MASAKI KASHIWARA
PIERRE SCHAPIRA

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Astérisque, tome 128 (1985)
<http://www.numdam.org/item?id=AST_1985__128__1_0>

128 ASTÉRISQUE

1985

MICROLOCAL STUDY OF SHEAVES

Masaki KASHIWARA Pierre SCHAPIRA

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

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INTRODUCTION

Let X be a real manifold, $\underline{F}$ a sheaf on X, or better an objet of $D^+(X)$, the derived category of the category of complexes bounded from below of sheaves on X. Let T^*X be the cotangent bundle to X. We associate to $\underline{F}$ a closed conic subset of T^*X, denoted $SS(\underline{F})$, the "micro-support of $\underline{F}$" , as follows :

Definition 0.1. : *Let* $p = (x_o, \xi_o) \in T^*X$. *Then* $p \notin SS(\underline{F})$ *if and only if there exists an open neighborhood* U *of* p *in* T^*X *such that for any* $(x_1, \xi_1) \in U$, *any real* C^1*-function* ϕ *on* X, *with* $\phi(x_1) = 0$, $d\phi(x_1) = \xi_1$, *we have* : $(\mathbb{R}\Gamma_{\{x;\phi(x)\geqslant 0\}}(\underline{F}))_{x_1} = 0$.

In other words the micro-support of $\underline{F}$ describes the set of codirections of X where $\underline{F}$, and its cohomology, "do not propagate".

This definition is motivated by the following situation.

Assume X is a complex manifold, and let $\mathcal{M}$ be a coherent module over the Ring $\mathcal{D}_X$ of (holomorphic, finire order) differential operators. Let $\mathrm{char}(\mathcal{M})$ be the characteristic variety of $\mathcal{M}$ in T^*X. Then we can interpret a well-known result of Zerner [1], Bony-Schapira [1], Kashiwara [5], through the formula :

$$(0.1) \qquad SS(\mathbb{R}\,\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \subset \mathrm{char}(\mathcal{M}) \quad .$$

A natural problem then arising in the theory of (micro-)differential equations, is to evaluate the set of codirections of propagation for the sheaf of hyperfunction or microfunction solutions of $\mathcal{M}$ (or more generally of a system of micro-differential equations). To be more precise, let M be a real analytic manifold of dimension n, X a complexification of M. Recall that the sheaf B_M (resp. C_M) of Sato's hyperfunctions on M (resp. Sato's microfunctions on T^*_MX, the conormal bundle to M in X) is defined by :

$$B_M = \mathbb{R}\Gamma_M(\mathcal{O}_X) \otimes \underline{\omega}_M \quad [n]$$

(resp. $C_M = \mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M \quad [n]$)

where $\underline{\omega}_M$ is the orientation sheaf on M, $[n]$ means the n-shift in $D^+(X)$, and $\mu_M(\cdot)$ is the functor of Sato's microlocalization along M (cf. Chapter 2).

Then the problem is : evaluate $SS(\mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, B_M))$ or $SS(\mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, C_M))$. Taking $\underline{F} = \mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ this is a particular case of the following problem : given $\underline{F} \in Ob(D^+(X))$, and M a real submanifold of X, calculate $SS(\mathbb{R}\Gamma_M(\underline{F}))$ or $SS(\mu_M(\underline{F}))$.

As we see, in this new formulation, we may forget that X is a complex manifold, and we do not study separately the $\mathcal{D}_X$-module $\mathcal{M}$ from one side and the sheaf $\mathcal{O}_X$ on the other side. On the contrary we work with the whole complex of solutions of $\mathcal{M}$ in $\mathcal{O}_X$. The only information that we keep is the geometrical data of the characteristic variety, which is interpreted in terms of micro-support (in fact we shall prove in Chapter 10 that the inclusion in (0.1) is an equality).

Now let us come back to the subject of this paper.

We study in Chapter 4 and 5 the functorial properties of the micro-support : behavior under direct or inverse images, functors $\mathbb{R}\underline{Hom}(\cdot,\cdot)$, $\cdot \overset{L}{\otimes} \cdot$, specialization, Fourier-Sato transformation, microlocalization (the construction of these functors are recalled in Chapter 1 and 2). But in order to manipulate micro-supports, the definition (0.1) given above is too much of a local one and one has to replace it by a more global criterium. This is achieved in Chapter 3, using a Mittag-Leffler procedure for sheaves (Theorem 1.4.3).

The calculations of Chapters 4 and 5 are all essentially based on the computation of the micro-support of the direct image of a sheaf by an open immersion. In this case the procedure, and the result, are very similar to those encountered in the theory of micro-hyperbolic systems (cf. our work [2]), and the set we obtain is defined as a "normal cone", (Theorem 4.3.1.). The preliminaries concerning such normal cones are presented in Chapter 1, §2.

The notion of micro-support allows us to work with sheaves "micro-locally", that is, locally in T^*X. In fact for a subset Ω of T^*X, we introduce the triangulated category $D^+(X;\Omega)$ obtained from $D^+(X)$ by localization on Ω, that is, by regarding as the zero object the sheaves whose micro-support do not meet Ω. A useful tool in the microlocal study of sheaves, is the "G-topology". The idea of the G-topology is the following : in order to work microlocally, let us say on $X \times U$ where X is open in a real vector space E and U is an open cone in the dual space E^*, the usual topology on X is too strong, and may be weakened by introducing a closed convex proper cone G in E whose polar set G^o is contained in $(-U) \cup \{0\}$, and by considering only those open sets Ω of X such that :

$$(0.2) \qquad \Omega = (\Omega + G) \cap X$$

Let X_G be the space X endowed with the G-topology (i.e. : the open subsets of X_G satisfy (0.2)) and let Φ_G be the continuous map $X \longrightarrow X_G$. Let Ω_0 and Ω_1 be two G-open subsets of E such that $\Omega_0 \subset \Omega_1$, $\Omega_1 \setminus \Omega_0 \subset\subset X$. then one proves (cf. Theorem 3.2.2.) that for $\underline{F} \in Ob(D^+(X))$ one has the isomorphism :

$$(0.3) \qquad \Phi_G^{-1} \mathbb{R}\Phi_{G*} \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_0}(\underline{F}) \xrightarrow{\sim} \underline{F} \text{ in } D^+(X;\mathrm{Int}(\Omega_1 \setminus \Omega_0) \times \mathrm{Int}(-G^o))$$

and moreover :

(0.4) $$SS(\Phi_G^{-1}\mathbb{R}\Phi_{G^*}\mathbb{R}\Gamma_{\Omega_1\setminus\Omega_0}(\underline{F}))\subset X\times(-G^o)$$

Thus the G-topology permits to cut off sheaves in the "dual variable", which can be compared to the cut off of differential operators by classical pseudo-differential operators. Remark that we had already introduced the G-topology in our work [2] in order to give a meaning to the action of microdifferential operators on the sheaf of holomorphic functions.

With all these tools in hand, it is now not too difficult to extend contact transformations to sheaves.

Let $\Phi : T^*Y \supset V \longrightarrow U \subset T^*X$ be a contact transformation and let Λ be the image of the graph of Φ by the antipodal map of T^*X. Let $\underline{K}$ be a sheaf on $X \times Y$, with $SS(\underline{K}) \subset \Lambda$ in a neighborhood of Λ, $\underline{K}$ satisfying some other suitable conditions (cf. Theorem 6.3.2.). Then we prove that the transformation :

(0.5) $$\Psi_{\underline{K}} : \underline{F} \longmapsto \mathbb{R}q_{2!}(\underline{K} \overset{\mathbb{L}}{\otimes} q_1^{-1}\underline{F})$$

gives an equivalence of categories between $D^+(X;U)$ and $D^+(Y;V)$. Here q_1 and q_2 denote the projections from $X \times Y$ to X and Y respectively. We prove that such transformations $\Psi_{\underline{K}}$ always exist locally, and are essentially unique up to the shift (i.e. : translation of the complexes in $D^+(X)$). Moreover $\Psi_{\underline{K}}$ "commutes" to microlocalization, up to shifts.

As a corollary, we obtain that micro-supports of sheaves are always involutive subsets of T^*X (Theorem 6.4.1.).

In order to calculate the shifts which naturally appear when performing a contact transformation, we introduce in Chapter 7 the notion of simple sheaves, (microlocally along a smooth Lagrangean manifold Λ of T^*X). This notion is analogous to that of "simple holonomic Modules" of M. Sato, M. Kashiwara, T. Kawaï [1] or to that of "Fourier distributions" of L. Hörmander [2]. This study requires classical computations on the index associated to three Lagrangean manifolds (the "Maslov index"), (cf. V.P. Maslov [1], J. Leray [2], L. Hörmander (loc. cit.)). For example if M is a submanifold of X, the sheaf $\underline{\mathbb{Z}}_M$ on X is simple on $T^*_M X$, with shift $\frac{1}{2}$ codim M. We study the functorial properties of simple sheaves, and in particular the shift obtained by interchanging the two operations of microlocalization and contact transformation.

Next we derive some applications of our theory, (Chapters 8,9,10).

First we study $\mathbb{R}$- or $\mathbb{C}$ - constructible sheaves. Those sheaves are characterized by some finitude properties and the fact that their micro-supports are Lagrangean sets (and subanalytic in the real case, $\mathbb{C}$-analytic in the complex case). Thus the functorial properties of constructible sheaves are derived from the functorial properties of micro-supports.

For example we may perform contact transformations on $\mathbb{R}$-constructible sheaves. Moreover $\mathbb{C}$-constructible sheaves (on a complex manifold X) are just those $\mathbb{R}$-constructible sheaves whose micro-support is stable by the action of $\mathbb{C}^\times$ on T^*X, and this allows us to state theorems on direct images for $\mathbb{C}$-constructible sheaves by a holomorphic map $f : Y \longrightarrow X$, in the non proper case. In particular such direct images are always $\mathbb{C}$-constructible when $\dim X = 1$, locally on Y.

Using the Riemann-Hilbert correspondence, we may translate our results on

$\mathbb{C}$-constructible sheaves to obtain results for regular holonomic systems (M. Kashiwara, T. Kawaï [6]), in particular results on direct images in the non proper case. We also give a microlocal construction of the Riemann-Hilbert correspondence which enables us to characterize perverse sheaves as those sheaves whose shift is zero (microlocally, and generically on their micro-support).

Then we study micro-differential systems on a complex manifold X, or more generally bounded complexes of free $\mathcal{E}_X^{\mathbb{R}}$-modules of finite rank, where $\mathcal{E}_X^{\mathbb{R}}$ is the ring of microlocal operators on X of Sato-Kashiwara-Kawaï (loc. Cit.). If $\mathcal{M}$ is such a complex we recall how the G-topology gives a meaning to the complex $\underline{F} = \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X^{\mathbb{R}}}(\mathcal{M}, \mathcal{O}_X)$ and we relate the support of $\mathcal{M}$ to the micro-support of $\underline{F}$. By this method we obtain the involutivity of Supp($\mathcal{M}$) which generalizes the corresponding theorem of Sato-Kashiwara-Kawaï (loc. cit.) for coherent $\mathcal{E}_X$-modules. When microlocalizing $\underline{F}$ along real submanifolds of X, and applying the results of Chapter 5, we immediately recover (and we even improve) the results of our previous work [2]. As another application of the relation between supp($\mathcal{M}$) and SS($\underline{F}$), we give a bound to the characteristic variety of the restriction to a complex submanifold Y of a coherent $\mathcal{D}_X$-module $\mathcal{M}$. This generalizes previous works on this subject (Kashiwara-Kawaï [5], Kashiwara-Schapira [3]). In a similar way we give a bound to the analytic wave front set of the distribution $\prod_j (f_j + io)^{\lambda_j}$. We end Chapter 10 with a new description of the action of $\mathcal{E}_X^{\mathbb{R}}$ on $\mathcal{O}_X$, by sending $\mathcal{E}_X^{\mathbb{R}}$ in the sheaf of microlocal homomorphisms of $\mathcal{O}_X$.

In the last chapter, we study the action of complex contact transformations on the sheaf $\mathcal{O}_X$ on a complex manifold X. More precisely if $\Psi_{\underline{K}}$ is the equivalence of categories defined in (0.5), and $\underline{K}$ is a simple sheaf with shift $n = \dim_{\mathbb{C}} X$ (along a complex Lagrangean submanifold of $T^*(X \times Y))$, then we prove that one can locally "quantize" $\Psi_{\underline{K}}$ as an isomorphism between $\Psi_{\underline{K}}(\mathcal{O}_X)$ and $\mathcal{O}_Y$ in $D^+(Y;V)$ (after shrinking V).

In particular if N and M are two real submanifolds of X such that a complex contact transformation Φ interchanges $T^*_N X \cap V$ and $T^*_M X \cap U$, then one can locally find an isomorphism beetween the sheaves $\mu_N(\mathcal{O}_X)|_V$ and $\mu_M(\mathcal{O}_X)|_U [d]$ where the shift d is explicitly calculated. This improves previous results of Sato-Kashiwara-Kawaï (loc. cit.) who considered the case where N and M are real analytic and X is a complexification of N and M, and results of Kashiwara-Kawaï [4] who considered the case where the Levi forms of N and M where non degenerate. In fact our method is essentially different, since we proceed with the sheaf $\mathcal{O}_X$ itself, without using the induced Cauchy-Riemann systems. As an application we obtain, almost without any computation, new theorems for the vanishing of the cohomology groups $H^j(\mu_M(\mathcal{O}_X))_p$.

The main results of this paper have been announced in our papers [4], [5].

Let us conclude by thanking Mrs Catherine Simon for her excellent typing and her great patience with our numerous corrections.

CHAPTER 1 - PRELIMINARIES

§1.1. Notations and conventions

1.1.1. Let X be a topological space, S a subset of X. We denote by $\overline{S}$ and $Int(S)$ the closure of S and the interior of S, respectively.

For a sequence $\{x_n\}$ in X, $x_n \xrightarrow[n]{} x$ means that this sequence converges to x.

In a metric space, the distance between two sets A and B is denoted $d(A,B)$.

Let E be a real vector space, Γ a cone with vertex at 0 in E. The polar cone, Γ^o, is defined by :

$$\Gamma^o = \{\theta \in E^* \; ; \; \langle v,\theta\rangle \geqslant 0 \quad \forall v \in \Gamma\}$$

where E^* is the dual space to E.

1.1.2. Let X be a real manifold of class C^α ($1 \leqslant \alpha \leqslant \infty$ or $\alpha = \omega$, which means that X is real analytic). We denote by TX (resp. T^*X) the tangent (resp. cotangent) bundle to X. If Y is a submanifold of X, T_YX (resp. T^*_YX) will denote the normal (resp. conormal) bundle to Y. In particular T_XX (resp. T^*_XX) denotes the zero section of TX (resp. T^*X). We usually denote by τ (resp. π) the projection from TX (resp. T^*X) to X.

If E is a vector bundle over X, $p : E \longrightarrow X$ its projection, we identify X with the zero section of E, and denote by $\dot{E}$ the bundle E with the zero section removed. We denote by $\dot{p}$ the projection $\dot{E} \longrightarrow X$, and by "a" the anti-podal map in E (or $\dot{E}$). If S is a subset of E, S^a is the image of S by this map.

1.1.3. Assume $\alpha \geqslant 2$. The canonical 1-form on T^*X is denoted by ω_X or simply ω. For a system of local coordinates $(x) = (x_1,\dots,x_n)$ on X, $(x\,;\,\xi dx) = (x_1,\dots,x_n\,;\,\sum_j \xi_j\,dx_j)$ on T^*X, we have :

$$\omega = \sum_{j=1}^{n} \xi_j\,dx_j\ .$$

The Hamiltonian isomorphism H from T^*T^*X to TT^*X associated to $d\omega$ is defined by :

$$(1.1.1)\qquad \langle\theta,v\rangle = \langle d\omega,\ v\wedge H(\theta)\rangle\ ,\ v\in TT^*X,\ \theta\in T^*T^*X\ .$$

We shall identify T^*T^*X and TT^*X by $-H$. Thus :

$$(1.1.2)\qquad -H(\langle\lambda,\ dx\rangle + \langle\mu,\ d\xi\rangle) = \langle\lambda,\ \frac{\partial}{\partial\xi}\rangle - \langle\mu,\ \frac{\partial}{\partial x}\rangle\ .$$

1.1.4. Let $f : Y \longrightarrow X$ be a map of class C^α. We denote by ρ_f and ϖ_f , or simply by ρ and ϖ , the maps associated to f , from $Y \times_X T^*X$ to T^*Y and T^*X respectively :

$$(1.1.3)\qquad T^*Y \xleftarrow[\rho_f]{} Y\times_X T^*X \xrightarrow[\varpi_f]{} T^*X\ .$$

We also define :

$$(1.1.4)\qquad T^*_YX = \mathrm{Ker}\ \rho_f$$

Assume $\alpha \geqslant 2$ and f is an embedding. The projection from T^*_YX to Y defines the embedding :

$$T^*Y \times_Y T^*_YX \hookrightarrow T^*T^*_YX\ .$$

Using the zero section of T^*_YX we get the embeddings :

$$(1.1.5)\qquad T^*Y \hookrightarrow T^*Y\times_Y T^*_YX \hookrightarrow T^*T^*_YX\ .$$

Remark that T^*_YX is a Lagrangean submanifold of T^*X, and the

Hamiltonian isomorphism (1.1.2) induces an isomorphism :

$$(1.1.6) \qquad T^*T^*_Y X \cong T_{T^*_Y X} T^* X$$

In particular , replacing X by T^*X, Y by $T^*_X X$, we find the embedding :

$$(1.1.7) \qquad T^*X \hookrightarrow TT^*X \quad .$$

§1.2. Normal cones

1.2.1. Let X be a real manifold of class C^α, and let S and V, be two subsets of X. The normal cone of S along V , denoted C(S,V) , is the closed cone of TX defined as follows (cf. Kashiwara-Schapira [2]).

Let $x \in X$, and choose a system of local coordinates in the neighborhood of x. We define the closed cone $C_x(S,V)$ of T_xX by :

$$(1.2.1) \quad \begin{cases} \theta \in C_x(S,V) \iff \text{there exists a sequence } \{(c_n,s_n,v_n)\} \\ \text{in } \mathbb{R}^+ \times S \times V \text{ such that } \{s_n\} \text{ and } \{v_n\} \text{ converge to} \\ x \text{ and } \{c_n(s_n - v_n)\} \text{ converges to } \theta \end{cases}$$

and we set :

$$(1.2.2) \qquad C(S,V) = \bigcup_{x\in X} C_x(S,V) \quad .$$

If X is a vector space, a vector $\theta \in T_xX$ does not belong to $C_x(S,V)$ if and only if there exists an open cone Γ , with $\theta \in \Gamma$, and a neighborhood U of x such that :

$$(1.2.3) \qquad ((U \cap V) + \Gamma) \cap U \cap S = \emptyset$$

If V is smooth, C(S,V) is invariant by TV. In this case we denote by $C_V(S)$ the image of C(S,V) in the normal bundle T_VX :

$$(1.2.4) \qquad C_V(S) = C(S,V)/TV$$

For example if X is a vector space, S a closed conic subset, V a linear subspace, we find $C(S,V) = S + V$.

1.2.2. Let X be a manifold of class C^α, $\alpha \geqslant 2$, Y a submanifold.

Proposition 1.2.1. (cf. Kashiwara-Schapira [3]).
Let (y,t) be a system of local coordinates on X such that $Y = \{(y,t) \,;\, t = 0\}$, and let $(y,t \,;\, \eta,\tau)$ be the associated coordinates on T^*X. Let A be a conic subset of T^*X. Then :

$$(y,\eta) \in T^*Y \cap C_{T^*_YX}(A) \iff \text{there exists a}$$

sequence $\{(y_n,t_n \,;\, \eta_n,\tau_n)\}$ in A such that :

$$(1.2.5)\quad \begin{cases} (y_n;\eta_n) \xrightarrow[n]{} (y;\eta) \\ |t_n| \xrightarrow[n]{} 0 \ , \ |t_n||\tau_n| \xrightarrow[n]{} 0 \end{cases}$$

(Remark that T^*Y is naturally embedded into $T_{T^*_YX}T^*X$ by (1.1.5) and (1.1.6)).

Proof

We set $\Lambda = T^*_YX$. First we remark that the isomorphism $-H$ between $T^*\Lambda$ and $T_\Lambda T^*X$ is given by :

$$\langle\eta,dy\rangle + \langle t,d\tau\rangle \longmapsto \langle\eta, \frac{\partial}{\partial\eta}\rangle - \langle t, \frac{\partial}{\partial t}\rangle$$

and hence T^*Y is embedded into $T_\Lambda T^*X$ by :

$$(y, \eta\, dy) \longmapsto ((y,0 \,;\, 0,0) \,;\, \eta \frac{\partial}{\partial\eta})$$

i) let $\{(y_n,t_n\ ;\ \eta_n,\tau_n)\}$ be a sequence satisfying the condition of the proposition. Remark that there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$ such that $t_n/\varepsilon_n \xrightarrow[n]{} 0$, $\varepsilon_n|\tau_n| \xrightarrow[n]{} 0$, and $\varepsilon_n \xrightarrow[n]{} 0$. In fact we take $\varepsilon_n = (|t_n|/|\tau_n|)^{1/2}$ for $|t_n||\tau_n| \neq 0$. Then we have :

$$(y_n,t_n\ ;\ \varepsilon_n\eta_n,\ \varepsilon_n\tau_n) \xrightarrow[n]{} (y,0\ ;\ 0,0)$$

$$(y_n,0\ ;\ 0,\ \varepsilon_n\tau_n) \xrightarrow[n]{} (y,0\ ;\ 0,0)$$

$$\varepsilon_n^{-1}(t_n;\ \varepsilon_n\eta_n) \xrightarrow[n]{} (0\ ;\ \eta)$$

ii) let $(y;\eta) \in C_\Lambda(A) \cap T^*Y$. There exist a sequence $\{(y_n,t_n\ ;\ \eta_n,\ \tau_n)\}$ in A, a sequence $\{(y_n',0\ ;\ 0,\ \tau_n')\}$ in Λ , and a sequence $\{c_n\}$ in $\mathbb{R}^+$ such that :

$$(y_n,\ t_n\ ;\ \eta_n,\ \tau_n) \xrightarrow[n]{} (y,0\ ;\ 0,0)$$

$$(y_n',\ 0\ ;\ 0,\ \tau_n') \xrightarrow[n]{} (y,0\ ;\ 0,0)$$

$$c_n(t_n\ ;\ \eta_n) \xrightarrow[n]{} (0;\eta)$$

The sequence $\{(y_n,\ t_n\ ;\ c_n\eta_n,\ c_n\tau_n)\}$ satisfies the condition of the proposition. □

Now let $\dot{\pi}$ denote the projection from $\dot{T}_Y^*X = T_Y^*X \setminus Y$ to Y, and let ρ and ϖ be the natural associated maps :

$$T^*Y \xleftarrow{\ \rho\ } T^*Y \underset{Y}{\times} \dot{T}_Y^*X \xhookrightarrow{\ \varpi\ } T^*\dot{T}_Y^*X \cong T_{\dot{T}_Y^*X}T^*X$$

Using the same notations as for Proposition 1.2.1. and a similar proof, we obtain :

<u>Proposition 1.2.2.</u> : <u>Let</u> A <u>be a conic subset in</u> T^*X. <u>Then</u> :

$(y;\eta) \in \rho\,\varpi^{-1} C_{\dot{T}_Y^*X}(A) \iff$ <u>there exists a sequence</u>

$\{(y_n,\ t_n\ ;\ \eta_n,\ \tau_n)\}$ <u>in</u> A <u>which satisfies (1.2.5) and also</u> :

(1.2.6) $$|\tau_n| \xrightarrow[n]{} \infty$$

Remark that if we identify X with the diagonal of $X \times X$ and T^*X with $T^*_X(X \times X)$ by the first projection, we get a natural projection:

$$\varpi^{-1} T^*\dot{T}^*X \xrightarrow[\rho]{} T^*X$$

This projection ρ may also been described as the projection $T^*X \times_X \dot{T}^*X \longrightarrow T^*X$ associated to $\dot{\pi} : \dot{T}^*X \longrightarrow X$.

<u>Definition 1.2.3.</u> : <u>For a pair</u> (A_1, A_2) <u>of two conic subsets of</u> T^*X <u>we set</u> :

$$A_1 \hat{+} A_2 = T^*X \cap C(A_1, A_2^a)$$

$$A_1 \underset{\infty}{\hat{+}} A_2 = \rho\, \varpi^{-1} C(A_1|_{\dot{T}^*X}, A_2^a|_{\dot{T}^*X}) .$$

<u>Corollary 1.2.4.</u> : <u>Let</u> A_1 <u>and</u> A_2 <u>be two conic subsets of</u> T^*X. <u>Then</u> :

a) $(x;\xi) \in A_1 \hat{+} A_2 \Longleftrightarrow$ <u>there exist sequences</u> $\{(x_n;\xi_n)\}$ <u>in</u> A_1 <u>and</u> $\{(x'_n;\xi'_n)\}$ <u>in</u> A_2 <u>such that</u> :

(1.2.7) $$\begin{cases} x_n \xrightarrow[n]{} x, \quad x'_n \xrightarrow[n]{} x, \quad \xi_n + \xi'_n \xrightarrow[n]{} \xi \\ |x_n - x'_n|\, |\xi_n| \xrightarrow[n]{} 0 \end{cases}$$

b) $(x,\xi) \in A_1 \underset{\infty}{\hat{+}} A_2 \Longleftrightarrow$ <u>there exist sequences</u> $\{(x_n,\xi_n)\}$ <u>in</u> A_1 <u>and</u> $\{(x'_n;\xi'_n)\}$ <u>in</u> A_2 <u>which satisfy (1.2.7) and also</u> :

(1.2.8) $$|\xi_n| \xrightarrow[n]{} \infty$$

Let $f ; Y \longrightarrow X$ be a map of class C^α, $\alpha \geqslant 2$, A a conic subset of T^*X. We may consider Y as a closed subset of $Y \times X$ by the graph map. Let ρ and ϖ be the maps naturally associated to the embedding $Y \hookrightarrow Y \times X$.

Definition 1.2.5. : Let $f : Y \longrightarrow X$ be a map of class C^{α}, $\alpha \geqslant 2$, and let A be a conic subset of T^*X. One sets :

$$f^{\#}(A) = T^*Y \cap C_{T^*_Y(Y\times X)}(T^*_Y Y \times A)$$
$$f^{\#}_{\infty}(A) = \rho\varpi^{-1} C_{\dot{T}^*_Y(Y\times X)}(T^*_Y Y \times A)$$

Proposition 1.2.6. : In the situation of Definition 1.2.5., let (y) (resp. (x)) be a system of local coordinates on Y (resp. X), $(y;\eta)$ (resp. $(x;\xi)$) the associated coordinates on T^*Y (resp. T^*X). Then :

i) $(y;\eta) \in f^{\#}(A) \iff$ there exists a sequence $\{(y_n;(x_n;\xi_n))\}$ in $Y \times A$ such that :

$$(1.2.9)\quad \begin{cases} y_n \xrightarrow[n]{} y\,, & x_n \xrightarrow[n]{} f(y)\,, \\ {}^t f'(y_n).\xi_n \xrightarrow[n]{} \eta, & |x_n - f(y_n)|\,|\xi_n| \xrightarrow[n]{} 0 \end{cases}$$

ii) $(y;\eta) \in f^{\#}_{\infty}(A) \iff$ there exists a sequence $\{(y_n;(x_n;\xi_n))\}$ in $Y \times A$ which satisfies (1.2.9) and also :

$$(1.2.10)\qquad |\xi_n| \xrightarrow[n]{} \infty\ .$$

The proof follows immediately from Proposition 1.2.1. .

Remark that if f is an embedding, we find by Proposition 1.2.1. and 1.2.2. :

$$f^{\#}(A) = T^*Y \cap C_{T^*_Y X}(A)$$

$$f_\infty^{\#}(A) = \rho\, \varpi^{-1} C_{\dot{T}_Y^*X}(A)$$

Remark 1.2.7.: i) Let A_1 and A_2 be two closed conic subset of T^*X, and assume $(A_1 \hat{+}_\infty A_2) \cap U = \emptyset$, for an open subset U of T^*X. Then we find :

$$(A_1 \hat{+} A_2) \cap U = (A_1 + A_2) \cap U$$

(Recall that $A_1 + A_2 = \{(x,\xi_1+\xi_2) ; (x,\xi_1) \in A_1, (x,\xi_2) \in A_2\}$).

