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GEOMETRICAL INEQUALITIES AND MIXED VOLUMES IN THE LOCAL THEORY OF BANACH SPACES

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1. Introduction. 2. Background; Search for Euclidean sections of a convex body. 3. Background; mixed volumes and geometrical inequalities. 4. Euclidean decomposition of an arbitrary normed (finite dimensional) space. 5. Projections onto Euclidean sections. 6. Computation of mixed volumes through a Levy mean approach. 7. Problems.

1. INTRODUCTION

In this paper we are studying finite dimensional linear normed space (so called Local Theory of Banach spaces). Since every such space is uniquely defined by some central symmetric compact convex body, that is its unit ball, our investigation is actually about this geometrical object. Therefore, in this paper we advocate to combine known analytical and combinatorial approaches in Local Theory with a pure geometrical classical study of convex bodies and the deep theory of geometrical inequalities. This is the main purpose of the paper.

However we must remember that a unit ball as a geometrical object is defined only up to affine transforms and that questions which arise are related to a linear structure of a space. These problems usually involve a description of some standard simple subspaces and projections on them. In this short introduction I will recall some known results especially those related to the main stream of our discussion.

1.1 Let X and Y be n-dimensional normed linear spaces. The distance (Banach-Mazur distance) $d(X,Y) = \inf\{||T||\cdot||T^{-1}||$ over all linear isomorphism $T : X \to Y\}$. Obviously $d(X,Y) \geq 1$ while $d(X,Y) \leq 1+\varepsilon$ means that X and Y are isometrically close (we say $(1+\varepsilon)$-isomorphic). In the geometrical language this means that two unit balls $K(X) = \{x \in X : ||x|| \leq 1\}$ and $K(Y)$ may be put by an affine transform (say $\varphi: Y \to X$) in the same linear space (say X) in a position

$$K(X) \subset \varphi(K(Y)) \subset d(X,Y)K(X) \subset (1+\varepsilon)K(X).$$

So after such an affine transform - two geometrical bodies $K(X)$ and $K(Y)$ become close in a geometrical sense.

We introduce the family of n-dimensional spaces which plays a special role in Local Theory, the so called ℓ_p-spaces (for $1 \leq p \leq \infty$).

ℓ_p^n is a linear n-dimensional space X with the norm for $p<\infty: ||(a_i)_{i=1}^n||_p = (\sum_{i=1}^n |a_i|^p)^{1/p}$ ($(a_i)_1^n = x \in X$) and for $p = \infty: ||(a_i)_1^n||_\infty = \max_{1\leq i\leq n} |a_i|$.

1.2 It is well known (F.John, 1948) that $d(X,\ell_2^n) \leq \sqrt{n}$ and if $d(X,\ell_2^n)$ is close to that extremal case, that is $d(X,\ell_2^n) \geq c\sqrt{n}$ (for some fixed constant $c>0$ and n large enough) then ([M.-W.]; see also [T]) X contains a subspace E, $\dim E = k$, such that i) $d(E,\ell_1^k) \leq 1+\varepsilon$ and ii) $k \geq (\ell n\ n)^{\alpha(c;\varepsilon)}$ where $\alpha(c;\varepsilon) > 0$ is some number depending only on c and $\varepsilon > 0$.

1.3 DVORETZKY THEOREM (real case [D]; a new proof which covers also the complex case [M]).

a) *For any* $\varepsilon > 0$ *and integer* k *there exists* $N(k;\varepsilon)$ *such that for each integer* $n \geq N(k;\varepsilon)$ *every n-dimensional normed space* X_n *contains a subspace* $E_k (\dim E_k = k)$, $(1+\varepsilon)$ *-isomorphic to an Euclidean space.*

b) [M] *The above number* $N(k;\varepsilon)$ *is bounded by the number* $\exp\{c(\varepsilon)k\}$ *where* $c(\varepsilon) > 0$ *depends only on* $\varepsilon > 0$ *and this estimate is exact: for the space* ℓ_∞^n *the above number* $N(k;\varepsilon) \geq \exp(ck\ell n\ 1/\varepsilon)$ *for some absolute constant* c.

(We advise to pay attention to a geometrical interpretation of both results 1.2 and 1.3, which states an existence of special central symmetric sections of a symmetric compact body).

The logarithmic estimate in 1.3. being precise in a general case can be improved significantly in most cases. We give a few examples of such an improvement. Let X be an n-dimensional normed space and X* be its dual. We denote $d(X,\ell_2^n) = d_n$, $k = k(X;\varepsilon)$ being the largest integer such that X contains an $(1+\varepsilon)$-isomorphic copy of a k-dimensional Euclidean space ℓ_2^k. Standardly, by the same letter c we will denote different absolute constants.

1.4 THEOREM [M]: a) $k(X_n,\varepsilon) \geq c(\varepsilon)\ n/d_n^2$ *where* $c(\varepsilon) \geq c\varepsilon^2/\ell n\ 1/\varepsilon$.

b) Important remark : *If a family of spaces* $\{X_n\}$ *has a uniformly bounded distance from* $\ell_2^n (d_n \leq K)$, *then* $k(X_n,\varepsilon)$ *is proportional to* n*;* by this reason we shall estimate only $k(X_n,1)$ which will simply be defined as $k(X_n)$.

1.5 THEOREM [FLM]: a) $k(X_n)\cdot k(X_n^*) \geq c\ n^2/d_n^2 \geq c\cdot n$ (because, by 1.2, $d_n \leq \sqrt{n}$).

In comparison with 1.4 it means that

either $k(X_n)$ *or* $k(X_n^*)$ *is at least* cn/d_n.

b) *There exists a function* $k = k(n;\alpha)$ *where* k *and* n *are integers and* $\alpha > 0$ *and* $k(n;\alpha) \to \infty$ *if* $n \to \infty$ *and* $\alpha \to 0$ *such that for every* X_n *- n-dimensional*

normed space either $k(X_n) \geq cn^{\alpha}$ *or* X_n *contains a* 2-*isomorphic copy of* ℓ_∞^k for $k = k(n;\alpha)$.
Note that a function $k(n;\alpha)$ may be precisely estimated using [Al.-M.].

1.6 ℓ_p-Spaces a) *for* $1 \leq p < 2$: $k(\ell_p^n) \geq cn$ [FLM]

b) *moreover, the space* ℓ_1^n *contains for every* $1 < p \leq 2$ $(1+\varepsilon)$-*isomorphic copy of* ℓ_p^k *for* $k \geq c(\varepsilon)n$ *where* $c(\varepsilon)$ *depends only on* $\varepsilon > 0$([J.-Sch];[P_1]).

c) Krivine Theorem [K] (the special case). *Fixed* p, $1 \leq p \leq \infty$. *For every* $T>1$ $\varepsilon > 0$ *and an integer* k *there exists an integer* $N(T;k;\varepsilon)$ *such that for every* $n > N$ *and* n-*dimensional normed space* X_n *with* $d(X_n,\ell_p^n) \leq T$ *contains* $(1+\varepsilon)$-*isomorphic copy of* ℓ_p^k (Of course, the special cases $p = 1;2;\infty$ were known previously).

It is known that for $p \neq 2$ the inverse function $k(T;n;\varepsilon) \leq c_1 n^{c_2\varepsilon/T}$ (for some absolute constants c_1 and c_2) and it is proved that $k \geq c_1(\varepsilon;T)n^{c_2\cdot(\varepsilon/T)^p}$ [A.-M_2].

Previous examples 1.2. - 1.5. could lead to the conclusion that power-type estimates on dimensions of subspaces which we are looking for are typical while a proportional type (which we had at 1.4. b and 1.5. a and b) is an exceptional one. (There is no mention here of the Figiel-Johnson [F.-J.] result, although strongly related to the matter, because it uses special operator norms which are not introduced here). However we show in this paper, that there exists a non-trivial possibility to develop a proportional theory and this will be our second purpose.

2. BACKGROUND; SEARCH FOR EUCLIDEAN SECTIONS OF A CONVEX BODY.

We will describe briefly in that section two methods for extracting an almost Euclidean subspace from a finite dimensional normed space. The first approach is based on the so called measure concentration phenomena (for more information we refer to the original papers [M],[FLM], [A-M_1], [G-M] or to the book [M-Sch]]). One new application of this method is also given (see 2.4-2.5). The second approach involves a volume computation and was applied up to now to a few special but important examples (see [Ka]-[Sz] or the Lectures [Pel]).

We start with a few definitions which are used throughout the paper. We consider an n-dimensional linear space $\mathbb{R}^n$ with an inner product (x,y) which induces the Euclidean norm $|x|$ $(x \in \mathbb{R}^n, y \in \mathbb{R}^n)$. Denote $D = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $S(E) = \{x \in E: |x| = 1\}$ for every subspace $E \subsetneq \mathbb{R}^n$.

Let: $S^{n-1} = \{x \in R^n : |x| = 1\}$ be the *Euclidean unit sphere* in $\mathbb{R}^n$, $O(n)$ be *the othogonal group*, and $G_{n,k}$ $(1 \leq k < n)$ be the homogeneous spaces of k-dimensional subspaces of $\mathbb{R}^n$ (so called *Grassmann manifolds*).

We denote by the same letter μ the normalized Haar measure on each of the above

manifolds.
We consider also another norm $||\cdot||$ in the same underlying linear space $\mathbb{R}^n$ and let for every $x \in \mathbb{R}^n$

(2.1) $$\frac{1}{a}|x| \leq ||x|| \leq b|x|.$$

The norm $||\cdot||$ and the Euclidean norm $|\cdot|$ standardly define the dual norm $||x||^* = \sup\{\frac{|(x,y)|}{||y||} : y \in \mathbb{R}^n \setminus \{0\}\}$. It is clear that

$$\frac{1}{b}|x| \leq ||x||^* \leq a|x|.$$

2.1 LEMMA (P. Levy [L]). *Let* $A \subset S^{n-1}$ *and* $\mu(A) \geq \frac{1}{2}$. *Define* $A_\varepsilon = \{x \in S^{n-1}:$ *distance* $(x,A) \leq \varepsilon\}$ *for a fixed* $\varepsilon > 0$. *Then* $\mu(A_\varepsilon) \geq 1 - \exp(-\varepsilon^2(n-2)/2)$.

