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METRIC THEOREMS ON UNIFORM DISTRIBUTION AND APPROXIMATION THEORY

by
Harald NIEDERREITER and Robert F. TICHY

A sequence $(\underline{x}_n)$, $n = 1,2,\dots$, of points in $[0,1)^k$ is called uniformly distributed (u.d.) if

$$\lim_{N\to\infty} N^{-1} \operatorname{card}\{1 \leq n \leq N: \underline{x}_n \in J\} = \operatorname{vol}(J)$$

for all subintervals J of $[0,1)^k$. A sequence (x_n) of numbers in $[0,1)$ is called completely uniformly distributed (c.u.d.) if for all $k \geq 1$ the sequence of points $\underline{x}_n = (x_n, x_{n+1}, \dots, x_{n+k-1})$, $n=1,2,\dots$, is u.d., and a sequence (x_n) of real numbers is called c.u.d. mod 1 if the sequence $(\{x_n\})$ of fractional parts is c.u.d.

Franklin [3] proved that the sequence (x^n) is c.u.d. mod 1 for almost all $x>1$ (in the sense of Lebesgue measure). Knuth [4] defined a sequence (x_n) in $[0,1)$ to be random if, for every sequence (b_n) of distinct positive integers obtained by an effective algorithm, the sequence (x_{b_n}) is c.u.d. and raised the problem of whether $(\{x^n\})$ satisfies this definition of randomness for almost all $x>1$. This problem was recently settled in the affirmative by the following result.

THEOREM 1 (Niederreiter and Tichy [7]). - *If (a_n) is any sequence of distinct positive integers, then the sequence (x^{a_n}) is c.u.d. mod 1 for almost all $x > 1$.*

The following lemmas are the key ingredients in the proof.

LEMMA 1 (Davenport, Erdös and LeVeque [2]). - *Let (u_n) be a sequence of Lebesgue-measurable functions on the interval $[a,b]$ and put*

$$I(N) = \int_a^b \left|\frac{1}{N}\sum_{n=1}^{N} e(u_n(x))\right|^2 dx, \quad \text{where } e(t) = e^{2\pi i t}.$$

If $\sum_{N=1}^{\infty} I(N)/N$ converges, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(u_n(x)) = 0 \text{ for almost all } x \in [a,b].$$

LEMMA 2 (van der Corput [8]). - Let the function u on [a,b] have a continuous and monotone first derivative that satisfies $|u'(x)| \geq L > 0$ for all $x \in [a,b]$. Then

$$\left|\int_a^b e(u(x))dx\right| \leq \frac{1}{L}.$$

LEMMA 3. - If P is a monic polynomial of degree $d \geqslant 1$, then for the supremum norm $\|P\|_\infty$ on [a,b] we have

$$\|P\|_\infty \geqslant 2^{1-2d}(b-a)^d. \tag{1}$$

Lemma 3 is a standard result in approximation theory (see [6,p.39]). The lower bound (1) is in fact best possible, as it is attained by the normalized Chebyshev polynomial on [a,b] of degree d. The question of whether the integer exponents in Theorem 1 can be replaced by real exponents leads to another problem in approximation theory, namely to establish a lower bound for $\|P\|_\infty$ when

$$P(x) = x^{\lambda_s} + \sum_{j=1}^{s-1} c_j x^{\lambda_j} \tag{2}$$

is a monic Müntz polynomial with real coefficients c_j and real exponents λ_j satisfying $0 \leqslant \lambda_1 < \lambda_2 < \dots < \lambda_s$, $s \geqslant 2$. The lower bound should not depend on the c_j. For [a,b] = [0,1] this is a classical problem considered by Müntz and Szász. By applying the transformation $x \to bx$, intervals [0,b] can also be handled. For [a,b] with $a > 0$ we cannot use the transformation trick since $P(a + (b - a)x)$ is not necessarily a Müntz polynomial. We now have the following result. We are grateful to Prof. J. Korevaar for providing the important reference [5].

LEMMA 4. - For any monic Müntz polynomial P given by (2) and any interval [a,b] with $0 < a < b$ we have

$$\|P\|_\infty \geqslant C_s a^{\lambda_s} \prod_{j=1}^{s-1} \frac{\lambda_s - \lambda_j}{\lambda_j - \lambda_1 + (1/\tau)},$$

where $\tau = \frac{1}{2}\log\frac{b}{a}$ and the constant $C_s > 0$ depends only on s.

