\tilde{q}_j$$

where the matrix $T:(T)_{ij} = T_{ij}$ *is the inverse of* $\tilde{T}$:

$$(\tilde{T})_{ij} = \int_{\gamma_j} d\tilde{q}_i .$$

In general, for a family of Hyperelliptic Riemann surfaces, it is not possible to compute explicitly the Abelian integrals as functions of the parameters. But they are solutions of a Picard-Fuchs differential equation [D], [M].

The same situation appears for the Milnor fibration where the Gauss-Manin connection provides a Regular Singular Differential System which is very useful to study the local monodromy [D], [M], [Gr]. The integrals are in that case related to the Birkhoff series of Hamiltonian Systems [F], [V].

We make more explicit the computation of these differential equations for the Kowalevskaya Top.

6. Picard-Fuchs Equations for the Kowalevskaya Top

We must, first of all, choose a system of generators for the homology of the real part of $\underline{H}^{-1}(\underline{c})$. The coordinates (w_1, w_2) represent a point on $\underline{H}^{-1}(\underline{c})$. If p, q; γ_1, γ_2 are real, then $x_1 = \bar{x}_2$ and (cf. (2.5)) w_1, $w_2 \in R$ (in fact $w_2 \in R_-$). With (w_1, w_2) we can parametrize an element $(z_1, w_1) - (z_2, w_2)$ of $\mathrm{Pic}_0(C_{\underline{c}})$.

The mapping $h_{w_2} : C_{\underline{c}} \to \mathrm{Jac}(C_{\underline{c}})$ defined by

$$h_{w_2} : (z_1, w_1) \to (z_1, w_1) - (z_2, w_2),$$

where w_2 is fixed, is a quasi-isomorphism.

So a system of generators for the Homology of $\underline{H}^{-1}(\underline{c})$ can be prescribed by paths in the w_1-plane.

For our case, the polynomial Φ (2.6) is of degree 5 and we know that $+\sqrt{K}$ and $-\sqrt{K}$ are two roots of Φ. So we can explicitly compute the three other roots. They will be denoted (e_1, e_2, e_3). The equation of the Discriminant locus of $C_{\underline{c}}$ is

$$\delta\delta' = 0 \tag{6.1}$$

where

$$\delta = -K + \left(H - 2\ell^2\right)^2 \tag{6.2}$$

$$\delta' = 4p^3 + 27q^2$$

with $p = \left(c^2 - K - H^2/3\right)$ and $q = \frac{2H}{3}\left(c^2 - K\right) - 2c^2\ell^2 + \frac{2H^2}{27}$.

Thus the Discriminant locus is the union of a Parabola and of a singular sextic (with four singular points in its affine part).

We need a quick analysis of the respective localization of each roots of ϕ. For instance if $\ell = 0$ then (cf. (2.6))

$$\phi(w) = (w + H)\left(w^2 + c^2 - K\right). \tag{6.3}$$

If K is small enough, there is only one real root $-H$. If $-H \ll -\sqrt{K}$, let e be the real root of ϕ which equals $-H$ for $\ell = 0$; for datas which are small perturbations of this situation, we get sign of ϕ

Hence, we can choose the segments $[-\infty, e]$ and $[\sqrt{K}, -\sqrt{K}]$ to have a basis of the real homology of $\mathrm{Jac}(\mathbf{C})$. We are concerned with the four integrals

$$(6.4)\qquad \begin{aligned} P_1 &= \int_{-\infty}^{e} \frac{dw}{\sqrt{\Phi(w)}}, \qquad P_2 = \int_{-\infty}^{e} \frac{wdw}{\sqrt{\Phi(w)}}, \\ Q_1 &= \int_{-\sqrt{K}}^{\sqrt{K}} \frac{dw}{\sqrt{\Phi(w)}}, \qquad Q_2 = \int_{-\sqrt{K}}^{\sqrt{K}} \frac{wdw}{\sqrt{\Phi(w)}} \end{aligned}$$

and their analytic extensions to any values of $\underline{H} = (K,H)$.