REMARK: It is clear that for each $x \in S^{n-1}$

$$\mu\{T \in O(n) : Tx \in A_\varepsilon\} = \mu(A_\varepsilon)$$

and therefore for every set $N = \{x_i\}_{i=1}^N \subset S^{n-1}$

$$\mu\{T \in O(n) : TN \subset A_\varepsilon\} \geq 1 - N\exp(-\varepsilon^2(n-2)/2).$$

So, if $N < \exp(\varepsilon^2(n-2)/4)$ then there exists a subset of $O(n)$ with measure exponentially close to 1 such that $TN \subset A_\varepsilon$ for any T from this subset. Choose now a δ-net ($\delta > 0$) of a fixed k-dimensional subspace (for $k = [\varepsilon^2 n/_{10\ell n}1/\delta]$) as the special set $N = \{x_i\}_1^N$. The above Remark and an estimate on the cardinality N of the δ-net prove the following lemma:

2.2 LEMMA ([M]). *Fix* $\varepsilon > 0$ *and* $k < [\varepsilon^2 n/_{10\ell n}1/\delta]$. *If* $A \subset S^{n-1}$ *and* $\mu(A) \geq \frac{1}{2}$ *then*

$$\mu\{E \in G_{n,k} : S(E) \subset A_{\varepsilon+\delta}\} \geq 1 - 2\exp(-\varepsilon^2 n/4)$$

Below, in sections 2.3-2.5, we take $\delta = \varepsilon$.

2.3 For every $A \subset S^{n-1}$ we define

$$B_{A,k} = \{E \in G_{n,k} : E \cap A \neq \emptyset\}.$$

and

$$I_{A,k} = \{E \in G_{n,k} : S(E) \subset A\}.$$

LEMMA: *Fix* $\varepsilon > 0$ *and* $k \leq [\varepsilon^2 n/_{10\ell n 1/\varepsilon}]$. *If* $\mu(B_{A,k}) > 2\exp(-\varepsilon^2 n/4)$ *then*

$$\mu(I_{A_{4\varepsilon},k}) \geq 1 - 2\exp(-\varepsilon^2 n/4).$$

PROOF: First we show that $\mu(A_{2\varepsilon}) \geq \frac{1}{2}$. Indeed, if not, then $\mu(A_{2\varepsilon}{}^c) > \frac{1}{2}$ and by

Lemma 2.2

$$\mu\{E \in G_{n,k} : S(E) \subset (A_{2\varepsilon})^c_{2\varepsilon} \subset A^c\} \geq 1 - 2\exp(-\varepsilon^2 n/4).$$

However this contradicts the estimate on $\mu(B_{A,k})$. Therefore $\mu(A_{2\varepsilon}) \geq \frac{1}{2}$ and again using Lemma 2.2 we conclude the proof.

2.4. We return to study the normed linear space $(\mathbb{R}^n, ||\cdot||)$ which is denoted by X. We also denote $X^* = (\mathbb{R}^n, ||\cdot||^*)$. For $E \in G_{n,k}$ we write $E \hookrightarrow X$ to indicate that we consider E as having the norm $||\cdot||$. Similarly, $E \hookrightarrow X^*$ or $E \hookrightarrow R^n$ mean that E has the norm $||\cdot||^*$ or $|\cdot|$ respectively. It is clear that $d(X, \ell_2^n) \leq d = a.b.$

THEOREM. *Fix* $\frac{1}{4} > \varepsilon > 0$ *and* $k \leq [\varepsilon^2 n/_{10\ell n 1/\varepsilon}]$. *For each decomposition* $d = a_1 \cdot b_1 (a_1 > 0)$ *there exists a subset* $\mathcal{E} \subset G_{n,k}$ *of a large measure*

$\mu(\mathcal{E}) \geq 1 - 2\exp(-\varepsilon^2 n/4)$ *such that*

either for each $E \in \mathcal{E}$

$$d(E \hookrightarrow X^*, E \hookrightarrow \mathbb{R}^n) \leq \frac{1}{1-4\varepsilon} a_1$$

or for each $E \in \mathcal{E}$

$$d(E \hookrightarrow X; E \hookrightarrow \mathbb{R}^n) \leq b_1.$$

It is sufficient, of course, to prove the theorem with the original $a_1 = a$ and $b_1 = b$. We prove first the following lemma:

LEMMA. *If* $x \in S^{n-1}$ (e.i. $|x| = 1$) *and* $||x||^* < (1 - 4\varepsilon)$ *then for every* $z \in S^{n-1}$ *such that* $|z - x| < 4\varepsilon$ *we have* $||z|| \geq 1$.

PROOF is obvious: $||z||(1-4\varepsilon) > ||x||^* ||z|| \geq |(x,z)| = |(x,x) + (x,z-x)| \geq 1-4\varepsilon$. ■
Return to the proof of the theorem. Let $A_* = \{x \in S^{n-1} : ||x||^* \geq 1 - 4\varepsilon\}$. Then by the preceding Lemma $(A_*^c)_{4\varepsilon} \subseteq A = \{z \in S^{n-1} : ||z|| \geq 1\}$. Using the notation of 2.3 we have

$$\text{either} \quad \mu(I_{A_*,k} \subset G_{n,k}) \geq 1 - 2\exp(-\varepsilon^2 n/4)$$

$$\text{or} \quad \mu(B_{A_*^c,k}) \geq 2\exp(-\varepsilon^2 n/4)$$

(these two sets are just complemented in $G_{n,k}$). In the first case we have found the large subset $I_{A_*,k} \subset G_{n,k}$ such that $(1-4\varepsilon)|x| \leq ||x||^* \leq a|x|$ for every $E \in I_{A_*,k}$ and every $x \in E$. It means that we may choose $\mathcal{E} = I_{A_*,k}$. In the second case, by Lemma 2.3 we have the set $I_{(A_*^c)_{4\varepsilon},k} \subset I_{A,k}$ such that $\mu(I_{A,k} \subset G_{n,k}) \geq 1 - 2\exp(-\varepsilon^2 n/4)$ and for every $E \in I_{A,k}$ and $x \in E$

$$|x| \leq ||x|| \leq b|x| \quad ■$$

We will mostly use this theorem with $\varepsilon = \frac{1}{8}$ and $k = [\lambda_0 n]$ for $\lambda_0 = 1/1300$.

2.5 COROLLARIES: a) *Let* $d(X, \ell_2^n) = d$. *Then for* $k = [\lambda_0 n] (\lambda_0 = 1/1300)$ *either* X *contains a k-dimensional subspace* E, $d(E, \ell_2^k) \leq \sqrt{d}$ *or* X^* *contains a k-dimensional subspace* F, $d(F, \ell_2^k) < 2\sqrt{d}$.

(This corollary shows that 1.5.a follows from 1.4 ; Moreover, it gives an interpretation of 1.5.a. in terms of a "proportional" theory : k in 2.5.a is proportional to n).

b) *Let* $d(X, \ell_2^n) = d_n$. *If there exists* $\alpha > 0$ *such that every subspace* $E \hookrightarrow X$, $\dim E = k \geq \lambda_0 n$, *satisfies* $d(E, \ell_2^k) \geq \alpha d_n$, *then there exists a subspace* $F \hookrightarrow X^*$, $k = \dim F \geq \lambda_0 n$, *such that* $d(F, \ell_2^k) \leq 2/\alpha$.
(It is the case, for example, for ℓ_q^n, $\infty > q > 2$, and it gives again 1.6.a).

c) STATEMENT: *For each* $\alpha > 0$ *there exists* $\lambda > 0$ *such that every n-dimensional normed space* X *with* $d(X, \ell_2^n) = d$ *contains a subspace* $E \hookrightarrow X$ *and* E^* *contains a subspace* $F \hookrightarrow E^*$ *such that* $k = \dim F \geq \lambda n$ *and* $d(F, \ell_2^k) \leq 2d^\alpha (\leq 2n^{\alpha/2})$.

To prove the statement, we apply Theorem 2.4 (or 2.5.a) repeatedly a number of times. (A much stronger result is proved in Section 4).

2.6 Let $f(x)$ be a real valued function on $S^{n-1}: f(x) \in C(S^{n-1})$, and $w_f(\varepsilon) = \sup\{|f(x) - f(y)| : |x-y| \leq \varepsilon\}$ be the modulus continuity of $f(x)$. For any $\ell \in \mathbb{R}$ define $A_\ell^+ = \{x \in S^{n-1} : f(x) \geq \ell\}$, $A_\ell^- = \{x \in S^{n-1} : f(x) \leq \ell\}$, and $A_\ell = \{x \in S^{n-1} : f(x) = \ell\}$ We say that L_f is *the Levy mean* of $f(x)$ if $\mu(A_{L_f}^+) \geq \frac{1}{2}$ and $\mu(A_{L_f}^-) \geq \frac{1}{2}$. Applying 2.1-2.2 to the intersection of the ε-neighbourhoods of A_{Lf}^+ and A_{Lf}^- we have:

LEMMA: *Fix* $\varepsilon > 0$ *and* $\delta > 0$. *For each* $k \leq \frac{\varepsilon^2}{10 \ln 1/\delta} n$ *there exists a k-dimensional subspace* $E \hookrightarrow \mathbb{R}^n$ *such that*

$$|f(x) - L_f| < w_f(\varepsilon) \tag{2.2}$$

for x *in some* δ*-net of* $S(E)$. *It means, in particular, that* $|f(x) - L_f| < w_f(\varepsilon + \delta)$ *for every* $x \in S(E)$.

2.7 However, ε and δ in 2.6, Lemma, differently influence an estimate on k and it becomes important when we apply this Lemma to the function $r(x) = ||x||$. In this case it follows from (2.1) that $w_r(\varepsilon) \leq b\varepsilon$. Take ε so small that $b\varepsilon \leq \varepsilon_0 \cdot L_r$ for some small but fixed ε_0 (! remember that b/L_r usually depends on n and may be very large). It is possible to show that there exist $\kappa(\delta; \varepsilon_0)$ such that

a) (2.2) implies $|r(x) - L_r| \lesssim \kappa(\delta;\varepsilon_0)L_r$ for every $x \in S(E)$

b) $\kappa(\delta;\varepsilon_0) \to 0$ ($\delta \to 0$ and $\varepsilon_0 \to 0$).