In particular if $A_1 \cap A_2^a \subset T_X^*X$, then $A_1 \hat{+} A_2 = A_1 + A_2$.

Similarly if A is a conic closed set in T^*X such that $f_\infty^{\#}(A) \cap V = \emptyset$, for an open set V of T^*Y , then :

$$f^{\#}(A) \cap V = \rho\, \varpi^{-1}(A) \cap V$$

In particular if $T_Y^*X \cap \varpi^{-1}(A)$ is contained in $Y \underset{X}{\times} T_X^*X$, then $f^{\#}(A) = \rho\, \varpi^{-1}(A)$.

ii) Let $g : Z \longrightarrow Y$ and $f : Y \longrightarrow X$ be two maps of class C^α, $\alpha \geqslant 2$. Let A be a conic subset of T^*X. Assume f is smooth. Then :

$$g^{\#}(f^{\#}(A)) = (f \circ g)^{\#}(A) \tag{1.2.11}$$

1.2.3. Now we recall the notion of conormal to a subset S (cf. Kashiwara-Schapira [2]). For a subset S of X, the strict normal cone is :

$$N(S) = TX \setminus C(X \setminus S, S) \tag{1.2.12}$$

This is an open convex cone in TX and by the definition $N_x(S)$

contains a vector θ if and only if there exists (for a choice of local coordinates in a neighborhood of x) an open cone Γ containing θ and a neighborhood U of x such that :

$$U \cap ((S \cap U) + \Gamma) \subset S$$

The "conormal cone to S", denoted $N^*(S)$, is the polar set to $N(S)$:

$$(1.2.13) \quad \begin{cases} N_x^*(S) = \{\theta \in T_x^*X \ ; \ \langle v,\theta\rangle \geq 0 \quad \forall v \in N_x(S)\} \\ N^*(S) = \bigcup_{x\in X} N_x^*(S) \end{cases}$$

This is a closed convex cone in T^*X which contains X. Remark that :

$$N_x(S) = T_xX \iff x \notin \mathrm{clos}(S) \quad \text{or} \quad x \in \mathrm{Int}(S)$$

$$N_x(S) = \emptyset \iff N_x^*(S) = T_x^*X$$

$N_x(S) \neq \emptyset \iff N_x^*(S)$ is a proper cone (a cone is called proper if it contains no line).

§1.3. Sheaves

1.3.1. In all this paper we fix a unitary ring A, and we shall work with sheaves of A-modules. If not otherwise specified, A-module means left A-module, but of course, if we write for example $M \otimes_A N$, M is supposed to be a right A-module.

We write for short $M \otimes N$ or $\mathrm{Hom}(M,N)$ instead of $M \otimes_A N$ or $\mathrm{Hom}_A(M,N)$, when there is no risk of confusion. In fact in many questions, as for the definition of the micro-support, it is equivalent to work with sheaves of A-modules or with sheaves of $\mathbb{Z}$-modules.

Let X be a topological space. We denote by $D(X)$ the derived category of the abelian category of sheaves of A-modules and $D^+(X)$

and $D^b(X)$ denote the full subcategories of $D(X)$ consisting of complexes with cohomology bounded from below, and bounded cohomology, respectively. If we need to specify A, we write $D(X;A)$ instead of $D(X)$, and similarly for $D^+(X;A)$, $D^b(X;A)$.

We shall usually denote an object of $D^+(X)$ by a underlined capital letter such as $\underline{F}$, or $\underline{G}$.

We denote by $\underline{F}[k]$ the object of $D(X)$ obtained by shifting $\underline{F}$ by k steps, and replacing the differential d by $(-1)^k d$.

Hence $H^j(\underline{F}[k]) = H^{j+k}(\underline{F})$.

We denote by $\underline{F} \longrightarrow \underline{F}' \longrightarrow \underline{F}'' \xrightarrow[+1]{} \dots$ a distinguished triangle in $D(X)$.

We shall identify a sheaf $\underline{F}$ with a complex of sheaves "concentrated in degree 0".

We use the classical notions for derived categories and sheaf cohomology, and we refer to Godement [1], Bredon [1], Hartshorne [1] , Verdier [2], Iversen [1].

In particular if Z is a closed subset of X, $\underline{F}$ a sheaf on X, $\Gamma_Z(X,\underline{F})$ denotes the group of sections of $\underline{F}$ on X supported by Z. We write $\Gamma(X,\underline{F})$ or sometimes $\underline{F}(X)$ instead of $\Gamma_X(X,\underline{F})$. If Z is locally closed in X and U is an open subset of X containing Z as a closed subset, one sets $\Gamma_Z(X,\underline{F}) = \Gamma_Z(U,F)$.

For $x \in X$, one sets $\underline{F}_x = \varinjlim_U \underline{F}(U)$, where U runs over the family of open neighborhoods of x. We denote by $\Gamma_Z(\underline{F})$ the sheaf $U \longrightarrow \Gamma_{Z\cap U}(X,\underline{F})$ on X. If $\underline{F}$ and $\underline{G}$ are two sheaves on X we denote by $\mathrm{Hom}_A(\underline{F},\underline{G})$ the group of sheaves homomorphisms from $\underline{F}$ to $\underline{G}$ and by $\underline{\mathrm{Hom}}_A(\underline{F},\underline{G})$ the sheaf $U \longmapsto \mathrm{Hom}_A(\underline{F}|_U , \underline{G}|_U)$, where $F|_U$ and $G|_U$ means the restriction of these sheaves to U, (U open in X).

We denote by $\underline{F} \otimes_A \underline{G}$ the sheaf associated to the presheaf

$U \longmapsto \underline{F}(U) \otimes_A \underline{G}(U)$. If $f : Y \longrightarrow X$ is a continuous map, and if $\underline{G}$ is a sheaf on Y, the direct image $f_* \underline{G}$ is the sheaf $U \longmapsto \underline{G}(f^{-1}(U))$, ($U$ open in X). The inverse image of a sheaf $\underline{F}$ on X, denoted $f^{-1}\underline{F}$, is the sheaf associated to the presheaf $V \longmapsto \varinjlim_{U \supset f(V)} \underline{F}(U)$, ($V$ open in Y, and U runs over the family of open neighborhoods of $f(V)$ in X). The functor $f^{-1}(\cdot)$ is exact.

If Y is a subspace of X, f the natural injection, on writes $\underline{F}|_Y$ instead of $f^{-1}\underline{F}$.

The functors $\Gamma_Z(X,\cdot)$, $\Gamma_Z(\cdot)$, $\mathrm{Hom}_A(\cdot,\cdot)$, $\underline{\mathrm{Hom}}_A(\cdot,\cdot)$ are left exact (i.e. : left exact in each of their arguments, for $\mathrm{Hom}(\cdot,\cdot)$ and $\underline{\mathrm{Hom}}(\cdot,\cdot)$). Since the category of sheaves of A-modules has enough injectives, one can define the right derived functors of the preceding functors. They are denoted by $\mathbb{R}\Gamma_Z(\cdot)$, $\mathbb{R}\Gamma_Z(X,\cdot)$, $\mathbb{R}\mathrm{Hom}_A(\cdot,\cdot)$, $\mathbb{R}\underline{\mathrm{Hom}}_A(\cdot,\cdot)$ respectively. We also set :

$$H^i_Z(X,\cdot) = H^i\,\mathbb{R}\Gamma_Z(X,\cdot)$$

$$H^i_Z(\cdot) = H^i\,\mathbb{R}\Gamma_Z(\cdot)$$

$$\mathrm{Ext}^i_A(\cdot,\cdot) = H^i\,\mathbb{R}\mathrm{Hom}_A(\cdot,\cdot)$$

$$\underline{\mathrm{Ext}}^i_A(\cdot,\cdot) = H^i\,\mathbb{R}\underline{\mathrm{Hom}}_A(\cdot,\cdot)$$

The functor $\cdot \otimes_A \cdot$ is right exact (in each of its arguments). When defining its left derived functor, we shall make the assumption that $\mathrm{wg\ell d}(A)$ is finite, (recall that $\mathrm{wg\ell d}(A)$, the weak global dimension of A, is the smallest integer $m \in \mathbb{N} \cup \{\infty\}$ such that every A-module admits a resolution of length at most m by flat modules). In that case every $\underline{F} \in \mathrm{Ob}(D^+(X))$ is isomorphic to a complex $\underline{F}'$ of sheaves bounded from below such that $\underline{F}'^{\,i}_x$ is flat for any x, any i. Then for $\underline{G} \in \mathrm{Ob}(D^+(X))$ the isomorphism class of $\underline{F}' \otimes_A \underline{G}$ in $D^+(X)$ is independent of the choice of $\underline{F}'$, and the left derived functor $\cdot \overset{\mathbb{L}}{\otimes}_A \cdot$ from $D^+(X;A) \times D^+(X;A)$ to $D^+(X;\mathbb{Z})$ and from $D^b(X;A) \times D^b(X;A)$ to $D^b(X;\mathbb{Z})$ is well defined. We set :

$$\underline{Tor}_i^A(\underline{F},\underline{G}) = H^{-i}(\underline{F} \overset{\mathbb{L}}{\otimes}_A \underline{G})$$

Let us repeat that we shall not always write "A" in these formulas.

Let $\underline{F} \in Ob(D^+(X))$. We set :

(1.3.1) $$\mathrm{supp}(\underline{F}) = \overline{\bigcup_j \mathrm{supp}(H^j(\underline{F}))}$$

where $\mathrm{supp}(H^j(\underline{F}))$ is the support of the sheaf $H^j(\underline{F})$.

1.3.2. We denote by $\Gamma_c(X,\underline{F})$ the submodule of $\Gamma(X,\underline{F})$ of sections with compact supports in X. The functor $\Gamma_c(X,\cdot)$ is left exact and its derived functor is denoted $\mathbb{R}\Gamma_c(X,\cdot)$. One also sets :

$$H_c^i(X,\cdot) = H^i\,\mathbb{R}\Gamma_c(X,\cdot)$$

Now let Y and X be two locally compact spaces, $f : Y \longrightarrow X$ a continuous map. Recall that f is proper if and only if the inverse image of a compact set of X is compact in Y.

Let $\underline{G}$ be a sheaf on Y. One defines $f_!\underline{G}$, the proper direct image of $\underline{G}$ by setting :

$$\Gamma(U, f_!\underline{G}) = \varinjlim_Z \Gamma_Z(f^{-1}(U), \underline{G})$$

where U is open in X, and where Z runs over the family of closed subsets of $f^{-1}(U)$ such that f is proper on Z.

The functor $f_!(\cdot)$ is left exact, and its derived functor is denoted $\mathbb{R}f_!(\cdot)$.

1.3.3. Let M be an A-module. We denote by $\underline{M}_X$ the constant sheaf on X with fiber M.

Let Z be a locally closed subset of X, $j : Z \longrightarrow X$ the injection of Z in X. Recall that to a sheaf $\underline{F}$ on X, one can naturally associate a sheaf $\underline{F}_Z$ on X, such that :

$$(\underline{F}_Z)_x = \underline{F}_x \quad \text{if } x \in Z$$
$$= 0 \quad \text{if } x \notin Z$$

If Z is closed in X, one has :

$$\underline{F}_Z = j_* j^{-1} \underline{F}$$

If Z is open in X, one has an exact sequence of sheaves on X :

$$0 \longrightarrow \underline{F}_Z \longrightarrow \underline{F} \longrightarrow \underline{F}_{(X \setminus Z)} \longrightarrow 0$$

If Z_1 and Z_2 are locally closed, one has :

$$\underline{F}_{Z_1 \cap Z_2} = (\underline{F}_{Z_1})_{Z_2}$$

If M is an A-module, one usually write $\underline{M}_Z$ instead of $(\underline{M}_X)_Z$. Hence the notation $\underline{M}_Z$ has two meanings : it can denote a sheaf on Z, or a sheaf on X (supported by Z).

Remark that for any sheaf $\underline{F}$ on X :

$$\underline{F}_Z \cong \underline{A}_Z \otimes_A \underline{F} \ .$$

1.3.4. Now we assume X is a C^0-manifold of dimension n. We denote by $\underline{\omega}_X$ the orientation sheaf on X. Recall that $\underline{\omega}_X$ is the sheaf associated to the presheaf $U \longrightarrow \mathrm{Hom}(H^n_c(U,\underline{\mathbb{Z}}_X), \underline{\mathbb{Z}}_X)$, and that $\underline{\omega}_X$ satisfies :

i) $\underline{\omega}_X$ is locally isomorphic to $\underline{\mathbb{Z}}_X$,

ii) $\underline{\omega}_X \otimes \underline{\omega}_X \cong \underline{\mathbb{Z}}_X$

iii) $\underline{\omega}_X \cong H^n_\Delta(\underline{\mathbb{Z}}_{X\times X})$, where Δ is the diagonal of $X \times X$, identified to X by the first projection.

Let $f : Y \longrightarrow X$ be a map of C^0-manifolds. One defines the relative orientation sheaf on Y, by setting :

$$\underline{\omega}_{Y/X} = \underline{\omega}_Y \otimes f^{-1} \underline{\omega}_X$$

We shall say that f is a topological embedding if f is a homeomorphism onto a locally closed subset of X. We shall say that f is a topological submersion of codimension ℓ if, locally on Y, f is isomorphic to the projection $\mathbb{R}^{n+\ell} \longrightarrow \mathbb{R}^n$.

Now recall the "Poincaré-Verdier duality theorem".

Theorem 1.3.1. (Verdier [1]). *There exists a functor* $f^! : D^+(X) \longrightarrow D^+(Y)$ *which satisfies* :

i) $f^!$ *is a right adjoint to* $f_!$, *i.e. for any* $\underline{G} \in Ob(D^b(Y))$, $\underline{F} \in Ob(D^+(X))$:

$$\mathbb{R}Hom(\mathbb{R}f_! \underline{G}, \underline{F}) \cong \mathbb{R}Hom(\underline{G}, f^! \underline{F}) ,$$

ii) *if* f *is a topological embedding, then for* $\underline{F} \in Ob(D^+(X))$:

$$f^! \underline{F} \cong f^{-1}(\mathbb{R}\Gamma_{f(Y)}(\underline{F})) ,$$

iii) *if* f *is a topological submersion of codimension* ℓ , *then* :

$$f^! \underline{\mathbb{Z}}_X \cong \underline{\omega}_{Y/X}[\ell]$$

and for any $\underline{F} \in Ob(D^+(X))$, *we have a natural isomorphism* :

$$f^{-1} \underline{F} \overset{\mathbb{L}}{\otimes} f^! \underline{\mathbb{Z}}_X \cong f^! \underline{F}$$

Since any map $Y \longrightarrow X$ may be decomposed by the graph map as an immersion followed by a submersion, one sees that $f^!(\cdot)$ is characterized by the properties ii) and iii).

Corollary 1.3.2. : Let X and Y be two C^0-manifolds, and let q_j $(j = 1,2)$ be the j-th projection on $X \times Y$. Let $\underline{F} \in Ob(D^b(X))$, $\underline{G} \in Ob(D^+(Y))$. Then :

$$\mathbb{R}\Gamma(X \times Y, \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F}, q_2^!\underline{G})) = \mathbb{R}Hom(\mathbb{R}\Gamma_c(X,\underline{F}), \mathbb{R}\Gamma(Y,\underline{G}))$$

Proof

Applying Theorem 1.3.1. to the map q_2, we get :

$$\mathbb{R}\Gamma(X \times Y, \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F}, q_2^!\underline{G}) \cong \mathbb{R}\Gamma(Y, \mathbb{R}\underline{Hom}(q_{2!}q_1^{-1}\underline{F}, \underline{G}) .$$

Put $M = \mathbb{R}\Gamma_c(X,\underline{F})$. Then $\mathbb{R}q_{2!}q_1^{-1}\underline{F}$ is the locally constant sheaf $\underline{M}_Y$ on Y, and one has :

$$\begin{aligned}\mathbb{R}\Gamma(Y, \mathbb{R}\underline{Hom}(\underline{M}_Y, \underline{G})) &\cong \mathbb{R}Hom(M, \mathbb{R}\Gamma(Y,\underline{G})) \\ &= \mathbb{R}Hom(\mathbb{R}\Gamma_c(X,\underline{F}), \mathbb{R}\Gamma(Y,\underline{G})). \quad \square\end{aligned}$$

1.3.5. Let us recall some useful formulas. Let $f : Y \longrightarrow X$ be a continuous map. Then :

$$\mathbb{R}Hom(f^{-1}\underline{F}, \underline{G}) \cong \mathbb{R}Hom(\underline{F}, \mathbb{R}f_* \underline{G})$$

$$\mathbb{R}f_* \mathbb{R}\underline{Hom}(f^{-1}\underline{F}, \underline{G}) \cong \mathbb{R}\underline{Hom}(\underline{F}, \mathbb{R}f_* \underline{G})$$

Now assume Y and X are C^0-manifolds.

Then for $\underline{F} \in Ob(D^b(X))$, $F' \in Ob(D^+(X))$:

$$f^! \mathbb{R}\underline{Hom}(\underline{F},\underline{F}') \cong \mathbb{R}\underline{Hom}(f^{-1}\underline{F}, f^!F') .$$

For $\underline{G} \in Ob(D^b(Y))$, $\underline{F} \in Ob(D^+(X))$:

$$\mathbb{R}f_* \, \mathbb{R}\underline{Hom}(\underline{G}, f^! \underline{F}) \cong \mathbb{R}\underline{Hom}(\mathbb{R}f_! \underline{G}, \underline{F})$$

For $\underline{G} \in \mathrm{Ob}(D^+(Y))$, $\underline{F} \in \mathrm{Ob}(D^+(X))$:

$$\underline{F} \overset{\mathbb{L}}{\otimes} \mathbb{R}f_! \, \underline{G} \cong \mathbb{R}f_!(f^{-1} F \overset{\mathbb{L}}{\otimes} \underline{G}) ,$$

(for this last formula, we assumed $\mathrm{wg\ell d}(A) < \infty$).

Finally consider a cartesian diagram :

$$\begin{array}{ccc} Y' & \xrightarrow{\quad f' \quad} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[\quad f \quad]{} & X \end{array}$$

(i.e. : $Y' \cong Y \underset{X}{\times} X'$). Let $\underline{G} \in \mathrm{Ob}(D^+(Y))$.

Then :

$$\mathbb{R}f'_* \, g'^{!}(\underline{G}) \cong g^! \, \mathbb{R}f_*(\underline{G})$$

$$\mathbb{R}f'_! \, g'^{-1}(\underline{G}) \cong g^{-1} \, \mathbb{R}f_!(\underline{G})$$

§1.4. An extension theorem for sheaves

1.4.1. First let us recall the so called "Mittag-Leffler condition" (cf. Grothendieck [2]). We shall write "M-L" instead of "Mittag-Leffler", for short.

Let $(E_n, \rho_{n,p})_{n \in \mathbb{N}}$ be a projective system of abelian groups. One says that the M-L condition is satisfied if for any $n \in \mathbb{N}$ the decreasing sequence of sub-groups of E_n, $(\rho_{n,p}(E_p))_{p \geq n}$, is stationary.

Let $(E_n^{\bullet})_n$ be a complex of projective systems of abelian groups, $E_\infty^{\bullet}$ the complex where $E_\infty^i = \varprojlim_n E_n^i$. The natural maps from $\varprojlim_n E_n^i$

to E_p^i define the morphism :

$$\phi_i : H^i(E_\infty^\bullet) \longrightarrow \varprojlim_n H^i(E_n^\bullet)$$

Proposition 1.4.1. (Grothendieck [2]) Assume that for any i the projective system $(E_n^i)_n$ satisfies the M-L condition. Then :

a) for any i, the morphism ϕ_i is surjective

b) assume moreover that $(H^{i-1}(E_n))_n$ satisfies the M-L condition. Then ϕ_i is bijective.

Proposition 1.4.2. (Kashiwara [5]) Let X be a topological space, $\underline{F} \in Ob(D^+(X))$, $\{U_n\}$ an increasing sequence of open subsets of X and $\{Z_n\}$ a decreasing sequence of closed subsets of X. Set $U = \bigcup_n U_n$ and $Z = \bigcap_n Z_n$.

a) For any k, $\phi_k : H_Z^k(U;\underline{F}) \longrightarrow \varprojlim_n H_{Z_n}^k(U_n ; \underline{F})$ is surjective.

b) If $\{H_{Z_n}^{k-1}(U_n ; \underline{F})\}_n$ satisfies the M-L condition, then ϕ_k is bijective.

Proof

We may assume F^k are flabby. If we denote by $E_n^\bullet$ the simple complex associated with the double complex :

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \Gamma(U_n;\underline{F}^{k-1}) & \longrightarrow & \Gamma(U_n;\underline{F}^k) & \longrightarrow & \Gamma(U_n;\underline{F}^{k+1}) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \Gamma(U_n \setminus Z_n;\underline{F}^{k-1}) & \longrightarrow & \Gamma(U_n \setminus Z_n;\underline{F}^k) & \longrightarrow & \Gamma(U_n \setminus Z_n;\underline{F}^{k+1}) & \longrightarrow \cdots
\end{array}$$

then $H_{Z_n}^k(U_n;\underline{F}) = H^k(E_n^\bullet)$ and $H_Z^k(U;\underline{F}) = H^k(\varprojlim_n E_n^\bullet)$.

Since $\{E_n^k\}_n$ satisfies the M-L conditions this proposition follows from the preceding one. □

1.4.2. The following result will be one of the main tool in this paper.

Theorem 1.4.3. (cf. Kashiwara [5]) *Let* $\{\Omega_t\}_{t\in\mathbb{R}}$ *be a family of open subsets of a Hausdorff space* X *and* $\underline{F} \in \mathrm{Ob}(D^+(X))$. *We assume the following conditions* :

(i) $\Omega_t = \bigcup_{s<t} \Omega_s$.

(ii) *For* $t \geq s$, $\overline{\Omega_t \setminus \Omega_s} \cap \mathrm{supp}(\underline{F})$ is compact .

(iii) *Setting* $Z_s = \bigcap_{t>s} \overline{\Omega_t \setminus \Omega_s}$, *we have for any* s, t *with* $s \leq t$ *any* $x \in Z_s \setminus \Omega_t$, $(\mathbb{R}\Gamma_{X\setminus\Omega_t}(\underline{F}))_x = 0$.

Then we have the isomorphism :

$$\mathbb{R}\Gamma(\bigcup_s \Omega_s \ ; \ \underline{F}) \xrightarrow{\sim} \mathbb{R}\Gamma(\Omega_t \ ; \ \underline{F})$$

for any t.

Proof

By the same argument as in (Kashiwara [5]) it is sufficient to show

$$(1.4.1) \qquad \varinjlim_{t>t_0} H^k(\Omega_t;\underline{F}) \longrightarrow H^k(\Omega_{t_0};\underline{F})$$

is an isomorphism for any $t_0 \in \mathbb{R}$ and any $k \in \mathbb{Z}$. Replacing X with $\mathrm{supp}\,\underline{F}$ we may assume from the begining that $\overline{\Omega_t \setminus \Omega_s}$ is compact for $t \geq s$. Let us denote by j_t the inclusion map $\Omega_t \hookrightarrow X$. Then (iii) implies :

$$\mathbb{R}\Gamma_{X\setminus\Omega_t}(\underline{F})\Big|_{Z_{t_0}} \xrightarrow{\sim} \mathbb{R}\Gamma_{X\setminus\Omega_{t_0}}(\underline{F})\Big|_{Z_{t_0}} = 0$$

for any $t > t_o$. Thus, we obtain

$$\mathbb{R}j_{t*}j_t^{-1}\mathbb{R}\Gamma_{X\setminus\Omega_{t_o}}(\underline{F})\Big|_{Z_{t_o}} = 0$$

Hence for any k :

$$0 = H^k(Z_{t_o}\ ;\ \mathbb{R}j_{t*}j_t^{-1}\mathbb{R}\Gamma_{X\setminus\Omega_{t_o}}(\underline{F}))$$

$$= \varinjlim_{U\supset Z_{t_o}} H^k(U\cap\Omega_t\ ;\ \mathbb{R}\Gamma_{X\setminus\Omega_{t_o}}(\underline{F}))$$

where U runs over a fundamental system of open neighborhoods of Z_{t_o}.

This implies

$$\varinjlim_{U\supset Z_{t_o}} H^k((U\cup\Omega_{t_o})\cap\Omega_t\ ;\ \underline{F}) \cong H^k(\Omega_{t_o}\ ;\ \underline{F})$$

Since for any open $U\supset Z_{t_o}$, there exists $t > t_o$ such that $U\cup\Omega_{t_o}\supset\Omega_t$, we have (1.4.1). □

§1.5. G-topology

1.5.1. We assume that X is open in a (finite dimensional) real vector space E.

Proposition 1.5.1. : Let $\underline{F}$ be a sheaf over X such that for any convex compact set K in X, the natural map from $\underline{F}(X)$ to $\underline{F}(K)$ is surjective. Then for any convex open set Ω in X we have :

$$H^i(\Omega\ ;\ \underline{F}) = 0 \qquad \forall i > 0$$

Proof

By replacing X with Ω we may assume X is convex. We proceed by induction on $\dim X$.

Lemma 1.5.2. : Let $I = [0,1]$, and let $\underline{F}$ be a sheaf over I such that the natural map $\underline{F}(I) \longrightarrow \underline{F}_t$ is surjective for any $t \in I$. Then :

$$H^j(I,\underline{F}) = 0 \quad \forall j > 0$$

Proof of the lemma

Let $j \geq 1$, and let s belong to $H^j(I;\underline{F})$. We shall prove $s = 0$. Let f_{t_1,t_2} be the natural map :

$$f_{t_1,t_2} : H^j(I;\underline{F}) \longrightarrow H^j([t_1,t_2];\underline{F})$$

Set :

$$A = \{t \in [0,1] \; ; \; f_{0,t}(s) = 0\}$$

Then $0 \in A$, and $0 \leq t' \leq t$, $t \in A$ implies $t' \in A$. Moreover A is open since we have :

$$\varinjlim_{t>t_0} H^j([0,t];\underline{F}) = H^j([0,t_0];\underline{F})$$

and $f_{0,t_0}(s) = 0$ implies $f_{0,t}(s) = 0$ for some $t > t_0$.

Consider the Mayer-Vietoris sequence associated to the decomposition $[0,t_0] = [0,t] \cup [t,t_0]$:

$$\longrightarrow H^j([0,t_0];\underline{F}) \longrightarrow H^j([0,t];\underline{F}) \oplus H^j([t,t_0];\underline{F}) \longrightarrow H^j(\{t\};\underline{F}) = 0$$

By applying our hypothesis we get for any $j > 0$, $\forall t$, $0 \leq t \leq t_0$:

$$H^j([0,t_0];\underline{F}) = H^j([0,t];\underline{F}) \oplus H^j([t,t_0];\underline{F})$$

Let $t_0 = \sup A$. Then :

$$H^j([0,t_0];\underline{F}) = H^j([t,t_0];\underline{F})$$

for any t such that $0 \leq t < t_0$. But $f_{t,t_0}(s) = 0$ for

$0 \leqslant t_o - t << 1$ since $\lim\limits_{\substack{\to \\ t<t_o}} H^j([t,t_o];\underline{F}) = 0$. Thus $A = [0,1]$. □

<u>End of the proof of Proposition 1.5.1</u>.

Let F be a vector space with $\dim F = \dim E-1$, p a surjective linear map from E to F, $Y = p(X)$.

Let K be a convex compact set in X and set $\tilde{p} = p|_K$. First we remark that :

$$\mathbb{R}\tilde{p}_*(\underline{F}|_K) = \tilde{p}_*(\underline{F}|_K)$$

since $(\mathbb{R}\tilde{p}_*(\underline{F}|_K))_y = \mathbb{R}\Gamma(\tilde{p}^{-1}(y) ; \underline{F}|_K)$ and $\tilde{p}^{-1}(y)$ being isomorphic to a closed interval we may apply the preceding lemma, since the map $(\underline{F}|_K)(\tilde{p}^{-1}(y)) \longrightarrow (\underline{F}|_K)_x = \underline{F}_x$ is surjective by the hypothesis.

Now let S be a compact convex set in Y. The map :

$$\Gamma(Y ; \tilde{p}_*(\underline{F}|_K)) \longrightarrow \Gamma(S ; \tilde{p}_*(\underline{F}|_K))$$

is surjective since $\Gamma(Y ; \tilde{p}_*(\underline{F}|_K)) = \Gamma(K ; \underline{F})$ and $\Gamma(S ; \tilde{p}_*(\underline{F}|_K))$ $= \Gamma(p^{-1}(S) \cap K ; \underline{F})$, and each of this space is the image of $\Gamma(X ; \underline{F})$.