The Picard-Fuchs equation does not depend on the generator of the homology so we can restrict ourselves to the path γ defined by going from $-\infty$ to e on the first sheet of $\mathbf{C}_{\underline{c}}$ then back from e to $-\infty$ on the second sheet of $\mathbf{C}_{\underline{c}}$. We have to deal with

$$P_i = \int_\gamma \frac{w^{i-1} dw}{\sqrt{\Phi(w)}} \quad \text{for} \quad i = 1, \ldots, 4.$$

For the Kowalevskaya top there is a nice simplification of the monodromy computation because there <u>is a vector field</u> X_0:

$$(6.5)\qquad X_0 = \frac{1}{2}\frac{\partial}{\partial H} - w\frac{\partial}{\partial K} - \frac{1}{2}\frac{\partial}{\partial w}$$

such that $X_0 \cdot \Phi(w) = 0$.

From this and the relation

(6.6) $$\int_\gamma \frac{w^4 dw}{\sqrt{\Phi}} = \int_\gamma \frac{w^4 dw}{\sqrt{\Phi}} - \frac{1}{5}\int_\gamma \frac{\Phi'}{\sqrt{\Phi}} dw$$

$$\int_\gamma \frac{w^4 dw}{\sqrt{\Phi(w)}} = -\frac{4}{5}HP_4 - \frac{3}{5}(c^2 - 2k)P_3 - \frac{2}{5}[(c^2 - 2k)H - 2c^2\ell^2]P_2$$

$$+ \frac{1}{5}K(c^2 - k)P_1.$$

We get

(6.7) $$\partial P_1/\partial H = 2\partial P_2/\partial K$$

$$\partial P_i/\partial H = 2\partial P_{i+1}/\partial K - iP_{i-1} \quad \text{for} \quad i = 1, 2, 3$$

$$\partial P_4/\partial H = \frac{8}{5}H \frac{\partial P_4}{\partial K} - \frac{6}{5}(c^2 - 2K)\frac{\partial P_3}{\partial K} - \frac{4}{5}((c^2 - 2K)H - 2c^2\ell^2)\frac{\partial P_2}{\partial K}$$

$$+ \frac{2}{5}K(c^2 - K)\frac{\partial P_1}{\partial K} - \frac{8}{5}P_3 + \frac{8H}{5}P_2 + \frac{1}{5}[2c^2 - 4K]P_1.$$

This allows to separate simply the Picard-Fuchs equations into two parts involving respectively only the partial derivatives relatively to H or K.

Let us see now, for instance, the system for the partial derivatives relatively to H. We follow here the usual way [D].

We start with

(6.8) $$\frac{\partial P_i}{\partial H} = -\frac{1}{2}\int_\gamma \frac{w^{i-1}}{\sqrt{\Phi(w)}} \frac{\Phi'_H}{\Phi} dw = -\frac{1}{2}\int \frac{w^{i-1}}{\sqrt{\Phi}} \frac{\phi'_H}{\phi} dw$$

we denote by χ, $\chi = w^2 - K$, then $\Phi = 2\chi \cdot \phi$

$$\phi'_H = w^2 - K + c^2 = \chi + c^2. \tag{6.9}$$

We can check that

$$1 = \lambda\phi + \mu\chi \tag{6.10}$$

with

$$\lambda = -\frac{w-(H-2\ell^2)}{c^2\delta}, \qquad \mu = \frac{w^2+2\ell^2(w+H)-H^2+2\ell^2H+c^2}{c^2\delta} \tag{6.11}$$

so

$$\int_\gamma \frac{w^{i-1}}{\sqrt{\Phi(w)}} \frac{\phi'_H}{\phi}dw = \int_\gamma \frac{w^{i-1}}{\sqrt{\Phi(w)}}c^2\lambda\, dw + \int_\gamma \frac{w^{i-1}}{\sqrt{\Phi}}(1 + c^2\mu)\frac{\chi}{\phi}dw. \tag{6.12}$$