By this reasoning we have proved the following statement.

STATEMENT ([M]; see also [FLM]). *Let a norm* $||\cdot||$ *be given in* $\mathbb{R}^n$, *and let it be connected with the Euclidean norm* $|\cdot|$ *by the inequalities* (2.1) (i.e. $a^{-1}|x| \leq ||x|| \leq b|x|$ *for every* $x \in \mathbb{R}^n$). *Let* L_r *be the Levy mean of the function* $r(x) = ||x||$ *and let a number* $\varepsilon > 0$ *be given. Then for each integer* $k \leq c\frac{\varepsilon^2}{\ell n 1/\varepsilon} \cdot n(\frac{L_r}{b})^2$ *(where* c *is some absolute constant) there exists a* k*-dimensional subspace* E *of* $\mathbb{R}^n$ *such that*

$$(2.3) \qquad (1-\varepsilon)L_r|x| \leq ||x|| \leq (1+\varepsilon)L_r|x|$$

for every $x \in E$. Moreover, subspaces which don't satisfy the above inequalities (2.3) have a measure at most $\exp\ [-\frac{\varepsilon^2 n}{4} \cdot (L_r/b)^2]$.

To use this Statement in order to estimate the dimension k of an "almost" Euclidean section of $X = (R^n, ||\cdot||)$ one has to estimate L_r/b from below. It is much easier to deal with

$$M_r = \int_{x \in S^{n-1}} ||x|| d\mu(x)$$

instead of L_r. The following remark is useful for this purpose.

REMARK [FLM]. *If* $b \leq \sqrt{n}$ *then there exists an absolute constant* C *such that* $|L_r - M_r| \leq C$.

Results 1.3 - 1.6a were obtained by estimating L_r/b or M_r/b from below. In [FLM] this estimate was connected with the so called "cotype" - condition. We pass now to a different way of estimating this quantity through a volume ratio.

2.8 FINITE VOLUME RATIO . Let Vol_n be the usual (n-dimensional) Lebesgue measure on $\mathbb{R}^n$ normalized (for example) so that the induced measure on S^{n-1} coincides with $\mu(x)$. It means that $\mathrm{Vol}_n D = \frac{1}{n}\mu(S^{n-1}) = 1/n$. Let $K = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ denote the unit ball of $X = (\mathbb{R}^n, ||\cdot||)$. Assume that $L_r = 1$. Then

$$\mathrm{Vol}\ K \cap D > \frac{1}{2} \mathrm{Vol}\ D \quad \text{(and, also, } \mathrm{Vol}[K \cap 3^{1/n}D] < 2\ \mathrm{Vol}\ D)$$

The following lemma is an easy consequence of this inequality.

LEMMA: *Let* $\mathrm{Vol}\ K \leq C^n\ \mathrm{Vol}\ D$. *Then* $L_r \geq 1/(2^{1/n}C)$.

Therefore, the Statement 2.7 implies:

Let $||x|| < |x|$ for every $x \in \mathbb{R}^n$ (i.e. $b = 1$) and $\text{Vol } K \leq C^n \text{Vol } D$ (we say, following [Sz], that X has a *finite volume ratio* - f.v.r.). Then for each $\varepsilon > 0$ there exists $\lambda(\varepsilon) > 0$ depending on ε only such that for each integer $k < \lambda(\varepsilon)n/C^2$ X contains an $(1+\varepsilon)$ - isomorphic copy of ℓ_2^k.
However in the case of f.v.r. a much stronger result is true: X *contains a* k-*dimensional* $\varphi(C,\lambda)$-*isomorphic copy of* ℓ_2^k *even for* $k \approx \lambda n$ *and* λ, *say, is equal to* 2/3 *(or any other number* <1). The above function $\varphi(C,\lambda)$ depends only on C and $\lambda < 1$. It was first observed by Kashin [Ka] for ℓ_1^n(ℓ_1^n has a f.v.r.) and later by Szarek [Sz] in a general case. We will sketch Szarek's proof with some minor additional information as it will be used in Section 4. The proof of the following three Lemmas may be found, e.g., in [Pe1], Lecture 1.

2.9 Assume $||x|| \leq |x|$ for every $x \in \mathbb{R}^n$ and that
$\text{v.r. } K \overset{\text{def.}}{=} (\text{Vol } K/\text{Vol } D)^{1/n} \leq A$.

LEMMA 1.: *Let* $Z_\rho = \{x \in S^{n-1} : ||x|| \leq \rho\}$. *Then*
$\mu(Z_\rho) \leq (A\rho)^n$.

LEMMA 2.: *For each integer* $k < n$ *and a Borel set* $B \subset S^{n-1}$ *we define*
$E_k = \{\xi \in G_{n,k} : \mu_{k-1}(B \cap \xi) \leq T\mu_{n-1}(B)\}$ (we write μ_{k-1} to emphasize that we consider measure on $S(\xi) = S^{k-1}$). *Then* $\mu\{\xi \in G_{n,k} : \xi \in E_k\} > 1 - 1/T$ *and* $\{\xi \in G_{n,k} : \xi \in E_k$ *and* $\xi^\perp \in E_{n-k}\} > 1 - 2/T$ (here $\xi^\perp$ **means** the (n-k)-dimensional subspace which is orthogonal to ξ).

LEMMA 3: *If* E *is a* k-*dimensional subspace of* X *and for some* ρ, $0 < \rho < 1$, *and* $\alpha > 0$

$$\mu_{k-1}\{x \in S^{n-1} \cap E : ||x|| \leq \rho\} < \alpha^{k-1}$$

then for every $x \in E$

$$(\rho - \frac{\pi\alpha}{2})\ |x| \leq ||x|| \leq |x|.$$

THEOREM ([Sz]): *Let* $\text{v.r.} K \leq A$.

1) *Fix* $0 < \lambda < 1$ *and* $t > 1$. *Then for each* $k \leq \lambda n$ *there exists a subspace* E, $\dim E = k$, *such that*

$$\frac{1}{2} \cdot \frac{1}{(t\pi A)^\theta}\ |x| \leq ||x|| \leq |x|$$

where $\theta = 1/(1-\lambda)$. *The normalized Haar measure* μ *of such subspaces in* $G_{n,k}$ *is at least* $1 - 1/t^n$.

2) There exists a subspace E, $\dim E=[n/2]$, *such that*

$$\frac{1}{2(t\pi A)^2}|x| \le ||x|| \le |x|$$

for every $x \in E$ *and every* $x \in E^{\perp}$. *The measure* μ *of such subspaces in* $G_{n,[n/2]}$ *is at least* $1 - 2/t^n$.

PROOF of 1). Use Lemma 1 of this section for $\rho = 1/(t\pi A)^{\theta}$. Then $\mu(Z_\rho) \le (A\rho)^n$. By Lemma 2, for $T = t^n$ and $k = [\lambda n]+1$ there exists a k-dimensional subspace E (and a large measure of such subspaces as described in Lemma 2) such that

$$\mu_{k-1}(Z_\rho \cap E) \le (tA\rho)^n = \left[\frac{tA}{(t\pi A)^{\theta}}\right]^{(1/\lambda)(k-1)}$$

Define $\beta = [tA/(t\pi A)^{\theta}]^{1/\lambda} = \frac{1}{\pi^{\theta/\lambda}} \cdot \frac{1}{(tA)^{\theta}}$. Then, by Lemma 3

$$(\rho - \tfrac{\pi}{2}\beta)|x| \le ||x|| \le |x| .$$

and, trivially $\frac{1}{2} \cdot \frac{1}{(t\pi A)^{\theta}} \le \rho - \frac{\pi}{2}\beta$.

PROOF of 2) is the same. We use only the suitable part of Lemma 2.

§3. BACKGROUND;MIXED VOLUMES AND GEOMETRIC INEQUALITIES.

In this Section we recall a few classical definitions and results which are well known to experts in Geometric Inequalities but not yet known enough to experts in Local Theory of Banach Spaces. This is the reason why full proofs of the results used later are given. To the number of the well known classical books on this subject we will add two relatively recent ones: Santalo [S_1] and Burago and Zalgaller [B-Z].

3.1 THEOREM (Minkowski, 1911) *Let* K_i, $i = 1,\dots,m$, *be convex compacts in* $\mathbb{R}^n$, $\lambda_i \ge 0$, *and* $m \ge n$. *Then* $\mathrm{Vol}(\lambda_1 K_1 + \dots + \lambda_m K_m)$ = *(a homogeneous polynom of* λ_i *of degree* $\le n$ *written in the form:)* $\sum_{1\le i_j\le m} \lambda_{i_1}\cdot\lambda_{i_2}\cdots\lambda_{i_n} V(K_{i_1}\dots K_{i_n})$ *(and such that the coefficients* $V(K_{i_1},\dots,K_{i_n})$ *do not depend on the order* $i_1,\dots,i_n$).

We say that $V(K_{i_1},\dots,K_{i_n})$ is the <u>mixed volume</u> of $K_{i_1},\dots,K_{i_n}$ (!Some or all of the indices i_j may be repeated a number of times). By construction it is not dependent on the order of sets $\{K_i\}$.
We will not deal with this general form of the theorem and therefore we will not discuss it, but for a few remarks:

The mixed volume $V(K_1,\dots,K_n)$ is the ***Symmetric polylinear form*** with respect to set addition $(V(K_1' + K_1'', K_2,\dots,K_n) = V(K_1',K_2,\dots,K_n) + V(K_1'',K_2,\dots,K_n)$ where $A + B = \{x+y : x \in A \text{ and } y \in B\})$ and homothety $V(\alpha_1 K_1,K_2,\dots K_n) = |\alpha_1|\cdot V(K_1,\dots,K_n)$; $\mathrm{Vol}\, K = V(K,\dots,K)$; Therefore the mixed volume is, in a sense, the *polylinearization* of Vol K and **Vol** K *is the diagonal of that form*; the mixed volume is a *monotone function:* $A_1 \subset A_2$ implies $V(A_1,K_2,\dots,K_n) \le V(A_2,K_2,\dots,K_n)$; Consequently $V(K_1,\dots,K_n) \ge 0.$

3.2 Now we turn our attention to a special case of Minkowski's theorem which was already considered in 1840 by Steiner. Let D be an Euclidean unit ball, and K be a convex compact and $\rho \ge 0$. Define $V(\underbrace{K,\dots K}_{m}, \underbrace{D,\dots D}_{n-m}) = V_m(K)$. Then $V_0(K) = \mathrm{Vol}\, D$ and $V_n(K) = \mathrm{Vol}\, K$.

