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The purpose of the present paper is to describe new results in transcendental number theory, which have been proved in the last few years with the help of the methods using the commutative algebra. These results concern the estimates for the multiplicities of zeros of polynomials in a solution of a system differential equations and the estimates for the measure of algebraic independence of the values some functions. These estimates were proved just the same way.

In the case of numbers it may be describe in brief as follows. For any polynomial P with complex coefficients H(P) will denote the maximum of the absolute values of coefficients of P and deg P - degree P, $t(P) = \deg P + \ln H(P)$. In general the problem may be put as follows. For the given point $\bar{\omega} = (\omega_o, \dots, \omega_m) \in \mathbb{C}^{m+1}$ we are searching for the lower estimate for $|P(\bar{\omega})|$, $P \not\equiv 0$, $P \in \mathbb{Z}[X_o, \dots, X_m]$, in terms of the H(P) and deg P, or in terms of t(P).

We shall use the notion of the rank for ideals. The rank of the prime ideal $\mathfrak{P} \subset \mathbb{Z}[X_o, \dots, X_m]$ is the maximal length of any increasing chain of prime ideals terminating with $\mathfrak{P}$. The rank of any ideal $\mathfrak{J}$ is the minimal rank of prime ideals $\mathfrak{P}$, containing $\mathfrak{J}$. The rank of an ideal $\mathfrak{J}$ will be denoted by $h(\mathfrak{J})$.

Any homogeneous unmixed ideal $\mathfrak{J} \subset \mathbb{Z}[X_o, \dots, X_m]$ may be characterized by numbers $N(\mathfrak{J})$, $H(\mathfrak{J})$, which are analogous to deg P, H(P) for polynomials $P \in \mathbb{Z}[X_o, \dots, X_m]$. One may also define $|\mathfrak{J}(\bar{\omega})|$, analogous to $|P(\bar{\omega})|$. These numbers $N(\mathfrak{J})$, $H(\mathfrak{J})$, $|\mathfrak{J}(\bar{\omega})|$ have the properties concerning their behaviour under decomposition of the ideal into primary ideals almost analogous to the properties of the corresponding characteristics of P concerning the decomposition of P into irreducible factors. It allows to reduce the problem of obtaining the lower bound for $|\mathfrak{J}(\bar{\omega})|$ in terms of $N(\mathfrak{J})$, $H(\mathfrak{J})$ to the same problem for prime ideals $\mathfrak{P} \subset \mathbb{Z}[X_o, \dots, X_m]$, $h(\mathfrak{P}) = h(\mathfrak{J})$. An assertion holds, reducing the estimate for $|\mathfrak{P}(\bar{\omega})|$ to the analogous estimate for ideals of higher rank, which allows to prove the estimates by induction by rank from m to 1. For the principal ideal $\mathfrak{J} = (P)$ its rank equals 1 and the quantities $|\mathfrak{J}(\bar{\omega})|$ and $|P(\bar{\omega})|$ are closely related, which gives us possibility to obtain finally the lower estimate for $|P(\bar{\omega})|$ in terms of the characteristics of P.

For example, this method leads to the proof of the following theorem 1, concerning the values of functions, satisfying of functional equations of special kind. In 1929 K. Mahler studied transcendental functions satisfying such and more general equations and proved the algebraic independence of their values. These results were extended later by Mahler himself, J. Loxton and A. van der Poorten, K. Kubota, D. Masser and others. The estimates of transcendence measures were stated in the first by A. Galochkin [1]. Concerning this subject we mention the papers by W. Miller [9] and S. Molchanov [3], [4]. S. Molchanov and A. Yanchenko [2] obtain a good estimation of measure of algebraic independence of values of two functions in p-adic case.