By the induction hypothesis we obtain :

$$H^j(K ; \underline{F}) = H^j(Y ; \tilde{p}_*(\underline{F}|_K)) = 0 \qquad \forall j > 0$$

Finally letting (K_n) be an increasing sequence of convex compact sets in Ω, with $\bigcup_n K_n = \Omega$ and applying Proposition 1.4.2. we obtain the desired vanishing of cohomology groups. □

<u>1.5.2</u>. Now let G be a closed convex cone in the real (finite dimensional) vector space E, with $0 \in G$. The G-topology over E is the topology for which open sets Ω are open in the usual topology and :

$$\Omega = \Omega + G$$

We say G-open or G-closed for open or closed in the G-topology. For X in E, X_G is the set X endowed with the topology induced by the G-topology (cf. Kashiwara-Schapira [2, §3]). We shall denote by ϕ_X or ϕ the natural continuous map :

$$\phi_X : X \longrightarrow X_G$$

We also sometimes write ϕ_G instead of ϕ to specify G.

Remark that $X_{\{0\}} = X$.

Theorem 1.5.3. : Assume X G-open. Let $\underline{F}$ be a sheaf over X_G. Then we have the natural isomorphism :

$$\underline{F} \xrightarrow{\sim} \mathbb{R}\phi_* \phi^{-1}(\underline{F})$$

Proof

We set $\tilde{\underline{F}} = \phi^{-1}(\underline{F})$.

Lemma 1.5.4. : For a convex open set U in X we have the natural isomorphism :

$$\underline{F}(U + G) \xrightarrow{\sim} \tilde{\underline{F}}(U)$$

Proof of lemma 1.5.4.

i) $\underline{F}(U + G) \longrightarrow \tilde{\underline{F}}(U)$ is injective. In fact let s be a section of $\underline{F}$ over $U + G$ such that $s_x = 0$ in $\tilde{\underline{F}}_x$ for any $x \in U$. There exists a G-open set W which contains x such that $s_y = 0$ $\forall y \in W$. Thus $s_{x+\gamma} = 0$ $\forall x \in U$, $\forall \gamma \in G$.

ii) $\underline{F}(U + G) \longrightarrow \tilde{\underline{F}}(U)$ is surjective. A section s of $\tilde{\underline{F}}$ over U is defined by a open covering $U = \bigcup_i U_i$, and sections s_i of $\underline{F}$ over $U_i + G$ with $s_x = s_{i,x}$ $\forall x \in U_i$. We shall show :

$$x \in (U_i + G) \cap (U_j + G) \Longrightarrow s_{i,x} = s_{j,x}$$

Let $x_i \in U_i \cap (x + G^a)$, $x_j \in U_j \cap (x + G^a)$,
$x_t = t\, x_i + (1-t)x_j \in U \cap (x + G^a)$. Set :

$A = \{t \in [0,1]$; for any k such that $x_t \in U_k$, we have $s_{i,x} = s_{k,x}\}$.

We can also define A by :

$A = \{t \in [0,1]$; there exists k such that $x_t \in U_k$ and $s_{i,x} = s_{k,x}\}$.

Then A is open and closed and contains 1. Thus $A = [0,1]$. □

Lemma 1.5.5. : We have the isomorphism :

$$\underline{F} \cong \phi_* \phi^{-1}(\underline{F})$$

Proof

Let U be G-open and convex. Then :

$$\phi_* \phi^{-1}(\underline{F})(U) = \phi^{-1}(\underline{F})(U) = \underline{F}(U). \qquad \square$$

End of the proof of Theorem 1.5.3.

To prove that $\mathbb{R}^j \phi_* \phi^{-1}(\underline{F}) = 0$ for $j > 0$ we may assume $\underline{F}$ flabby. In that case the map $\tilde{\underline{F}}(\Omega) \longrightarrow \tilde{\underline{F}}(K)$ is surjective for $K \subset \Omega$, K compact, Ω open, K and Ω convex, since :

$$\tilde{\underline{F}}(\Omega) = \underline{F}(\Omega+G),\ \tilde{\underline{F}}(K) = \varinjlim_{U \supset K} \underline{F}(U+G) .$$

By Proposition 1.4.2. we get $H^j(\Omega,\tilde{\underline{F}}) = 0$ $\forall j > 0$, $\forall\Omega$ convex, thus

$$(\mathbb{R}^j \phi_* \phi^{-1}(\underline{F}))_x = \varinjlim_{U} H^j(U,\tilde{\underline{F}}) = 0$$

where U runs over the set of G-open and convex neighboroods of x. □

As an immediate consequence we get :

Corollary 1.5.6. : For any open convex subset U of X and

$\underline{F} \in D^+(X_G)$ the natural morphism :

$$\mathbb{R}\Gamma(U+G\ ;\ \underline{F}) \longrightarrow \mathbb{R}\Gamma(U, \phi_G^{-1}\ \underline{F})$$

is an isomorphism.

1.5.3. Let E be a real finite dimensional vector space.

We shall need the following result.

Proposition 1.5.7. : Let $\underline{F} \in \mathrm{Ob}(D^+(E))$, let Ω_1 be a G-open subset and Ω a G^a-open subset of E. Assume $\Omega_1 \cap \Omega \subset\subset E$. Then we have :

$$H^j(\Omega_1\ ;\ \underline{F}_\Omega) \cong \varinjlim_K H^j_{K\cap\Omega_1}(\Omega_1\ ;\ \underline{F})$$

$$\cong \varinjlim_K H^j_{K\cap\Omega_1}(\Omega_1\ ;\ \mathbb{R}\phi_{G*}\underline{F})$$

where K runs over the set of G-closed subsets of E such that $K \cap \Omega_1 \subset \Omega$.

In particular if $\Omega \cap (K+G) \subset\subset E$ for any compact set K, $\mathbb{R}\phi_{G*}\underline{F} = 0$ implies $\mathbb{R}\phi_{G*}\underline{F}_\Omega = 0$.

Proof

We have :

$$H^j(\Omega_1\ ;\ \underline{F}_\Omega) \cong \varinjlim_K H^j_K(\Omega_1\ ;\ \underline{F})$$

where K runs over the set of closed subsets of Ω_1 contained in Ω.

For such a K, $K' = \overline{K} + G^a$ is closed in E, $\overline{K}$ denoting the closure of K in E.

We have :

$$K' \cap \Omega_1 \subset \Omega .$$

In fact if $x = y + \gamma \in \Omega_1$, with $y \in \overline{K}$, $\gamma \in G^a$, then :

$$y = x - \gamma \in (\Omega_1 + G) \cap \overline{K} = \Omega_1 \cap \overline{K} = K.$$

Therefore :

$$x \in y + G^a \subset \Omega + G^a = \Omega \quad .$$

Since $K \cap \Omega_1 \subset K' \cap \Omega_1$, we have :

$$\underset{K}{\underrightarrow{\lim}}\; H^j_K(\Omega_1 ; \underline{F}) = \underset{K'}{\underrightarrow{\lim}}\; H^j_{K' \cap \Omega_1}(\Omega_1 ; \underline{F})$$

where K' runs over the set of G-closed subsets such that $K' \cap \Omega_1 \subset \Omega$. □

CHAPTER 2 - MICROLOCALIZATION

The main results of this chapter are due to Sato [2] and Sato-Kashiwara-Kawaï [1] ; (cf. also Malgrange [1] and Brylinski-Malgrange-Verdier [1] for §.2.1.).

§2.1. Fourier-Sato transformation

For the reader's convenience we recall here the main results concerning the Fourier-Sato transformation, without proofs, and we refer to Sato-Kashiwara-Kawaï [1], Kashiwara-Kawaï [3], Malgrange [1] and Brylinski-Malgrange-Verdier [1].

2.1.1. Let Z be a locally compact topological space, and let $E \longrightarrow Z$ be a vector bundle. We shall denote by $D^+_{conic}(E)$ the full subcategory of $D^+(E)$ of complexes whose cohomology groups are locally constant on any half-line of E. Let E^* denote the dual vector bundle of E. We denote by q_1 and q_2 the projections from E and E^* onto Z and by p_1 and p_2 the projections from $E \underset{Z}{\times} E^*$ onto E and E^*, respectively. We set

$$P = \{(x,y) \in E \underset{Z}{\times} E^* \;;\quad \langle x,y\rangle \geqslant 0\}$$

$$P' = \{(x,y) \in E \underset{Z}{\times} E^* \;;\quad \langle x,y\rangle \leqslant 0\}$$

Proposition 2.1.1. : For $\underline{F} \in Ob(D^+_{conic}(F))$, we have :

$$\mathbb{R}p_{2*}\,\mathbb{R}\Gamma_P\,(p_1^{-1}\underline{F}) \;\cong\; \mathbb{R}p_{2!}((p_1^{-1}\underline{F})_{P'}) .$$

Definition 2.1.2. : We set, for $\underline{F} \in Ob(D^+_{conic}(E))$,

$$\underline{F}^{\wedge} = \mathbb{R}p_{2*}\,\mathbb{R}\Gamma_P\,(p_1^{-1}\underline{F}) \cong \mathbb{R}p_{2!}(p_1^{-1}\underline{F}_{P'})$$

and call it the Fourier-Sato transform of $\underline{F}$.

For $\underline{G} \in Ob(D^+_{conic}(E^*))$, we set :

$$\underline{G}^\vee = \mathbb{R}p_{1*}\mathbb{R}\Gamma_{P'}(p_2^! \underline{G}) \cong \mathbb{R}p_{1!}((p_2^! \underline{G})_P)$$

One can see easily :

$$\mathbb{R}q_{2*} \underline{F}^\wedge \cong \mathbb{R}q_{1!} \underline{F}$$

$$\mathbb{R}q_{1!} \underline{G}^\vee \cong \mathbb{R}q_{2*} \underline{G}$$

Theorem 2.1.3. : i) The functors $\wedge$ from $D^+_{conic}(E)$ to $D^+_{conic}(E^*)$ and $\vee$ from $D^+_{conic}(E^*)$ to $D^+_{conic}(E)$ are inverse to each other.

ii) Let $\underline{F} \in Ob(D^b_{conic}(E))$, $\underline{F}' \in Ob(D^+_{conic}(E))$.

Then :

$$\mathbb{R}\,Hom(\underline{F},\underline{F}') \cong \mathbb{R}\,Hom(\underline{F}^\wedge, \underline{F}'^\wedge) .$$

iii) Let $\underline{F} \in Ob(D^b_{conic}(E))$. Then :

$$\mathbb{R}\,\underline{Hom}(\underline{F}, \underline{A}_E)^\wedge \cong \mathbb{R}\,\underline{Hom}(\underline{F}^\wedge, \underline{A}_{E^*}) \overset{\mathbb{L}}{\otimes} q_2^{-1}\mathbb{R}q_{1*}\underline{A}_E .$$

One can describe easily the sections of $\underline{F}^\wedge$ on convex sets.

Proposition 2.1.4. : i) Let U be an open convex subset of E^*. Then for $\underline{F} \in Ob(D^+_{conic}(E))$:

$$\mathbb{R}\Gamma(U,\underline{F}^\wedge) \cong \mathbb{R}\Gamma_{U^\circ}(E, \underline{F})$$

(Recall that U° is the polar set to U).

ii) Let A be a closed convex subset of E^*. Then :

$$\mathbb{R}\Gamma_A(E^*,\underline{F}^\wedge) \cong \mathbb{R}\Gamma(Int(A^\circ), \underline{F}) \otimes \underline{\omega}_{E/Z}\,[-\ell]$$

where ℓ is the dimension of the fiber, and $\underline{\omega}_{E/Z}$ the relative orientation sheaf on E.

2.1.2. Let $f : E_1 \longrightarrow E_2$ be an homomorphism of vector bundles over Z and let $f^* : E_2^* \longrightarrow E_1^*$ be the dual homomorphism.

Proposition 2.1.5. : i) Let $\underline{F} \in \mathrm{Ob}(D^+_{conic}(E_1))$. Then :

$$({}^t f^*)^{-1} \underline{F}^\wedge \cong (\mathbb{R}f_! \underline{F})^\wedge .$$

ii) Let $\underline{F} \in \mathrm{Ob}(D^+_{conic}(E_2))$. Then :

$$(f^! \underline{F})^\wedge \cong \mathbb{R}({}^t f^*)_* (\underline{F}^\wedge) .$$

Now let $Z' \xrightarrow[h]{} Z$ be a continuous map, E a vector bundle over Z, $E' = Z' \times_Z E$. Let $f : E' \longrightarrow E$ and $g : E'^* \longrightarrow E^*$ be the natural maps associated to h .

Proposition 2.1.6. : i) Let $\underline{F} \in \mathrm{Ob}(D^+_{conic}(E))$. Then :

$$(f^! \underline{F})^\wedge \cong g^! (\underline{F}^\wedge) ,$$

$$(f^{-1}\underline{F})^\wedge \cong g^{-1} (\underline{F}^\wedge).$$

ii) Let $\underline{F} \in \mathrm{Ob}(D^+_{conic}(E'))$. Then :

$$(\mathbb{R}f_* \underline{F})^\wedge \cong \mathbb{R}g_* (\underline{F}^\wedge) ,$$

$$(\mathbb{R}f_! \underline{F})^\wedge \cong \mathbb{R}g_! (\underline{F}^\wedge) .$$

§2.2. Specialization

2.2.1. Let X be a real manifold of class C^α ($\alpha \geqslant 2$), M a submanifold $i : M \hookrightarrow X$ the embedding of M in X. We construct a new manifold as follows. Let $X = \bigcup_i U_i$ be an open covering, and $\phi_i : U_i \longrightarrow \mathbb{R}^n$ be an open embedding such that $U_i \cap M = \phi_i^{-1}(\{0\}^\ell \times \mathbb{R}^{n-\ell})$.
Set :

$$V_i = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n;\ (tx', x'') \in U_i\}$$

where $x = (x',x'') \in \mathbb{R}^{\ell} \times \mathbb{R}^{n-\ell}$. We glue the V_i's as follows. On $V_i \underset{U_i}{\times} (U_i \cap U_j)$, the first ℓ components of $\phi_j(\phi_i^{-1}(tx',x''))$ vanish when $t = 0$. Define $\psi_{j,i}(t,x) = (y',y'')$ by :

$$(ty',y'') = \phi_j\ \phi_i^{-1}(tx',x'') .$$

Then we identify $(t_i,x_i) \in V_i$ and $(t_j,x_j) \in V_j$ if $t_i = t_j$ and $x_j = \psi_{j,i}(t_i,x_i)$.

Let $\tilde{X}_M$ be the manifold so obtained. Then we have a map $p : \tilde{X}_M \longrightarrow X$ given by $V_i \ni (t,x) \longmapsto \phi_i^{-1}(tx',x'')$, and a map $t : \tilde{X}_M \longrightarrow \mathbb{R}$ given by $(t,x) \longmapsto t$. Hence we have a map $: \tilde{X}_M \longrightarrow X \times \mathbb{R}$. By this map, $p^{-1}(X \setminus M)$ is isomorphic to $(X \setminus M) \times (\mathbb{R}\setminus\{0\})$, $t^{-1}(c)$ is isomorphic to X for $c \in \mathbb{R}\setminus\{0\}$, and $t^{-1}(0)$ is isomorphic to the normal bundle T_MX to M. Hereafter we identify $t^{-1}(0)$ with T_MX. Note that the multiplicative group $\mathbb{R}\setminus\{0\}$ acts on $\tilde{X}_M$ by

$$c(t,x',x'') \longmapsto (c^{-1}t,\ cx',\ x'') .$$

Let Ω be the open set obtained as the inverse image of $\mathbb{R}^+ = \{c \in \mathbb{R}, c > 0\}$ by the projection $t : \tilde{X}_M \longrightarrow \mathbb{R}$, and let j be the inclusion map $\Omega \hookrightarrow \tilde{X}_M$. We denote by τ (resp. π) the projection from T_MX (resp. T_M^*X) to M.

<u>Definition 2.2.1.</u> : <u>Let</u> $\underline{F} \in \mathrm{Ob}(D^+(X))$. <u>One sets</u> :

$$\nu_M(\underline{F}) = (\mathbb{R}j_* j^{-1}\ p^{-1}\ \underline{F})|_{T_MX}$$

<u>and says</u> $\nu_M(\underline{F})$ <u>is the specialization of</u> $\underline{F}$ <u>along</u> M.

Let us mention that $\nu_M(\underline{F})$ may also be obtained as follows. One endows $(X \setminus M) \cup T_M X$ with its natural topology of blown-up space. This topology induces the usual topology on $X \setminus M$ and on $T_M X$, the set $X \setminus M$ is open and a subset V of $(X \setminus M) \cup T_M X$ is a neighborhood of a subset Z of $T_M X$ if $V \cap T_M X$ is a neighborhood of Z in $T_M X$ and $V \cap (X \setminus M)$ contains an open set U such that $C_M(X \setminus U) \cap Z = \emptyset$. Let k be the embedding $X \hookrightarrow (X \setminus M) \cup T_M X$. Then :

$$(2.2.1) \qquad \nu_M(\underline{F}) \cong (\mathbb{R}k_* \underline{F})|_{T_M X}$$

By the definition of $\nu_M(\underline{F})$ one easily obtains :

<u>Proposition 2.2.1.</u> : <u>Let</u> $\underline{F} \in \mathrm{Ob}(D^+(X))$. <u>Then</u> :

i) $\nu_M(\underline{F}) \in \mathrm{Ob}(D^+_{conic}(T_M X))$.

ii) $\nu_M(\underline{F})|_M \cong \underline{F}|_M \cong \mathbb{R}\tau_* \nu_M(\underline{F})$

$\mathbb{R}\tau_!(\nu_M(\underline{F})) \cong i^! \underline{F}$ $(= \mathbb{R}\Gamma_M(\underline{F})|_M)$.

iii) <u>Let</u> $v \in T_M X$. <u>Then</u> :

$$H^j(\nu_M(\underline{F}))_v = \varinjlim_U H^j(U,\underline{F})$$

<u>where</u> U <u>runs over the family of open subsets of</u> X <u>such that</u> $v \notin C_M(X \setminus U)$.

iv) <u>Let</u> V <u>be a conic open subset of</u> $T_M X$. <u>Then</u> :

$$H^j(V, \nu_M(\underline{F})) = \varinjlim_U H^j(U,\underline{F})$$

<u>where</u> U <u>runs over the family of open subsets of</u> X <u>such that</u> $C_M(X \setminus U) \cap V = \emptyset$.

v) <u>Let</u> A <u>be a closed conic subset of</u> $T_M X$. <u>Then</u> :

$$H^j_A(T_M X, \nu_M(\underline{F})) = \varinjlim_{Z,U} H^j_Z(U,\underline{F})$$

where Z *runs over the family of closed subsets of* X *such that* $C_M(Z) \subset A$, *and* U *runs over the family of open neighborhoods of* M *in* X.

2.2.2. Let $f : Y \longrightarrow X$ be a map of class C^{α} $(\alpha \geqslant 2)$. Let M (resp. N) be a closed submanifold of X (resp. Y). We assume $f(N) \subset M$, and we denote by f' the map : $T_N Y \longrightarrow T_M X$. Remark that f' is the composite of the maps : $T_N Y \longrightarrow N \underset{M}{\times} T_M X \longrightarrow T_M X$.

Proposition 2.2.2. : *Let* $\underline{G} \in \mathrm{Ob}(D^+(Y))$. *There exists a canonical morphism* :

$$\mathbb{R}f'_! \, \nu_N(\underline{G}) \longrightarrow \nu_M(\mathbb{R}f_! \underline{G})$$

If moreover f *is transversal to* M, *and proper over* $\mathrm{supp}(\underline{G})$, *this morphism is an isomorphism.*

Proof

Let p_X, t_X, j_X be the maps associated to the blowing up of M in X, let $\Omega_X = t_X^{-1}(\mathbb{R}^+)$ and similarly for p_Y, t_Y, j_Y, Ω_Y . Let f' denote the map $T_N Y \longrightarrow T_M X$, $\tilde{f}'$ the map $\tilde{Y}_N \longrightarrow \tilde{X}_M$, $\tilde{f}$ the map $\Omega_Y \longrightarrow \Omega_X$, $\tilde{p}_X$ (resp. $\tilde{p}_Y$) the restriction of p_X (resp. p_Y) to Ω_X (resp. Ω_Y). Consider the diagram (2.2.2) below where all the square are cartesian :

$$(2.2.2)\qquad \begin{array}{ccccccc} T_N Y & \overset{s_Y}{\hookrightarrow} & \tilde{Y}_N & \overset{j_Y}{\hookleftarrow} & \Omega_Y & \overset{\tilde{p}_Y}{\longrightarrow} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \tilde{f}'} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ T_M X & \overset{s_X}{\hookrightarrow} & \tilde{X}_M & \overset{j_X}{\hookleftarrow} & \Omega_X & \overset{\tilde{p}_X}{\longrightarrow} & X \end{array}$$

We have :

$$\nu_M(\mathbb{R}f_! \underline{G}) = s_X^{-1} \, \mathbb{R}j_{X*} \tilde{p}_X^{-1} \mathbb{R}f_! \underline{G} \;\cong\; s_X^{-1} \, \mathbb{R}j_{X*} \mathbb{R}\tilde{f}_! \tilde{p}_Y^{-1} \underline{G}$$

The morphism : $\mathbb{R}j_{X*}\tilde{f}_!(\cdot) \longleftarrow \mathbb{R}\tilde{f}'_!\,\mathbb{R}j_{Y*}(\cdot)$ defines :

$$\nu_M(\mathbb{R}f_!\,\underline{G}) \longleftarrow s_X^{-1}\,\mathbb{R}\tilde{f}'_!\,\mathbb{R}j_{Y*}\tilde{p}_Y^{-1}\underline{G} \cong \mathbb{R}f'_!\,s_Y^{-1}\,\mathbb{R}j_{Y*}\tilde{p}_Y^{-1}\underline{G}$$

$$\cong \mathbb{R}f'_!\,\nu_N(\underline{G})$$

Now assume f is transversal to M and proper over $\mathrm{supp}\,\underline{G}$. Then $\tilde{f}$ is proper over $\mathrm{supp}\,\tilde{p}_Y^{-1}\,\underline{G}$, and $\mathbb{R}j_{X*}\tilde{f}_!(\tilde{p}_Y^{-1}\,\underline{G}) \cong \mathbb{R}\tilde{f}'_!\,\mathbb{R}j_{Y*}(\tilde{p}_Y^{-1}\,\underline{G})$, hence all the morphisms are isomorphisms. $\square$

<u>Proposition 2.2.3.</u> : <u>Let</u> X, Y, M, N <u>be as precedingly. Let</u> $\underline{F} \in \mathrm{Ob}(D^+(X))$. <u>There are canonical morphisms</u> :

$$\alpha \;:\; f'^{-1}\,\nu_M(\underline{F}) \longrightarrow \nu_N(f^{-1}\,\underline{F})$$

<u>and</u>

$$\beta \;:\; \nu_N(f^{!}\,\underline{F}) \longrightarrow f'^{!}\,\nu_M(\underline{F})$$

<u>such that the diagram below commutes</u> :

$$\begin{array}{ccc}
f^{!}\underline{\mathbb{Z}}_X \otimes f'^{-1}\,\nu_M(\underline{F}) & \xrightarrow{\;f^{!}\underline{\mathbb{Z}}_X \otimes \alpha\;} & f^{!}\underline{\mathbb{Z}}_M \otimes \nu_N(f^{-1}\,\underline{F}) \\
\Vert & & \Big\downarrow \\
f'^{!}\underline{\mathbb{Z}}_{T_M X} \otimes f'^{-1}\,\nu_M(\underline{F}) & & \\
\Big\downarrow & & \\
f'^{!}\,\nu_M(\underline{F}) & \xleftarrow[\;\beta\;]{} & \nu_N(f^{!}\,\underline{F})
\end{array}$$

<u>Moreover if</u> $f : Y \longrightarrow X$ <u>and its restriction</u> $f|_N : N \longrightarrow M$ <u>are smooth, then</u> α <u>and</u> β <u>are isomorphisms</u>.

<u>Proof</u>

Consider the diagram (2.2.2). We have :

$$f'^{-1}\nu_M(\underline{F}) = f'^{-1}s_X^{-1}\mathbb{R}j_{X*}\tilde{p}_X^{-1}\,\underline{F} \cong s_Y^{-1}\tilde{f}'^{-1}\mathbb{R}j_{X*}\tilde{p}_X^{-1}\,\underline{F}$$

and

$$\nu_N(f^{-1}\underline{F}) = s_Y^{-1}\,\mathbb{R}j_{Y*}\tilde{p}_Y^{-1}\,f^{-1}\,\underline{F} = s_Y^{-1}\,\mathbb{R}j_{Y*}\tilde{f}^{-1}\,\tilde{p}_X^{-1}\,\underline{F}\ .$$

Hence α is given by the morphism

$$\tilde{f}'^{-1}\ \mathbb{R}j_{X*}(\cdot) \longrightarrow \mathbb{R}j_{Y*}\tilde{f}^{-1}(\cdot)\ .$$

Similarly we have :

$$f'^{!}\ \nu_M(\underline{F}) = f'^{!}\ s_X^{-1}\ \mathbb{R}j_{X*}\tilde{p}_X^{-1}\ \underline{F}$$

$$\nu_N(f^{!}\ \underline{F}) = s_Y^{-1}\ \mathbb{R}j_{Y*}\ \tilde{p}_Y^{-1}\ f^{!}\underline{F} = s_Y^{-1}\ \mathbb{R}j_{Y*}\ \tilde{f}^{!}\ \tilde{p}_X^{-1}\ \underline{F}$$

$$= s_Y^{-1}\ \tilde{f}'^{!}\ \mathbb{R}j_{X*}\ \tilde{p}_X^{-1}\underline{F}$$

The morphism $s_Y^{-1}\ \tilde{f}'^{!}(\cdot) \longrightarrow f'^{!}\ s_X^{-1}(\cdot)$ gives β. The commutativity follows immediately.

If $Y \longrightarrow X$ and $N \longrightarrow M$ are smooth, then $\tilde{f}'$ is smooth. Therefore $\tilde{f}'^{-1}\ \mathbb{R}j_{X*}\underline{F} \longrightarrow \mathbb{R}j_{Y*}\tilde{f}^{-1}\ \underline{F}$ and $s_Y^{-1}\ \tilde{f}'^{!}\ \underline{F} \longrightarrow f'^{!}\ s_X^{-1}\ \underline{F}$ are isomorphisms, and hence so are α and β . □

§3. Sato's microlocalization

2.3.1. Let X be a manifold of class C^{α} $(\alpha \geqslant 2)$, M a submanifold, $i : M \hookrightarrow X$ the embedding of M in X.

Definition 2.3.1. : Let $\underline{F} \in Ob(D^{+}(X))$. One sets :

$$\mu_M(\underline{F}) = (\nu_M(\underline{F}))^{\wedge}$$

and says $\mu_M(\underline{F})$ is the (Sato's) microlocalization of $\underline{F}$ along M.

Applying Propositions 2.2.1. and 2.1.4. one gets :

Proposition 2.3.2. : i) $\mu_M(\underline{F}) \in Ob(D^{+}_{conic}(T^{*}_{M}X))$.

ii) $\mu_M(\underline{F})|_M \cong i^!\underline{F}\ (\cong \mathbb{R}\Gamma_M(\underline{F})|_M) \cong \mathbb{R}\pi_* \mu_M(\underline{F})$

$\mathbb{R}\pi_! \mu_M(\underline{F}) \cong i^{-1}\underline{F} \overset{\mathbb{L}}{\otimes} i^! \underline{\mathbb{Z}}_X .$

iii) Let $p \in T^*_M X$. Then :

$$H^j(\mu_M(\underline{F}))_p = \varinjlim_{Z} H^j_Z(\underline{F})_{\pi(p)} ,$$

where Z runs over the family of closed subsets of X such that :

$$C_M(Z)_{\pi(p)} \subset \{v \in (T_M X)_{\pi(p)} ;\ \langle v,p\rangle > 0\} \cup \{0\} .$$

iv) Let U be a convex open subset of $T^*_M X$. Then :

$$H^j(U,\mu_M(\underline{F})) = \varinjlim_{Z,V} H^j_Z(V,\underline{F}) ,$$

where (Z,V) runs over the family of closed subsets Z of X and open subsets V of X such that :

$$V \cap M = \pi(U) ,\quad C_M(Z) \subset U^{\circ} .$$

v) Let Z be a closed convex subset of $T^*_M X$. Then :

$$H^j_Z(T^*_M X, \underline{F}) = \varinjlim_{U} H^{j-\ell}(U,\underline{F}) \otimes \underline{\omega}_{M/X}$$

where U runs over the family of open subsets of X such that $C_M(X \setminus U) \cap \operatorname{Int} Z^{\circ} = \emptyset$, and $\ell = \operatorname{codim} M$.