STEINER'S FORMULA:

(St.) $$\mathrm{Vol}(K + \rho D) = \sum_{i=0}^{n} \binom{n}{i} V_{n-i}(K)\rho^i .$$

We will prove this formula along with the following well known and important interpretation of the mixed volume $V_m(K)$:

STATEMENT: *Let* $G_{n,m}$ *be the Grassmann manifold as in Section 2 and let* $\mu(\xi)$ *be the normalized Haar measure on* $G_{n,m}$. *Let* P_ξ *be an orthogonal projection onto* ξ, D_m *be the unit m-dimensional Euclidean ball, and* Vol_m *be m-dimensional Volume.*

(V_m) $$V_m(K) = \frac{\mathrm{Vol}\, D_n}{\mathrm{Vol}_m D_m} \int_{\xi\in G_{n,m}} \mathrm{Vol}_m(P_\xi K)\, d\mu(\xi).$$

(St.) and (V_m) will be proved by induction on dimension n. At first, we define functions $V_m(K)$ by the formula (V_m) and we prove (St.) with these numbers.

a) For $n=1$, (St) is trivial: length $(K+\rho D)$ = length(K) + 2ρ = $= V_1(K) + V_0(K)\rho$. $(V_0(K) = V(D_1) = 2)$.

b) If we will prove (St) + (V_m) with some coefficients $a_{n,i}$ instead of $\binom{n}{i}$ then immediately $a_{n,i} = \binom{n}{i}$ (take $K = D$). So, in our inductive proof we disregard coefficients independent of K and ρ.

c) The special case of (V_m): Cauchy formula (1841)

(C) Area of K $\overset{\text{def}}{=} S(K) = n\cdot V_{n-1}(K)$

(Recall: definition of $V_{n-1}(K)$ see at (V_m) for $m = n-1$).

PROOF: We prove the above formula first for K being any convex compact polytop (with, say, faces f_i and Area $f_i \equiv \mathrm{Vol}_{n-1} f_i = S_i$) and subsequently we obtain a general case by an approximation argument. Let ξ be an arbitrary $(n-1)$-dimensional subspace and θ_i be an angle between ξ and a face f_i. Then $\mathrm{Vol}_{n-1} P_\xi f_i = S_i |\cos\theta_i|$ and $\int \mathrm{Vol}_{n-1} P_\xi f_i \, d\nu(\xi) = S_i \int |\cos\theta_i| d\nu(\xi) = a_n S_i$ (where a_n depends only on n). Therefore $a_n S(K) = a_n \sum_{f_i} S_i = \int_{\xi \in G_{n,n-1}} (\Sigma \mathrm{Vol}_{n-1} P_\xi f_i) d\nu(\xi) =$ $= 2 \int_\xi \mathrm{Vol}_{n-1} P_\xi K \, d\nu$ and we have proved that $S(K) = c_n V_{n-1}(K)$. To compute the number c_n, take $K = D$.

d) Assume that (St) + (V_m) is proved for $n-1$. Let $\xi \in G_{n,n-1}$. Then

$$\mathrm{Vol}(P_\xi(K+\rho D)) = \sum_{i=0}^{n-1} \binom{n-1}{i} V_{n-1-i}(P_\xi K)\rho^i$$

(and $V_{n-1-i}(P_\xi K)$ is defined by (V_m) in $(n-1)$-dimensional space). Averaging over $\xi \in G_{n,n-1}$ gives (using (C) and definition (V_m) but now in the n-dimensional space)

$$S(K+\rho D) = \sum_0^{n-1} a_{n,i} V_{n-1-i}(K)\rho^i$$

for some numbers $a_{n,i}$ depending on n and i only. Integrating by ρ from 0 to r gives

$$\mathrm{Vol}(K+rD) - \mathrm{Vol}(K) = \sum_0^{n-1} \frac{a_{n,i}}{i+1} V_{n-(i+1)}(K) r^{i+1}$$

Change $i+1 \to i$ and pay attention that $V_n(K) = \mathrm{Vol}\,K$. This ends our proof of (St) (use b) to define coefficients $\binom{n}{i}$) and the description of $V_{n-i}(K)$ given by (V_m).

3.3 BRUNN-MINKOWSKI INEQUALITIES

The following family of inequalities generalizes the isoperimetric inequality for $\mathbb{R}^n$:

For each m, $n \geq m \geq 1$, and every compact sets A and B (not necessarily convex)

(Br.-M) $$V_m(A+B)^{1/m} \geq V_m(A)^{1/m} + V_m(B)^{1/m}$$

For $m=n$ we have

$$\mathrm{Vol}(A+B)^{1/n} \geq \mathrm{Vol}\,A^{1/n} + \mathrm{Vol}\,B^{1/n}$$

which implies the isoperimetric property of the Euclidean ball. Indeed, if we take $B = \rho D$, then

$$(\rho) \qquad \mathrm{Vol}(A+\rho D)^{1/n} \geq \mathrm{Vol}\ A^{1/n} + \rho(\mathrm{Vol}\ D)^{1/n}$$

Now, if Vol A = Vol D then $\mathrm{Vol}(A+\rho D) \geq (1+\rho)^n \mathrm{Vol}\ D$ and *inf* for Vol (A+ρD) is attained at A = D∎

Divide (ρ) by $(\mathrm{Vol}\ D)^{1/n}$, take a power n from both sides and use (St):

$$\sum_{i=0}^{n} \binom{n}{i} \frac{V_{n-i}(A)}{\mathrm{Vol}\ D} \rho^i \geq \left[\left(\frac{\mathrm{Vol}\ A}{\mathrm{Vol}\ D}\right)^{1/n} + \rho\right]^n = \sum_{i=0}^{n} \binom{n}{i} \left[\frac{\mathrm{Vol}\ A}{\mathrm{Vol}\ D}\right]^{\frac{n-i}{n}} \rho^i,$$

for every $\rho \geq 0$. Because 0-term and n-term are equal on both sides of the inequality, we obtain inequalities for 1-and (n-1)-terms:

$$\frac{V_{n-1}(A)}{\mathrm{Vol}\ D} \geq \left(\frac{\mathrm{Vol}\ A}{\mathrm{Vol}\ D}\right)^{\frac{n-1}{n}} \qquad \text{and}$$

$$(U) \qquad \frac{V_1(A)}{\mathrm{Vol}\ D} \geq \left(\frac{\mathrm{Vol}\ A}{\mathrm{Vol}\ D}\right)^{1/n}.$$

The second inequality is Urysohn inequality [U] and the first one brings us back to the isoperimetric one. Note here that both inequalities are the partial cases of the more general Alexandrov inequalities [A]

$$(A) \qquad \left(\frac{V_m(K)}{\mathrm{Vol}\ D}\right)^{1/m} \geq \left(\frac{V_j(K)}{\mathrm{Vol}\ D}\right)^{1/j} \quad \text{for each } 1 \leq m < j \leq n.$$

(All of them may be similarly obtained from the general case of (Br.-M.) using a generalization of (St.) for $V_m(K)$: $V_m(K+\rho D) = \sum_{i=0}^{m} \binom{m}{i} V_{m-i}(K)\rho^i$ which is an easy formal consequence of (St) - see $[S_1]$).

To complete a proof of Urysohn inequality (U), intensively used in this paper, we are now going to sketch a proof of (Br.-M.) for the case m=n.

By an approximation argument it is enough to show (Br.-M.) for such A and B which are finite unions of parallelograms with non-zero volumes with faces which are parallel to coordinate (pair-wise orthogonal) planes. We shall refer to this parallelograms further as "particles". The proof is by induction on a number k of particles in A and B together. Let k = 2 (i.e. A and B are just parallelograms with edges of the lengths $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$). It is worthwhile to apply parallel shifts of A and B such that each will have a corner at the origin; such shifts do not change the volumes of the bodies. Then the inequality which has to be proved is the following one

$$\prod_{i=1}^{n} (a_i+b_i)^{1/n} \geq (\Pi a_i)^{1/n} + (\Pi b_i)^{1/n}.$$

This last inequality is the consequence of the inequality between geometrical and arithmetical means:

$$\left(\Pi \frac{a_i}{a_i+b_i}\right)^{1/n} \leq \frac{1}{n} \sum_1^n \frac{a_i}{a_i+b_i} \quad \text{and} \quad \left(\Pi \frac{b_i}{a_i+b_i}\right)^{1/n} \leq \frac{1}{n} \sum_1^n \frac{b_i}{a_i+b_i} \quad .$$

By induction, we may assume now that A contains at least 2 particles. Then there exist a parallel shift of A and a coordinate plane P which devides A (after the shift) to two sets A' and A'' each of them having a strictly smaller number of particles than A. Let $\text{Vol } A' = \lambda \text{Vol } A$ and therefore $\text{Vol } A'' = (1-\lambda)\text{Vol } A$. We shift also B to such a position that the plane P devides B to the parts B' and B'' with the same volume proportion ($\text{Vol } B' = \lambda \text{Vol } B$ and $\text{Vol } B'' = (1-\lambda) \text{ Vol } B$). Then

$$\text{Vol}(A+B) \geq \text{Vol}(A'+B') + \text{Vol}(A''+B'') \geq \text{(by induction)}$$

$$[(\text{Vol } A')^{1/n} + (\text{Vol } B')^{1/n}]^n + [(\text{Vol } A'')^{1/n} + (\text{Vol } B'')^{1/n}]^n = \lambda[(\text{Vol } A)^{1/n} +$$

$$+ (\text{Vol } B)^{1/n}]^n + (1-\lambda)[(\text{Vol } A)^{1/n} + (\text{Vol } B)^{1/n}]^n = [(\text{Vol } A)^{1/n} + (\text{Vol } B)^{1/n}]^n \quad \blacksquare$$