THEOREM 1. Let $f_1(z), \dots, f_m(z)$ be power series, convergent in a neighbourhood $\mathcal{U}$ of $z=0$, with coefficients from algebraic fields K, $[K:\mathbb{Q}]<\infty$, satisfying the functional equations $f_i(z^d) = a_i(z) f_i(z) + b_i(z)$, $a_i(z)$, $b_i(z) \in K(z)$, $1 \le i \le m$, where d is an integer, $d \ge 2$, and algebraically independent over $\mathbb{C}(z)$. Suppose that α is an algebraic number, $\alpha \in \mathcal{U}$, $0<|\alpha|<1$, numbers $\alpha, \alpha^d, \alpha^{d^2}, \dots$ differ from the singular points of $a_i(z)$, $b_i(z)$. Then for any numbers $s \ge 1$, $H \ge H_0(s, \alpha, f_i(z))$ and any polynomial $P \in \mathbb{Z}[X_1, \dots, X_m]$, $P \not\equiv 0$, $\deg P \le s$, $H(P) \le H$ the inequality

$$|P(f_1(\alpha), \dots, f_m(\alpha))| \ge H^{-\gamma_1 s^m},$$

holds, where $\gamma_1 = \gamma_1(\alpha, f_i) > 0$.

COROLLARY. Let α be algebraic number, $0<|\alpha|<1$, d be integer, $d \ge 2$, $\varphi(z) = \sum_{n=0}^{\infty} z^{d^n}$. Then for any polynomial $P \in \mathbb{Z}[X_1, \dots, X_{d-1}]$, $P \not\equiv 0$, the inequality

$$|P(\varphi(\alpha), \varphi(\alpha^2), \dots, \varphi(\alpha^{d-1}))| \ge C . H(P)^{-\gamma_2 s^{d-1}}$$

holds, where $\gamma_2 = \gamma_2(\alpha, d) > 0$, $C = C(\alpha, d, s) > 0$, $s = \deg P$.

In order to establish the corollary it is sufficient to apply the theorem 1 to functions $\varphi(z^i)$, $i = 1, \dots, d-1$.

Theorem 1 is deduced from the next statement, which gives the lower estimate for $|\mathfrak{J}(\bar{\omega})|$ with $\bar{\omega} = (1, f_1(\alpha), \dots, f_m(\alpha))$.

THEOREM 2. Suppose that the conditions of theorem 1 are satisfied, $\mathcal{B} \ge 1$ and $H \ge 1$. Let $\mathfrak{J}$ be a homogeneous unmixed ideal of $\mathbb{Z}[X_0, \dots, X_m]$, such that $\mathfrak{J} \cap \mathbb{Z} = (0)$, $r = m+1-h(\mathfrak{J}) \ge 1$,

$$N(\mathfrak{J}) \le \lambda^{m-r} \mathcal{B}^{m-r+1}, \quad \ln H(\mathfrak{J}) \le \lambda^{m-r} \mathcal{B}^{m-r} \ln H,$$

where $\lambda \geq \lambda_o(\alpha, K, f_1, \dots, f_m) > 0$. If H exceeds a boundary, depending on λ and ϑ, then

$$\ell n\,|\mathfrak{J}(\bar{\omega})| \geq -\lambda^r\,(\vartheta\,\ell n H(\mathfrak{J}) + N(\mathfrak{J})\,\ell n H)\,\vartheta^{r-1}.$$

Let us deduce theorem 1 from theorem 2. If P is a polynomial from theorem 1, $Q = X_o^{\deg P}\,P\left(\frac{X_1}{X_o}, \dots, \frac{X_m}{X_o}\right)$ – a homogenisation of P, $\mathfrak{J} = (Q)$ – a principal ideal in $\mathbb{Z}[X_o, \dots, X_m]$, then one may prove that

$$N(\mathfrak{J}) = \deg P \quad , \quad H(\mathfrak{J}) \leq H(P)\, e^{m^2 \deg P}$$

and

$$|P(f_1(\alpha), \dots, fm(\alpha))| \geq |\mathfrak{J}(\bar{\omega})| \,.\, |\bar{\omega}|^{\deg P} \,.\, (m+1)^{-2m \deg P} \qquad (1)$$

From theorem 2 with $r = m$, $\vartheta = s$ and $H e^{m^2 s}$ instead of H it follows, that

$$|\mathfrak{J}(\bar{\omega})| \geq -2\lambda^m(\ell n H + m^2 s)\, s^m \geq -3\lambda^m s^m\, \ell n\, H.$$

This inequalities and (1) prove theorem 1. Theorem 2 is proved by induction on r from 1 to m.