Proof.- We first assume that $\lambda_1 = 0$. By a result in [5] there exists a sequence (γ_n), $n = 0,1,\dots$, with $\gamma_n > 0$ and

$\sum_{n=0}^{\infty} \gamma_n = 1$ such that $\prod_{n=0}^{\infty} \cos(\gamma_n z)$ is entire and

$$\prod_{n=0}^{\infty} |\cos(\gamma_n u)| \leqslant C e^{-\sqrt{u}} \quad \text{for all } u \geq 0, \tag{3}$$

where C is an absolute constant. Then the complex function

$$F(z) = \prod_{j=1}^{s-1} (z - i\lambda_j) \prod_{n=0}^{\infty} \cos(\gamma_n \tau z)$$

is entire and of exponential type τ, i.e. $|F(z)| = O_\varepsilon (e^{(\tau+\varepsilon)|z|})$ for all $\varepsilon > 0$, and the restriction of F to $\mathbb{R}$ is in $L^2(\mathbb{R})$. By the Paley-Wiener theorem (see [1,p.103]) we have

$$F(z) = \int_{-\tau}^{\tau} f(t)e^{-izt}dt$$

for some $f \in L^2[-\tau,\tau]$. With $\sigma = \frac{1}{2}\log(ab)$ we get

$$G(z) := e^{-i\sigma z}F(z) = \int_{-\tau+\sigma}^{\tau+\sigma} f(t-\sigma)e^{-izt}dt = \int_{\log a}^{\log b} g(t)e^{-izt}dt$$

with $g \in L^2[\log a, \log b]$. By Fourier inversion we obtain

$$(4) \qquad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(u)e^{itu}du \text{ for } t\in[\log a, \log b].$$

Put $h(x) = x^{-1}g(\log x)$ for $x \in [a,b]$. Then with $x = e^t$,

$$(5) \qquad \int_a^b x^{\lambda_j}h(x)dx = \int_{\log a}^{\log b} g(t)e^{\lambda_j t}dt = G(i\lambda_j) \text{ for } 1 \leqslant j \leqslant s.$$

For P in (2) we use (5) and $G(i\lambda_j) = 0$ for $1 \leqslant j \leqslant s - 1$ to get

$$(6) \qquad \left|\int_a^b x^{\lambda_s}h(x)dx\right| = \left|\int_a^b P(x)h(x)dx\right| \leqslant \|P\|_\infty \int_a^b |h(x)|dx.$$

Then by (4), (5) and (6),

$$(7) \qquad |G(i\lambda_s)| \leqslant \|P\|_\infty \int_a^b \frac{dx}{x}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(u)|du = \frac{\tau}{\pi}\|P\|_\infty \int_{-\infty}^{\infty}|G(u)|du.$$

Now

$$|G(i\lambda_s)| = e^{\sigma\lambda_s}|F(i\lambda_s)| = e^{\sigma\lambda_s}\prod_{j=1}^{s-1}(\lambda_s - \lambda_j)\prod_{n=0}^{\infty}\cosh(\gamma_n\tau\lambda_s)$$

$$(8) \quad = b^{\lambda_s}\prod_{j=1}^{s-1}(\lambda_s - \lambda_j)\prod_{n=0}^{\infty}\frac{1}{2}(1 + e^{-2\gamma_n\tau\lambda_s}) \geqslant b^{\lambda_s}\prod_{j=1}^{s-1}(\lambda_s-\lambda_j)\prod_{n=0}^{\infty} e^{-2\gamma_n\tau\lambda_s}$$

$$= a^{\lambda_s}\prod_{j=1}^{s-1}(\lambda_s - \lambda_j).$$

Using (3) we get

$$\int_{-\infty}^{\infty}|G(u)|du \leqq 2C\int_0^{\infty}\prod_{j=1}^{s-1}(u + \lambda_j)e^{-\sqrt{\tau u}}du$$

$$= \frac{2C}{\tau} \int_0^\infty \prod_{j=1}^{s-1} \left(\frac{v}{\tau} + \lambda_j\right) e^{-\sqrt{v}} dv \leqslant \frac{C_s}{\tau} \prod_{j=1}^{s-1} \left(\frac{1}{\tau} + \lambda_j\right) ,$$

and together with (7) and (8) the result for $\lambda_1 = 0$ follows. In the general case the lemma follows from

$$\|P\|_\infty \geqslant a^{\lambda_1} \|x^{\lambda_s - \lambda_1} + \sum_{j=1}^{s-1} c_j x^{\lambda_j - \lambda_1}\|_\infty . \square$$