Let us consider the first integral

$$-\int_\gamma \frac{w^i dw}{\delta\sqrt{\phi}} + \frac{(H-2\ell^2)}{\delta}\int_\gamma \frac{w^{i-1}dw}{\sqrt{\Phi(w)}} \tag{6.13}$$

and so for $i = 1, 2, 3$ we find it is

$$-\frac{1}{\delta}P_{i+1} + \frac{H-2\ell^2}{\delta}P_i. \tag{6.14}$$

For $i = 4$, with (6.6), we have

$$\frac{H-2\ell^2}{\delta}P_4 - \frac{1}{\delta}\int_\gamma \frac{w^4\,dw}{\sqrt{\phi}} = \frac{\frac{9}{5}H-2\ell^2}{\delta}P_4 + \frac{\frac{3}{5}(c^2-2K)}{\delta}P_3 + \frac{\frac{2}{5}((c^2-2K)H-2c^2\ell^2)}{\delta}P_2 \tag{6.15}$$

$$- \frac{\frac{1}{5}K(c^2-K)}{\delta}P_1.$$

The second integral in (6.12) is slightly harder to compute. First of all, we have:

$$1 + c^2\mu = \frac{1}{\delta}\left[w^2 + 2\ell^2 w + 4\ell^4 - 2\ell^2 H + c^2 - K\right] = \frac{R}{\delta}. \tag{6.16}$$

With the notations of (6.3) and $z = w + \frac{H}{3}$

$$\phi(z) = z^3 + Pz + q. \tag{6.17}$$

We write now

$$R = z^2 + uz + v \tag{6.18}$$

with

$$u = -\frac{2H}{3} + 2\ell^2$$

$$v = \frac{H^2}{9} - \frac{8\ell^2 H}{3} + 4\ell^4 + c^2 - K. \tag{6.19}$$

Then we have

$$R = L\phi + M\phi' \tag{6.20}$$

with

$$L = \frac{1}{\delta'}[-3rz - 3s] \tag{6.21}$$

$$M = \frac{1}{\delta'}[rz^2 + sz + t]$$

and

$$\begin{vmatrix} r \\ s \\ t \end{vmatrix} = \begin{vmatrix} 2p^2 & -9q & -6p \\ -3qp & -2p^2 & 9q \\ -9q^2 & -6pq & -4p^2 \end{vmatrix} \begin{vmatrix} 1 \\ -u \\ v \end{vmatrix} \tag{6.22}$$

and were $\delta' = -4p^3 - 27q^2$ is the notation of (6.2).

So we can write

$$\int_\gamma \frac{w^{i-1}}{\sqrt{\Phi(w)}}(1 + c^2\mu)\frac{\chi}{\phi}dw = \frac{1}{\delta}\int_\gamma \frac{w^{i-1}}{\sqrt{\Phi(w)}}(L\phi + M\phi')\frac{\chi}{\phi}dw \tag{6.23}$$

$$= \frac{1}{\delta}\int_\gamma \frac{w^{i-1}}{\sqrt{\Phi}}L\chi dw + \frac{1}{\delta}\int_\gamma \frac{w^{i-1}M\sqrt{\chi}}{\phi^{3/2}}\phi' dw.$$

The first integral gives

$$(6.24) \qquad \frac{1}{\delta\delta'}\int_{\gamma} \frac{w^{i-1}(-3rw-rH-3s)(w^2-K+c^2)}{\sqrt{\Phi}}dw$$

$$= \frac{1}{\delta\delta'}[-3r\int_{\gamma} \frac{w^{i+2}}{\sqrt{\Phi}}dw - (rH + 3s)\int_{\gamma} \frac{w^{i+1}}{\sqrt{\Phi}}dw$$

$$- 3r(-K + c^2)\int_{\gamma} \frac{w^{i}}{\sqrt{\Phi}}dw - (rH + 3s)(c^2 - K)\int_{\gamma} \frac{w^{i-1}}{\sqrt{\Phi}}dw].$$