Second group of results concern the values of exponential function at transcendental points. After a classical result by Gelfond and Schneider many of interesting assertions were proved here by A. Shmelev, R. Tijdeman, D. Brownawell, G. Chudnovsky, M. Waldschmidt, E. Reyssat, P. Philippon and others. The last results by P. Philippon [11] on criteria of algebraic independence and algebraic independence of values of exponential function are worth to be mentioned.

THEOREM 3. Let α, β be algebraic numbers, $\alpha \neq 0, 1$, degree of β equals to d, $d \geq 2$; let τ be a real number, $0 < \tau < \frac{d+1}{2}$. Then there exists a constant $\gamma_3 = \gamma_3(\alpha, \beta) > 0$ with the following property : for all $P_1, \dots, P_N$ in $\mathbb{Z}[X_1, \dots, X_{d-1}]$ of $t(P_j) \leq T$, which generate an ideal of rank $d-r$, $0 \leq r < \tau$, we have

$$\max_{1 \leq j \leq N} |P_j(\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}})| \geq \exp\left(-\gamma_3\, T^{(d-r)\frac{\tau}{\tau-r}}\right).$$

From this theorem is follows, for example, that among the numbers

$$\alpha^\beta, \alpha^{\beta^2}, \dots, \alpha^{\beta^{d-1}} \qquad (2)$$

there are at least $[\frac{d}{2}]$ algebraically independent over $\mathbb{Q}$. Indeed, let the number of algebraically independent among them equal to ℓ. Then prime ideal $\mathfrak{P}$, consisting of all polynomials in $\mathbb{Z}[X_1, \dots, X_{d-1}]$, which vanish at the point (2), has a rank $d-1-\ell$. Therefore the number $r = \ell+1$ is not less than $\frac{d+1}{2}$ and $\ell \geq [\frac{d}{2}]$. Theorem 3 follows from the next statement, which gives the lower estimate for $|\mathfrak{J}(\bar{\omega})|$.

THEOREM 4. Let $a_1, \dots, a_p$, $b_1, \dots, b_q$ be complex numbers, satisfying for any $\epsilon > 0$ and for any vectors $\bar{K} = (K_1, \dots, K_p) \in \mathbb{Z}^p$, $\bar{\ell} = (\ell_1, \dots, \ell_q) \in \mathbb{Z}^q$ the inequalities

$$|K_1 a_1 + \dots K_p a_p| > \exp(-|\bar{K}|^{\epsilon}), \quad |\ell_1 b_1 + \dots + \ell_q b_q| > \exp(-|\bar{\ell}|^{\epsilon})$$

for $|\bar{K}|$, $|\bar{\ell}|$ exceeding a boundary, depending from ϵ, a_i, b_i. Let $\omega_1, \dots, \omega_m$ be the set of numbers

$$a_1, \dots, a_p, e^{a_1 b_1}, \dots, e^{a_p b_q}, \quad m = p + pq,$$

ordered in any way, and $\omega_o = 1$, $\bar{\omega} = (\omega_o, \dots, \omega_m)$.

If r is an integer and τ is a real number, $1 \le r < \tau < \frac{p+pq}{p+q}$, then for any homogeneous unmixed ideal $\mathfrak{J} \subset \mathbb{Z}[X_o, \dots, X_m]$, $h(\mathfrak{J}) = m+1-r$, the estimate

$$|\mathfrak{J}(\bar{\omega})| \ge \exp\left(-\mu_r t(\mathfrak{J})^{\frac{\tau}{\tau - r}}\right)$$

holds, where $\mu_r = \mu(r, \tau, a_i, b_j) > 0$, $t(\mathfrak{J}) = N(\mathfrak{J}) + \ell n H(\mathfrak{J})$.