Using Hölder's inequality in (6), the method above yields for the L^p-norm $(1 < p < \infty)$

$$\|P\|_p \geqslant C_s \quad \tau a^{\lambda_s} \frac{(q-1)^{1/q}}{(a^{1-q} - b^{1-q})^{1/q}} \prod_{j=1}^{s-1} \frac{\lambda_s - \lambda_j}{\lambda_j - \lambda_1 + (1/\tau)} ,$$

where $p^{-1} + q^{-1} = 1$. For the L^1-norm we get

$$\|P\|_1 \geqslant C_s \tau a^{\lambda_s + 1} \prod_{j=1}^{s-1} \frac{\lambda_s - \lambda_j}{\lambda_j - \lambda_1 + (1/\tau)} .$$

THEOREM 2.- <u>If</u> $\gamma \neq 0$ <u>and</u> (δ_n) <u>is any sequence of reals with</u> $\inf \delta_n > -\infty$ <u>and</u> $\inf_{m \neq n} |\delta_m - \delta_n| > 0$, <u>then the sequence</u> (γx^{δ_n}) <u>is c.u.d. mod 1 for almost all</u> $x > 1$.

<u>Proof</u>. We take $\gamma = 1$ to simplify the writing. We indicate the changes that are necessary in the proof of Theorem 1 (see [7]) to obtain Theorem 2. Dropping finitely many initial terms, if necessary, we can assume that all $\delta_n \geq 1$. As in [7] it suffices to show that if the lattice point $\underline{h} = (h_0, \ldots, h_{k-1}) \neq \underline{0}$ and $1 < \alpha < \beta$ are fixed, then with

$$I(N) = \int_\alpha^\beta \left|\frac{1}{N} \sum_{n=1}^{N} e\left(\sum_{j=0}^{k-1} h_j x^{\delta_{n+j}}\right)\right|^2 dx$$

the series $\sum_{N=1}^{\infty} I(N)/N$ converges. We take $N \geqslant 3$. The cases (i) and (ii) in [7] require no changes, if we note that the number of zeros of a Müntz polynomial in $[\alpha, \beta]$ can be bounded by [6, p.49, Hilfssatz]. In case (iii) the function $R_s(x)$ in [7, equ. (2.12)] is now a Müntz polynomial with $\lambda_s = d \leqslant (\log N)^{\sigma(s)} \leqslant \log N$, $\lambda_1 = 0$, $2 \leqslant s \leqslant k$, and

$$|\lambda_i - \lambda_j| \geqslant \delta \ := \inf_{m \neq n} |\delta_m - \delta_n| \text{ for } i \neq j .$$

Instead of the choice $\varepsilon = (\log N)^{-2d}$ in [7] we take $\varepsilon = (\log N)^{-2(s-1)-d}$. For a subinterval $[a,b]$ of $[\alpha, \beta]$ with $|R_s(x)| \leqslant \varepsilon$ for all $x \in [a,b]$ we have then by Lemma 4,

$$\varepsilon \geqslant C_s a^{\lambda_s} \prod_{j=1}^{s-1} \frac{\lambda_s - \lambda_j}{\lambda_j + (1/\tau)} \geqslant C_s \delta^{s-1} \left(\lambda_s + \frac{1}{\tau}\right)^{1-s} ,$$

hence

$$\frac{1}{\tau} \geq C'_s\ \delta(\log N)^2 - \lambda_s \geq C(\underline{h},\delta)(\log N)^2 \text{ for } N \geq N_0(\underline{h},\delta).$$

From the definition of τ it follows that

$$\log \frac{b}{a} \leq C_1(\underline{h},\delta)(\log N)^{-2} \text{ for } N \geq N_0(\underline{h},\delta) \ ,$$

and so

$$b - a \leq [\exp(C_1(\underline{h},\delta)(\log N)^{-2}) - 1]\ \beta$$
$$\leq C(\underline{h},\delta,\beta)(\log N)^{-2} \text{ for } N \geq N_1(\underline{h},\delta).$$

Therefore, the Lebesgue measure of $V = \{x \in [\alpha,\beta] : |R_s(x)| \leq \varepsilon\}$ is $O((\log N)^{-2})$. If W is the closure of the complement of V in $[\alpha,\beta]$, then for $x \in W$ we have

$$|R_s(x)| \geq (\log N)^{-2(s-1)-d} \geq (\log N)^{-2(\log N)^{\sigma(s)}} \text{ for } N \geq N_2(\underline{h}) \ ,$$

and so the argument following (2.13) in [7] can be applied. This yields $I(N) = O((\log N)^{-2})$, and the proof is complete. □

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