Applying Proposition 2.3.2. with $Z = M$, we get :

Corollary 2.3.3. : There exists a distinguished triangle in $D^+(X)$:

$$F|_M \otimes \underline{\omega}_{M/X}[-\ell] \longrightarrow \mathbb{R}\Gamma_M(\underline{F}) \longrightarrow \mathbb{R}\dot{\pi}_* \mu_M(\underline{F}) \xrightarrow[{[1]}]{} \cdots$$

Remark that if we denote by j the embedding $M \longrightarrow X$, we have :

$$\underline{F}|_M \otimes \underline{\omega}_{M/X}[-\ell] \cong j^{-1}\underline{F} \overset{\mathbb{L}}{\otimes} j^! \underline{\mathbb{Z}}_X .$$

2.3.2. Let $f : Y \longrightarrow X$ and let $N \subset Y$, $M \subset X$ with $f(N) \subset M$, be as in §2.2.2. . We denote by ϖ and ρ the natural maps from $Y \times_X T^*X$ to T^*X and T^*Y respectively, as well as the induced maps on $N \times_M T^*_M X$:

$$T^*_N Y \xleftarrow{\rho} N \times_M T^*_M X \xrightarrow{\varpi} T^*_M X \quad .$$

Proposition 2.3.4. : Let $\underline{G} \in \mathrm{Ob}(D^+(Y))$. There exists a canonical morphism :

$$\mathbb{R}\varpi_! \, \rho^{-1} \, \mu_N(\underline{G}) \longrightarrow \mu_M(\mathbb{R} f_! \, \underline{G}) \; .$$

If moreover f is transversal to M and proper over $\mathrm{supp}(\mathbf{G})$, this morphism is an isomorphism.

Proof

Apply Propositions 2.2.2., 2.1.5. and 2.1.6. . □

Proposition 2.3.5. : Let $\underline{F} \in \mathrm{Ob}(D^+(X))$. There are canonical morphisms and a commutative diagram :

$$\begin{array}{ccc} \mathbb{R}\rho_! \, \varpi^{-1} \, \mu_M(\underline{F}) \overset{L}{\otimes} \varpi^! \, \underline{\mathbb{Z}}_{N \times_M T^*_M X} & \longrightarrow & \mu_N(f^{-1}\underline{F}) \overset{L}{\otimes} f^! \, \underline{\mathbb{Z}}_X \\ \downarrow & & \downarrow \\ \mathbb{R}\rho_* \, \varpi^! \, \mu_M(\underline{F}) & \longleftarrow & \mu_N(f^! \, \underline{F}) \end{array}$$

Moreover if $Y \longrightarrow X$ and $N \longrightarrow M$ are smooth, all these morphisms are isomorphisms.

Proof

Apply Propositions 2.2.3., 2.1.5. and 2.1.6. . □

CHAPTER 3 - MICRO-SUPPORT

§3.1. Equivalent definitions for $SS(\underline{F})$

3.1.1. Let E be a real finite dimensional vector space, X an open set in E, $\underline{F} \in Ob(D^+(X))$.

Theorem 3.1.1. : *Let* $p = (x_o ; \xi_o) \in T^*X$. *Then for any* α, $1 \leqslant \alpha \leqslant \infty$ *or* $\alpha = \omega$, *the following conditions are equivalent.*

$(1)_\alpha$ *There exists an open neighborhood* U *of* p *such that for any* $x_1 \in X$, *any real function* f *of class* C^α, *defined in a neighborhood of* x_1, *with* $f(x_1) = 0$, $df(x_1) \in U$, *we have* :

$$(\mathbb{R}\Gamma_{\{x; f(x) \geqslant 0\}}(\underline{F}))_{x_1} = 0 .$$

(2) *There exist a proper closed convex cone* G *in* E, *with* $0 \in G$, *and* $\underline{F}' \in Ob(D^+(E))$ *such that* :

(a) $G \setminus \{0\} \subset \{\gamma ; \langle\gamma, \xi_o\rangle < 0\}$

(b) $F'|_U \cong F|_U$ *for a neighborhood* U *of* x_o

(c) $\mathbb{R}\phi_{G*}F' = 0$

(3) *There exist a neighborhood* U *of* x_o, *an* $\varepsilon > 0$ *and a proper closed convex cone* G *with* $0 \in G$, *satisfying the condition* (a) *of* (2) *such that if we set* :

$$H = \{x ; \langle x-x_o ; \xi_o\rangle \geqslant -\varepsilon\}, \quad L = \{x ; \langle x-x_o ; \xi_o\rangle = -\varepsilon\}$$

we have :

$$\mathbb{R}\Gamma(H \cap (x+G) ; \underline{F}) \xrightarrow{\sim} \mathbb{R}\Gamma(L \cap (x+G) ; \underline{F})$$

for all $x \in U$.

Proof

If $\xi_o = 0$ the three conditions are equivalent to $\underline{F} = 0$ in a neighborhood of x_o . Assume $\xi_o \neq 0$.

$\underline{(2) \Longrightarrow (1)_\alpha}$. We may assume $\underline{F} = \underline{F}'$ and $X = E$. We set $U = X \times \text{Int}\, G^{oa}$. For any function f with $f(x_1) = 0$, $df(x_1) \in U$, there exists a G-open set Ω such that Ω and $\{x \,;\, f(x) < 0\}$ coïncide on a neighborhood of x_1. Moreover when Ω' runs over a system of G-neighborhoods of x_1 , $\Omega' \setminus \Omega$ forms a system of neighborhoods of x_1 in $X \setminus \Omega$ in the usual topology. Thus we obtain :

$$(\mathbb{R}\Gamma_{\{x; f(x) \geqslant 0\}}(\underline{F}))_{x_1} = \mathbb{R}\Gamma_{(X_G \setminus \Omega)}(\mathbb{R}\phi_{G*}F)_{x_1} = 0$$

$\underline{(3) \Longrightarrow (2)}$ We may assume X is G-open and $U \subset H \setminus L$. Let Ω_o and Ω_1 be two G-open sets such that $\Omega_o \subset \Omega_1$, $x_o \in \text{Int}(\Omega_1 \setminus \Omega_o)$, $\Omega_1 \setminus \Omega_o \subset\subset U$. We have :

$$\mathbb{R}\Gamma((x + G) \cap H \,;\, \underline{F}) \cong \mathbb{R}\Gamma((x + G) \,;\, \underline{F}_H) \tag{3.1.1}$$

and similarly with H replaced by L.

From (3.1.1), we get :

$$(\mathbb{R}\phi_{G*}\, \underline{F}_{H \setminus L})\big|_{U_G} = 0 \quad . \tag{3.1.2}$$

Applying Theorem 1.5.3. , we obtain :

$$\begin{aligned}
\mathbb{R}\phi_{G*}\, \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}(\underline{F}_{H \setminus L}) &\cong \mathbb{R}\Gamma_{\Omega_{1G} \setminus \Omega_{oG}}\, \mathbb{R}\phi_{G*}\, \underline{F}_{H \setminus L} \\
&\cong \mathbb{R}\Gamma_{\Omega_{1G} \setminus \Omega_{oG}}\, \mathbb{R}\phi_{G*}\, \phi_G^{-1}\, \mathbb{R}\phi_{G*}\, \underline{F}_{H \setminus L} \\
&\cong \mathbb{R}\phi_{G*}\, \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}\, \phi_G^{-1}\, \mathbb{R}\phi_{G*}\, \underline{F}_{H \setminus L} \\
&= 0
\end{aligned}$$

$\underline{(1)_\omega \Longrightarrow (3)}$. We may assume $\xi_o = (1,0,\dots,0)$ and $x_o = 0$. Set $x = (x_1, x')$, $x' = (x_2,\dots,x_n)$. We take $\varepsilon > 0$ small enough and define H and L as in (3). We take $\delta > 0$ small enough and set

$$G = \{x \ ; \ x_1 \leqslant -\delta |x'|\}$$

We take a G-open subset Ω_1 such that $0 \in \Omega_1$, and :

$$(\Omega_1 \cap H) \times G^{oa} \subset U .$$

Now, we shall show :

$$\mathbb{R}\Gamma(H \cap (x+G) \ ; \ \underline{F}) \xrightarrow{\sim} \mathbb{R}\Gamma(L \cap (x+G) \ ; \ \underline{F}) \tag{3.1.3}$$

for any $x \in \Omega_1 \cap H$.

For $a \in \Omega_1 \cap H$, we shall construct a family $\{\Omega_t(a)\}_{t\in\mathbb{R}^+}$ such that:

(3.1.4)
- (i) $\Omega_t(a) \subset a + \text{Int } G$
- (ii) $\Omega_t(a) \cap L = (a + \text{Int } G) \cap L$
- (iii) $\Omega_t(a) = \bigcup_{s<t} \Omega_s(a)$
- (iv) $\Omega_t(a)$ has a real analytic smooth boundary
- (v) $Z_t(a) = (\bigcap_{s>t} (\overline{\Omega_s(a) \setminus \Omega_t(a)}) \cap H$, is contained in $\partial\Omega_t(a)$ and the conormal of $\Omega_t(a)$ at $Z_t(a)$ is contained in U.
- (vi) $(\bigcup_{t>o} \Omega_t(a)) \cap H = (a + \text{Int } G) \cap H$
- (vii) $(\bigcap_{t>o} \Omega_t(a)) \cap H = L \cap (a + \text{Int } G)$

For example, it is enough to take $\Omega_t(a) = \{x \ ; \ x_1 < a_1$ and

$$(x_1 - a_1)^2 > \delta^2(|x' - a'|^2 + \frac{(\delta^2|x'-a'|^2 - (\varepsilon+a_1)^2)^2}{\sqrt{t + (\delta^2|x'-a'|^2 - (\varepsilon+a_1)^2)^2}})\} .$$

Now, by setting $\tilde{\Omega}_t(a) = \Omega_t(a) \cup ((a + \mathrm{Int}\, G) \setminus H)$, we shall show :

(3.1.5) $$\mathbb{R}\Gamma(a + \mathrm{Int}\, G ; \underline{F}) \xrightarrow{\sim} \mathbb{R}\Gamma(\tilde{\Omega}_t(a) ; \underline{F})$$

for any t.

Letting j be the embedding $a + \mathrm{Int}\, G \hookrightarrow X$, and applying Theorem 1.4.3. it is enough to show :

$$\mathbb{R}\Gamma_{X \setminus \tilde{\Omega}_t(a)}(\mathbb{R}j_* j^{-1}\underline{F})_y = 0 \text{ for any } y \in Z_s(a) \setminus \tilde{\Omega}_t(a) \text{ for } s \leqslant t .$$

Since $Z_s(a) \setminus \tilde{\Omega}_t(a) \subset Z_t(a)$, we may assume $y \in Z_t(a) \setminus \tilde{\Omega}_t(a)$.

If y belongs to $\mathrm{Int}\, H = H \setminus L$, then this a consequence of (1).

Now we assume $y \in Z_t(a) \cap L = L \cap \partial(a + \mathrm{Int}\, G)$. Then since :

$$\mathbb{R}\Gamma_{X \setminus \Omega_t(a)}(\underline{F})_y = \mathbb{R}\Gamma_{X \setminus (a + \mathrm{Int}\, G)}(\underline{F})_y = 0$$

by (1),

$$\mathbb{R}\Gamma_{X \setminus \Omega_t(a)}(\mathbb{R}j_* j^{-1}\underline{F})_y = 0 .$$

On the other hand :

$(a + \mathrm{Int}\, G) \setminus \Omega_t(a)$ is the disjoint union of $((a + \mathrm{Int}\, G) \setminus \tilde{\Omega}_t(a))$

and $(a + \mathrm{Int}\, G) \setminus (\Omega_t(a) \cup H)$.

Hence $\mathbb{R}\Gamma_{X \setminus \tilde{\Omega}_t(a)}(\mathbb{R}j_* j^{-1}\underline{F})_y$ is a direct summand of

$\mathbb{R}\Gamma_{X \setminus \Omega_t(a)}(\mathbb{R}j_* j^{-1}\underline{F})_y$ and hence this vanishes.

This shows (3.1.5) for any t, and hence we have :

(3.1.6) $$\begin{aligned}\mathbb{R}\Gamma((a + \mathrm{Int}\, G) \cap H ; \underline{F}|_H) &\xrightarrow{\sim} \mathbb{R}\Gamma(\tilde{\Omega}_t(a) \cap H ; \underline{F}|_H) \\ &= \mathbb{R}\Gamma(\Omega_t(a) \cap H ; \underline{F}|_H)\end{aligned}$$

Now, we shall show, for any $x \in \Omega_1 \cap H$

(3.1.7) $$\mathbb{R}\Gamma((x+G) \cap H ; \underline{F}|_H) \xrightarrow{\sim} \mathbb{R}\Gamma((x+G) \cap L ; \underline{F}|_H)$$

Set $v = (1,0,\dots,0)$. Then $(x + \rho v + \mathrm{Int}\, G) \cap H$, $(\rho > 0)$ form a neighborhood system of $(x+G) \cap H$ (in H) and $\Omega_t(x + \rho v) \cap L$ $(\rho > 0,\ t > 0)$ form a neighborhood system of $(x+G) \cap L$ (in H). Thus (3.1.7) follows from (3.1.6). □

Definition 3.1.2. : Let X be a real manifold of class C^α $(1 \leqslant \alpha \leqslant \infty$ or $\alpha = \omega)$, $\underline{F} \in \mathrm{Ob}(D^+(X))$.

i) The micro-support of $\underline{F}$, denoted $SS(\underline{F})$ is the subset of T^*X defined by :

$$p \notin SS(\underline{F}) \iff \text{condition } (1)_\alpha \text{ of Theorem 3.1.1. is satisfied.}$$

ii) Let $u : \underline{F} \longrightarrow \underline{F}'$ be a morphism in $D^+(X)$ and let A be a conic subset of T^*X. We say that u is an isomorphism on A if u is embedded in a distinguished triangle $\underline{F} \xrightarrow{u} \underline{F}' \longrightarrow \underline{F}'' \xrightarrow{+1} \dots$ with $SS(\underline{F}'') \cap A = \emptyset$.

ii) is equivalent to saying that there is an open conic neighborhood Ω of A such that for $x_1 \in X$ and any C^α function f on X with $df(x_1) \in \Omega$, $f(x_1) = 0$, we have :

$$(\mathbb{R}\Gamma_{\{f \geqslant 0\}}(\underline{F}))_{x_1} \xrightarrow{\sim} (\mathbb{R}\Gamma_{\{f \geqslant 0\}}(\underline{F}'))_{x_1}$$

3.1.2. It follows immediately from the definition that $SS(\underline{F})$ is a closed conic set in T^*X, and $SS(\underline{F}) \cap T^*_X X = \mathrm{supp}(\underline{F})$.

Moreover if we have a distinguished triangle in $D^+(X)$ as precedingly, then we have the "triangular inequalities" :

$$(3.1.8)\quad \begin{cases} SS(\underline{F}) \subset SS(\underline{F}') \cup SS(F'') \\ (SS(\underline{F}') \setminus SS(\underline{F}'')) \cup (SS(F'') \setminus SS(F')) \subset SS(\underline{F}) \end{cases}$$

<u>3.1.3. Examples</u> : Let Y be a locally closed set in X. To calculate the micro-support of the sheaf $\underline{A}_Y$ we apply the "triangular inequalities" and the exact sequence associated to a closed set Z :

$$0 \longrightarrow \underline{A}_{Y \setminus Z} \longrightarrow \underline{A}_Y \longrightarrow \underline{A}_{Y \cap Z} \longrightarrow 0$$

Here $X = \mathbb{R}^n$ for $n \geqslant 1$, $n = 2$ after 4) and $\xi' = (\xi_2, \ldots, \xi_n)$.

1) $Y = \mathbb{R}^n$, $SS(\underline{A}_Y) = \{(x,\xi) \in T^*\mathbb{R}^n \; ; \; \xi = 0\}$,

2) $Y = \{x \in \mathbb{R}^n ; \; x_1 \geqslant 0\}$,

$SS(\underline{A}_Y) = \{(x,\xi) \; ; \; x_1 \geqslant 0, \xi = 0\} \cup \{(x,\xi) ; x_1 = 0, \; \xi_1 \geqslant 0, \; \xi' = 0\}$

3) More generally let G be a closed convex cone with vertex at 0 in $\mathbb{R}^n$. Then :

$$\pi^{-1}(0) \cap SS(\underline{A}_G) = G^{\circ}$$

4) $Y = \{x \in \mathbb{R}^n ; \; x_1 > 0\}$,

$SS(\underline{A}_Y) = \{(x,\xi) \; ; \; x_1 \geqslant 0, \; \xi = 0\} \cup \{(x,\xi) ; \; x_1 = 0, \; \xi_1 \leqslant 0, \; \xi' = 0\}$

5) $Y = \{x \in \mathbb{R}^2 ; \; x_1 \, x_2 \geqslant 0\}$

$\pi^{-1}(0) \cap SS(\underline{A}_Y) = \{\xi \in \mathbb{R}^2 ; \; \xi_1 \, \xi_2 \leqslant 0\}$

6) $Y = \{x \in \mathbb{R}^2 ; \; x_1 \geqslant 0, \; x_2 \geqslant 0, \; x \neq 0\}$

$\pi^{-1}(0) \cap SS(\underline{A}_Y) = \{\xi \in \mathbb{R}^2 ; \; \xi_1 \leqslant 0\} \cup \{\xi \in \mathbb{R}^2 ; \; \xi_2 \leqslant 0\}$

7) $Y = \{x \in \mathbb{R}^2 ; \; x_1 \geqslant 0, \; x_2 > 0\}$

$\pi^{-1}(0) \cap SS(\underline{A}_Y) = \{\xi \in \mathbb{R}^2 ; \; \xi_1 \geqslant 0, \; \xi_2 \leqslant 0\}$

8) $Y = \{x \in \mathbb{R}^2 ; \; x_1^3 \geqslant x_2^2\}$

$\pi^{-1}(0) \cap SS(\underline{A}_Y) = \{\xi \in \mathbb{R}^2 ; \; \xi_1 \geqslant 0\}$

9) $Y = \{(x_1, x_2) \; ; \; 0 < x_1 \leqslant x_2^{\alpha}\}$ with $\alpha > 1$

$\pi^{-1}(0) \cap SS(\underline{A}_Y) = \{\xi \in \mathbb{R}^2 ; \; \xi_1 = 0, \; \xi_2 \leqslant 0\}$

3.1.4. Example : Let $(X, \mathcal{O}_X)$ be a complex manifold and let $\mathcal{M}$ be a coherent Module over the sheaf $\mathcal{D}_X$ of holomorphic differential operators on X. Then we shall prove in Chapter 10 that :

$$SS(\mathbb{R}\,\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = char(\mathcal{M})$$

where $char(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$ in T^*X.

§3.2. Propagation theorems

3.2.1. Let E be a real finite dimensional vector space, X an open subset of E.

Proposition 3.2.1. : Let V be an open set in X, and let G be a closed convex proper cone in E, with $\operatorname{Int} G \neq \emptyset$. Let $\underline{F} \in Ob(D^+(X))$, and assume $SS(\underline{F}) \cap (V \times \operatorname{Int} G^{oa}) = \emptyset$. Let $x_o \in V$, $\xi_o \in \operatorname{Int} G^{oa}$, $c > 0$ and set $\Omega_o = \{x ; \langle x-x_o, \xi_o\rangle < -c\}$. Then for any G-open set Ω_1, with $\Omega_o \subset \Omega_1 \subset \Omega_o \cup V$ we have :

$$(\mathbb{R}\phi_*(\mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F})))|_{\Omega_1} = 0$$

where ϕ is the map $X \longrightarrow X_G$.

Proof

It is enough to show that $\mathbb{R}\Gamma(\Omega, \mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F})) = 0$ for any closed proper convex cone G' properly containing G and any G'-open subset Ω such that $\Omega\setminus\Omega_o \subset \Omega_1$. Hence we may assume from the beginning $SS(\underline{F}) \cap (V \times G^{oa}\setminus\{0\}) = \emptyset$. In order to see that $\mathbb{R}\phi_*(\mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F}))|_{\Omega_1} = 0$, it is sufficient to show that for any G-open subset Ω with $\overline{\Omega} \subset \Omega_1$, $\mathbb{R}\phi_*\mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F})|_{\Omega} = 0$. Therefore we may assume $\overline{\Omega_1\setminus\Omega_o} \subset V$. Then by extending $\underline{F}$ to an object of $D^+(E)$ we may assume $X = E$. For $x_1 \in \Omega_1$ we set :

$$U(\varepsilon, x_1) = \{x ; d(x, x_1 + G) < \varepsilon\}$$

We remark that the function $d(x,x_1+G)$ is C^1 on $X \setminus (x_1+G)$, and its differential $d(d(x,x_1+G))$ belongs to G^{oa} on this set.

Let :

$$W_t = \{s \in \mathbb{R}^2 ; s_1 < t\} \cup \{s ; s_2 < 0\} \cup \{s ; s_1 < \varepsilon, s_2 < \varepsilon - t, (s_1-\varepsilon)^2 + (s_2-\varepsilon+t)^2 > (\varepsilon-t)^2\}$$

for $0 < t < \varepsilon$.

We set :

$$\ell(x) = \langle x-x_o, \xi_o\rangle + c \quad (\text{then } d\ell = \xi_o \in G^{oa})$$

$$V_t = \{x ; (d(x,x_1+G), \ell(x)) \subset W_t\}$$

and choose ε small enough such that $\overline{U(\varepsilon,x_1)}$ is contained in Ω_1.

We may apply Theorem 1.4.3. to the family $(V_t)_{0<t\leqslant\varepsilon}$ and we obtain :

$$\mathbb{R}\Gamma(U(\varepsilon,x_1) \cup \Omega_o ; \underline{F}) \cong \mathbb{R}\Gamma(V_t ; \underline{F})$$

thus :

$$\mathbb{R}\Gamma(U(\varepsilon,x_1) ; \mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F})) \cong \mathbb{R}\Gamma(V_t; \mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F}))$$

and taking the inductive limit for $t > 0$:

$$\mathbb{R}\Gamma(U(\varepsilon,x_1) ; \mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F})) \cong \mathbb{R}\Gamma((x_1+G)); \mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F}))$$

Now we prove that for x in Ω_1 we have in fact :

$$\mathbb{R}\Gamma((x+G)) ; \mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F})) = 0$$

Choose $\gamma \in \mathrm{Int}(G)$ and set :

$$x_t = x + t\gamma$$

$$I = \{t \geqslant 0 ; \mathbb{R}\Gamma((x_t+G) ; \mathbb{R}\Gamma_{X\setminus\Omega_o}(\underline{F})) = 0\}$$

Then I is non empty since $x_t \in \Omega_o$ for $t >> 0$.

Let $t_o = \inf I$.

Let $\varepsilon > 0$ such that $U(\varepsilon, x_{t_o})$ is contained in Ω_1, and let $t > t_o$ such that $t \in I$ and

$$x_{t_o} + G \subset U(\varepsilon, x_t) \subset U(\varepsilon, x_{t_o})$$

We have, setting $\underline{\tilde{F}} = \mathbb{R}\Gamma_{X \setminus \Omega_o}(\underline{F})$:

$$\begin{array}{ccc} \mathbb{R}\Gamma(U(\varepsilon, x_{t_o}) ; \underline{\tilde{F}}) & \xrightarrow{\sim} & \mathbb{R}\Gamma((x_{t_o}+G) ; \underline{\tilde{F}}) \\ & \searrow \quad \nearrow & \\ & \mathbb{R}\Gamma(U(\varepsilon, x_t) ; \underline{\tilde{F}}) = 0 & \end{array}$$

Thus t_o belongs to I.

If t_o would be strictly positive, we find $t > 0$, $t < t_o$, $\varepsilon > 0$ such that :

$$x_t + G \subset U(\varepsilon, x_{t_o}) \subset U(\varepsilon, x_t) \subset \Omega_1$$

We get :

$$\begin{array}{ccc} \mathbb{R}\Gamma(U(\varepsilon, x_t) ; \underline{\tilde{F}}) & \xrightarrow{\sim} & \mathbb{R}\Gamma((x_t + G) ; \underline{\tilde{F}}) \\ & \searrow \quad \nearrow & \\ & \mathbb{R}\Gamma(U(\varepsilon, x_{t_o}) ; \underline{\tilde{F}}) = 0 & \end{array}$$

and hence $t \in I$. Thus $t_o = 0$, which achieves the proof. □

<u>3.2.2.</u> Let Y be a C^α-manifold, $X = Y \times E$ where E is a finite dimensional vector space. Let G be a (not necessarily proper) closed convex cone with $0 \in G$. We set $X_G = Y \times E_G$ and denote by ϕ the continuous map $X \longrightarrow X_G$.

Proposition 3.2.2. : a) For $\underline{F} \in Ob(D^+(X))$, $SS(\underline{F})$ is contained in $T^*X \times (E \times G^{oa})$ if and only if the morphism $\phi^{-1}\mathbb{R}\phi_*\underline{F} \longrightarrow \underline{F}$ is an isomorphism.

b) For $\underline{F} \in Ob(D^+(X))$, $\phi^{-1}\mathbb{R}\phi_*\underline{F} \longrightarrow \underline{F}$ is an isomorphism on $T^*Y \times (E \times \text{Int } G^{oa})$.

Proof

We may assume Y is affine. By replacing G with $\{0\} \times G$, we may assume $X = E$ from the beginning.

Assuming first $\phi^{-1}\mathbb{R}\phi_*\underline{F} \cong \underline{F}$ we shall show $SS(\underline{F}) \subset X \times G^{oa}$.

For $\xi_o \notin G^{oa}$ there exists a proper convex closed cone G' such that $G' \setminus \{0\} \subset \{\gamma, \langle\gamma, \xi_o\rangle < 0\}$ and $G' + G = X$. Then for any G'-open non empty convex subset Ω of X, we have :

$$\mathbb{R}\Gamma(\Omega;\underline{F}) = \mathbb{R}\Gamma(\Omega + G ; \underline{F}) = \mathbb{R}\Gamma(X ; \underline{F})$$

This implies that for any pair of two convex G'-open subsets $\Omega_o \subset \Omega_1$ with $\Omega_1 \setminus \Omega_o \subset\subset X$,

$$\mathbb{R}\phi_{G'*}\mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}(\underline{F}) = 0$$

which implies $SS(\underline{F}) \cap (X \times \{\xi_o\}) = \emptyset$.

Conversely assume $SS(\underline{F}) \subset X \times G^{oa}$. In order to prove the isomorphism $\phi^{-1}\mathbb{R}\phi_*\underline{F} \cong \underline{F}$ it is sufficient to show that for any relatively compact open convex subset Ω of X, the restriction morphism :

$$\mathbb{R}\Gamma(\Omega + G ; \underline{F}) \longrightarrow \mathbb{R}\Gamma(\Omega ; \underline{F})$$

is an isomorphism.

By Proposition 1.4.2. it is enough to prove that for any relatively compact open convex subset Ω' of X, $\Omega' \supset \Omega$, we have :

$$\mathbb{R}\Gamma(\Omega' \cap (\Omega + G) ; \underline{F}) \cong \mathbb{R}\Gamma(\Omega ; \underline{F})$$

Let $\mathcal{G}$ be the set of open convex subsets V of $\Omega' \cap (\Omega + G)$ such that $V \supset \Omega$ and $\mathbb{R}\Gamma(V; \underline{F}) \longrightarrow \mathbb{R}\Gamma(\Omega ; \underline{F})$ is an isomorphism. It is easy to see that $\mathcal{G}$ is inductively ordered.

Let V be a maximal element of $\mathcal{G}$. We shall show that $V = \Omega' \cap (\Omega + G)$ by contradiction.

Assume there exists $x_o \in \Omega' \cap (\Omega + G)$, $x_o \notin V$.