This theorem, like the theorem 2, is proved by induction on r. Induction is successed not to $m = p+pq$, as in the theorem 2, but only to $\frac{m}{p+q}$, which is connected with analytic possibilities of constructing polynomials small enough at the point $\bar{\omega}$.

We give auxilliary assertion, which connects lower estimates for ideals and polynomials at the point $\bar{\omega} = (1, \omega_1, \dots, \omega_m) \in \mathbb{C}^m$ and allows us to deduce the theorem 3 from the theorem 4.

PROPOSITION 1. Let the transcendence base of the field $\mathbb{Q}(\omega_1, \dots, \omega_m)$ be contained among the numbers $\omega_1, \dots, \omega_K$, $K \le m$; let $R_1, \dots, R_N \in \mathbb{Z}[X_o, \dots, X_m]$ be homogeneous polynomials $t(R_j) \le T$, $a = (R_1, \dots, R_N)$ is ideal in $\mathbb{Z}[X_o, \dots, X_m]$, $r = K + 1 - h(a)$. Then there exists a homogeneous unmixed ideal $\mathfrak{J} \subset \mathbb{Z}[X_o, \dots, X_m]$, $h(\mathfrak{J}) = m+1-r$, $t(\mathfrak{J}) \le \gamma_4 T^{K-r+1}$ such that

$$\max_{1 \le j \le N} |R_j(\bar{\omega})| \ge |\mathfrak{J}(\bar{\omega})| \, e^{-\gamma_5 T^{K-r+1}},$$

when $\gamma_i = \gamma_i(\bar{\omega}) > 0$, $i = 4, 5$.

Suppose in the theorem 4, that

$$p = q = d, \quad a_i = \beta^{i-1}, \quad b_i = \beta^{i-1} \ell n \alpha, \qquad i = 1, \dots, d,$$

$$\omega_1 = \alpha^{\beta}, \dots, \omega_{d-1} = \alpha^{\beta^{d-1}}, \quad K = d-1.$$

Then the theorem 4 and the proposition 1 give us the statement of theorem 3. We remark, that the estimate mentioned above for the quantity of algebraically independent numbers among (2) is a simple consequence from the theorem 4.

Let us describe in brief the proof of theorem 4. Suppose, that the assertion theorem 4 is valid for all homogeneous unmixed ideals $\mathfrak{J}\subset \mathbb{Z}[X_o,\dots,X_m]$, $h(\mathfrak{J}) > m+1-r$. The step of induction contains two stages.

1. Reduction of the estimate of $|\mathfrak{I}(\bar\omega)|$ to the analogous estimate for prime ideals.

If the assertion of the theorem 4 is valid for prime ideals $\mathfrak{P}$, $h(\mathfrak{P})=m+1-r$, then there is constant $\mu>0$ such that

$$\ell n|\mathfrak{P}(\bar\omega)| \geq -\mu\ t(\mathfrak{P})^{\frac{\tau}{\tau-r}} \tag{3}$$

We define for a homogeneous unmixed ideal $\mathfrak{I}$ with $h(\mathfrak{I})=m+1-r$ prime ideals $\mathfrak{P}_1,\dots,\mathfrak{P}_s$ and natural numbers $K_1,\dots,K_s$ according to the proposition 2 [7]. Then this proposition and (3) give us

$$m^3 t(\mathfrak{I}) + \ell n|\mathfrak{I}(\bar\omega)| \geq -\mu \sum_{\ell=1}^{s} K_\ell\, t(\mathfrak{P}_\ell)^{\frac{\tau}{\tau-r}} \geq -\mu\Big(\sum_{\ell=1}^{s} K_\ell\, t(\mathfrak{P}_\ell)\Big)^{\frac{\tau}{\tau-r}}$$
$$\geq -\mu(m+1)^{\frac{2\tau}{\tau-r}}\ t(\mathfrak{I})^{\frac{\tau}{\tau-r}}.$$

This proves theorem 4 with $\mu_r=\mu(m+1)^{\frac{2\tau}{\tau-r}}+m^3$.