The second integral of (6.23) gives

$$\frac{1}{\delta}\int_{\gamma} \frac{2[(i-1)w^{i-2}M+w^{i-1}M']\chi+w^{i-1}M\chi'}{\sqrt{\Phi}}dw$$

and then

$$(6.25) \qquad \frac{1}{\delta\delta'}[2(i + 2)r\int_{\gamma} \frac{w^{i+2}}{\sqrt{\Phi}}dw + 2(i + 1)(\frac{2Hr}{3} + s)\int_{\gamma} \frac{w^{i+1}}{\sqrt{\Phi}}dw$$

$$+ [2i(r\frac{H^2}{9} + \frac{sH}{3} + t) + 2(i + 1)r(-K - c^2)]\int_{\gamma} \frac{w^{i}dw}{\sqrt{\Phi(w)}}$$

$$+ 2i(\frac{2Hr}{3} + s)(-K + c^2)\int_{\gamma} \frac{w^{i-1}}{\sqrt{\Phi}}dw$$

$$+2(i - 1)[\frac{rH^2}{9} + s\frac{H}{3} + t][-K + c^2]\int_{\gamma} \frac{w^{i-2}}{\sqrt{\Phi}}dw].$$

Now we have to express the integrals

$$\int_\gamma \frac{w^4 dw}{\sqrt{\Phi}}, \qquad \int_\gamma \frac{w^5 dw}{\sqrt{\Phi}}, \qquad \int_\gamma \frac{w^6 dw}{\sqrt{\Phi}}$$

as a combination of the Abelian integrals of the first and the second kinds. This is a classical computation.

For the first one, we have the formula (6.6). For the others, we use the formula

$$(6.26) \qquad \frac{b_0 w^m + b_1 w^{m-1} + \ldots + bm}{\sqrt{\Phi}} - \frac{b_0}{k + \frac{5}{2}} \frac{d}{dw}[w^k \sqrt{\Phi}] \frac{S}{\sqrt{\Phi}}$$

where S is of degree lower than m, choosing a k in such a way, that $m = k + 4$.

So if we denote

$$\Phi(w) = w^5 + \sigma_1 w^4 + \sigma_2 w^3 + \sigma_3 w^2 + \sigma_4 w + \sigma_5,$$

we find

$$(6.27) \qquad \int_\gamma \frac{w^6 dw}{\sqrt{\Phi}} = -\frac{2}{9}\left[4\sigma_1 \int_\gamma \frac{w^5 dw}{\sqrt{\Phi}} + \frac{7}{2}\sigma_2 \int_\gamma \frac{w^4 dw}{\sqrt{\Phi}} + 3\sigma_3 P_4 + \frac{5}{2}\sigma_4 P_3 + 2\sigma_5 P_2\right]$$

and

$$(6.28) \qquad \int_\gamma \frac{w^5 dw}{\sqrt{\Phi}} = -\frac{2}{7}\left[3\sigma_1 \int_\gamma \frac{w^4 dw}{\sqrt{\Phi}} + \frac{5}{2}\sigma_2 P_4 + 2\sigma_3 P_3 + \frac{3}{4}\sigma_4 P_2 + \sigma_5 P_1\right].$$

Finally, if we put together (6.14), (6.15), (6.24), (6.25) and (6.6), (6.26), (6.27), we find explicitly the Picard-Fuchs equation in the form

$$\frac{\partial P_i}{\partial H} = \sum_{j=1}^{4} = [\frac{\alpha_{ij}}{\delta} + \frac{\beta_{ij}}{\delta\delta'}]P_j \tag{6.29}$$

where α_{ij} are given by (6.14) and (6.15) and β_{ij} are derived from (6.24), (6.25). The α_{ij} and β_{ij} are simple polynomial expressions of $\underline{H} = (K,H)$.

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