Lemma 3.2.3. : Let U be an open convex subset and $x_o \in X$. Let S be the cone generated by $\{u - x_o ; u \in U\}$, $S' = \overline{S} \cup \{0\}$ and U_1 be the interior of the convex hull of $U \cup \{x_o\}$. Then $U_1 \setminus U$ is locally closed with respect to the S'-topology.

Proof of Lemma 3.2.3.

a) We have

$$U_1 = \{(1-t)x_o + u ; u \in U , 0 < t \leqslant 1\} .$$

Set :

$$U_2 = \{(1-t)x_o + tu ; u \in U, 0 < t\} = x_o + S .$$

Then U_2 is S'-open. Set :

$$U_3 = \{(1-t)x_o + tu ; u \in U, 1 < t\} .$$

Then U_3 is also S'-open . Hence it is sufficient to show that $U_2 \setminus U_3 = U_1 \setminus U$. Since we have :

$$U \subset U_1 \subset U_2$$

$$U \subset U_3 \subset U_2$$

$$U_2 \subset U_1 \cup U_3$$

it is enough to show $U_1 \cap U_3 \subset U$. For $x \in U_1 \cap U_3$ we may write :

$$x = (1-t)x_o + tu = (1-s)x_o + sv$$

for $0 < t \leqslant 1$, $1 < s$, $u, v \in U$. Then we have $(s-t)x = (s-1)tu + (1-t)sv$, thus $x \in U$. □

End of the proof of Proposition 3.2.2.

By the Lemma, if we denote by V' the interior of the convex hull of $V \cup \{x_o\}$ and by G' the closed cone generated by $\{v - x_o ; v \in V\}$ and 0, then $V' \setminus V$ is locally G'-closed. On the other hand we have $\mathrm{Int}\, G' \ni \gamma$ for some $\gamma \in G^a$. Hence $G'^{oa} \cap G^{oa} \subset \{0\}$. This implies by Proposition 3.2.1. :

$$\mathbb{R}\phi_{G'*}\mathbb{R}\Gamma_{V'\setminus V}(\underline{F})|_{V'} = 0 .$$

Thus we obtain $V' \in \mathcal{G}^{\sim}$, which contradicts the choice of V.

b) Let $\phi^{-1}\mathbb{R}\phi_*\underline{F} \longrightarrow \underline{F} \longrightarrow \underline{F}' \xrightarrow[+1]{} \ldots$

be a distinguished triangle. By applying the functor $\mathbb{R}\phi_*$ we obtain $\mathbb{R}\phi_*\underline{F}' = 0$, which implies $SS(\underline{F}') \cap X \times \mathrm{Int}\, G^{oa} = \emptyset$. □

CHAPTER 4 - FUNCTORIAL PROPERTIES OF MICRO-SUPPORTS I

§4.1. Proper direct images and inverse images for smooth maps

4.1.1. Let Y and X be two manifolds of class C^{α} ($\alpha \geqslant 1$), f a map from Y to X, ρ and ϖ the natural associated maps from $Y \underset{X}{\times} T^*X$ to T^*Y and T^*X, respectively.

Proposition 4.1.1. : Let $\underline{G} \in \mathrm{Ob}(D^+(Y))$ and assume f is proper on $\mathrm{supp}(\underline{G})$. Then :

i) $SS(\mathbb{R}f_*(\underline{G})) \subset \varpi\rho^{-1}(SS(\underline{G}))$

ii) if f is a closed immersion the equality holds in a).

Proof

i) It is enough to show that fo any function ϕ defined on a neighborhood of $x \in X$ such that $\phi(x) = 0$ and such that $d(\phi \circ f)(y) \notin SS(\underline{G})$ $\forall y \in f^{-1}(x)$ we have :

$$(\mathbb{R}\Gamma_{\{x;\phi(x)\geqslant 0\}}(\mathbb{R}f_*(\underline{G})))_x = 0$$

But :

$$\begin{aligned}(\mathbb{R}\Gamma_{\{\phi\geqslant 0\}}(\mathbb{R}f_*(\underline{G})))_x &= (\mathbb{R}f_*(\mathbb{R}\Gamma_{\{\phi\circ f\geqslant 0\}}(\underline{G})))_x \\ &= \mathbb{R}\Gamma(f^{-1}(x)\,;\,\mathbb{R}\Gamma_{\{\phi\circ f\geqslant 0\}}(\underline{G})) \\ &= 0\end{aligned}$$

ii) We may assume that Y and X are vector spaces an f is linear. Suppose $p \notin SS(\mathbb{R}f_*(\underline{G}))$ and take H, L, G, U which satisfy condition (3) of Theorem 3.1.1. for the sheaf $\mathbb{R}f_*\underline{G}$ on X. Then for $x \in U \cap Y$:

$$\mathbb{R}\Gamma(H \cap (x+G)\,;\,\mathbb{R}f_*\underline{G}) = \mathbb{R}\Gamma(L \cap (x+G)\,;\,\mathbb{R}f_*\underline{G})$$

but :

$$\mathbb{R}\Gamma(H \cap (x+G)\ ;\ \mathbb{R}f_*\underline{G}) = \mathbb{R}\Gamma((Y \cap H) \cap (x+G \cap Y)\ ;\ \underline{G})$$

$$\mathbb{R}\Gamma(L \cap (x+G)\ ;\ \mathbb{R}f_*\underline{G}) = \mathbb{R}\Gamma((Y \cap L) \cap (x+G \cap Y)\ ;\ \underline{G})$$

thus $p \notin SS(\underline{G})$. □

Remark : Even when f is finite on $\mathrm{supp}(\underline{G})$ the inclusion in a) may be strict. For example take $X = Y = \mathbb{R}$, $f(x) = x^3$ and $\underline{G} = \underline{A}_Y$. Then $T^*_{\{0\}}X$ is contained into $\varpi\rho^{-1}(SS(\underline{G}))$, but $\mathbb{R}f_*(\underline{G}) = \underline{A}_X$.

4.1.2. Let us study inverse images in the smooth case.

Proposition 4.1.2. : Assume $f : Y \longrightarrow X$ is smooth.

i) Let $\underline{F} \in \mathrm{Ob}(D^+(X))$. Then :

$$SS(f^{-1}(\underline{F})) = \rho\varpi^{-1}(SS(\underline{F}))$$

ii) Let $\underline{G} \in \mathrm{Ob}(D^+(Y))$. Then all the $H^j(\underline{G})$ are locally constant on the fibers of f if and only if $SS(\underline{G})$ is contained in $\rho(Y \times_X T^*X)$.

Proof

i) First we prove the inclusion

$$SS(f^{-1}(\underline{F})) \subset \rho\varpi^{-1}(SS(\underline{F}))$$

We may assume $Y = \mathbb{R}^n \times \mathbb{R}^\ell$, $X = \mathbb{R}^n$, f being the projection $(x,y) \longmapsto x$.

Take $p = (x_0, y_0\ ;\ \xi_0, \eta_0) \notin \rho\varpi^{-1}(SS(\underline{F}))$. If $\eta_0 \neq 0$ we take $v \in \mathbb{R}^\ell$ with $\langle v, \eta_0\rangle < 0$ and set $G = \{(0,tv)\ ;\ t \geqslant 0\}$. For $\varepsilon > 0$ arbitrary, set :

$$H = \{(x,y)\ ;\ \langle y,\eta_0\rangle \geqslant -\varepsilon\}$$

$$L = \{(x,y)\ ;\ \langle y,\eta_o\rangle = -\varepsilon\}$$

For any $z \in \mathrm{Int}(H)$:

$$\mathbb{R}\Gamma(H\cap(z+G)\ ;\ f^{-1}(\underline{F})) \cong \mathbb{R}\Gamma(L\cap(z+G)\ ;\ f^{-1}(\underline{F}))$$

since $f^{-1}(\underline{F})|_{(z\ +\ \mathbb{R}(0,v))}$ is a constant sheaf.

Now assume $\eta_o = 0$, $(x_o;\xi_o) \notin SS(\underline{F})$. There exist H, L, G, U which satisfy condition (3) of Theorem 3.1.1. for the sheaf $\underline{F}$ on X.

Set $\tilde{G} = G \times \{0\}$. Then for any $z \in f^{-1}(U)$:

$$\mathbb{R}\Gamma(f^{-1}(H)\cap(z+\tilde{G})\ ;\ f^{-1}(\underline{F})) \cong \mathbb{R}\Gamma(H\cap(f(z)\ +\ G)\ ;\ \underline{F})$$

$$\mathbb{R}\Gamma(f^{-1}(L)\cap(z+\tilde{G})\ ;\ f^{-1}(\underline{F})) \cong \mathbb{R}\Gamma(L\cap(f(z)\ +\ G)\ ;\ \underline{F})$$

To prove the converse inclusion we use condition $(1)_\alpha$ of Theorem 3.1.1. and remark that for a function ϕ on X and $x \in X$ we have :

$$(\mathbb{R}\Gamma_{\{\phi\geqslant 0\}}(\underline{F}))_x = (\mathbb{R}\Gamma_{\{\phi\circ f\geqslant 0\}}(f^{-1}(\underline{F})))_y$$

for any $y \in Y$ such that $f(y) = x$.

ii) Let $\underline{G}$ be a sheaf on Y, locally constant on the fibers of f : then locally on Y, $\underline{G} = f^{-1}(\underline{F})$ for a sheaf $\underline{F}$ on X, and $SS(\underline{G}) \subset \rho\varpi^{-1}(T^*X)$ by i). To prove the converse we may assume $Y = X \times E$, where E is a vector space, f being the projection on X. Then we apply Proposition 3.2.2. with $G = E$. □

§4.2. Tensor product and $\underline{Hom}(\cdot,\cdot)$

4.2.1. Let X and Y be two manifolds, q_1 and q_2 the projections from $X \times Y$ to X and Y, respectively.

If $\underline{F}$ is a sheaf of right A-modules over X and $\underline{G}$ a sheaf of left A-modules over Y, the sheaf $q_1^{-1}\underline{F} \otimes q_2^{-1}\underline{G}$ is well defined as a sheaf of $\mathbb{Z}$-modules (A-modules if A is commutative).

Proposition 4.2.1. : Assume $wg\ell d(A) < \infty$. Let $\underline{F} \in Ob(D^+(X))$, $\underline{G} \in Ob(D^+(Y))$. Then :

$$SS(q_1^{-1}\underline{F} \overset{L}{\otimes} q_2^{-1}\underline{G}) \subset SS(\underline{F}) \times SS(\underline{G}) .$$

Proof

We may assume X and Y are vector spaces. Let $(x_o, y_o; \xi_o, \eta_o) \in T^*(X \times Y)$, and assume for example $(x_o; \xi_o) \notin SS(\underline{F})$. Take H, L, U, G satisfying condition (3) of Theorem 3.1.1. for the sheaf $\underline{F}$ on X and set $\tilde{G} = G \times \{0\}$.

For $z = (x,y) \in U \times Y$ we have :

$$\mathbb{R}\Gamma((H \times Y) \cap (z + \tilde{G}) ; \underline{F} \overset{L}{\otimes} \underline{G}) = \mathbb{R}\Gamma(H \cap (x+G) ; \underline{F} \overset{L}{\otimes} \underline{G}_y)$$

$$\mathbb{R}\Gamma((L \times Y) \cap (z + \tilde{G}) ; \underline{F} \overset{L}{\otimes} \underline{G}) = \mathbb{R}\Gamma(L \cap (x+G) ; \underline{F} \overset{L}{\otimes} \underline{G}_y)$$

Now it is enough to remark, for a compact set K in X :

$$\mathbb{R}\Gamma(K ; \underline{F} \overset{L}{\otimes} \underline{G}_y) = \mathbb{R}\Gamma(K ; \underline{F}) \overset{L}{\otimes} \underline{G}_y . \quad \square$$

4.2.2. Proposition 4.2.2. : Let $\underline{F} \in Ob(D^b(X))$, $\underline{G} \in Ob(D^+(Y))$. Then :

$$SS(\mathbb{R}\underline{Hom}(q_1^{-1}\underline{F} , q_2^{-1}\underline{G})) \subset (SS(\underline{F}))^a \times SS(\underline{G}) .$$

Proof

We assume that X and Y are vector spaces.

We have, by the Poincaré duality formula (Corollary 1.3.2.) :

$$\mathbb{R}\Gamma(X \times Y ; \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F}, q_2^{-1}\underline{G}) = \mathbb{R}hom(\mathbb{R}\Gamma_c(X;\underline{F}), \mathbb{R}\Gamma(Y;\underline{G})) \quad [-\dim X]$$

Let $(x_o, y_o ; -\xi_o, \eta_o) \notin (SS(\underline{F}))^a \times SS(\underline{G})$

i) $(y_o ; \eta_o) \notin SS(\underline{G})$. The proof is similar to that of Proposition 4.2.1. : take H, L, U, G satisfying condition (3) of Theorem 3.1.1. for the sheaf $\underline{G}$ on Y and set $\tilde{G} = \{0\} \times G$. For $z = (x,y) \in X \times U$ we have :

$$H^j(\mathbb{R}\Gamma((X \times H) \cap (z + \tilde{G}) ; \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F} , q_2^{-1}\underline{G})))$$

$$= \varinjlim_{W \ni x} H^j \, \mathbb{R}Hom(\mathbb{R}\Gamma_c(W ; \underline{F}), \mathbb{R}\Gamma(H \cap (y+G) ; \underline{G})) \quad [-\dim X] \quad \forall j ,$$

and a similar formula with H replaced by L, which achieves the proof in that case.

ii) $(x_o ; \xi_o) \notin SS(\underline{F})$. Take G satisfying condition (2) of Theorem 3.1.1. for the sheaf $\underline{F}$ on X .

We may assume $\mathbb{R}\phi_{G_*}(\underline{F}) = 0$. By the Poincaré duality formula it is enough to show that for any G^a-open sets Ω and Ω' in X with $\Omega' \subset \Omega$ and $\Omega \setminus \Omega' \subset\subset X$, we have :

$$\mathbb{R}\Gamma_c(\Omega' ; \underline{F}) \cong \mathbb{R}\Gamma_c(\Omega ; \underline{F})$$

We may assume $\Omega \cap (K+G) \subset\subset X$ for any compact K of X . We have :

$$\mathbb{R}\Gamma_c(\Omega ; \underline{F}) \cong \mathbb{R}\Gamma_c(X ; \underline{F}_\Omega)$$

and similarly with Ω' instead of Ω. Thus it is enough to show $\mathbb{R}\Gamma_c(X ; \underline{F}_{\Omega'}) \cong \mathbb{R}\Gamma_c(X ; \underline{F}_\Omega)$ or equivalently $\mathbb{R}\Gamma_c(X ; \underline{F}_{\Omega \setminus \Omega'}) = 0$. Since the support of $\underline{F}_{\Omega \setminus \Omega'}$ is compact, $\mathbb{R}\Gamma_c(X ; \underline{F}_{\Omega \setminus \Omega'}) \cong \mathbb{R}\Gamma(X ; \underline{F}_{\Omega \setminus \Omega'})$. On the other hand Proposition 1.5.7. implies :

$$\mathbb{R}\phi_{G*}(\underline{F}_\Omega) = \mathbb{R}\phi_{G*}(\underline{F}_{\Omega'}) = 0 .$$

Hence we obtain :

$$\mathbb{R}\Gamma(X ; \underline{F}_\Omega) = \mathbb{R}\Gamma(X ; \underline{F}_{\Omega'}) = 0$$

and the result follows. $\square$

<u>4.2.3</u>. The preceding results permit to give a complement to Proposition 3.2.2. .

Let E be a vector space, G a closed convex proper cone in E, X a G-open subset of E, ϕ the natural map $X \longrightarrow X_G$.

<u>Proposition 4.2.3.</u> : <u>Let</u> $\underline{F} \in Ob(D^+(X))$, $\underline{F}' = \phi^{-1}\mathbb{R}\phi_*\underline{F}$. <u>Let</u> $x \in X$ <u>and assume that for a compact neighborhood</u> K <u>of</u> x, $(K+G) \cap supp(\underline{F})$ <u>is compact. Let</u> $\xi \in E^*$ <u>such that</u> $(x+G;\xi) \cap SS(\underline{F}) = \emptyset$. <u>Then</u> $(x;\xi) \notin SS(\underline{F}')$.

<u>Proof</u>

Consider the maps :

$$X \xleftarrow{\ s\ } X \times X, \qquad X \xleftarrow{\ q_1\ } X \times X \xrightarrow{\ q_2\ } X$$

where q_j is the j-th projection, and

$$s(y,x) = x-y$$

Set :

$$\underline{F}'' = \mathbb{R}s_*(q_1^{-1}\underline{A}_G \overset{L}{\otimes} q_2^{-1}\underline{F})$$

First we shall show that $\underline{F}''$ is isomorphic to $\underline{F}'$ in a neighborhood of x.

Let L be a convex compact set contained in K. Then :

$$\mathbb{R}\Gamma(L,\underline{F}'') = \mathbb{R}\Gamma(s^{-1}L\,;\,q_1^{-1}\underline{A}_G \overset{L}{\otimes} q_2^{-1}\underline{F})$$

$$= \mathbb{R}\Gamma((s^{-1}L \cap q_1^{-1}G)\,;\,q_2^{-1}\underline{F})$$

$$= \mathbb{R}\Gamma(q_2(s^{-1}L \cap q_1^{-1}G);\underline{F})$$
$$= \mathbb{R}\Gamma(L+G;\underline{F})$$
$$= \mathbb{R}\Gamma(L;\underline{F}')$$

Now we have :

$$SS(\underline{F}'') \subset \{(x,\xi) ; \exists y,\ (y,x+y) ; -\xi,\xi) \in SS(\underline{A}_G) \times SS(\underline{F})\}$$

and the result follows. □

§4.3. Direct images for open immersion

4.3.1. Let us begin by the "non characteristic" case .

Proposition 4.3.1. : Let Ω be an open set in X, j the injection $\Omega \hookrightarrow X$ and let $x_o \in \partial\Omega$. Let $\underline{F} \in Ob(D^+(X))$. We assume :

$$SS(\underline{F}) \cap N^*_{x_o}(\Omega)^a \subset \{0\}$$

Then :

$$SS(\mathbb{R}j_*j^{-1}(\underline{F})) \cap \pi^{-1}(x_o) \subset N^*_{x_o}(\Omega) + (SS(\underline{F}) \cap \pi^{-1}(x_o)).$$

Proof

We may assume that X is a vector space. If $N^*_{x_o}(\Omega) = T^*_{x_o}X$, then the proposition is trivial. Hence we may assume that $N^*_{x_o}(\Omega)$ is a closed proper convex cone.

Let $\xi_o \notin N^*_{x_o}(\Omega) + (SS(\underline{F}) \cap \pi^{-1}(x_o))$; then by the hypothesis :

$$(N^*_{x_o}(\Omega) + \overline{\mathbb{R}^-\xi_o}) \cap (SS(F)^a \cap \pi^{-1}(x_o)) \subset \{0\}$$

and there exists a closed proper convex cone K such that :

$$N^*_{x_o}(\Omega) + \overline{\mathbb{R}^-\xi_o} \subset Int(K) \cup \{0\}$$
$$K^a \cap (SS(\underline{F}) \cap \pi^{-1}(x_o)) \subset \{0\}$$

Let G be the polar cone to K :

$$G = K^{\circ}$$

This is a closed convex proper cone with non empty interior, which satisfies :

$$G \subset N_{x_o}(\Omega) \cup \{0\}$$

$$G \subset \{\gamma \; ; \; <\gamma \; , \; \xi_o> \; < \; 0\} \cup \{0\}$$

$$G^{oa} \cap (SS(\underline{F}) \cap \pi^{-1}(x_o)) \subset \{0\}$$

Hence there exists an open neighborhood U of x_o such that :

$$\begin{cases} (U \times G^{oa}) \cap SS(\underline{F}) \subset T^*_X X \\ U \cap \Omega = U \cap \Omega' \text{ , for a G-open set } \Omega' \end{cases}$$

thus we may assume Ω is G-open.

Take $\Omega_o = \{x \; ; \; <x-x_o, \; \xi_o> \; < \; -c\}$, where $c > 0$ is small enough so that $\Omega_o \cup U$ is a G-open neighborhood of x_o. We know by Proposition 3.2.1. that for any G-open set Ω_1 such that $\Omega_o \subset \Omega_1 \subset \Omega_1 \cup U$ we have :

$$(\mathbb{R}\phi_* \mathbb{R}\Gamma_{X \setminus \Omega_o}(\underline{F}))|_{\Omega_1} = 0$$

This achieves the proof since, Ω being G-open, $\mathbb{R}\phi_*$ and $\mathbb{R}j_* j^{-1}$ commute. □

<u>Proposition 4.3.2.</u> : <u>Let</u> Ω, j, x_o, $\underline{F}$ <u>be as in Proposition 4.3.1</u>. <u>We assume</u> :

$$SS(\underline{F}) \cap N^*_{x_o}(\Omega) \subset \{0\}$$

<u>Then</u> :

$$SS(\mathbb{R}j_! j^{-1}(\underline{F})) \cap \pi^{-1}(x_o) \subset N^*_{x_o}(\Omega)^a + (SS(\underline{F}) \cap \pi^{-1}(x_o))$$

Proof

We proceed as for Proposition 4.3.1. .

Let $\xi_o \notin N^*_{x_o}(\Omega)^a + (SS(\underline{F}) \cap \pi^{-1}(x_o))$. We may find a closed convex proper cone G, a neighborhood U of x_o such that :

$$\begin{cases} G \subset \{\gamma \; ; \; \langle\gamma,\xi_o\rangle < 0\} \cup \{0\} \\ U \times G^{oa} \cap SS(\underline{F}) \subset T^*_X X \end{cases}$$

and we may assume Ω G^a-open.

By Proposition 3.2.1. there exist G-open subsets Ω_o and Ω_1 such that :

$$\Omega_o \subset \Omega_1 \, ,$$

$$\Omega_1 \setminus \Omega_o \text{ is a neighborhood of } x_o \, ,$$

$$\mathbb{R}\phi_* \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}(\underline{F}) = 0$$

Set $\underline{F}' = \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}(\underline{F})$. By replacing Ω , we may assume $\Omega \cap (K+G) \subset\subset X$ for any compact K of X. Then we can apply Proposition 1.5.7. and obtain :

$$\mathbb{R}\phi_* \underline{F}'_\Omega = 0$$

Since $\underline{F}'$ is isomorphic to $\underline{F}$ on a neighborhood of x_o, we obtain $(x_o,\xi_o) \notin SS(\underline{F}_\Omega)$.

Recall that $\underline{F}_\Omega \cong \mathbb{R}j_! j^{-1}(\underline{F})$. □

Corollary 4.3.3. : Let Z be a closed subset of X, $x \in \partial Z$ and assume $N^*_x(Z) \neq T^*_x X$. Let $\underline{F} \in Ob(D^+(X))$ such that $SS(\underline{F}) \cap N^*_x(Z) \subset \{0\}$. Then $\mathbb{R}\Gamma_Z(\underline{F})_x = 0$.

Proof

We have $SS(\mathbb{R}\Gamma_Z(\underline{F})) \cap \pi^{-1}(x) \subset SS(\underline{F}) + N^*_x(Z)^a$.

Since $(SS(\underline{F}) + N_Z^*(Z)^a) \cap N_X^*(Z) \subset \{0\}$, there exists a closed convex cone K such that $(SS(\underline{F}) + N_X^*(Z)^a) \cap K \subset \{0\}$ and $N_X^*(Z) \subset \mathrm{Int} K \cup \{0\}$. Hence if we set $G = K^{oa}$ we may assume Z is G-closed, and hence there exist G-open subsets $\Omega_o \subset \Omega_1$ such that $x \in \mathrm{Int}(\Omega_1 \setminus \Omega_o)$, and $\mathbb{R}\phi_{G*} \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}(\mathbb{R}\Gamma_Z(\underline{F}|_{\Omega_1})) = 0$. Moreover the family $\{\Omega \cap Z\}$ forms a neighborhood system of x when Ω runs over the G-open neighborhoods of x . □

4.3.2. Now we study the general case.

Theorem 4.3.4. : Let X be a manifold of class C^α, $\alpha \geq 2$, Ω an open set in X , j the open embedding $\Omega \hookrightarrow X$. Let $\underline{F} \in \mathrm{Ob}(D^+(\Omega))$. Then :

a) $SS(\mathbb{R}j_*(\underline{F})) \subset SS(\underline{F}) \hat{+} N^*(\Omega)$

b) $SS(\mathbb{R}j_!(\underline{F})) \subset SS(\underline{F}) \hat{+} N^*(\Omega)^a$

Proof

We may assume X is a vector space.

Let $(x_o;\xi_o) \notin SS(\underline{F}) \hat{+} N^*(\Omega)$.

We may assume $\xi_o \neq 0$, $x_o \in \partial\Omega$ thus $(x_o;\xi_o) \notin \overline{SS(\underline{F})}$.

Since $(x_o;\xi_o)$ does not belong to $N_{x_o}^*(\Omega)$ we may assume Ω G-open, for a closed convex proper cone G such that $\mathrm{Int}(G) \neq \emptyset$ and $\xi_o \notin G^o$.

Let $\gamma \in \mathrm{Int}(G)$, $s > 0$, $t > 0$, with $1+t\langle\gamma,\xi_o\rangle > 0$, $\langle\xi_o,\gamma\rangle < 0$. Set :

$$H_s = \{x \ ; \ \langle x-x_o , \xi_o\rangle > -s\}$$

$$\Omega_{t,s} = \{x \ ; \ x-t(\langle x-x_o , \xi_o\rangle + s)\gamma \in \Omega\}$$

Then :

$$\overline{\Omega_{t,s}} \cap H_s \subset \Omega$$

$$\Omega_{t,s} \cap H_s \subset \Omega_{t',s} \qquad 0 < t' \leq t$$

$$(\bigcup_t \Omega_{t,s}) \cap H_s = \Omega \cap H_s$$

For $x \in H_s$, if we set :

$$y = x - t(\langle x - x_o, \xi_o\rangle + s)\gamma$$

$$\xi = \eta - t \langle \eta,\gamma\rangle \xi_o$$

we have :

$$(x;\xi) \in N^*_x(\Omega_{t,s}) \iff (y;\eta) \in N^*_y(\Omega)$$

Now we shall prove that there exists an open neighborhood $U \times W$ of (x_o,ξ_o), $\varepsilon > 0$, such that if we set $U_s = U \cap H_s$ we have for $0 < t < \varepsilon$, $0 < s < \varepsilon$:

$$\begin{cases} \text{i)} & SS(\underline{F}) \cap N^*(\Omega_{t,s})^a \cap \pi^{-1}(U_s) \subset \{0\} \\ \text{ii)} & (SS(\underline{F}) + N^*(\Omega_{t,s})) \cap (U_s \times W) = \emptyset \end{cases}$$

If i) or ii) is false we find sequences $\{t_n\}$, $\{s_n\}$, $\{x_n\}$, $\{\xi_n\}$, $\{\zeta_n\}$ such that :

$$\begin{cases} t_n \xrightarrow[n]{} 0,\ s_n \xrightarrow[n]{} 0,\ t_n > 0,\ s_n > 0,\ x_n \xrightarrow[n]{} x_o \\ \xi_n \in N^*_{x_n}(\Omega_{t_n,s_n}) \setminus \{0\} \\ (x_n;\zeta_n) \in SS(F) \\ \xi_n + \zeta_n = c\,\tilde{\xi}_n,\quad \tilde{\xi}_n \xrightarrow[n]{} \xi_o \quad \text{where} \\ c = 0 \text{ or } c = 1 \quad (c = 0 \text{ for i)},\ c = 1 \text{ for ii)}). \end{cases}$$

We define $(y_n;\eta_n) \in N^*(\Omega)$ by :

$$y_n = x_n - t_n(\langle x_n - x_o,\xi_o\rangle + s_n)\gamma$$

$$\xi_n = \eta_n - t_n \langle \eta_n,\gamma\rangle \xi_o$$

and we set :

$$\rho_n = \zeta_n + \eta_n = c\,\tilde{\xi}_n + t_n \langle \eta_n,\gamma\rangle \xi_o$$

We have :

$$\langle \eta_n, \gamma \rangle > 0, \quad \langle \xi_o, \gamma \rangle < 0, \quad \langle \tilde{\xi}_n, \gamma \rangle < 0$$

$$|\eta_n| \leqslant c' \langle \eta_n, \gamma \rangle \quad \text{for some} \quad c' > 0$$

$$c''|\rho_n| \geqslant -\langle \rho_n, \gamma \rangle \quad \text{for some} \quad c'' > 0$$

$$\geqslant -c\langle \tilde{\xi}_n, \gamma \rangle - t_n \langle \eta_n, \gamma \rangle \langle \xi_o, \gamma \rangle$$

$$\geqslant t_n |\langle \eta_n, \gamma \rangle| \; |\langle \xi_o, \gamma \rangle|$$

$$\geqslant c''' t_n |\eta_n| \quad \text{for some} \quad c''' > 0$$

thus $|\eta_n|(|\rho_n|^{-1})t_n$ is bounded, and $|\eta_n|(|\rho_n|^{-1})|x_n - y_n| \xrightarrow[n]{} 0$. Since $\rho_{n/|\rho_n|}$ converges to $\xi_{o/|\xi_o|}$ this contradicts the hypothesis.