2. Increase of the rank of ideal.

In order to establish the inequality (3) it is sufficient to prove, that a set of real numbers S, such that there exist prime homogeneous ideal $\mathfrak{P}\subset\mathbb{Z}[X_o,\dots,X_m]$, $\mathfrak{P}\cap\mathbb{Z}=(0)$, $h(\mathfrak{P})=m-r+1$ with conditions

$$|\mathfrak{P}(\bar\omega)| < \exp\Big(-S^{\frac{\tau}{\tau-r}}\Big),\quad t(\mathfrak{P})\leq S \tag{4}$$

is bounded.

The proof is based on the next proposition.

PROPOSITION 2. Let $Q\in\mathbb{Z}[X_o,\dots,X_m]$, $Q\neq 0$, be a homogeneous polynomial ; $\mathfrak{P}\subset\mathbb{Z}[X_o,\dots,X_m]$ be a prime homogeneous ideal, $\mathfrak{P}\cap\mathbb{Z}=(0)$, $r=m+1-h(\mathfrak{P})\geq 1$, $\bar\omega=(\omega_o,\dots,\omega_m)\in\mathbb{C}^{m+1}$, $\bar\omega\neq 0$,

$$|\mathfrak{P}(\bar\omega)|\leq e^{-X},\ X>0,\ |Q(\bar\omega)|.|\bar\omega|^{-\deg Q}\leq t(Q)^{-2m-2}.$$

Suppose that for $\sigma\geq 1$

$$\min\Big(X,\frac{1}{2}\,\ell n\frac{1}{\rho}\Big) = -\sigma\ \ell n\Big(|Q(\bar\omega)|.|\bar\omega|^{-\deg Q}\Big),$$

where ρ is minimal of the distances between $\bar\omega$ and zeros of ideal $\mathfrak{P}$. Then for $r\geq 2$ there exist homogeneous unmixed ideal $\mathfrak{J}\subset\mathbb{Z}[X_o,\dots,X_m]$, $h(\mathfrak{J})=m-r+2$ such that

1) $t(\mathfrak{J})\leq 2m^2 t(\mathfrak{P})\,t(Q)$,

2) $\ln|\mathcal{G}(\bar{\omega})| \leq -\frac{1}{2\sigma} X + 8\, m^2\, t(\mathfrak{P})\, t(Q)$.

If $r = 1$, then the right-hand side of the last inequality is not negative.

Let $\mathfrak{P}$ be prime homogeneous ideal, satisfying inequalities (4) for sufficiently large S, and δ be a small positive number. Let us define number T by the equality

$$T^{m+\delta} = \min\left(S^{\frac{\tau}{\tau-r}},\ \frac{1}{2}\ln\frac{1}{\rho}\right). \tag{5}$$

It may be proved with the help of analytical construction from the Gelfonds method, that there exist homogeneous polynomial $Q \in \mathbb{Z}[X_o, \ldots, X_m]$ satisfying inequalities

$$-T^{m+\delta} \leq \ln\left(|Q(\bar{\omega})| \cdot |\bar{\omega}|^{-\deg Q}\right) \leq -T^{m-\delta},\quad t(Q) \leq T^{p+q+\delta}. \tag{6}$$

This construction uses, following Philippon, function on many complex variables, "small value lemma" for exponential polynomials, due to Tijdeman, and interpolation formula in $\mathbb{C}^n$.

