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Central limit theorems for additive functionals of reversible Markov chains and applications

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Claude Kipnis and S. R. S. Varadhan

1. Introduction

This article deals with applications of a central limit theorem for general additive functions of reversible Markov chains. Although the results are valid in the context of continuous time processes and we have in mind several possible applications we will concentrate here on the simple case of a Markov chain and a single application to a random walk among random scatterers. We will omit the details of the proof which will appear elsewhere.

2. The Problem

Let Z^d be the lattice of d-dimensional vectors with integral components. Let Ω be the space of all possible subsets of Z^d. We have a translation invariant measure P_0 on Ω. We will assume that P_0 is ergodic with respect to translations on Z^d and has in addition some mild regularity assumptions. A point $\omega \in \Omega$ will be thought of as fixed random assignment of scatterers among the sites in Z^d. Translation invariance of P_0 means that the random scatterers form a medium that is statistically homogeneous.

We consider a particle that starts from the origin at time 0 and travels through Z^d in time with both time and space steps being discrete. Initially we pick at random with equal probability one of 2d possible coordinate directions and the particle moves in that direction taking one step per unit time. It does not change its course until it arrives at a site that is the location of a scatterer. When that happens it picks a new direction independently of everything and with equal probabilities from among the same set of 2d directions.

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The particle wanders in this manner through Z^d changing its direction every now and then when it comes across a scattering location. If X_n is the position of the particle after n time steps we want to show that the distribution of $\frac{1}{\sqrt{n}} X_n$ is approximately Gaussian. We would also like to show that $\frac{1}{\sqrt{n}} X_{[nt]}$ converges to a Brownian motion (with some covariance matrix) in distribution.

The fact that the scattering sites are fixed forever introduces long term correlations and standard methods of central limit theorems for dependent random variables do not apply directly. The method we will outline is relatively straightforward and applies directly without any need for hard computations.

3. Additive Functionals of Markov Chains

Let $(X\ \Sigma)$ be a measurable space and $q(x,dy)$ a transition probability on (X, Σ). We assume that $q(x,dy)$ is reversible with respect to a probability measure λ on (X,Σ) and that the stationary Markov chain P_λ with λ as initial distribution is ergodic. Let $\phi(x)$ be a function on X such that

$$\int \phi(x)\ \lambda(dx) = 0 \tag{3.1}$$

and

$$\int \phi^2(x)\ \lambda(dx) < \infty \tag{3.2}$$

Let $X_1, X_2, \ldots, X_n, \ldots$ be the Markov chain with distribution P_λ. We are interested in proving the central limit theorem for

$$Y_n = \sum_{j=1}^{n} \phi(X_j)\ . \tag{3.3}$$

If we denote by $\mathcal{F}_j$ the σ-field generated by $X_1, X_2, \ldots, X_j$, then we construct $\mathcal{F}_j$ measurable random variables ξ_j such that

$$E[\xi_j \mid \mathcal{F}_{j-1}] = 0 \quad \text{a.e. } P_\lambda \tag{3.4}$$

and if

$$Z_n = \sum_{j=1}^{n} \xi_j \tag{3.5}$$

then

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sup_{1\leq j\leq n} |Z_j - Y_j| = 0 \tag{3.6}$$

in probability with respect to P_λ. Moreover ξ_j vary covariantly with time translations. In particular Z_n is a homogeneous additive functional which is a Martingale. Standard techniques see for instance [1] apply to this case. The problem then is reduced to proving the following theorem:

Theorem 3.1. Suppose $\phi(x)$ satisfies

$$\int \phi(x)\, d\lambda(x) = 0$$

$$\int \phi^2(x)\, d\lambda(x) < \infty$$

and for all $\psi \in L_2(\lambda)$, there is a constant C independent of ψ such that

$$\left| \int \phi(x)\, \psi(x)\, \lambda(dx) \right| \leq C\left[\iint [\psi(x) - \psi(y)]^2\, q(x,dy)\, \lambda(dx)\right]^{1/2}. \tag{3.7}$$

Then there exists a homogeneous additive functional Z_n satisfying (3.4), (3.5) and (3.6) such that

$$E^{P_\lambda}[\xi_j^2] < \infty$$

In particular the functional form of the central limit theorem is valid for Y_n.