3.4 Urysohn and Santalo Inequalities.

a) Let K be a unit ball of an n-dimensional normed space X and D be the Euclidean ball. We consider also in the same affine underlying space the dual norm $||x||^*$ with respect to the duality defined by D. Let K^* be the unit ball in this dual norm . The geometrical interpretation of the dual norm implies immediately that

$$V_1(K)/\text{Vol } D = \int_{S^{n-1}} ||x||^* d\mu(x) \overset{\text{def.}}{=} M_{r^*}$$

where $S^{n-1} = \partial D$ is the Euclidean sphere and $\mu(x)$ is the normalized Haar measure on S^{n-1}. So Urysohn inequality (U) may be rewritten in the following form:

$$\text{(U')} \qquad (\text{Vol } K/\text{Vol } D)^{1/n} \leq M_{r^*}.$$

b) Santalo Inequality

([S_2]; for a new and very nice proof see [R]):

$$\text{(S)} \qquad \operatorname{Vol} K \cdot \operatorname{Vol} K^* \leq (\operatorname{Vol} D)^2$$

Note that Urysohn inequality is an easy consequence from (S) (I learned this from Y. Gordon). Indeed:

$$\left(\frac{\operatorname{Vol} K^*}{\operatorname{Vol} D}\right)^{1/n} = \left[\int_{S^{n-1}} \left(\frac{1}{||x||^*}\right)^n d\mu(x)\right]^{1/n} \geq \int_{S^{n-1}} (||x||^*)^{-1} d\mu(x)$$

On the other side

$$M_{r^*} = \int_{S^{n-1}} ||x||^* d\mu(x) \geq 1/\int_{S^{n-1}} (||x||^*)^{-1} d\mu(x) \geq \left(\frac{\operatorname{Vol} D}{\operatorname{Vol} K^*}\right)^{1/n}.$$

Use now (S) to obtain (U).

4. EUCLIDEAN DECOMPOSITION OF AN ARBITRARY NORMED (FINITE DIMENSIONAL) SPACE.

The following Problems are investigated in this Section.

PROBLEM 1. *Is it true that for every* $\varepsilon > 0$ *there exists* $\lambda(\varepsilon) > 0$ *such that every* n-*dimensional normed space* X *contains an* m-*dimensional subspace* $E \subset X$ *such that* E^* *contains a* k-*dimensional subspace* $F \subset E^*$ *such that* $k \geq \lambda(\varepsilon)n$ *and* $d(F, \ell_2^k) \leq 1+\varepsilon$?

PROBLEM 2. *Is it true that there exists an absolute constant* C *such that every* n-*dimensional normed space* X *may be decomposed in a direct sum of four subspaces* $X = E_1 \dot{+} E_2 \dot{+} E_3 \dot{+} E_4$ *such that* $\dim E_i = n_i \geq [n/4]$ *for each* $i = 1,2,3,4$ *and for every* $i_1 \neq i_2$

$$d(E_{i_1} \dot{+} E_{i_2} / E_{i_1}, \ell_2^{n_{i_2}}) \leq C?$$

These problems have positive solutions for a large family of spaces and, in the general case, we will prove these results up to a logarithmic factor. Some variations of the problems will be discussed.

We use the same notations as in 2., 2.8, and 3.4. So, e.g., $X = (\mathbb{R}^n, ||\cdot||)$, K and K^* are the unit balls of X and X^* respectively, D is the Euclidean unit ball of $(\mathbb{R}^n, |\cdot|)$ and the orthogonality is understood with respect to the Euclidean norm $|\cdot|$ in $\mathbb{R}^n$.

4.1 THEOREM. *Let*

$$\operatorname{Vol}(\operatorname{Conv} K \cup D) \leq v_1^n \operatorname{Vol} D \quad \textit{and}$$

$$\operatorname{Vol}(\operatorname{Conv} K^* \cup D) \leq v_2^n \operatorname{Vol} D.$$

Then

i) *For each* α, $0 < \alpha < 1$, *there exist* $c(\alpha) > 0$ *and* $\lambda(\alpha) > 0$ *such that* X *contains an* m-*dimensional subspace* $E \hookrightarrow X$ *and* E^* *contains an* k-*dimensional subspace* $F \hookrightarrow E^*$ *such that*

$$k \geq \lambda(\alpha)n \quad \textit{and} \quad d(F,\ell_2^k) \leq C(\alpha)[v_1 \cdot v_2]^{\alpha}$$

In particular this means (using 1.4), that ***for every*** $\varepsilon > 0$ *and every* $\alpha > 0$ *there exists* $\lambda(\alpha;\varepsilon) > 0$ *such that* E^* *contains a* $(1+\varepsilon)$ *isomorphic copy of* ℓ_2^k *for* $k \geq \lambda(\alpha;\varepsilon)\frac{n}{(v_1 \cdot v_2)^{\alpha}}$.

ii) *For each* $0 < \lambda < 1$ *and* $0 < \mu < 1$ *there exists* $C(\lambda;\mu)$ *depending on* $\lambda < 1$ *and* $\mu < 1$ *only such that for every* $m = [\lambda n]$ *and* $k = [\mu m]$ *there exists an* m-*dimesnional subspace* $E \hookrightarrow X$ *and a* k-*dimensional subspace* $F \hookrightarrow E^*$ *such that*

$$d(F,\ell_2^k) \leq C(\lambda;\mu) \quad v_1^{1/(1-\lambda)(1-\mu)} \cdot v_2^{1/\lambda(1-\mu)}.$$

Moreover,

iii) *there exists an orthonormal (in the sense of* $(\mathbb{R}^n, |\cdot|)$*) basis* $e = \{e_i\}_{i=1}^n \subset X$ *and a constant* $C(\lambda,\mu)$ *depending on* $\lambda < 1$ *and* $\mu < 1$ *only such that for every* $A \subset [1,\dots,n]$, $m = |A| = [\lambda n]$ *and every subset* $B \subset A$, $k = |B| = [\mu m]$ *we have*

$$d(\operatorname{span}\{e_i\}_{i\in A}/\operatorname{span}\{e_i\}_{i\in A\setminus B}, \ell_2^k) \leq C(\lambda;\mu)v_1^{1/(1-\lambda)(1-\mu)}v_2^{1/\lambda(1-\mu)}$$

In other words

$$d(\operatorname{span}_{i\in B}\{e_i\} \hookrightarrow [\operatorname{span}_{i\in A}\{e_i\} \hookrightarrow X]^*;\ \ell_2^k) \leq C(\lambda;\mu)v_1^{1/(1-\lambda)(1-\mu)}v_2^{1/\lambda(1-\mu)}.$$

(recall, that, as in 2.4, we use the notation $E \hookrightarrow Y$ to indicate that the subspace E is considered in the Y-norm

iv) DECOMPOSITION: ***there exists*** *an orthogonal decomposition of* $X = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ *and* $\dim E_i = n_i \geq [n/4]$ *for every* $i = 1,2,3,4$, *such that for every* $i_1 \neq i_2$

$$d(E_{i_1} \oplus E_{i_2}/E_{i_1}, \ell_2^{n_{i_2}}) \leq cv_1^4 \cdot v_2^4$$

for some absolute constant c. (Of course, iv) is just a partial case of iii))

We prove first ii). Step a). Define $K_1 = \operatorname{Conv} K \cup D$ and consider the norm

$||\cdot||_1$ such that K_1 is the unit ball in this norm. Then for every $x \in \mathbb{R}^n$.

$$||x||_1 \leq ||x|| \quad \text{and} \quad ||x||_1 \leq |x|.$$

Using Theorem 2.9 for $||\cdot||_1$, t and λ we find (for $m=[\lambda n]$) an m-dimensional subspace $E \subsetneq R^n$ such that for every $x \in E$

$$\frac{1}{2} \cdot \frac{1}{(t\pi v_1)^\theta} |x| \leq ||x||_1 (\leq ||x||) \tag{4.1}$$

where $\theta = 1/(1-\lambda)$. Moreover, one has a large measure of such subspaces. Note that to prove ii) it is enough to take $t=2$. However we will keep t because it will be important in a proof of iii).

Step b). Consider $E^* = X^*/E^\perp$. It is clear that the unit ball $K(E^*)$ of E^* is the orthogonal projection of K^* onto $E \subsetneq R^n$. By (4.1), for every $x \in E^*$

$$||x||_E^* \leq 2(t\pi v_1)^\theta |x|. \; (||\cdot||_E^* \text{ denotes the dual norm to the norm on } E).$$

One has to use now the second volume condition. Then

$$\frac{\mathrm{Vol}(K^*+\rho D)}{\mathrm{Vol}\ D} \leq (1+\rho)^n \frac{\mathrm{Vol}(\mathrm{Conv}\ (K^* \cup D))}{\mathrm{Vol}\ D} \leq v_2^n (1+\rho)^n. \tag{4.2}$$

From the other side, using Steiner formula (St) from 3.2 we obtain

$$\frac{\mathrm{Vol}(K^*+\rho D)}{\mathrm{Vol}\ D} = \sum_{i=0}^{n} \binom{n}{i} \frac{V_{n-i}(K^*)}{\mathrm{Vol}\ D} \rho^i \geq \binom{n}{m} \frac{V_m(K^*)}{\mathrm{Vol}\ D} \rho^{n-m} . \tag{4.3}$$

Define $\left(\frac{V_m(K^*)}{\mathrm{Vol}\ D}\right)^{1/m} = B$. It follows from (4.2) and (4.3)

$$B^\lambda \rho^{1-\lambda} \leq V_2(1+\rho)$$

Take $\rho = 1$; we have proved

LEMMA: $$B = \left(\frac{V_m(K^*)}{\mathrm{Vol}\ D}\right)^{1/m} \leq 2^{1/\lambda} v_2^{1/\lambda}$$

Step c). Recall now the formula (V_m) from 3.2 and define $f(\xi)=\mathrm{Vol}_m(P_\xi K^*)/\mathrm{Vol}_m D_m$ where $\xi \in G_{n,m}$ and P_ξ is the orthogonal projection onto ξ. It is trivial that

LEMMA: *If* $\int f(\xi) d\mu(\xi) \leq a$ *then*

$$\mu_T = \mu\{\xi \in G_{n,m}: f(\xi) \leq Ta\} \geq 1 - 1/T.$$

So, taking $T = t^n$ and using Step b) and the previous Lemma, we obtain the

STATEMENT. *Let* $E = \{\xi \in G_{n,m} : (\mathrm{Vol}_m(P_\xi K^*)/\mathrm{Vol}_m D_m)^{1/m} \le (2tv_2)^{1/\lambda}\}$. *Then* $\mu(E) \ge 1 - 1/t^n$.