Now let $j_{t,s}$ denote the injection $\Omega_{t,s} \hookrightarrow X$, and set for s fixed, $0 < s < \varepsilon$:

$$\underline{F}_t = \mathbb{R}j_{t,s*}(\underline{F}|_{\Omega_{t,s}})$$

By Proposition 4.3.1. :

$$SS(\underline{F}_t) \cap (U_s \times W) = \emptyset .$$

Hence there exist a closed proper cone G' and G'-open subsets $\Omega_o \subset \Omega_1$ such that $G' \subset \{\gamma \; ; \; \langle \gamma, \xi_o \rangle < 0\} \cup \{0\}$ and $H_s \supset \mathrm{Int}(\Omega_1 \setminus \Omega_o) \ni x_o$, with $\mathbb{R}\phi_{G'*} \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}(\underline{F}_t) = 0 \quad \forall t > 0$. Thus we obtain $\mathbb{R}\phi_{G'*} \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}(\mathbb{R}_{j*}\underline{F}) = 0$ by Proposition 1.4.2. .

The proof of b) is similar with the help of Proposition 4.3.2. instead of Proposition 4.3.1. . □

Corollary 4.3.5. : Let Ω be open in X , j the injection $\Omega \longrightarrow X$ and assume that $Y = \partial\Omega$ is a C^2-hypersurface, Ω locally on one side of $\partial\Omega$. Let $\underline{F} \in \mathrm{Ob}(D^+(\Omega))$. Then :

$$SS(\mathbb{R}j_*(\underline{F})|_Y) \subset C_{T_Y^*X}(SS(\underline{F})) \cap T^*Y .$$

Proof

Set $\underline{G} = (\mathbb{R}j_*\underline{F})|_Y$ and let i be the injection $Y \xrightarrow{i} X$. We have a distinguished triangle :

$$\mathbb{R}j_!\underline{F} \longrightarrow \mathbb{R}j_*\underline{F} \longrightarrow \mathbb{R}i_*\underline{G} \xrightarrow{+1} \cdots$$

thus by applying Theorem 4.3.4. :

$$SS(\mathbb{R}i_*\underline{G}) \subset C(SS(\underline{F}), T^*_YX) \cap T^*X$$

Since i is a closed immersion :

$$SS(\mathbb{R}i_*\underline{G}) = \varpi(\rho^{-1}(SS(\underline{G}))$$

which achieves the proof. □

Remark 4.3.6. : The micro-support is invariant by C^1-transformations on X, but $\cdot \hat{+} \cdot$ and $C_{T^*_YX}(\cdot) \cap T^*_YX$ are not invariant by C^1-transformations. This means that Theorem 4.3.4. and Corollary 4.3.5. are not the best possible results.

§4.4. Direct images for non proper maps

4.4.1. Let f be a map from Y to X, and let $\underline{G} \in Ob(D^+(Y))$.

Theorem 4.4.1. : Assume that there exists a family $(Y_s)_{s>0}$ of open subsets of Y such that :

i) $\bigcup_s Y_s = Y$, $\bigcup_{r<s} Y_r = Y_s$, $\bigcap_{t>s} Y_t \subset \overline{Y}_s$, $N^*_y(Y_s) \neq T^*_yY$ for any $y \in Y$,

ii) f is proper over $\overline{Y}_s \cap \mathrm{supp}(\underline{G})$ for all s,

iii) $N^*(Y_s)^a \cap \overline{(SS(\underline{G}) + \rho(Y \times_X T^*X))} \subset T^*_YY$

$$\text{(resp. } N^*(Y_s) \cap \overline{(SS(\underline{G}) + \rho(Y \times_X T^*X))} \subset T^*_Y Y)).$$

Let i_s be the open embedding $Y_s \hookrightarrow Y$. Then :

1) $\mathbb{R}f_*(\underline{G}) = \mathbb{R}(f \circ i_s)_*(i_s^{-1}(\underline{G}))$ for any $s > 0$

(resp. $\mathbb{R}f_!(\underline{G}) = \mathbb{R}(f \circ i_s)_!(i_s^{-1}(\underline{G}))$ for any $s > 0$)

2) $SS(\mathbb{R}f_*(\underline{G})) \subset \varpi\rho^{-1}(SS(\underline{G}))$

(resp. $SS(\mathbb{R}_!(\underline{G})) \subset \varpi\rho^{-1}(SS(\underline{G})))$.

Proof

Since the assertion about $f_!$ can be proved similarly we shall prove only the assertion about f_*.

First, we assume f is smooth.

Set $f_s = f \circ i_s$.

Let $x \in X$, and let W be a relatively compact open neighborhood of x in X. Let j be the ambedding $f^{-1}(W) \hookrightarrow Y$.

$$\mathbb{R}\Gamma(W ; \mathbb{R}f_{s*}(i_s^{-1}(\underline{G}))) = \mathbb{R}\Gamma(f^{-1}(W) \cap Y_s ; \underline{G}) = \mathbb{R}\Gamma(Y_s ; \mathbb{R}j_*(j^{-1}\underline{G}))$$

We remark that :

$$SS(\mathbb{R}j_*(j^{-1}\underline{G})) \subset SS(\underline{G}) \hat{+} \rho(Y \times_X T^*X)$$

and the set on the right hand side is nothing but $\overline{SS(\underline{G}) + \rho(Y \times_X T^*X)}$.

Thus by the hypothesis :

$$\mathbb{R}\Gamma_{Y \setminus Y_s}(\mathbb{R}j_*(j^{-1}(\underline{G})))|_{\partial Y_s} = 0$$

and we may apply Proposition 1.4.3. to the family Y_s to obtain

$$\mathbb{R}\Gamma(Y_s \cap f^{-1}(W) ; \underline{G}) \xleftarrow{\sim} \mathbb{R}\Gamma(f^{-1}(W) ; \underline{G}) .$$

We get, by applying Proposition 4.1.1. and Theorem 4.3.4. :

$$SS(\mathbb{R}f_{s*}(\underline{G})) \subset \varpi\rho^{-1}(SS(\underline{G}) \hat{+} N^*(Y_s))$$

but hypothesis iii) implies :

$$\rho^{-1}(SS(\underline{G}) \hat{+} N^*(Y_s)) \subset \rho^{-1}\, SS(\underline{G})$$

To treat the general case, we decompose f by $Y \xrightarrow[\tilde{f}]{} Y \times X \longrightarrow X$, where $\tilde{f}(y) = (y, f(x))$. Set $\tilde{\underline{G}} = \mathbb{R}\tilde{f}_* \underline{G}$. Then by Proposition 4.1.1. we have :

$$SS(\tilde{\underline{G}}) = \{(y,x\,;\,\eta,\xi) \in T^*(Y\times X)\,;\, x = f(y), (y\,;\,\eta + {}^tf'(y)\cdot\xi) \in SS(\underline{G})\}$$

Thus :

$$\overline{SS(\tilde{\underline{G}}) + Y \times T^*X} \subset \overline{SS(G) + (Y \underset{X}{\times} T^*X)} \times T^*X$$

and since $N^*(Y_s \times X) = N^*(Y_s) \times T^*_X X$, we see that the hypotheses of the Theorem are still satisfied for $\tilde{G}$ on $Y \times X$, for the family $\{Y_s \times X\}_s$. □

<u>4.4.2</u>. We shall also need a "microlocal" version of the direct image theorem.

<u>Theorem 4.4.2.</u> : <u>Let</u> X <u>and</u> Y <u>be two manifolds, and let</u> f <u>be the projection</u> $X \times Y \longrightarrow X$ <u>and</u> h <u>the projection</u> $T^*(X \times Y) \longrightarrow (T^*X) \times Y$. <u>Let</u> Ω <u>be an open subset of</u> T^*X <u>and let</u> $\underline{F} \in Ob(D^+(X \times Y))$. <u>Assume</u> :

(4.4.1) $\overline{h(SS(\underline{F}))} \cap (\Omega \times Y)$ <u>is proper over</u> Ω

<u>Then we have</u> :

i) $SS(\mathbb{R}f_* \underline{F}) \cap \Omega \subset \varpi\rho^{-1}(SS(\underline{F}))$

ii) $SS(\mathbb{R}f_! \underline{F}) \cap \Omega \subset \varpi\rho^{-1}(SS(\underline{F}))$

iii) $\mathbb{R}f_!(\underline{F}) \longrightarrow \mathbb{R}f_*(\underline{F})$ <u>is an isomorphism on</u> Ω .

Here ϖ and ρ denote as usual the natural maps from $Y \times T^*X$ to T^*X and $T^*(Y \times X)$ respectively.

Proof

By a closed embedding of Y in $\mathbb{R}^n$, we may assume $Y = \mathbb{R}^n$. Since Y is isomorphic to the open ball in $\mathbb{R}^n$, we may assume Y is the open ball of $Y' = \mathbb{R}^n$. Let j denote the embedding $X \times Y \longrightarrow X \times Y'$, and let g denote the projection $X \times Y' \longrightarrow X$. Then we have :

$$\mathbb{R}f_* \underline{F} = \mathbb{R}g_* \mathbb{R}j_* \underline{F}$$

$$\mathbb{R}f_! \underline{F} = \mathbb{R}g_* \mathbb{R}j_! \underline{F}$$

Setting $Z = X \times \partial Y$, and applying Theorem 4.3.4. we get :

$$SS(\mathbb{R}j_* \underline{F}) \setminus SS(\underline{F}) \subset T^*_Z(X \times Y') \hat{+} SS(\underline{F})$$

$$SS(\mathbb{R}j_! \underline{F}) \setminus SS(\underline{F}) \subset T^*_Z(X \times Y') \hat{+} SS(\underline{F})$$

$$SS((\mathbb{R}j_* \underline{F})_Z) \subset T^*_Z(X \times Y') \hat{+} SS(\underline{F})$$

Hence it is sufficient to show :

$$(4.4.2) \qquad \varpi\rho^{-1}(T^*_Z(X \times Y') \hat{+} SS(\underline{F})) \cap \Omega = \emptyset$$

For a compact subset K of Ω, $(K \times Y) \cap \overline{h(SS(\underline{F}))}$ is compact. Hence $\overline{\pi(SS(\underline{F}) \cap K \times T^*Y)}$ is a compact set. Therefore $\overline{\pi(SS(\underline{F}) \cap K \times T^*Y)} \cap Z = \emptyset$ which implies (4.4.2). □

CHAPTER 5 - FUNCTORIAL PROPERTIES OF MICRO-SUPPORT II

§5.1. Fourier-Sato transformation

5.1.1. Let $\tau : E \longrightarrow Z$ be a real finite dimensional vector bundle over the manifold Z. We have already defined in Chapter 2 the full subcategory $D^+_{conic}(E)$ of $D^+(E)$, consisting of complexes whose cohomology groups are locally constant on the orbits of the action of $\mathbb{R}^+$.

Let us denote by S_E the canonical hypersurface of T^*E, the characteristic variety of the Euler field on E. In a system of coordinates (z,x) on E, (x being the fiber coordinates), $(z,x ; \zeta,\xi)$ on T^*E,

$$(5.1.1) \qquad S_E = \{(z,x ; \zeta,\xi) ; \langle x,\xi\rangle = 0\}$$

Proposition 5.1.1. : Let $\underline{F} \in Ob(D^+(E))$. Then :

$$\underline{F} \in Ob(D^+_{conic}(E)) \iff SS(\underline{F}) \subset S_E$$

Proof

It is sufficient to prove this equivalence outside the zero section of E. Then $\dot{E}/\mathbb{R}^+$ is a manifold and we may apply Proposition 4.1.2. ii). Then one only has to check the equality :

$$(5.1.2) \qquad T^*(\dot{E}/\mathbb{R}^+) \underset{\dot{E}/\mathbb{R}^+}{\times} \dot{E} \cong S_E \underset{E}{\times} \dot{E} . \quad \square$$

Let ρ and ϖ be the maps from $E \underset{Z}{\times} T^*Z$ to T^*E and T^*Z respectively, associated to the projection $\tau : E \longrightarrow Z$. We may identify T^*Z to a subset of T^*E by the embeddings :

$$T^*Z \hookrightarrow E \underset{Z}{\times} T^*Z \underset{\rho}{\hookrightarrow} T^*E$$

Now if $\underline{F} \in Ob(D^+_{conic}(E))$, $SS(\underline{F})$ is homogeneous with respect to the action of $\mathbb{R}^+$ on E, and we have :

$$(5.1.3) \qquad \varpi\rho^{-1}\, SS(\underline{F}) = T^*Z \cap SS(\underline{F})$$

<u>Proposition 5.1.2.</u> : <u>Let</u> $\underline{F} \in Ob(D^+_{conic}(E))$. <u>Then</u> :

i) $SS(\mathbb{R}\tau_*(\underline{F})) \subset T^*Z \cap SS(\underline{F})$

ii) $SS(\mathbb{R}\tau_!(\underline{F})) \subset T^*Z \cap SS(\underline{F})$

iii) $SS(\mathbb{R}\dot{\tau}_*(\underline{F}|_{\dot{E}})) \subset \varpi\rho^{-1}\, SS(\underline{F}|_{\dot{E}})$

iv) $SS(\mathbb{R}\dot{\tau}_!(\underline{F}|_{\dot{E}})) \subset \varpi\rho^{-1}\, SS(\underline{F}|_{\dot{E}})$

<u>Proof</u>

Let (z,x) be a system of coordinates, (x) denoting the fiber coordinates. Then the Proposition is a particular case of Theorem 4.4.1. when taking $Y_s = \{(z,x)\ ;\ |x| < s\}$ for i) or ii) and $Y_s = \{(z,x)\ ;\ s^{-1} < |x| < s\}$ for iii) and iv). □

<u>5.1.2.</u> Before studying relations beetwen conic sheaves on E, and conic sheaves on E^*, the dual vector bundle to E, remark the following.

Let $\theta_E : T^*E \longrightarrow \mathbb{R}$ be the principal symbol of the Euler vector field. We have the canonical homomorphism $\tau^*E \longrightarrow TE$ of vector bundles over E (where τ^*E is the sub-bundle of TE consisting of vector fields which project to zero in TZ). Taking its dual, we obtain $\phi_E : T^*E \longrightarrow \tau^*E^* \longrightarrow E^*$.

We denote by ω_E the canonical 1-form on T^*E.

<u>Proposition 5.1.3.</u> : i) <u>There exists a unique map</u> $\Phi_E : T^*E \longrightarrow T^*E^*$ <u>such that</u> $\omega_E - d\theta_E = \Phi_E^*\omega_{E^*}$ <u>and that the composition</u>

$T^*E \xrightarrow{\Phi_E} T^*E^* \longrightarrow E^*$ is ϕ_E.

ii) $\theta_{E^*} \circ \Phi_E = -\theta_E$

iii) $\Phi_{E^*} \circ \Phi_E = a^*$, where a^* is the automorphism of T^*E induced by the antipodal map of E.

Proof

Take a coordinate system z of Z, (z,x) of $E, (z,x\ ;\ \zeta,\xi)$ of T^*E, (z,y) of E^* and $(z,y\ ;\ \zeta,\eta)$ of T^*E^* such that the canonical pairing of E and E^* is given by $\langle x,y\rangle = \Sigma\, x_j\, y_j$ and $\omega_E = \langle\zeta,dz\rangle + \langle\xi,dx\rangle$, $\omega_{E^*} = \langle\zeta,dz\rangle + \langle\eta,dy\rangle$. Then $\theta_E = \langle x,\xi\rangle$ and $\theta_{E^*} = \langle y,\eta\rangle$. The map ϕ_E is given by $(z,x\ ;\ \zeta,\xi) \longmapsto (z;\xi)$. Now, we have $\omega_E - d\theta_E = \langle\zeta,dz\rangle - \langle x,d\xi\rangle$. Hence $\Phi_E : (z,x\ ;\ \zeta,\xi) \longmapsto (z,\xi\ ;\ \zeta,-x)$ satisfies the desired condition. The uniqueness of Φ_E follows from the definition of the cotangent bundle. ii) follows from this formula.

Since Φ_{E^*} is given by $(z,y\ ;\ \zeta,\eta) \longmapsto (z,\eta\ ;\ \zeta,-y)$, we have $\Phi_{E^*} \circ \Phi_E(z,x\ ;\ \zeta,\xi) = (z,-x\ ;\zeta,-\xi)$, which shows iii). □

Remark : Φ_E is a symplectic transformation (i.e. preserves $d\omega$) but not a homogeneous symplectic tranformation.

5.1.3. Let $\underline{F} \in Ob(D^+_{conic}(E))$, and let $\underline{F}^\wedge$ be its Fourier-Sato transform.

Theorem 5.1.4. : With the identification of T^*E and T^*E^* by Φ_E, we have :

$$SS(\underline{F}) = SS(\underline{F}^\wedge)$$

Proof

By the Fourier inversion formula, it is enough to prove the inclusion $SS(\underline{F}^\wedge) \subset SS(\underline{F})$. We choose coordinates as in Proposition 5.1.3. .

First we evaluate $(SS(\underline{F}^\wedge)) \cap \{(z,y\ ;\ \zeta,\eta)\ ;\ y \neq 0,\ \eta \neq 0\}$.

Let $P = \{(z,x,y) \in E \underset{Z}{\times} E^* \ ;\ \langle x,y\rangle \geqslant 0\}$ .Since $\mathbb{R}\Gamma_P(\pi^{-1}\underline{F})$ is a conic sheaf on $E \underset{Z}{\times} E^*$ considered as a vector bundle over E^*, we can apply Proposition 5.1.2. :

$$SS(\underline{F}^\wedge) \subset \{(z,y\ ;\ \zeta,\eta)\ ;\ (z,x,y\ ;\ \zeta,\xi,\eta) \in SS\,\mathbb{R}\Gamma_P(\pi^{-1}\underline{F}) \text{ with } x=0,\ \xi=0\}$$

Now, by Theorem 4.3.4. we have :

$$SS\,\mathbb{R}\Gamma_P(\pi^{-1}\underline{F}) \subset SS(\pi^{-1}\underline{F}) \mathbin{\hat{+}} \mathbb{R}^- d\langle x,y\rangle\ .$$

Hence, if $(z,0,y\ ;\ \zeta,0,\eta) \in SS\,\mathbb{R}\Gamma_P(\pi^{-1}\underline{F})$ with $y \neq 0,\ \eta \neq 0$, then there exist sequences $\{(z_n,x_n\ ;\ \zeta_n,\xi_n)\} \in SS(\underline{F})$, $\{(x'_n,y_n)\}$ and $\{c_n\}$ $c_n > 0$ such that $x_n \xrightarrow[n]{} 0$, $x'_n \xrightarrow[n]{} 0$, $z_n \xrightarrow[n]{} z$, $y_n \xrightarrow[n]{} y$, $\zeta_n \xrightarrow[n]{} \zeta$, $\zeta_n - c_n y_n \xrightarrow[n]{} 0$, $-c_n x'_n \xrightarrow[n]{} \eta$ and $c_n|x_n - x'_n| \xrightarrow[n]{} 0$.

Thus $c_n x_n \xrightarrow[n]{} -\eta$, and $c_n \xrightarrow[n]{} \infty$. We have $(z_n, c_n x_n;\ \zeta_n, c_n^{-1}\xi_n) \in SS(\underline{F})$ and $c_n^{-1}\xi_n \xrightarrow[n]{} y$. This shows $(z,\ -\eta\ ;\ \zeta, y) \in SS(\underline{F})$.

To calculate $SS(\underline{F}^\wedge)$ at points $(z,y\ ;\ \zeta,\eta)$ where $y = 0$ or $\eta = 0$ we remark with Malgrange [1] that :

$$(\underline{F} \otimes \underline{\mathbb{Z}}_{\mathbb{R}} \otimes \underline{\mathbb{Z}}_{\{0\}})^\wedge = \underline{F}^\wedge \times \underline{\mathbb{Z}}_{\{0\}} \otimes \underline{\mathbb{Z}}_{\mathbb{R}}[-1]$$

Since $SS(\underline{F}^\wedge \otimes \underline{\mathbb{Z}}_{\{0\}} \otimes \underline{\mathbb{Z}}_{\mathbb{R}}) = SS(\underline{F}^\wedge) \times T^*_{\{0\}}\mathbb{R} \times T^*_{\mathbb{R}}\mathbb{R}$ (Propositions 4.1.1. and 4.1.2.) the result follows. □

§5.2. Specialization and microlocalization

5.2.1. Let X be a manifold of class C^α , $\alpha \geqslant 2$, Y a submanifold. We identify T^*T_YX and $T^*T^*_YX$ by ϕ_{T_YX} and we identify $T^*T^*_YX$

and $T_{T_Y^*X} T^*X$ by $-H$.

Theorem 5.2.1. : Let $\underline{F} \in Ob(D^+(X))$. Then :

i) $SS(\nu_Y(\underline{F})) \subset C_{T_Y^*X}(SS(\underline{F}))$

ii) $SS(\mu_Y(\underline{F})) = SS(\nu_Y(\underline{F}))$

iii) $supp(\mu_Y(\underline{F})) \subset SS(\underline{F}) \cap T_Y^*X$

iv) $SS(\underline{F}|_Y) \subset T^*Y \cap C_{T_Y^*X}(SS(\underline{F}))$

v) $SS((\mathbb{R}\Gamma_Y(F))|_Y) \subset T^*Y \cap C_{T_Y^*X}(SS(\underline{F}))$

Proof

Let us first assume i). Then ii) follows from Theorem 5.1.4., iii) follows from ii) and iv) and v) follow from Proposition 5.1.2. since :

$$\underline{F}|_Y \cong \mathbb{R}\tau_*(\nu_Y(\underline{F}))$$

$$(\mathbb{R}\Gamma_Y(\underline{F}))|_Y \cong \mathbb{R}\pi_* \mu_Y(\underline{F})$$

where τ (resp. π) denotes the projection $T_YX \longrightarrow Y$ (resp. $T_Y^*X \longrightarrow Y$). Now we prove i).

We take a system of local coordinates (y,t) on X with $Y = \{t = 0\}$ and a coordinate s on $\mathbb{R}$. We set as in §2.2. :

$$\tilde{X} = \{(s,y,t) \in \mathbb{R} \times X\}$$

and define

$$p : \tilde{X} \longrightarrow X \text{ by } p(s,y,t) = (y,st)$$

We identify T_YX with the set $\{(s,y,t) \,;\, s = 0\}$ of $\tilde{X}$.

Let $\Omega = \{(s,y,t) \in \tilde{X} \,;\, s > 0\}$ and let j be the injection $\Omega \overset{j}{\hookrightarrow} \tilde{X}$.

Then

$$\nu_Y(\underline{F}) = (\mathbb{R}j_*(j^{-1}p^{-1}(\underline{F})))|_{T_YX}$$

We know that :

$$SS(j^{-1}p^{-1}(\underline{F})) \subset \{(s,y,t\,;\,<t,\tau>,\eta,s\tau)\ ;\ (y,st\,;\,\eta,\tau) \in SS(\underline{F})\}$$

$$= \{(s,y,t\,;\,\sigma,\eta,\tau)\ ;\ (y,st\,;\,s\eta,\tau) \in SS(\underline{F})\ ,\ s\sigma = <t,\tau>\}\ .$$

Then applying Corollary 4.3.5. :

$$SS(\mathbb{R}j_*j^{-1}p^{-1}(\underline{F}))|_{T_YX} \subset \{(y_o,t_o;\eta_o,\tau_o)\ ;$$

$$\exists\ \{(s_n,y_n,t_n\,;\,\sigma_n,\eta_n,\tau_n)\} \subset SS(j^{-1}p^{-1}(\underline{F}))\ ,$$

$$s_n \xrightarrow[n]{} 0,\ s_n\sigma_n \xrightarrow[n]{} 0,\ y_n \xrightarrow[n]{} y_o,\ t_n \xrightarrow[n]{} t_o\ ,$$

$$\eta_n \xrightarrow[n]{} \eta_o,\ \tau_n \xrightarrow[n]{} \tau_o\}$$

$$= \{(y_o,\ t_o\ ;\ \eta_o,\ \tau_o)\ ;\ \exists\{(y_n,\ \tilde{t}_n\ ;\ \tilde{\eta}_n,\ \tau_n)\} \subset SS(\underline{F})\ ,$$

$$\{s_n\} \subset \mathbb{R}^+,\ \ y_n \xrightarrow[n]{} y_o,\ \tilde{t}_n \xrightarrow[n]{} 0,\ \tilde{\eta}_n \xrightarrow[n]{} 0,\ \tau_n \xrightarrow[n]{} \tau_o\ ,$$

$$\frac{1}{s_n}\ (\tilde{t}_n,\ \tilde{\eta}_n) \xrightarrow[n]{} (t_o,\ \eta_o)\ ,\ <t_o,\ \tau_o> = 0\}$$

(we have set $\tilde{t}_n = s_n t_n$, $\tilde{\eta}_n = s_n\eta_n$ and used the fact that $s_n\sigma_n = <t_n,\tau_n>$).

Thus $SS(\nu_Y(\underline{F}))$ is contained in $C_{T_YX}^{*}(SS(\underline{F}))$. □

<u>5.2.2</u>. By applying Theorem 5.2.1. iv) to the diagonal of $X \times X$, together with Propositions 4.2.1. and 4.2.2. we get :

<u>Theorem 5.2.2</u>. : <u>Let</u> $\underline{F} \in Ob(D^+(X))$, $\underline{G} \in Ob(D^+(X))$.

i) <u>Assume</u> $wg\ell d(A) < \infty$. <u>Then</u>

$$SS(\underline{F} \overset{L}{\otimes} \underline{G}) \subset SS(\underline{F})\ \hat{+}\ SS(\underline{G})$$

ii) <u>Assume</u> $\underline{F} \in Ob(D^b(X))$. <u>Then</u> :

$$SS(\mathbb{R}\underline{Hom}(\underline{F},\underline{G})) \subset SS(\underline{F})^a \hat{+} SS(\underline{G})$$

§5.3. Inverse images

5.3.1. Let $f : Y \longrightarrow X$ be a map of class C^α, $\alpha \geqslant 2$, and let ρ and ϖ be the associated maps from $Y \times_X T^*X$ to T^*Y and T^*X respectively.

Definition 5.3.1. : Let V be a subset of T^*Y, A a closed conic subset of T^*X.

i) We say that f is non characteristic for A on V if

$$f_\infty^{\#}(A) \cap V = \emptyset$$

ii) We say that f is non characteristic for A if f is non characteristic on T^*Y.

iii) Let $\underline{F} \in Ob(D^+(X))$. We say that f is non characteristic for $\underline{F}$ on V if f is non characteristic for $SS(\underline{F})$ on V.

Remark that if f is non characteristic for A on V, then the map $\rho : \rho^{-1}(V) \cap \varpi^{-1}(A) \longrightarrow V$ is proper.

Remark also that f is non characteristic for A if and only if $\varpi^{-1}(A) \cap T_Y^*X \subset Y \times_X T_X^*X$ (recall that T_Y^*X is the kernel of ρ in $Y \times_X T^*X$).

5.3.2. : Let $\underline{F} \in Ob(D^+(X))$.

Proposition 5.3.2. : One has :

i) $SS(f^{-1}\underline{F}) \subset f^{\#}(SS(\underline{F}))$

ii) $SS(f^{!}\,\underline{F}) \subset f^{\#}(SS(\underline{F}))$

Assume f non characteristic for $\underline{F}$ on an open set $V \subset T^*Y$. Then :

iii) $SS(f^{-1}\underline{F}) \cap V \subset \rho\varpi^{-1}(SS(\underline{F}))$

iv) $SS(f^{!}\underline{F}) \cap V \subset \rho\varpi^{-1}(SS(\underline{F}))$

v) the morphism $f^{-1}\underline{F} \overset{L}{\otimes} f^{!}\underline{\mathbb{Z}}_X \longrightarrow f^{!}\underline{F}$ is an isomorphism on V.