It is easy to prove, that in proposition 2 with $X = S^{\frac{\tau}{\tau-r}}$ inequalities $1 \leq \sigma \leq T^{2\delta}$ hold. From the inductive hypothesis and proposition 2 we find

$$\ln|\mathcal{G}(\bar{\omega})| \geq -\mu_{r-1}\, t(\mathcal{G})^{\frac{\tau}{\tau-r+1}} \geq -\mu_{r-1}(2m^2 S\, T^{p+q+\delta})^{\frac{\tau}{\tau-r+1}},$$

$$\ln|\mathcal{G}(\bar{\omega})| \leq -\frac{1}{2} T^{-2\delta} S^{\frac{\tau}{\tau-r}} + 8m^2 S\, T^{p+q+\delta}$$

These estimates lead for sufficiently large S to the inequality

$$S^{\frac{\tau}{\tau-r}} < T^{m+\delta},$$

which contradicts the equality (5).

If we could replace T^{δ} in the inequalities (6) by some power of $\ln T$, we should probably obtain the bound $[\frac{d+1}{2}]$ for the number of algebraically independent numbers among (2). The proof of the algebraical independence of these numbers demand the construction of the polynomials Q for which

$$\ln|Q(\bar{\omega})| \sim -\gamma\, t(Q)^{\varkappa}, \qquad \varkappa > d-1.$$

This method may be used for to prove some estimates for the orders of zeros of polynomials in a solution of a system of differential equations.

THEOREM 5. Suppose that $\xi_1, \ldots, \xi_q$ are different complex numbers, the functions $f_1(z), \ldots, f_m(z)$ are algebraically independent over $\mathbb{C}(z)$, and constitute solution of the system of differential equations

$$y_j' = q_{jo} + \sum_{i=1}^{m} q_{ji}\, y_i,\quad j = 1, \ldots, m,\qquad q_{ji} \in \mathbb{C}(z),$$

and are analytic at $\xi_1, \dots, \xi_q$. Then for any polynomial $P \in \mathbb{C}[z, X_1, \dots, X_m]$, $P \not\equiv 0$,

$$\sum_{j=1}^{q} \operatorname{ord}_{\xi_j} P(z, f_1(z), \dots, f_m(z)) \le \gamma_5 (\deg_z P + q)(\deg_X P)^m$$

holds, where $\gamma_5 = \gamma_5(f_i) > 0$.

THEOREM 6. Let $\xi_1, \dots, \xi_q$ be different complex numbers, the functions $f_o(z), \dots, f_m(z)$ constitute a solution of the system of differential equations

$$y_j' = R_j(y_o, \dots, y_m), \qquad j = 0, 1, \dots, m,$$

where $R_j \in \mathbb{C}[y_o, \dots, y_m]$ are homogeneous polynomials. Suppose that these functions are analytic at $\xi_1, \dots, \xi_q$, all vectors $\left(f_o(\xi_\ell), \dots, f_m(\xi_\ell)\right)$, $1 \le \ell \le q$, are different, and maximal number of homogeneous algebraically independent over $\mathbb{C}$ among $f_o(z), \dots, f_m(z)$ equals to $K+1$. Then there exists a constant $\gamma_6 = \gamma_6(f_i) > 0$ such that for any homogeneous polynomial $P \in \mathbb{C}[y_o, \dots, y_m]$, $P(f_o, \dots, f_m) \not\equiv 0$, inequality

$$\sum_{j=1}^{q} \operatorname{ord}_{\xi_j} P(\bar{f}) \le \gamma_6^K \left(\mathcal{D}^K + \sum_{j=1}^{K-1} a_j \mathcal{D}^j \right)$$

holds, where $\mathcal{D} = \deg P$ and a_j is a maximal number of points $\bar{f}(\xi_\ell)$, $1 \le \ell \le q$, lying on an irreducible variety in $\mathbb{P}^m$ of dimension $K-j$ with degree at most $\gamma_6^{j-1} \mathcal{D}^j$.