Step d) An intersection of E with the subspaces obtained in the Step a) gives a set $E_0 \subset G_{n,m}$ (and of a large measure) such that every $\xi \in E_0$ has the properties

$$||x||_\xi^* \le 2(t\pi v_1)^{1/1-\lambda}|x| \qquad \text{for every} \quad x \in \xi$$

and

$$(\mathrm{Vol}_m K(\xi)^* / \mathrm{Vol}_m D_m)^{1/m} \le (2tv_2)^{1/\lambda} .$$

After introducing the new Euclidean norm $|\cdot|_1 = 2(t\pi v_1)^{1/1-\lambda}|\cdot|$, we may apply Theorem 2.9 for a space E^* (for $E \in E_0$) (i.e. for $m = [\lambda n]$ instead of n and μ instead of λ). Again, it is enough in this part of the theorem to take $t = 2$, but for further purposes it will be necessary to use instead of t the number $t_1 = t^{1/\lambda}$. By that Theorem, there exists a subspace $F \hookrightarrow E^*$ such that $k = \dim F = [\mu[\lambda n]]$ and for a constant $C(t;\lambda;\mu)$ depending on $\lambda < 1$, $\mu < 1$ and $t > 1$ only

$$(4.4) \qquad d(F, \ell_2^k) \le C(t;\lambda;\mu) v_1^{1/(1-\lambda)(1-\mu)} \cdot v_2^{1/\lambda(1-\mu)} .$$

Take $t = 2$ to finish the proof.

PROOF i). Take in the preceeding proof $\lambda = \mu = \frac{1}{2}$; then, apply Statement 2.5.c to the space F.

PROOF iii). is a manipulation with large measures of subspaces obtained in the proof of ii). Fix integers $m < n$ and $k < m$. In the part ii) of the Theorem we were looking for a pair $(F;E)$ of subspaces of $\mathbb{R}^n$ such that $\dim E = m, F \hookrightarrow E$ and $\dim F = k$ and such that this pair **has** the described in ii) properties: if E is considered as the subspace of X (i.e. with the norm $||\cdot||$) and $F \hookrightarrow E^*$ (i.e. with the norm of E^*) then $d(F,\ell_2^k) \le$ a formula as described in (4.4); it will become clear later that now we have to put $t = 6$ in the $C(t;\lambda;\mu)$. All of such pairs form a subset $S_{m;k}$ of the manifold V which we describe below.

Let $\xi \in G_{n,m}$ (i.e. ξ is an m-dimensional subspace of $\mathbb{R}^n$). Denote $G_{m,k}(\xi)$ be the Grassmann manifold $G_{m,k}$ of all k-dimensional subspaces of ξ.

We consider the following manifold of pairs $V = \{(\eta;\xi)$, where $\xi \in G_{n,m}$ and $\eta \in G_{m,k}(\xi)\}$. It is clear that V is a homogeneous space under the action of $\mathcal{O}(n)$. Therefore, if μ_V denotes the normalized Haar measure on V then for every Borel set $S \subset V$ and for each fixed $(\eta_0;\xi_0) \in V$

$$\mu_V(S) = \mu\{T \in \mathcal{O}(n): T(\eta_0;\xi_0) \in S\}.$$

Further $d\mu_V = d\mu_{G_{n,m}} \cdot d\mu_{G_{m,k}}$.

We have to estimate a measure $\mu_V(S_{m;k})$. For this purpose we have to go once more through the proof of ii). At the step d) of that proof we built a set $\mathcal{E}_0 \subset G_{n,m}$ as an intersection of the sets from a) and c). So, its measure

$$\mu(\mathcal{E}_0) \geq 1 - 2/t^n.$$

For every $E \in \mathcal{E}_0$ we found (by Theorem 2.9) a set $\mathcal{F} \subset G_{m,k}(E)$ of subspaces $\{F \hookrightarrow E^*\}$ which satisfy the desired inequality (4.4). The measure of this set may be estimated again by Theorem 2.9.

$$\mu\{\mathcal{F} \subset G_{m,k}(E)\} \geq 1 - 1/t_1^m \simeq 1 - 1/t^n.$$

Therefore

$$\mu_V(S_{m,k}) \geq (1 - 2/t^n)(1 - 1/t^n) \geq 1 - 3/t^n. \tag{4.5}$$

Fix now any orthonormal basis $e_0 = \{e_{i0}\}_{i=1}^n$ in $(\mathbb{R}^n, |\cdot|)$ and define $E_A = \operatorname{span}\{e_{i,0}\}_{i\in A}$ for every $A \subset [1,\dots,n]$. For every pair $(B \subset A \subset [1,\dots,n])$, $|B| = k$ and $|A| = m$, denote $\mathcal{O}S_{A;B} = \{T \in \mathcal{O}(n) : (TE_B, TE_A) \in S_{m,k}\}$. Then $\mu(\mathcal{O}S_{A;B}) = \mu_V(S_{m,k})$ where $|A| = m$ and $|B| = k$. It is clear that the cardinality N of all pairs (B,A) such that $B \subset A \subset [1,\dots,n]$ is equal to $\Sigma_{m=0}^n \Sigma_{k=0}^m \binom{n}{m}\binom{m}{k} = \Sigma_0^n 2^m \binom{n}{m} < 4^n$. Therefore, using (4.5):

$$\mu\{\bigcap_{B;A} \mathcal{O}S_{A;B} \subset \mathcal{O}(n): \textbf{for every } A \subset [1,\dots,n] \text{ and every } B \subset A\} \geq$$
$$1 - 3\cdot 4^n/t^n > 0 \text{ for, say, } t = 6 \text{ and } n \geq 3.$$

(The case $n < 3$ is trivial). It means that there exists an orthogonal operator T_0 such that for every $B \subset A \subset [1,\dots,n]$ the space $T_0E_B \hookrightarrow (T_0E_A, \|\cdot\|)^*$, i.e. T_0E_B considered in the norm of $(T_0E_A, \|\cdot\|)^*$, satisfies (4.4) with $t = 6$, $\lambda = |A|/n$ and $\mu = |B|/|A|$. So, the basis $\{T_0e_{i,0} = e_i\}_{i=1}^n$ satisfies the conditions of iii).

4.2 Define

$$M_r = \int_{S^{n-1}} ||x|| d\mu(x) \quad \text{and} \quad M_{r^*} = \int_{S^{n-1}} ||x||^* d\mu(x).$$

LEMMA: $\mathrm{Vol}(\mathrm{Conv}\ K \cup D) \leq (1+M_{r^*})^n \mathrm{Vol}\ D$ *and similarly,*
$\mathrm{Vol}(\mathrm{Conv}\ K^* \cup D) \leq (1+M_r)^n \mathrm{Vol}\ D.$

PROOF: As in the Step a) of the proof of Theorem 4.1, we introduce the new norm $||\cdot||_1$ with its unit ball $K_1 = \mathrm{Conv}\ K \cup D$. Clearly $K_1^* = K^* \cap D$. Therefore

$$M_{r_1^*} = \int_{S^{n-1}} ||x||_1^* d\mu(x) = \int \max(||x||^*, 1) d\mu(x) \leq M_{r^*} + 1$$

By Urysohn inequality 3.4, (U'),

$$(\mathrm{Vol}\ K_1/\mathrm{Vol}\ D)^{1/n} \leq M_{r^*} + 1.$$

The second inequality is proved similarly.

COROLLARY. *If* $M_r \cdot M_{r^*} \leq T$ *then one may choose an Euclidean norm* $(\mathbb{R}^n, |\cdot|)$ *such that in Theorem* 4.1 $v_1 = 2$ *and* $v_2 = T + 1$ (normalize the original Euclidean norm such that $M_{r^*}=1$).

4.3 A few well known facts:

a) It was proved by [F.-T.] that *the Euclidean norm* $(\mathbb{R}^n, |\cdot|)$ *may be choosen in a such way that*

$$M_r \cdot M_{r^*} \leq C_1 ||\mathrm{Rad}_X||$$

where an absolute constant $C_1 \leq 27$ and $||\mathrm{Rad}_X||$ is the norm of the so called Rademacher projection of X (definition and properties see, e.g., [P_2])

b) This quantity $||\mathrm{Rad}_X||$ is very important in Local Theory and was investigated by Pisier [P_2]. He has proved that

i) *for every* n*-dimensional normed space* X

$$||\mathrm{Rad}_X|| \leq C_2 \ln(n+1)$$

for some absolute constant C_2*; and*

for each integer k *there exists a constant* $C_2(k)$ *such that for every normed space* X*:*

$||\mathrm{Rad}_X|| \geq C(k)$ *implies that* X *contains a 2-isomorphic copy of* ℓ_1^k.
We bring together all these facts in order to estimate v_1 and v_2 in Theorem 4.1.

4.4 COROLLARY 1.: *For every* n*-dimensional normed space* X *one may always assume that in Theorem 4.1* $v_1=2$ *and* $v_2 \leq C\ln(n+1)$ *for some absolute constant* C.

COROLLARY 2: *For each integer* k *there exists a constant* $C(k)$ *depending on* k *only, such that for every finite dimensional normed space* X *which does not contain a 2-isomorphic copy of* ℓ_1^k *all conclusions of Theorem 4.1 are satisfied for* $v_1=2$ *and* $v_2 = C(k)$.

(Proofs of both Corollaries follow immediately from 4.2, Corollary and 4.3)

4.5 The previous Corollary 2 indicates a large family of spaces which admit an exist nce of an Euclidean norm $|\cdot|$ such that the constants v_1 and v_2 defined by Theorem 4.1 are uniformly bounded. It is curious to observe that the family of ℓ_1^n-spaces which is the worst one in the sense of the condition of Corollary 2 any way has the uniformly bounded constant v_1 and v_2. Because of this, and Theorem 4.1, the following problem arises naturally.