Idea of Proof: Suppose $\phi(x)$ is of the form

$$\phi(x) = f(x) - (Qf)(x)$$

where

$$(Qf)(x) = \int f(y)\, q(x,dy)$$

for some bounded measurable function $f(\cdot)$. Then

$$\begin{aligned}\phi(X_j) &= f(X_j) - (Qf)(X_j)\\ &= f(X_j) - f(X_{j+1}) + f(X_{j+1}) - (Qf)(X_j)\\ &= \eta_j + \xi_{j+1}\,.\end{aligned}$$

Clearly

$$E[f(X_{j+1}) \mid F_j] = (Qf)(X_j)$$

and therefore

$$E[\xi_{j+1} \mid F_j] = 0$$

$$\sum_{j=1}^{n} \phi(X_j) - \sum_{j=1}^{n} \xi_j = f(X_{n+1}) - f(X_1) - \xi_1 + \xi_{n+1}$$

and (3.6) follows from the boundedness of $f(\cdot)$. The general proof involves using (3.7) and reversibility to approximate ϕ by functions of the form $[I - Q]f$ and obtaining good estimates. This will be carried out elsewhere.

4. Reduction of the Problem

The problem we started out with can be reduced to the following situation. Let Ω_0 be the subset of Ω consisting of those subsets that contain the origin. For each $\omega \in \Omega_0$ and $v \in V$ where V is the set of $2d$ possible directions let $d(\omega,v)$ be the distance to the nearest scattering site from the origin in the direction v. Let us consider the transformation T_v in Ω_0 which corresponds to translating the set by $-d(\omega,v)v$ so that the new scattering site becomes the origin. We now have a family $\{T_v \ v \in V\}$ of measure preserving transformations of Ω_0. Although these transformations do not commute $T_v T_{-v}$ = identity for all $v \in V$. We take for our state space $X = \Omega_0 \times V$, and a Markov chain on X with transition probability

$$(\omega,v) \rightarrow (T_v\omega, v') \quad \text{with probability } \frac{1}{2d}$$

for each $v' \in V$. Let ℓ be a fixed vector in R^d. We need to prove the central limit theorem for

$$Y_n = \sum_{j=1}^{n} \psi(X_j)$$

where

$$\psi(x) = d(\omega,v)\langle v,\ell\rangle ,$$

and x is the pair (ω, v).

If we look at the induced process on Ω_0 alone it is also a Markov chain with transition probabilities

$$\omega \to T_v \omega \quad \text{with probability} \quad \frac{1}{2d}$$

for each $v \in V$. Moroever this chain is reversible relative to the measure $\bar{P}_0$ obtained by restricting P_0 to Ω_0 and normalizing.

Let us define $\phi(\omega)$ by

$$\phi(\omega) = \frac{1}{2d} \sum_{v \in V} d(\omega, v) \; \langle v, \ell \rangle \tag{4.1}$$

Then

$$\psi(X_{j+1}) = \psi(X_{j+1}) - \phi(\omega_j) + \phi(\omega_j)$$

It turns out that $\psi(X_{j+1}) - \phi(\omega_j)$ are Martingale differences and one uses the theorem of Section 3 to replace $\phi(\omega_j)$ by a Martingale difference. In order to complete the proof we need only prove the estimate (3.7) for $\phi(\omega)$.

Remark. The time has been mixed up. The real time has been changed to the number of scattering sites visited. But this is easily taken care of by using the ergodic theorem for the sum

$$\lim_{n \to \infty} \frac{1}{n} \sum d(\omega_j, v_j)$$

which affects the time scale in the end by a constant factor. At this point we need to assume that the Markov chain on $\Omega_0 \times V$ with invariant measure $\bar{P}_0 \times \pi$ where π is the uniform distribution on the 2d directions is ergodic.

5. The Estimate

Finally we obtain the estimate (3.7) for the function ϕ defined by (4.1).

$$E^{\bar{P}_0}[\phi(\omega)\psi(\omega)]$$

$$= E^{\bar{P}_0}[\psi(\omega) \cdot \frac{1}{2d} \sum_v d(\omega, v) \; \langle v, \ell \rangle]$$

$$= - E^{\bar{P}_0}[\psi(\omega) \cdot \frac{1}{2d} \sum_v d(\omega,-v) \ \langle v,\ell\rangle]$$

$$= \frac{1}{4d} E^{\bar{P}_0}[\psi(\omega) \sum_v (d(\omega,v) - d(\omega,-v)) \ \langle v,\ell\rangle]$$

$$= \frac{1}{4d} E^{\bar{P}_0}[\psi(\omega) \sum_v (d(\omega,v) - d(T_{-v}\omega,v)) \ \langle v,\ell\rangle]$$

$$= \frac{1}{4d} E^{\bar{P}_0}[\sum_v d(\omega,v) \ (\psi(\omega) - \psi(T_v\omega)) \ \langle v,\ell\rangle]$$

$$\leq C\left(E^{\bar{P}_0} \sum_v [\psi(T_v\omega) - \psi(\omega)]^2\right)^{1/2}$$

which is the estimate we need. We had to assume that

$$E^{\bar{P}_0} \sum_v [d(\omega,v)]^2 < \infty .$$

6. References

[1] Hall, P, and Heyde, C. C., Martingale Limit Theory and Its Applications, Academic Press, 1980.

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