In particular if f is non characteristic for $\underline{F}$, then $f^{-1}\underline{F} \overset{L}{\otimes} f^{!}\underline{\mathbb{Z}}_X \longrightarrow f^{!}\underline{F}$ is an isomorphism.

Proof

By Remark 1.2.7. we may decompose f by the graph map as an immersion followed by a submersion. If f is a submersion the result follows from Proposition 4.1.2. and if f is an immersion, assertions i) - iv) follow from Theorem 5.2.1. and Remark 1.2.7. . Hence it remains to prove v) when f is an embedding. We remark that :

$$\mathbb{R}\pi_*\, \mu_Y(\underline{F}) \cong f^{!}\underline{F}$$

$$\mathbb{R}\pi_!\, \mu_Y(\underline{F}) \cong f^{-1}\underline{F} \overset{L}{\otimes} f^{!}\underline{\mathbb{Z}}_X$$

Hence we have a distinguished triangle :

$$f^{-1}\underline{F} \overset{L}{\otimes} f^{!}\underline{\mathbb{Z}}_X \longrightarrow f^{!}\underline{F} \longrightarrow \mathbb{R}\dot{\pi}_*(\mu_Y(\underline{F})|_{\dot{T}^*_Y X}) \xrightarrow{+1} \dots$$

Applying Theorem 5.2.1., Proposition 5.1.2. and Proposition 1.2.2. we get the result. □

Remark 5.3.3. : It is not possible to weaken the hypothesis in Proposition 5.3.2. by assuming only that ρ is proper over $\rho^{-1}(V) \cap \varpi^{-1} SS(\underline{F})$, as shown by the following example. Let $X = \mathbb{R}^2$ with coordinates (x,y), $Y = \{(x,y)\ ;\ x = 0\}$, $Z = \{(x,y)\ ;\ x = y^2\}$,

$V = \{(y,\eta) \in T^*Y \ ; \ \eta > 0\}$, $\underline{F} = \underline{\mathbb{Q}}_Z$. Then $f^{-1}(\underline{F}) = \underline{\mathbb{Q}}_{\{0\}}$, but $\varpi^{-1}(SS(\underline{F})) \cap \rho^{-1}(V) = \emptyset$.

§5.4. Comparison of inverse images and microlocalization

5.4.1. We come back to the situation of Proposition 2.3.5. .

Theorem 5.4.1. : Let $f : Y \longrightarrow X$ be a map from Y to X, $\underline{F} \in Ob(D^+(X))$, and let N and M be submanifolds of Y and X respectively with $f(N) \subset M$. Let U be an open subset of $T^*_N Y$. We assume :

i) $\rho^{-1}(U) \cap \varpi^{-1}(SS(\underline{F})) \subset Y \underset{X}{\times} T^*_M X$

ii) f is non characteristic for $\underline{F}$ on U

iii) $\varpi : \rho^{-1}(U) \cap N \underset{M}{\times} T^*_M X \longrightarrow T^*_M X$ is non characteristic for $C_{T^*_M X}(SS(\underline{F}))$.

Then we have isomorphisms on U :

$$(5.4.1) \quad (\mu_N(f^{-1}\underline{F}) \overset{L}{\otimes} f^! \underline{\mathbb{Z}}_X)\big|_U \cong (\mathbb{R}\rho_! \varpi^{-1} \mu_M(\underline{F}) \overset{L}{\otimes} \varpi^! \underline{\mathbb{Z}}_{N \underset{M}{\times} T^*_M X})\big|_U$$

$$\cong (\mathbb{R}\rho_* \varpi^{-1} \mu_M(\underline{F}) \overset{L}{\otimes} \varpi^! \underline{\mathbb{Z}}_{N \underset{M}{\times} T^*_M X})\big|_U$$

$$(5.4.2) \quad (\mu_N(f^! \underline{F}))\big|_U \cong (\mathbb{R}\rho_! \varpi^! \mu_M(\underline{F}))\big|_U$$

$$\cong (\mathbb{R}\rho_* \varpi^! \mu_M(\underline{F}))\big|_U$$

Proof

The condition ii) implies that $\rho^{-1}(U) \cap \varpi^{-1}(SS(\underline{F})) \longrightarrow U$ is a proper map, hence $\mathbb{R}\rho_*$ and $\mathbb{R}\rho_!$ coïncide in (5.4.1) and (5.4.2).

By Proposition 5.3.2., hypothesis ii) implies that $f^{-1}\underline{F} \overset{L}{\otimes} f^! \underline{\mathbb{Z}}_X \longrightarrow f^!\underline{F}$ is an isomorphism on U, which gives the isomorphism :

$$(\mu_N(f^{-1}\underline{F}) \overset{\mathbb{L}}{\otimes} f^! \underline{\mathbb{Z}}_X)\big|_U \cong \mu_N(f^{-1}\underline{F})\big|_U$$

Similarly, iii) implies :

$$\varpi^{-1}\mu_M(\underline{F}) \overset{\mathbb{L}}{\otimes} \varpi^! \underline{\mathbb{Z}}_{T_M^*X} \cong \varpi^! \mu_M(\underline{F})$$

Thus (5.4.1) and (5.4.2) are equivalent.

Following Proposition 2.2.3. , we shall consider the diagram :

$$\begin{array}{ccccccc}
T_NY & \xhookrightarrow{\;s_Y\;} & \tilde{Y}_N & \xhookleftarrow{\;j_Y\;} & \Omega_Y & \xrightarrow{\;\tilde{p}_Y\;} & Y \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \tilde{f}'} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\
T_MX & \xhookrightarrow{\;s_X\;} & \tilde{X}_M & \xhookleftarrow{\;j_X\;} & \Omega_X & \xrightarrow{\;\tilde{p}_X\;} & X
\end{array}$$

Then, as we have seen in the proof of Proposition 2.2.3., we have :

$$f'^{-1}\nu_M(\underline{F}) = s_Y^{-1}\tilde{f}'^{-1}\,\mathbb{R}j_{X*}\tilde{p}_X^{-1}(\underline{F})$$

$$\nu_N(f^{-1}\underline{F}) = s_Y^{-1}\,\mathbb{R}j_{Y*}\tilde{f}^{-1}\tilde{p}_X^{-1}(\underline{F})$$

If we have a distinguished triangle :

$$\tilde{f}'^{-1}\,\mathbb{R}j_{X*}\tilde{p}_X^{-1}(\underline{F}) \xrightarrow[\lambda]{} \mathbb{R}j_{Y*}\tilde{f}^{-1}\,\tilde{p}_X^{-1}(\underline{F}) \longrightarrow \underline{H} \xrightarrow[+1]{} \cdots$$

then $\mathrm{supp}(\underline{H}) \subset T_NY$, and we obtain a distinguished triangle :

$$f'^{-1}\nu_M(\underline{F}) \xrightarrow[\alpha]{} \nu_N f^{-1}(\underline{F}) \longrightarrow s_Y^{-1}(\underline{H}) \xrightarrow[+1]{} \cdots$$

In order to prove Theorem 5.4.1., it is sufficient to show that $\hat{\alpha}$, the Fourier-Sato transform of α , is an isomorphism on U :

$$\hat{\alpha} : \mathbb{R}\rho_*\varpi^{-1}\mu_M(F) \overset{\mathbb{L}}{\otimes} \mathbb{R}\underline{Hom}(f^!\underline{\mathbb{Z}}_X, \varpi^!\underline{\mathbb{Z}}_{N\times_M T^*X}) \longrightarrow \mu_N(f^{-1}\underline{F})$$

By Theorem 5.1.4. this is equivalent to saying that α is an isomorphism on $\phi_{T_NY}^{-1}(U)$, where $\phi_{T_NY} : T^*T_N^*Y \longrightarrow T^*T_NY$ is defined in Proposition 5.1.3. . This is again equivalent to saying that λ is an isomorphism at any point above $\phi_{T_NY}^{-1}(U)$.

Now we consider the diagram :

$$\begin{array}{ccc}
f^!\underline{\mathbb{Z}}_X \otimes \tilde{f}'^{-1}\mathbb{R}j_{X*}\tilde{p}_X^{-1}(\underline{F}) & \xrightarrow{\quad\beta\quad} & \tilde{f}'^!\mathbb{R}j_{X*}\tilde{p}_X^{-1}(\underline{F}) \\
\Big\downarrow f^!\underline{\mathbb{Z}}_X \otimes \lambda & & \Big\downarrow \wr \\
f^!\underline{\mathbb{Z}}_X \otimes \mathbb{R}j_{Y*}\tilde{f}^{-1}\tilde{p}_X^{-1}(\underline{F}) & \xrightarrow{\quad\gamma\quad} & \mathbb{R}j_{Y*}\tilde{f}^!\tilde{p}_X^{-1}(\underline{F})
\end{array}$$

Hence it is sufficient to show that β and γ are isomorphisms on $\phi_{T_NY}^{-1}(U)$.

Now we shall take a local coordinate system (x_o,y_o) on X, (x_1,y_1) on Y, such that $M = \{(x_o,y_o) ; x_o = 0\}$ and $N = \{(x_1,y_1) ; x_1 = 0\}$. Then $\tilde{p}_X(t,x_o,y_o) = (tx_o,y_o)$, $\tilde{p}_Y(t,x_1,y_1) = (tx_1,y_1)$, and if we write $f(x_1,y_1) = (g(x_1,y_1)$, $h(x_1,y_1))$ and set $g(tx_1,y_1) = t\,\tilde{g}(t,x_1,y_1)$, $h(tx_1,y_1) = \tilde{h}(t,x_1,y_1)$, then $\tilde{f}'$ is given by :

$$\tilde{f}' : (t,x_1,y_1) \longmapsto (\tilde{g}(t,x_1,y_1) \ , \ \tilde{h}(t,x_1,y_1))$$

We denote by $(x_o,y_o ; \xi_o, \eta_o)$ and $(x_1,y_1 ; \xi_1,\eta_1)$ the coordinates on T^*X and T^*Y, respectively, and by $(t,x_o,y_o ; \tau,\xi_o,\eta_o)$ and $(t,x_1,y_1 ; \tau,\xi_1,\eta_1)$ the coordinates on $T^*\tilde{X}_M$, and $T^*\tilde{Y}_N$ respectively.

Let us choose $p = (0,y_1^o ; \xi_1^o,0)$ in U. We shall show that β and γ are isomorphisms at $\tilde{p} = (0, 0, y_1^o ; \tau^o,\xi_1^o,0)$. The calculation for β and γ being almost similar, we shall only prove the result for β.

If β is not an isomorphism at $\tilde{p}$, then by Proposition 5.3.2., $\tilde{f}'$ is not non characteristic for $SS(\mathbb{R}j_{X*}\tilde{p}_X^{-1}(\underline{F}))$ at $\tilde{p}$. Hence, there exist a sequence $\{(t_n', x_1^n, y_1^n)\}$ in $\tilde{Y}_N$ and a sequence $\{(t_n, x_o^n, y_o^n\ ;\ \tau_n, \xi_o^n, \eta_o^n)\}$ in $SS(\mathbb{R}j_{X*}\tilde{p}_X^{-1}(\underline{F}))$ such that :

$$(5.4.3)\quad \begin{cases} (t_n', x_1^n, y_1^n) \xrightarrow[n]{} (0, 0, y_1^o) \\ (t_n, x_o^n, y_o^n) \xrightarrow[n]{} (0, 0, f(0, y_1^o)) \end{cases}$$

$$(5.4.4)\quad \tau_n + \tilde{g}_t(t_n', x_1^n, y_1^n)\xi_o^n + \tilde{h}_t(t_n', x_1^n, y_1^n)\eta_o^n \xrightarrow[n]{} \tau^o$$

$$(5.4.5)\quad \begin{cases} \tilde{g}_{x_1}(t_n', x_1^n, y_1^n).\xi_o^n + \tilde{h}_{x_1}(t_n', x_1^n, y_1^n).\eta_o^n \xrightarrow[n]{} \xi_1^o \\ \tilde{g}_{y_1}(t_n', x_1^n, y_1^n).\xi_o^n + \tilde{h}_{y_1}(t_n', x_1^n, y_1^n).\eta_o^n \xrightarrow[n]{} 0 \end{cases}$$

$$(5.4.6)\quad |\tau_n| + |\xi_o^n| + |\eta_o^n| \xrightarrow[n]{} \infty$$

By (5.4.4) and (5.4.6) we get :

$$(5.4.7)\quad |\xi_o^n| + |\eta_o^n| \xrightarrow[n]{} \infty$$

First we shall assume $t_n > 0$. Then :

$$(t_n, x_o^n, y_o^n\ ;\ \tau_o^n, \xi_o^n, \eta_o^n) \in SS(\tilde{p}_X^{-1}(\underline{F}))\ ,$$

hence $(t_n x_o^n, y_o^n\ ;\ \xi_o^n,\ t_n\eta_o^n) \in SS(\underline{F})$.

We have :

$$(df)(\xi_o^n, t_n\eta_o^n) = (\tilde{g}_{x_1}.\xi_o^n + \tilde{h}_{x_1}.\eta_o^n,\ \tilde{g}_{y_1}.\xi_o^n + \tilde{h}_{y_1}.\eta_o^n) \xrightarrow[n]{} (\xi_1^o, 0)$$

$$(t_n x_o^n, y_o^n) \xrightarrow[n]{} f(0, y_1^o)$$

Since f is non characteristic for $SS(\underline{F})$ on U, $|\xi_o|$ and

$|t_n \eta_o^n|$ are bounded . Hence :

(5.4.8) $$|\eta_o^n| \xrightarrow[n]{} \infty$$

If $(\xi_o^n, t_n\eta_o^n) \xrightarrow[n]{} (\tilde{\xi}_o, \tilde{\eta}_o)$, then $(0, y_1^o ; \tilde{\xi}_o, \tilde{\eta}_o) \in \varpi^{-1}(SS(\underline{F})) \cap \rho^{-1}(U)$.
By the condition i) , $\tilde{\eta}_o = 0$. Therefore :

(5.4.9) $$|t_n \eta_o^n| \xrightarrow[n]{} 0$$

Assume $\eta_o^n / |\eta_o^n| \xrightarrow[n]{} \bar{\eta}_o$. Then $\frac{1}{|\eta_o^n|}(t_n x_o^n, \eta_o^n) \xrightarrow[n]{} (0, \bar{\eta}_o)$ by (5.4.8), which implies :

(5.4.10) $$(f(0,y_1^o) \; ; \; \tilde{\xi}_o \; ; \; \langle \bar{\eta}_o \, , \, \frac{\partial}{\partial \eta} \rangle) \in C_{T_M^* X}(SS(\underline{F}))$$

(since $|\xi_o^n|$ is bounded, we may assume $\xi_o^n \xrightarrow[n]{} \tilde{\xi}_o$) .

By the condition iii), $\tilde{h}_{y_1}(0,0,y_1^o).\bar{\eta}_o$ is different from zero.
On the other hand, condition (5.4.5) implies :

$$\tilde{g}_{y_1}(t_n', x_1^n, y_1^n).\xi_o^n / |\eta_o^n| + \tilde{h}_{y_1}(t_n', x_1^n, y_1^n).\eta_o^n / |\eta_o^n| \xrightarrow[n]{} 0$$

Since $|\xi_o^n|$ is bounded we obtain $\tilde{h}_{y_1}(0,0,y_1^o).\bar{\eta}_o = 0$, which is a contradiction.

Now let us assume $t_n = 0$.
Since $(0, x_o^n, y_o^n \; ; \; \tau_o^n, \xi_o^n, \eta_o^n) \in SS(\mathbb{R}j_{X*}\tilde{p}_X^{-1}(\underline{F}))$, Theorem 4.3.4. guarantes the existence of a double sequence $\{(t_{n,m}, x_o^{n,m}, y_o^{n,m} \; ; \; \tau_o^{n,m}, \xi_o^{n,m}, \eta_o^{n,m})\}$ in $SS(\tilde{p}_X^{-1}(\underline{F}))$ such that :

$$(t_{n,m}, x_o^{n,m}, y_o^{n,m} ; \tau_o^{n,m}, \xi_o^{n,m}, \eta_o^{n,m}) \xrightarrow[m]{} (t_n, x_o^n, y_o^n ; \tau_o^n, \xi_o^n, \eta_o^n)$$

and $t_{n,m}$ is positive. Hence one can choose a subsequence which satisfies (5.4.3), (5.4.5), (5.4.7) and the same reasoning gives us the contradiction. □

Corollary 5.4.2. : Let $f, Y, X, N, M, U, \underline{F}$ be as in Theorem 5.4.1., and assume the same conditions i), ii), iii). Then if M is transversal to f (i.e. : f is non characteristic for $T^*_M X$), and $N = f^{-1}M$, then :

$$\mu_N(f^{-1}(\underline{F}))\big|_U \cong \mathbb{R}\rho_* \varpi^{-1} \mu_M(\underline{F})\big|_U$$

Note that is this case, ρ is an isomorphism from $Y \underset{X}{\times} T^*_M X$ to $T^*_N Y$.

Remark 5.4.3. : When f is smooth then ii) is automatically satisfied. When $f\big|_N : N \longrightarrow M$ is smooth, i) and iii) are automatically satisfied.

§5.5. The functor $\mu hom(\cdot,\cdot)$

5.5.1. Let f be a map of class C^α ($\alpha \geqslant 2$) from Y to X. We denote as usual by ρ and ϖ the associated maps from $Y \underset{X}{\times} T^*X$ to T^*Y and T^*X respectively. We shall identify T^*X (resp. T^*Y) with $T^*_{\Delta_X}(X \times X)$ (resp. $T^*_{\Delta_Y}(Y \times Y)$) by the first projection where Δ_X (resp. Δ_Y) is the diagonal of $X \times X$ (resp. $Y \times Y$). We denote by Δ the graph of f in $X \times Y$ and identify $T^*_\Delta(X \times Y)$ with $Y \underset{X}{\times} T^*X$. Thus we have the maps :

$$T^*_{\Delta_Y}(Y \times Y) \xleftarrow{\ \rho\ } T^*_\Delta(X \times Y) \xrightarrow{\ \varpi\ } T^*_{\Delta_X}(X \times X)$$

We denote by q_j ($j = 1,2$) the j-th projection defined on $X \times Y$. We shall consider the diagram :

$$\begin{array}{ccccc} Y \times Y & \xrightarrow{\ f_1\ } & X \times Y & \xrightarrow{\ f_2\ } & X \times X \\ \cup & & \cup & & \cup \\ \Delta_Y & = & \Delta & \xrightarrow{\ f\ } & \Delta_X \end{array}$$

where the square on the right-hand side is cartesian, and Δ is transversal to f_2 .

Definition 5.5.1. : i) Let $\underline{G} \in Ob(D^b(Y))$, $\underline{F} \in Ob(D^+(X))$. We set :

$$\mu hom(\underline{G} \longrightarrow \underline{F}) = \mu_\Delta\ \mathbb{R}\underline{Hom}(q_2^{-1}\ \underline{G},\ q_1^!\ \underline{F})$$

ii) Let $\underline{G} \in Ob(D^+(Y))$, $\underline{F} \in Ob(D^b(X))$. We set :

$$\mu hom(\underline{F} \longrightarrow \underline{G}) = \mu_\Delta\ \mathbb{R}\underline{Hom}(q_1^{-1}\ \underline{F}\ ,\ q_2^!\ \underline{G})$$

iii) When $Y = X$ and f is the identity, $\underline{F} \in Ob(D^+(X))$, $\underline{G} \in Ob(D^b(X))$, we set :

$$\mu hom(\underline{G},\underline{F}) = \mu hom(\underline{G} \longrightarrow \underline{F})$$

Remark that :

$$(5.5.1) \qquad supp\ \mu hom(\underline{G} \longrightarrow \underline{F}) \subset \varpi^{-1}(SS(\underline{F})) \cap \rho^{-1}(SS(\underline{G}))$$

$$(5.5.2) \qquad supp\ \mu hom(\underline{F} \longleftarrow \underline{G}) \subset \varpi^{-1}(SS(\underline{F})) \cap \rho^{-1}(SS(\underline{G}))$$

In particular when $X = Y$

$$(5.5.3) \qquad supp\ \mu hom(\underline{F}\ ,\ \underline{G}) \subset SS(\underline{F}) \cap SS(\underline{G})$$

Let π denote the canonical projection from $T^*_\Delta(X \times Y)$ to $\Delta \cong Y$.

Proposition 5.5.2. : We have canonical isomorphisms :

$$\mathbb{R}\pi_*\ \mu hom(\underline{G} \longrightarrow \underline{F}) = \mathbb{R}\underline{Hom}(\underline{G}, f^!\underline{F})$$

$$\mathbb{R}\pi_*\ \mu hom(\underline{F} \longleftarrow \underline{G}) = \mathbb{R}\underline{Hom}(f^{-1}\underline{F}, \underline{G})$$

In particulay when $Y = X$,

$$\mathbb{R}\pi_*\ \mu hom(\underline{G},\ \underline{F}) = \mathbb{R}\underline{Hom}(\underline{G},\underline{F})$$

Proof

We have :

$$\mathbb{R}\pi_*\ \mu_\Delta\ \mathbb{R}\underline{Hom}(q_2^{-1}\ \underline{G},\ q_1^!\ \underline{F}) = \mathbb{R}q_{2*}\,\mathbb{R}\Gamma_\Delta\,\mathbb{R}\underline{Hom}(q_2^{-1}\ \underline{G},\ q_1^!\ \underline{F})$$

$$= \mathbb{R}q_{2*}\,\mathbb{R}\underline{Hom}((q_2^{-1}\underline{G})_\Delta, \mathbb{R}\Gamma_\Delta(q_1^!\underline{F}))$$
$$= \mathbb{R}\underline{Hom}(\underline{G}, f^!\,\underline{F})$$

Similarly :

$$\mathbb{R}\pi_*\,\mu_\Delta\,\mathbb{R}\underline{Hom}(q_1^{-1}\underline{F}, q_2^!\,\underline{G}) = \mathbb{R}q_{2*}\,\mathbb{R}\underline{Hom}((q_1^{-1}\underline{F})_\Delta, \mathbb{R}\Gamma_\Delta(q_2^!\underline{G}))$$
$$= \mathbb{R}\underline{Hom}(\mathbb{R}q_{2!}((q_1^{-1}\underline{F})_\Delta), \underline{G})$$
$$= \mathbb{R}\underline{Hom}(f^{-1}\underline{F}, \underline{G}) . \quad \square$$

<u>Proposition 5.5.3.</u> : <u>Let</u> Y <u>be a closed submanifold of</u> X. <u>Then</u> :

$$\mu hom(\underline{A}_Y, \underline{F}) \cong \mu_Y(\underline{F}) .$$

<u>Proof</u>

Let f be the injection $Y \hookrightarrow X$. We have :

$$\mu_{\Delta_X}\,\mathbb{R}\underline{Hom}(q_2^{-1}\underline{A}_Y, q_1^!\,\underline{F}) \cong \mu_{\Delta_X}\,\mathbb{R}f_{2*}\,f_2^!\,q_1^!\,\underline{F}$$
$$\cong \mathbb{R}\varpi_*\,\mu_\Delta\,f_2^!\,q_1^!\,\underline{F}$$
$$\cong \mu_\Delta\,(q_1^!\,\underline{F})$$
$$\cong \mu_Y(\underline{F})$$

by applying Propositions 2.3.4. and 2.3.5. . $\square$

<u>5.5.2</u>. We shall compare the functors $\mu hom(\underline{G},\underline{F})$, $\mu hom(\underline{G}, f^!\,\underline{F})$, $\mu hom(f^{-1}F, \underline{G})$, etc ...

<u>Proposition 5.5.4.</u> : <u>In the situation of Definition 5.5.1. we assume</u> f <u>is proper on</u> supp $\underline{G}$. <u>Then</u> :

a) $\mu hom(\mathbb{R}f_!\underline{G},\underline{F}) \cong \mathbb{R}\varpi_*\,\mu hom(\underline{G} \longrightarrow \underline{F})$

b) $\mu hom(\underline{F},\mathbb{R}f_*\underline{G}) \cong \mathbb{R}\varpi_*\,\mu hom(\underline{F} \longleftarrow \underline{G})$

Proof

a) $\mu hom(\mathbb{R}f_! \underline{G}, \underline{F}) = \mu_{\Delta_X} \mathbb{R}\underline{Hom}(q_2^{-1} \mathbb{R}f_! \underline{G}, q_1^! \underline{F})$

$= \mu_{\Delta_X} \mathbb{R}\underline{Hom}(\mathbb{R}f_{2!} q_2^{-1} \underline{G}, q_1^! \underline{F})$

$= \mu_{\Delta_X} \mathbb{R}f_{2*} \mathbb{R}\underline{Hom}(q_2^{-1} \underline{G}, q_1^! \underline{F})$

It remains to apply Proposition 2.3.4..

b) $\mu hom(\underline{F}, \mathbb{R}f_* \underline{G})^a = \mu_{\Delta_X} \mathbb{R}\underline{Hom}(q_1^{-1} \underline{F}, q_2^! \mathbb{R}f_* \underline{G})$

$= \mu_{\Delta_X} \mathbb{R}\underline{Hom}(q_1^{-1} \underline{F}, \mathbb{R}f_{2*} q_2^! \underline{G})$

$= \mu_{\Delta_X} \mathbb{R}f_{2*} \mathbb{R}\underline{Hom}(q_1^{-1} \underline{F}, q_2^! \underline{G})$

and we apply Proposition 2.3.4. .

Proposition 5.5.5. : In the situation of Definition 5.5.1., we assume that f is non characteristic for $\underline{F}$ on $U \cap SS(\underline{G})$, where U is an open subset of T^*Y. Then we have :

$$\mu hom(\underline{G}, f^! \underline{F})\Big|_U \cong \mathbb{R}\rho_* \mu hom(\underline{G} \longrightarrow \underline{F})\Big|_U$$

$$\mu hom(f^{-1}\underline{F}, \underline{G})\Big|_U \cong \mathbb{R}\rho_* \mu hom(\underline{F} \longleftarrow \underline{G})\Big|_U$$

Proof

Since $f_1^! \mathbb{R}\underline{Hom}(q_2^{-1} \underline{G}, q_1^! \underline{F}) = \mathbb{R}\underline{Hom}(q_2^{-1} \underline{G}, q_1^! f^! \underline{F})$ and f_1 is non characteristic on U, we can apply Theorem 5.4.1. to obtain

$$\mathbb{R}\rho_* \mu_\Delta \mathbb{R}\underline{Hom}(q_2^{-1} \underline{G}, q_1^! \underline{F}) = \mu_{\Delta_Y} \mathbb{R}\underline{Hom}(q_2^{-1} \underline{G}, q_1^! f^! \underline{F})$$

This shows the first isomorphism. In order to prove the second one, we apply the same theorem to $\mathbb{R}\underline{Hom}(q_1^{-1} \underline{F}, q_2^! \underline{G})$.

Corollary 5.5.6. : In the situation of Definition 5.5.1. assume :

i) f is smooth

ii) f is proper on supp(G)

Then we have natural isomorphisms on T^*X :

a) $\mu\,hom(\mathbb{R}f_!\underline{G},\underline{F}) \cong \mathbb{R}\varpi_* \rho^{-1} \mu\,hom(\underline{G}, f^!\underline{F})$

$\cong \mathbb{R}\varpi_* \mu\,hom(\underline{G} \longrightarrow \underline{F})$

b) $\mu\,hom(\underline{F}, \mathbb{R}f_*\underline{G}) \cong \mathbb{R}\varpi_* \rho^{-1} \mu\,hom(f^{-1}\underline{F},\underline{G})$

$\cong \mathbb{R}\varpi_* \mu\,hom(\underline{F} \longleftarrow \underline{G})$

Corollary 5.5.7. : In the situation of Definition 5.5.1., assume :

(i) f is an embedding

(ii) f is non characteristic for $\underline{F}$ on $U \cap SS(\underline{G})$ where U is an open subset of T^*Y.