PROBLEM: *Is it correct that there exist absolute constants* v_1 *and* v_2 *such that for every finite dimensional space* $X = (R^n, ||\cdot||)$ *there exists an Euclidean norm* $(R^n, |\cdot|)$ *such that the conditions of Theorem 4.1 are satisfied ?*

§5. PROJECTIONS ONTO EUCLIDEAN SECTIONS.

We use notations and definitions from the previous sections 1.1, 2., 2.7, 2.8 and 3.2. So, $X = (\mathbb{R}^n, ||\cdot||)$, $1/a\,|x| \leq ||x|| \leq b|x|$, $K = K(X)$, P_ξ is the orthogonal projection onto a subspace $\xi \in G_{n,m}$. Throughout this section we assume normalization of the Euclidean norm $|\cdot|$ such that $M_r = 1$.

5.1. Let, as in 3.2, $V_m(K)$ be the m-th mixed volume of K. Define $(V_m(K)/\mathrm{Vol}\ D)^{1/m} = A_m$.

THEOREM. *There exists an absolute constant* $c > 0$ *such that for every integer* $k \leq c\cdot\min\{n/(A_k b)^2,\ n(A_k/a)^2\}$ *there exists a subspace* $E_0 \in G_{n,k}$ *such that* $1/2|x| \leq ||x|| \leq 2|x|$ *for* $x \in E_0$ (i.e. $d(E_0,\ell_2^k) \leq 4$) *and* $||P_{E_0}: X \to E_0|| \leq 4A_k$ (!Remember the normalization $M_r = 1$ which implies, by the way, $A_m \geq 1$ for every $n \geq m \geq 1$).

REMARK. The main application which we mean for this Theorem is for a case of a uniformly (independent of n) bounded $A_k \leq$ Const. for some $k(n)$ satisfying the condition of the Theorem.

Of course, the existence of a set $A \subset G_{n,k}$ of a large measure (say $\mu(A) > \frac{1}{2}$) of k-dimensional subspaces satisfying the above inequalities between the norms $||\cdot||$ and $|\cdot|$ on them is a consequence of the statement from 2.7. The additional information about $||P_{E_0}||$ is obtained from the geometrical fact which is proved below that $P_{E_0} K \subset 2A_k(D \cap E_o)$.

5.2. A proof of this fact uses a concentration measure phenomena on the following manifold of pairs

$$V = \{\xi \in G_{n,k}, \quad x \in S(\xi)\}.$$

It is clear that V is a homogeneous space under the action of SO(n). It means V = SO(n)/G for some subgroup G. We identify every element $e \in SO(n)$ with an orthonormal basis $e = (e_1, \ldots, e_n)$. Then define $i: SO(n) \to V$ by the formula $ie = (\xi = \text{span}(e_1, \ldots e_k); e_1) \in V$. Introduce also a metric on $V: \rho_V((\xi,x);(\eta,y)) = \inf\{\rho_{SO(n)}(T_1;T_2): iT_1 = (\xi,x)$ and $iT_2 = (\eta,y)\}$; ($\rho_{SO(n)}$ is the standard bi-invariant Riemannian metric on SO(n); equivalently, $\rho_{SO(n)}$ can be taken as the Hilbert-Schmidt operator metric, which is uniformly equivalent to the other one). The normalized Haar measure μ_V on V may be defined by

$$\mu_V(A \subset V) = \mu_{SO(n)}\{i^{-1}A \subset SO(n)\}.$$

For every subset A of a metric space (M,ρ) let $A_\varepsilon = \{x \in M: \rho(x,A) \leq \varepsilon\}$. It is clear that $i^{-1}A_\varepsilon \supset (i^{-1}A)_\varepsilon$. Therefore if $\mu_V(A \subset V) \geq 1/2$ then $\mu(i^{-1}A \subset SO(n)) \geq 1/2$ and this implies

$$\mu_V(A_\varepsilon) \geq \mu_{SO(n)}((i^{-1}A)_\varepsilon) \geq 1 - \exp(-\varepsilon^2 n/8)$$

(the last inequality for SO(n) is known - see [G], [G.-M.]). So, the following lemma has been proved:

LEMMA. *For every closed subset* $A \subset V$ *with* $\mu(A) \geq 1/2$ *and for every* $\varepsilon > 0$

$$\mu(A_\varepsilon) \geq 1 - \exp(-\varepsilon^2 n/8).$$

5.3 STATEMENT, *Let* M *be a Borel subset of* V, $A \subset G_{n,k}$, $\mu(A) \geq 1/2$ *and let*

$\varepsilon > 0$, $\delta > 0$. *If for every* $\xi \in A \subset G_{n,k}$ *there exists* $(\xi,x) \in M$ *and* $k \leq \varepsilon^2 n/5\ln 1/\delta$ *then there exists* $\xi^0 \in G_{n,k}$ *such that* $M_{2(\varepsilon+\delta)} \supset (\xi^0,x)$ *for every* $x \in S(\xi^0)$.

PROOF. Use Lemma from 5.2 in a standard way. We will not repeat it because a similar reasoning has been used in 2.3. (Do not pay any attention to the numbers as 5 in this Statement or 10 in 2.3)

REMARK: Note that the proof of the Statement gives a large measure of such $\xi^0 \in G_{n,k}$ as claimed in the Statement.

5.4 Return to the proof of Theorem 5.1. Define $P_E K = K_E$ and let $||\cdot||_{K_E}$ be the norm in the subspace E with the unit ball K_E. Let

$$_tM = \{(\xi,x) \in V: \exists y \in K \text{ and } P_\xi y = \lambda x \quad \text{for } \lambda \geq t\}.$$

It is clear that $(E,x) \in {}_tM$ means that $||x||_{K_E} \leq 1/t$.

LEMMA. *Fix* $\kappa > 0$. *There exists a constant* $C(\kappa) > 0$ *depending on* κ *only such that if* $\varepsilon < C(\kappa)\cdot\min\{1/tb; t/a\}$, *then* $({}_tM)_\varepsilon \subset {}_{(1-\kappa)t}M$.

PROOF: Let $(E,x) \in {}_tM$; it means that there exists y, $||y|| = 1$, and $P_E y = z = \lambda x$ for $\lambda \geq t$. We divide the proof into two steps

a) Let $x' \in S(E)$ and $|x-x'| \leq \varepsilon_1$. It is clear that $\Big| ||x'||_{K_E} - ||x||_{K_E} \Big| \leq \leq ||x' - x||_{K_E} \leq b|x'-x| \leq b\varepsilon_1$. Therefore $||x'||_{K_E} \leq ||x||_{K_E} + b\varepsilon_1 \leq 1/t + b\varepsilon_1 \leq \leq (1+\kappa)/t$ for $\varepsilon_1 < \kappa/tb$. So we have proved

$$\{(E,x) \in {}_tM \textit{ and } |x-x'| \leq \kappa/tb\} \textit{ implies } (E,x') \in {}_{t/1+\kappa}M.$$

b) Take $E' \in G_{n,k}$ such that $\rho(E,E') \leq \varepsilon_2$ (which means that $|P_E - P_{E'}| \leq \varepsilon_2$). There exists $T \in SO(n)$ such that : $|T-Id| \leq \varepsilon_2$ and $T^{-1}P_{E'}T = P_E$. We want to compute $|P_{E'}y|$ (where y was defined above in terms of x; recall also that $P_E y = z$ and $|Tz| = |z| \geq t$).

$$(5.1) \qquad \Big| |P_{E'}y| - |z| \Big| \leq |P_{E'}y - TP_E y| = |P_{E'}(y-Ty)| \leq |y-Ty| \leq \varepsilon_2|y| \leq \varepsilon_2 a$$

(because $||y|| = 1$ implies $|y| \leq a$). Therefore $|P_{E'}y| \geq |z| - \varepsilon_2 a \geq t - \varepsilon_2 a$. Take $\varepsilon_2 \leq \kappa t/a$ and we obtain $|P_{E'}y| \geq (1-\kappa)t$. So

$$\{(E,x) \in {}_tM \textit{ and } \rho(E,E') \leq \varepsilon_2 \leq \kappa t/a\} \textit{ implies } (E', P_{E'}y/|P_{E'}y|) \in {}_{(1-\kappa)t}M.$$

(add also that, by (5.1) $\left|P_{E'}y/|P_{E'}y| - x\right| \leq 2\kappa$). The steps a) and b) imply Lemma in a trivial way.

5.5 To prove now Theorem 5.1 we use Lemma 5.4 for $t_0 = 2A_k$ and $\kappa = 1/4$. We start from the set $A \subset G_{n,k}$ from 5.1. If for each $E \in A$ there exists $x \in S(E)$ and $(E,x) \in {}_{t_0}M$ then, by Lemma 5.4, $({}_{t_0}M)_\varepsilon \subset {}_{3t_0/4}M$. On the other side, by Statement 5.3, there exists E_0 such that $(E_0,x) \in {}_{3/4t_0}M$ for every $x \in S(E_0)$ and so $\mathrm{Vol}_k K_{E_0} \geq (3/2\ A_k)^k \cdot \mathrm{Vol}_k D_k$. Because of the Remark in 5.3, we may assume that there exists a set $E \subset G_{n,k}$ of large measure of subspaces, each having the same property as E_0 above. Therefore, the average of $\mathrm{Vol}_k K_E$ over $G_{n,k}$ has to be larger than it is allowed by the formula (V_m) from 3.2 for the mixed Volume A_k. This contradiction shows that there exists $E_0 \in A$ such that $(E_0,x) \notin {}_{t_0}M$ for every $x \in S(E_0)$ It means precisely what we have to prove, i.e. $P_{E_0}K \subset 2A_k(D \cap E_0)$

5.6. Compare Theorem 5.1 with the following well known result (! remember, that we assume $M_r = 1$)

THEOREM [FLM]. *There exists an absolute constant* c *such that* X *contains a k-dimensional subspace* E, $d(E,\ell_2^k) \leq 2$, $k \geq c \min(n/b^2, n/a^2)$, *and such that the orthogonal projection* $P_E : X \to E$ *has the norm* $||P_E|| \leq 2M_{r*}$. (It is worthwhile again to recall section 4.3).