For $K = 3$ this statement was proved by D. Brownawell [8] (see the bibliography of this paper too).

Any unmixed homogeneous for $X_o, \dots, X_m$ ideal $\mathfrak{J}$ in $\mathbb{C}[z, X_o, \dots, X_m]$ may be characterized by numbers $N(\mathfrak{J})$, $B(\mathfrak{J})$, $\operatorname{ord}_\xi \mathfrak{J}(\bar{f})$, which are analogous to $\deg_X P$, $\deg_z P$, $\operatorname{ord}_\xi P(\bar{f})$ for $P \in \mathbb{C}[z, X_o, \dots, X_m]$. The following assertion, generalizing theorem 5 is true.

THEOREM 7. Let $\xi_1, \dots, \xi_q$ be different complex numbers, the functions $f_o(z), \dots, f_m(z)$ be not connected by any homogeneous algebraic equation over $\mathbb{C}(z)$, constitute a solution of the system of differential equations

$$y_j' = \sum_{i=0}^{m} q_{ji} y_i, \quad j = 0, 1, \dots, m, \quad q_{ji} \in \mathbb{C}(z) \tag{7}$$

and be analytic at $\xi_1, \dots, \xi_q$. Let $\mathfrak{J}$ be an ideal of the ring $\mathbb{C}[z, X_o, \dots, X_m]$, which is homogeneous in the variables $X_o, \dots, X_m$, $r = m+1-h(\mathfrak{J}) \ge 1$. Then

$$\sum_{j=1}^{q} \operatorname{ord}_{\xi_j} \mathfrak{J}(\bar{f}) \le (6m)^{2m^2 r} B(\mathfrak{J}) N(\mathfrak{J})^{\frac{r}{m-r+1}} + \gamma_7^{3m(m+r)} q\, N(\mathfrak{J})^{\frac{m}{m-r+1}},$$

where $\gamma_7 = \gamma_7(f_i) > 0$.

This theorem is proved by induction on r from 1 to m. Theorem 5 follows from theorem 7 for $r = m$. The common scheme of induction is analogous to the one in the case of numbers. But instead of the analytical construction of polynomial Q, as it is made in the proof of the theorem 4, the next lemma is used.

LEMMA. Let $\mathfrak{P}$ be a prime ideal of the ring $\mathbb{C}[z, X_o, \dots, X_m]$, which is homogeneous in the variables $X_o, \dots, X_m$, $\mathfrak{P} \cap \mathbb{C}(z) = (0)$, $r = m+1-h(\mathfrak{P}) \geq 1$ and

$$\sum_{j=1}^{q} \operatorname{ord}_{\xi_j} \mathfrak{P}(\bar{f}) > B(\mathfrak{P}) + \gamma_8 N(\mathfrak{P}),$$

where $\gamma_8 = \gamma_8(f_i) > 0$. Then there exist homogeneous in the variables $X_o, \dots, X_m$ polynomials $R \in \mathfrak{P}$, $Q \notin \mathfrak{P}$ such that $Q(\bar{f}) = t(z) \frac{d}{dz} R(\bar{f})$, when $t(z)$ is the least common denominator of the coefficients q_{ji} in (7), and

$$\deg_X Q \leq 3(6m)^m N(\mathfrak{P})^{\overline{m-r+1}^{1}},$$

$$\deg_z Q \leq 3(6m)^m B(\mathfrak{P}) N(\mathfrak{P})^{-\frac{m'-r}{m-r+1}} + \gamma_8,$$

where $\gamma_8 = \gamma_8(f_i) > 0$.

The proof of lemma is connected with the theorem 2 from [6]. In order to establish theorem 5 and 6 we improve the method of [5], where result not so strong as theorem 7 was proved. In the case of numbers this method was stated for the first time in [10], [6], [7], where we proved that at least $[\log_2 (d+1)]$ numbers among (2) are algebraically independent over $\mathbb{Q}$.

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