By Alexandrov inequalities 3.3., (A), the sequence of mixed volumes $\{A_m\}_{m=1}^n$ is decreasing: $A_n \leq A_m \leq A_k \leq A_1 = M_{r*}$ for every $1 \leq k \leq m \leq n$ and therefore the assumption $M_{r*} \leq \mathrm{Const.}$ implies $A_k \leq \mathrm{Const.}$ (but not, generally, vice versa).

§6. COMPUTATION OF MIXED VOLUMES THROUGH LEVY MEAN APPROACH.

6.1. Let, as before, $1/a|x| \leq ||x|| \leq b|x|$ for $x \in \mathbb{R}^n$ and $K = \{x : ||x|| \leq 1\}$, K^*, $M_K = M_r$, $M_{K^*} = M_{r^*}$, $V_m(K)$ have the same sense as in the previous sections. Denote also by D_m the m-dimensional Euclidean ball and by $\mathrm{Vol}_m D_m$ the m-dimensional volume of D_m. Let $E \in G_{n,k}$. It is clear that $P_E K$ is the unit ball of the space $X/E^\perp$ and this space is the dual one to $E \hookrightarrow X^*$ (again, as before, it means that E inherits the norm of X^*). Therefore $d(X/E^\perp, \ell_2^k) = d(E \hookrightarrow X^*, \ell_2^k)$. Let

$$M(k;\varepsilon) = \{E \in G_{n,k} : \frac{1}{1+\varepsilon} M_{K^*}|x| \leq ||x||^* \leq M_{K^*}(1+\varepsilon)|x|, \quad \forall x \in E\}.$$

Apply Statement 2.7 to the dual norm $||\cdot||^*$. Then, for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that for each $k \leq c(\varepsilon)n(M_{K^*}/a)^2$

$$\mu(M(k;\varepsilon)) \geq 1 - \exp\left(-\frac{\varepsilon^2}{5} n \left(\frac{M_{K^*}}{a}\right)^2\right) .$$

Therefore by the monotone property of V_k and the formula (V_m) from 3.2

$$(6.1) \qquad V_k(K)/\mathrm{Vol}D \leq \frac{1}{\mathrm{Vol}_k D_k} \left[(1+\varepsilon)^k M_{K^*}^k \cdot \mathrm{Vol}_k D_k + a^k \cdot \exp\left(-\frac{\varepsilon^2}{5} \frac{n}{a^2} M_{K^*}^2\right) \cdot \mathrm{Vol}_k D_k\right]$$

(we use the trivial fact that on the set $M(k;\varepsilon)^c$ which has an exponentialy small measure, as on every subspace $E \in G_{n,k}$, $K_E \subset a\cdot D\cap E$). On the other side

$$(6.2) \qquad V_k(K)/\mathrm{Vol}D \geq \frac{1}{(1+\varepsilon)^k} \cdot M_{K^*}^k \left[1 - \exp\left(-\frac{\varepsilon^2}{5} \frac{n}{a^2} M_{K^*}^2\right)\right] .$$

The estimates (6.1) and (6.2) prove the following Statement.

THEOREM. *For every* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that for each* $k \leq c(\varepsilon) n/(a/M_{K^*})^2 \cdot \ln a/M_{K^*}$ *we have*

$$\frac{1}{1+\varepsilon} M_{K^*} \leq (V_k(K)/\mathrm{Vol}D)^{1/k} \leq (1+\varepsilon) M_{K^*}.$$

Note, that a quantity a/M_{K^*} may be estimated using the cotype 2 constant of X^* (see [FLM]). For example, if $K = [1,1]^n$ is the cube (i.e. that $X = \ell_\infty^n$ and $X^* = \ell_1^n$) and $D = \{x \in \mathbb{R}^n : \sum_1^n x_i^2 \leq 1\}$ is the standard Euclidean ball, then $a = \sqrt{n}$ and $M_{K^*} = c_n\sqrt{n}$ where $c_n \to \sqrt{2/\pi}(n \to \infty)$. So $a/M_{K^*} = 1/c_n \to \sqrt{\pi/2}(n \to \infty)$.

6.2 Directly generalizing Theorem 6.1, mixed volumes $V(K_1, K_2, \dots K_t, \underbrace{D, \dots, D}_{n-t})$ for central symmetric bodies K_i, $1 \leq i \leq t$, may be considered. Begin with the following known fact.

6.2.a. LEMMA [F.] *Let* $\xi \in G_{n,m}$ *be an m-dimensional subspace of* $\mathbb{R}^n$ *and* $\xi^\perp$ *be the orthogonal complemented to* ξ (n-m)-*dimensional subspace of* $\mathbb{R}^n$. *Let* $K_i \subset \mathbb{R}^n$, $i = 1, \dots, m$, *be arbitrary convex sets and* $A_j \subset \xi^\perp$, $j = m+1, \dots, n$ *be arbitrary* (n-m)-*dimensional convex sets. Then*

$$(6.3) \qquad \binom{n}{m} V(K_1, \dots, K_m, A_{m+1}, \dots, A_n) = V(P_\xi K_1, \dots, P_\xi K_m) \cdot V(A_{m+1}, \dots, A_n)$$

where P_ξ *is the orthogonal projection onto* ξ.

6.2.b. COROLLARY: *Let* $K_i \subset \mathbb{R}^n$, ξ *be as in Lemma 6.2.a and* $\mathrm{Vol}_m D_m$ *be the m-dimensional volume of the m-dimensional Euclidean ball* D_m. *Then*

(6.4) $$V(K_1,\dots,K_m,\underbrace{D,\dots,D}_{n-m}) = \frac{\mathrm{Vol}\ D}{\mathrm{Vol}_m D_m}\cdot \int_{G_{n,m}} V(P_\xi K_1,\dots,P_\xi K_m)\,d\mu(\xi)$$

PROOF. We argue by induction over $n - m = k = 1,2,\dots$. Denote by $D(\xi)$ the Euclidean ball of the subspace ξ. Because $P_\xi D = D(\xi)$, it is enough to prove the statement for every $n \geq 2$ and $m = n-1$. So, we apply (6.3) for $m = n-1$ and $A_n = D(\xi)$. Because the mixed volume is a linear function of A_n (with respect to the set addition - see 3.1), integrating (6.3) over $\xi \in G_{n,n-1}$ with $A_n = D(\xi)$ gives (6.4); (note that $\int D(\xi)d\mu(\xi) = D$ and $\int V(K_1,\dots,K_{n-1},D(\xi))d\mu(\xi) = V(K_1,\dots,K_{n-1}, \int D(\xi))$.

6.2.c STATEMENT. *Let* $K_i \subset \mathbb{R}^n$, $i = 1,\dots,m$, *be unit balls of norms* $\|\cdot\|_i$ *and let-for each* $a_i > 0$ *and* $b_i > 0$-, $1/a_i|x| \leq \|x\|_i \leq b_i|x|$ *for* $x \in \mathbb{R}^n$ *and* $i = 1,\dots,m$. *For every* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that for each* $t \leq c(\varepsilon) \min_{1\leq i\leq m} \{n/(a_i/M_{K_i^*})^2\cdot \ln a_i/M_{K_i^*}\}$ *we have*

$$\frac{1}{(1+\varepsilon)^t}\prod_{i=1}^t M_{K_i^*}\cdot \mathrm{Vol}\ D \leq V(K_1,\dots K_t,\underbrace{D\dots D}_{n-t}) \leq (1+\varepsilon)^t \prod_{i=1}^t M_{K_i^*}\cdot \mathrm{Vol}\ D.$$

PROOF. Repeat the argument from 6.1 using Corollary 6.2.b instead of formula (V_m) from 3.2.

6.2.d. The next Statement is a generalization of 6.2.c. and has the same proof. We use the same notation, as in 6.2.c.

STATEMENT. *Let* $t < c(\varepsilon)\cdot n/(a_1 M_{K_1^*})^2 \ \ln a_1/M_{K_1^*}$. *Then*

$$\frac{1}{1+\varepsilon} \leq V(K_1,\dots,K_t,\underbrace{D,\dots D}_{n-t})/M_{K_1^*}\cdot V(K_2,\dots,K_t,\underbrace{D,\dots D}_{n-t+1}) \leq 1+\varepsilon .$$

§7, PROBLEMS. In addition to some problems which were discussed in Section 4 we would like to raise a few questions in the direction of a "proportional" theory.

PROBLEM 1. *Is it true that there exist absolute constants* $\lambda > 0$ *and* $C > 0$ *such that every finite dimensional normed linear space* X *contains a subspace* E *such that* $\dim E > \lambda \dim X$ *and* E^* *has a cotype* 2 *with the cotype* 2 *constant* $C_2(E^*) \leq C$?

If this problem has the positive answer then a number of open problems in Local Theory would be solved (e.g. if X has a cotype $q < \infty$ with the cotype constant

C_q then, by Pisier's Theorem $[P_3]$, E has a type 2 with an upper bound on the type 2 constant $T_2(E)$ depending on λ, C, q and C_q but not on the dimension of X).

PROBLEM 2. *Is it correct that for every* $\varepsilon > 0$ *and* $C > 0$ *there exists* $\lambda = \lambda(\varepsilon,C) > 0$ *such that for each* n *and every* $X_1 = (R^n, ||\cdot||_1)$ and $X_2 = (R^n, ||\cdot||_2)$ *with* $d(X_1,X_2) \leq C$ *there exists a k-dimensional subspace* $E \subset \mathbb{R}^n$ *such that* $k \geq \lambda n$ *and* $d(E \hookrightarrow X_1, E \hookrightarrow X_2) \leq 1 + \varepsilon$?

It seems reasonable to assume that the positive solution on this problem is connected with a cotype condition on X_1 (i.e. λ depends also on $q < \infty$ and cotype q constant $C_q(X)$). However, a counterexample is unknown to me even for the case of $X_1 = \ell_\infty^n$.

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Added in proof. Since this paper was submitted, I have proved Problem 1, Section 4, in the affirmative. The proof will appear soon.