Then we have natural isomorphisms:

a) $\mu hom(\underline{G}, f^!\underline{F})|_U \cong \mathbb{R}\rho_* \varpi^{-1} \mu hom(\mathbb{R}f_!\underline{G},\underline{F})|_U$

b) $\mu hom(f^{-1}\underline{F},\underline{G})|_U \cong \mathbb{R}\rho_* \varpi^{-1} \mu hom(\underline{F}, \mathbb{R}f_*\underline{G})|_U$

5.5.3. We shall calculate the stalk of $\mu hom(\underline{F},\underline{G})$.

Proposition 5.5.8. : Assume X is a vector space. Let $\underline{F} \in Ob(D^b(X))$, $\underline{G} \in Ob(D^+(X))$, and let $(x_o;\xi_o) \in T^*X$. Then :

$$H^j \mu hom(\underline{F},\underline{G})_{(x_o;\xi_o)} = \varinjlim_{U,G} H^j\, \mathbb{R}\Gamma(U;\mathbb{R}\underline{Hom}(\phi_G^{-1}\mathbb{R}\phi_{G*}\underline{F}_U, \underline{G}))$$

where U runs over the family of open neighborhoods of x_o , and G runs over the family of closed convex proper cones such that :

$$G \subset \{\gamma \in X \; ; \; \langle\gamma,\xi_o\rangle < 0\} \cup \{0\} \qquad (5.5.4)$$

Here ϕ_G denotes the continuous map $X \longrightarrow X_G$.

Proof

Let G be a closed proper convex cone of X. Set :

$$Z_G = \{(x,x') \in X \times X \, ; \, x' - x \in G\} \tag{5.5.5}$$

Then :

$$H^j \, \mu hom(\underline{F},\underline{G})_{(x_o;\xi_o)} = \varinjlim_{U,V,G} H^j \, \mathbb{R}\Gamma_{Z_G}(U\times V, \mathbb{R}\underline{Hom}(q_2^{-1}\underline{F}, q_1^! \underline{G}))$$

where U and V run over the family of open neighborhoods of x_o, and G runs over the family of cones satisfying (5.5.4).

Then we have :

$$\mathbb{R}\Gamma_{Z_G}(U\times V, \mathbb{R}\underline{Hom}(q_2^{-1}\underline{F}, q_1^!\underline{G})) \cong \mathbb{R}\Gamma(U\times V, \mathbb{R}\underline{Hom}((q_2^{-1}\underline{F})_{Z_G}, q_1^!\underline{G})$$
$$\cong \mathbb{R}\Gamma(U, \mathbb{R}\underline{Hom}(\mathbb{R}q_{1V!}(q_2^{-1}\underline{F})_{Z_G}, \underline{G}))$$

where q_{1V} denotes the projection $X\times V \longrightarrow X$. Hence it remains to prove that for a relatively compact open subset V of X, we have :

$$\mathbb{R}q_{1V!}(q_2^{-1}\underline{F})_{Z_G} \cong \phi_G^{-1}\mathbb{R}\phi_{G*}\underline{F}_V \tag{5.5.6}$$

Let K be a compact convex subset of X. Then :

$$\begin{aligned}
\mathbb{R}\Gamma(K, \phi_G^{-1}\,\mathbb{R}\phi_{G*}\,\underline{F}_{\overline{V}}) &\cong \mathbb{R}\Gamma((K+G)\cap\overline{V}\ ;\ \underline{F}) \\
&\cong \mathbb{R}\Gamma(q_2(K\times\overline{V})\cap Z_G;\ \underline{F}) \\
&\cong \mathbb{R}\Gamma((K\times\overline{V})\cap Z_G;\ q_2^{-1}\underline{F}) \\
&\cong \mathbb{R}\Gamma(K\times\overline{V}\ ;\ (q_2^{-1}\underline{F})_{Z_G}) \\
&\cong \mathbb{R}\Gamma(K;\mathbb{R}q_{1\overline{V}*}(q_2^{-1}\underline{F})_{Z_G})
\end{aligned}$$

Moreover we have similar formulas with $\overline{V}$ replaced by ∂V. Then (5.5.6) follows by considering the distinguished triangles :

$$\underline{F}_V \longrightarrow \underline{F}_{\overline{V}} \longrightarrow \underline{F}_{\partial V} \xrightarrow[+1]{} \cdots$$

and

$$\mathbb{R}q_{1V!}(\cdot) \longrightarrow \mathbb{R}q_{1\overline{V}*}(\cdot) \longrightarrow \mathbb{R}q_{1\partial V*}(\cdot) \xrightarrow[+1]{} \cdots$$

Corollary 5.5.9. : Let K be a closed convex subset of X, and let $\underline{F} \in Ob(D^+(X))$. Let $(x_o;\xi_o) \in SS(\underline{A}_K)$. Then :

$$H^j\, \mu hom(\underline{A}_K;\underline{F})_{(x_o;\xi_o)} = \varinjlim_{U,G} H^j\, \mathbb{R}\Gamma_{(K+G^a)\cap U}(U;\underline{F})$$

where U runs over the family of open neighborhoods of x_o, G over the family of closed proper convex cones satisfying (5.5.4) and $G^a = -G$.

Proof

We may assume $\xi_o \neq 0$. Then :

$$K \subset \{x\ ;\ \langle x-x_o, \xi_o\rangle \geqslant 0\}$$

Let G be a closed convex proper cone satisfying (5.5.4). We may find an open neighborhood U of x_o such that :

$$K \cap U = K \cap (U + G) .$$

Then :

$$(\phi^{-1}\, \mathbb{R}\phi_{G*}\underline{A}_{K\cap U})\big|_U \simeq (\phi^{-1}\, \mathbb{R}\phi_{G*}\underline{A}_K)\big|_U .$$

Finally we remark that K being convex and compact :

$$(5.5.7) \qquad \phi^{-1}\, \mathbb{R}\phi_{G*}\underline{A}_K \cong \underline{A}_{K+G^a} .$$

This completes the proof. $\square$

§5.6. Cohomologically constructible sheaves

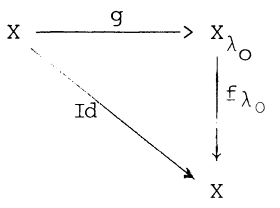
5.6.1. First recall the following about ind-objects and pro-objects.

Let C be a category $\{X_\lambda, f_{\lambda,\mu}\}$ an inductive system, i.e. :
$f_{\lambda,\mu} : X_\mu \longrightarrow X_\lambda$, $\mu \leqslant \lambda$.

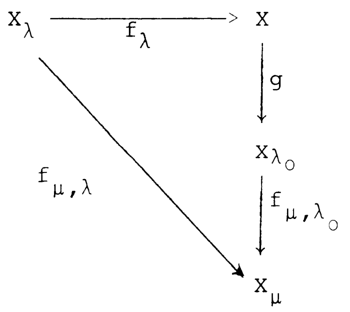
Then $\text{"}\varinjlim\text{"}\, X_\lambda = X$ iff :

1) $\exists\, f_\lambda : X_\lambda \longrightarrow X$ s.t. $f_\lambda \cdot f_{\lambda\mu} = f_\mu \quad \mu \leqslant \lambda$

2) $\exists\, \lambda_0$ and $g : X \longrightarrow X_{\lambda_0}$ s.t. $f_{\lambda_0} \circ g = \mathrm{Id}_X$



and $\forall\lambda\ \exists\mu > \lambda, \lambda_0$ s.t.



e.g. : assume C additive

then $\text{"}\varinjlim\text{"}\, X_\lambda = 0$ iff $\forall\lambda\ \exists\mu \geqslant \lambda$ s.t. $f_{\mu\lambda} = 0$.

The similar remark holds for $\text{"}\varprojlim\text{"}$ by reversing the arrows.

5.6.2. In order to compare the functors $\mathbb{R}\underline{\mathrm{Hom}}(\cdot,\cdot)$ and $\cdot \overset{L}{\otimes} \cdot$ we introduce:

Definition 5.6.1. : *For* $\underline{F} \in Ob(D^b(X))$ *we say that* $\underline{F}$ *is cohomologically constructible if the following conditions are satisfied.*

i) *For any* $x \in X$, $\underset{U \ni x}{\text{"}\varinjlim\text{"}} \mathbb{R}\Gamma(U; \underline{F})$ *is represented by a perfect complex (i.e. : a bounded complex of projective* $\underline{A}$*-modules of finite rank).*

ii) *For any* $x \in X$, $\underset{U \ni x}{\text{"}\varprojlim\text{"}} \mathbb{R}\Gamma_c(U; \underline{F})$ *is represented by a perfect complex.*

Here U *ranges over a system of neighborhoods of* x.

This definition is slightly different from that of Verdier [1].

The following proposition is immediate.

Proposition 5.6.2. : i) *If* $\underline{F} \in Ob(D^b(X))$ *is cohomologically constructible, then* $\mathbb{R}\underline{Hom}(\underline{F}, \underline{A}_X)$ *is also cohomologically constructible,* $\underline{F} \longmapsto \mathbb{R}\underline{Hom}(\mathbb{R}\underline{Hom}(\underline{F}, \underline{A}_X), \underline{A}_X)$ *is an isomorphism, and* $SS(\mathbb{R}\underline{Hom}(\underline{F}, \underline{A}_X)) = SS(\underline{F})^a$.

ii) *If* $\underline{F} \in Ob(D^b(X))$ *is cohomologically constructible, then for* $\underline{G} \in Ob(D^+(Y))$ *we have* :

$$\mathbb{R}\underline{Hom}(q_1^{-1}\underline{F}, q_2^{-1}\underline{G}) = q_1^{-1}\mathbb{R}\underline{Hom}(\underline{F}, A) \overset{L}{\otimes} q_2^{-1}\underline{G}$$

where q_j *is the j-th projection from* $X \times Y$, $(j = 1,2)$.

Proposition 5.6.3. : *Let* $\underline{F} \in Ob(D^b(X))$, $\underline{G} \in Ob(D^+(X))$. *If* $\underline{F}$ *is cohomologically constructible, then* :

$$\mathbb{R}\pi_!\, \mu hom(\underline{F}, \underline{G}) \cong \mathbb{R}\underline{Hom}(\underline{F}, \underline{A}_X) \overset{L}{\otimes} \underline{G}$$

Proof

Let $j : X \hookrightarrow X \times X$ be the diagonal inclusion and q_j the j-th projection from $X \times X$ $(j = 1,2)$. Then we have by Proposition 2.3.2. :

$$\mathbb{R}\pi_! \mu hom(\underline{F},\underline{G}) = j^{-1} \mathbb{R}\underline{Hom}(q_2^{-1} \underline{F}, q_1^! \underline{G}) \overset{\mathbb{L}}{\otimes} j^! \underline{\mathbb{Z}}_{X\times X}$$

On the other hand we have :

$$j^{-1} \mathbb{R}\underline{Hom}(q_2^{-1} \underline{F}, q_1^! \underline{G}) = j^{-1} (q_2^{-1} \mathbb{R}\underline{Hom}(\underline{F},\underline{A}_X) \overset{\mathbb{L}}{\otimes} p_1^! \underline{G})$$

$$= \mathbb{R}\underline{Hom}(\underline{F},\underline{A}_X) \overset{\mathbb{L}}{\otimes} \underline{G} \overset{\mathbb{L}}{\otimes} \mathbb{R}\underline{Hom}(j^! \underline{\mathbb{Z}}_{X\times X}, \underline{\mathbb{Z}}_X)$$

Corollary 5.6.4. : Assume $\underline{F}$ is cohomologically constructible. Then the morphism $\mathbb{R}\underline{Hom}(\underline{F},\underline{A}_X) \overset{\mathbb{L}}{\otimes} \underline{G} \longrightarrow \mathbb{R}\underline{Hom}(\underline{F},\underline{G})$ is an isomorphism on $T^*X \setminus (SS(\underline{F})^a \underset{\infty}{\hat{+}} SS(\underline{G}))$.

Proof

Let us denote by $\dot{\pi}$ the projection from $\dot{T}^*X$ to X. We have a distinguished diagram :

$$\mathbb{R}\pi_! \mu hom(\underline{F},\underline{G}) \longrightarrow \mathbb{R}\pi_* \mu hom(\underline{F},\underline{G}) \longrightarrow \mathbb{R}\dot{\pi}_* \mu hom(\underline{F},\underline{G})\big|_{\dot{T}^*X} \xrightarrow{+1} \cdots$$

Since $SS(\mu hom(\underline{F},\underline{G}))$ is contained into $C_{T^*X}(SS(\underline{G}), SS(\underline{F}))$ by Theorem 5.2.1. the result follows from Proposition 5.1.2. and Corollary 1.2.4. .

Corollary 5.6.5. : If $\underline{F}$ is cohomologically constructible and $SS(\underline{F})^a \cap SS(\underline{G}) \subset T^*_X X$, then :

$$\mathbb{R}\underline{Hom}(\underline{F},\underline{A}_X) \overset{\mathbb{L}}{\otimes} \underline{G} \cong \mathbb{R}\underline{Hom}(\underline{F},\underline{G}) .$$

§5.7. Micro-support and support of the microlocalization

5.7.1. Let X be a manifold of class C^α , $\alpha \geqslant 2$, Y a submanifold. In general the inclusion $supp(\mu_Y(\underline{F})) \subset SS(\underline{F}) \cap T^*_Y X$ is strict. However :

Theorem 5.7.1. : Let Y be a submanifold of X and let $\underline{F} \in Ob(D^+(X))$. Then :

$$T^*_Y X \cap SS(\underline{F}) = supp(\mu_Y(\underline{F})) \cup (\overline{SS(\underline{F}|X \setminus Y)} \cap T^*_Y X)$$

Proof

We may assume X and Y affine. Let $(x_o;\xi_o) \in T^*_Y X$ with $(x_o;\xi_o) \notin supp(\mu_Y(\underline{F}))$ and $(x_o;\xi_o) \notin \overline{SS(\underline{F})|X \setminus Y}$. We shall prove that $(x_o;\xi_o) \notin SS(\underline{F})$. There exists a neighborhood U of x_o and a closed convex proper cone G with non empty interior such that :

i) $\langle \xi_o, G \setminus \{0\}\rangle < 0$

ii) $x \in U \setminus Y$, $\langle \xi, G\setminus\{0\}\rangle < 0 \Longrightarrow (x;\xi) \notin SS(\underline{F})$

iii) $x \in U \cap Y$, $\langle \xi, G\setminus\{0\}\rangle < 0 \Longrightarrow (x;\xi) \notin supp(\mu_Y(\underline{F}))$

Let H be the half-space :

$$H = \{x ; \langle x-x_o, \xi_o\rangle < -a\}$$

with $a > 0$, $a << 1$.

Take $x_1 \in U$, $|x_1-x_o| << 1$, and set :

$$\Omega_1 = x_1 + Int(G)$$

$$\Omega_2 = \{x \in \Omega_1 ; (x+G) \cap Y = \emptyset\}$$

To prove that $(x_o;\xi_o)$ does not belong to $SS(\underline{F})$ we shall show :

$$\mathbb{R}\Gamma_{\Omega_1 \setminus H}(\Omega_1 ; \underline{F}) = 0$$

Let Ω be a G-open set, with $\Omega_1 \cap H \subset \Omega \subset \Omega_2$. Then $\mathbb{R}\Gamma_{\Omega \setminus H}(\Omega ; F) = 0$ since $(\mathbb{R}\phi_{G*} \mathbb{R}\Gamma_{X \setminus H}(\underline{F}))|\Omega_2 = 0$. Now recall (Proposition 2.3.2.) that :

$$H^j((\Omega_1 \cap Y) \times \operatorname{Int} G^{oa} \; ; \; \mu_Y(\underline{F})) = \varinjlim_{V,Z} H^j_Z(V;\underline{F})$$

where V is open, Z is closed in V, and :

$$V \supset \Omega_1 \cap Y$$

$$C_Y(Z) \subset (G^a+Y)/Y \quad (\text{in } T_Y X)$$

Let $\tilde{G}$ be a closed proper convex cone contained into $\operatorname{Int} G \cup \{0\}$, such that assertion i), ii) and iii) are still satisfied with $\tilde{G}$ instead of G. We may ask $Z \supset (V \cap (Y + \tilde{G}^a))$, and thus we may assume :

$$(\Omega_2 \cup (\Omega_1 \cap Y)) \subset V \subset \Omega_1 \;, \quad V \; G\text{-open}$$

$$(Z + \tilde{G}^a) \cap V = Z$$

that is $V \setminus Z$ is $\tilde{G}$-open .

Then :

$$\begin{cases} H^j_{V \setminus Z \setminus H}(V \setminus Z \; ; \; \underline{F}) = 0 & \forall j \;, \\ \varinjlim_{V,Z} H^j_Z(V; \underline{F}) = 0 & \forall j \;, \end{cases}$$

and since $H \cap V = H \cap (V \setminus Z)$:

$$\varinjlim_{V} H^j_{V \setminus H}(V \; ; \; \underline{F}) = 0 \quad \forall j \;,$$

and it remains to remark that ii) implies :

$$\mathbb{R}\Gamma_{\Omega_1 \setminus V}(\Omega_1 \; ; \; \underline{F}) = 0$$

for any $\tilde{G}$-open set V such that $(\Omega_1 \cap Y) \cup \Omega_2 \subset V \subset \Omega_1$.

This proves one of the inclusion of the theorem, and the other one is proved in Theorem 5.2.1. . □

5.7.2. Remark : One could have expected to have $T_Y^*X \cap SS(\underline{F}) = \mathrm{supp}(\mu_Y(\underline{F}))$, but the following example shows that such a result is false in general.

Take $X = \mathbb{R}^2$ with coordinates (x,y), $Y = \{(x,y) ; x = 0\}$, $Z = \{(x,y) ; x > 0, -x < y \leqslant x\}$, $\underline{F} = \underline{\mathbb{C}}_Z$.

Then $\mu_Y(\underline{F}) = 0$, but :

$$SS(\underline{F}) \cap \pi^{-1}(\{0\}) = \{(\xi,\eta) ; \xi+\eta \geqslant 0, \xi-\eta \not\geqslant 0\} .$$

At the same time this shows that we cannot replace in the statement of Theorem 5.2.1. i) the inclusion by an equality.

CHAPTER 6 - CONTACT TRANSFORMATIONS FOR SHEAVES

§6.1. The category $D^+(X;\Omega)$

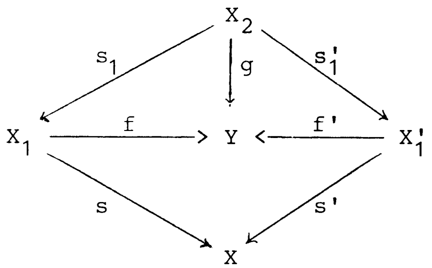
6.1.1. We shall use the same terminology for localization of categories as Hartshorne [1 , §3].

In particular, for a multiplicative system S of a category C, the localization C_S is defined as follows :

$$Ob(C_S) = Ob(C)$$

for $X,Y \in Ob(C)$, $Hom_{C_S}(X,Y) = \{(f,s,X_1) \ ; \ s \in Hom_C(X_1,X)\}$,

s belongs to S, $f \in Hom_C(X_1,Y)\} / \sim$ where $(f,s,X_1) \sim (f',s',X_1')$ if there exists a commutative diagram with $ss_1 = s's_1' \in S$:



Now let C be a triangulated category and N a collection of objects of C. We call N a null system if it satisfies the following axioms (N_1) - (N_3) .

(N_1) $0 \in N$

(N_2) $X \in N$ if and only if $X[1] \in N$

(N_3) Let $X \longrightarrow X' \longrightarrow X'' \xrightarrow[+1]{} \ldots$ be a distinguished triangle. If $X', X \in N$ then $X'' \in N$.

Let S be a collection of arrows $f : X' \longrightarrow X$ which is embedded into a distinguished triangle :

$$X' \xrightarrow{f} X \longrightarrow X'' \xrightarrow[+1]{} \ldots$$

with $X'' \in N$. Then S is a multiplicative system and we can define C_S, which we shall denote by C_N. The following lemma is immediate.

<u>Lemma 6.1.1.</u> : C_N <u>has a structure of triangulated category and the natural transformation</u> $Q : C \longrightarrow C_N$ <u>is a morphism of triangulated categories which satisfies</u> :

i) $Q(X) = 0$ <u>for any</u> $X \in N$

ii) <u>any functor</u> $F : C \longrightarrow D$ <u>of triangulated categories such that</u> $F(X) = 0$ <u>for any</u> $X \in N$ <u>factors uniquely through</u> Q .

<u>6.1.2.</u> Now let X be a real manifold, and let Ω be a subset of T^*X . Let $N(\Omega)$ be the collection of objects $\underline{F}$ of $D^+(X)$ such that $SS(\underline{F}) \cap \Omega = \emptyset$. We write $D^+(X;\Omega)$ for $D^+(X)_{N(\Omega)}$. For $p \in T^*X$ we write $D^+(X;p)$ instead of $D^+(X;\{p\})$ (cf. Brynlinski [2] for a similar construction in the framework of $\mathbb{C}$-constructible sheaves, when $\Omega = \dot{T}^*X$). In §5.5. we defined the functor $\mu hom(\underline{F},\underline{G})$ from $D^b(X)^\circ \times D^+(X)$ to $D^+(T^*X)$. We know by Theorem 5.2.1. and Proposition 4.2.2. that if $\underline{F} \in N(\Omega)$ or $\underline{G} \in N(\Omega)$ then $\mu hom(\underline{F},\underline{G})|_\Omega = 0$. Thus $\mu hom(\underline{F},\underline{G})|_\Omega$ can be considered as a functor from $D^b(X;\Omega)^\circ \times D^+(X;\Omega)$ to $D^+(\Omega)$.

Now we have (Proposition 5.5.2) :

$$H^\circ(T^*X ; \mu hom(\underline{F},\underline{G})) = \mathrm{Hom}_{D^+(X)}(\underline{F},\underline{G})$$

Thus we obtain an homomorphism :

$$\mathrm{Hom}_{D^+(X;\Omega)}(F,G) \longrightarrow H^\circ(\Omega, \mu hom(F,G))$$

In general this morphism is not an isomorphism, but we have :

Proposition 6.1.2. : For any $p \in T^*X$, the natural morphism :

$$\mathrm{Hom}_{D^+(X;p)}(\underline{F},\underline{G}) \longrightarrow H^0(\mu hom(\underline{F},\underline{G}))_p$$

is an isomorphism.

Proof

If $p \in T^*_X X$, there is nothing to prove. Now assume X is a vector space and $p = (x_o;\xi_o) \in \dot{T}^*X$. We keep the notations of Proposition 5.5.8. . Then :

$$H^0\ \mu hom(\underline{F},\underline{G})_p = \varinjlim_{V,G} H^0(\mathbb{R}\Gamma(V;\mathbb{R}\underline{Hom}(\phi_G^{-1}\mathbb{R}\phi_{G*}\underline{F}_V,\underline{G})))$$

Now Proposition 3.2.2. implies :

$$\phi_G^{-1}\mathbb{R}\phi_{G*}\underline{F}_V \longrightarrow \underline{F}$$

is an isomorphism in $D^+(X;p)$.

This proves the injectivity of the morphism in Proposition 6.1.2. . In fact let $u \in \mathrm{Hom}_{D^+(X)}(\underline{F},\underline{G})$ which vanishes in $H^0(\mu hom(\underline{F},\underline{G}))_p$. Then there exist V, G such that the composite :

$$\phi_G^{-1}\mathbb{R}\phi_{G*}\underline{F}_V \xrightarrow[u]{} \underline{G}|_V$$

vanishes.

The surjectivity is similarly proven. □

§6.2. Study of sheaves in a neighborhood of an involutive manifold

6.2.1. Let X be a manifold of class C^α, $\alpha \geqslant 2$, Y a submanifold, and denote by $j : Y \hookrightarrow X$ the embedding of Y in X.

Proposition 6.2.1. : Let $p \in T^*_Y X$ and let $\underline{F} \in Ob(D^+(X))$. Assume $SS(\underline{F}) \subset \pi^{-1}(Y)$ in a neighborhood of p. Then there exists $\underline{G} \in Ob(D^+(Y))$ such that $\underline{F} \cong \mathbb{R}j_*\underline{G}$ in $D^+(X\,;p)$.

Proof

If p belongs to $T^*_X X$ there is nothing to prove. Assume $p \in \dot{T}^*_Y X$. By induction on the codimension of Y in X we may assume Y is a hypersurface. Let $\{f = 0\}$ be an equation of Y, with $x_o \in Y$, $p = df(x_o)$. Set $\Omega^{\pm} = \{x\,;\,f(x) \lessgtr 0\}$, and let i (resp. i') be the injection $\Omega^- \hookrightarrow X$ (resp. $\Omega^+ \hookrightarrow X$). Applying Theorem 4.3.4. we find $p \notin \mathbb{R}i_*\, i^{-1}\,\underline{F}$. Hence $\mathbb{R}\Gamma_{\{f \geqslant 0\}}(\underline{F}) \cong \underline{F}$ in $D^+(X\,;p)$, and we may assume from the beginning that $supp(\underline{F}) \subset \{f \geqslant 0\}$. Since $\underline{F}_{\Omega^+} \cong \mathbb{R}i'_!\, i'^{-1}(\underline{F})$, we find again by Theorem 4.3.4. that $p \notin SS(\underline{F}_{\Omega^+})$. Thus $\underline{F}$ is isomorphic to $\underline{F}_Y$ in $D^+(X,p)$. Finally we remark that $\underline{F}_Y \cong j_*j^{-1}(\underline{F}_Y)$. □

Proposition 6.2.2. : Let Y be a submanifold of X, $p \in T^*_Y X$, $\underline{F} \in Ob(D^+(X))$ and assume $SS(\underline{F}) \subset T^*_Y X$ in a neighborhood of p. Then there exists a complex $M^\bullet$ of A-modules such that $\underline{F}$ is isomorphic to $\underline{M}^\bullet_Y$ in $D^+(X\,;p)$.

Proof

By Proposition 6.2.1., $\underline{F} \cong \mathbb{R}j_*\underline{G}$ in $D^+(X\,;p)$.

Hence Proposition 4.1.1. b) implies $SS(\underline{G}) \subset T^*_Y Y$, and we can apply Proposition 4.1.2.. □

6.2.2. Now let $f : Y \longrightarrow X$ be a smooth map of manifolds of class C^α ($\alpha \geqslant 2$). We identify $Y \times_X T^*X$ to a submanifold of T^*Y.

Proposition 6.2.3. : Let $p \in Y \times_X T^*X$ and let $\underline{G} \in Ob(D^+(Y))$.

Assume $SS(\underline{G})$ is contained in $Y \times_X T^*X$ in a neighborhood of p. Then there exists $\underline{F} \in Ob(D^+(X))$ such that $\underline{G} \cong f^{-1}(\underline{F})$ in $D^+(Y;p)$.

Proof

We may assume $Y = \mathbb{R}^n$, $X = \mathbb{R}^{n-1}$, f denoting the projection $(x_1, x') \longmapsto (x')$, and $p = (0;\xi_o)$, with $\xi_o = (0, \xi_o')$. If $\xi_o = 0$ the result has already been proved (Proposition 4.1.2.), thus we assume $\xi_o \neq 0$.

Let G be a closed convex proper cone such that G^{oa} is a neighborhood of ξ^o and that $(U \times G^{oa}) \cap SS(\underline{F})$ is contained in $Y \times_X T^*X$ for a neighborhood U of 0.

Let ϕ be the natural map from Y to Y_G and let H be the half space $\{x ; \langle x, \xi_o\rangle > -\varepsilon\}$. Then $\underline{F}' = \phi^{-1} \mathbb{R}\phi_*(\underline{F}_H)$ is isomorphic to $\underline{F}$ in $D^+(Y;p)$ by Proposition 3.2.2. .

Now $SS(\underline{F}_H) \cap (U \times G^{oa})$ is contained in $Y \times_X T^*X$ since $((Y \times_X T^*X) \hat{+} N^*(H)^a) \subset Y \times_X T^*X$ and $((SS(\underline{F}) \setminus (U \times G^{oa})) \hat{+} N^*(H)^a) \cap (U \times \{\xi_o\}) = \emptyset$. If we choose ε and G^{oa} small enough we get by Propositions 3.2.2. and 4.2.3. that $SS(\underline{F}')$ is contained in $Y \times_X T^*X$. It remains to apply Proposition 4.1.2. . $\square$

§6.3. Contact transformations

In this section we shall assume for the sake of simplicity that A is commutative and $wg\ell d(A)$ is finite.

6.3.1. Let X and Y be two manifolds of class C^α $(\alpha \geqslant 2)$. We denote by q_j the j-th projection from $X \times Y$, and by p_j the j-th projection from $T^*(X \times Y) = T^*X \times T^*Y$. We set $p_j^a = p_j \circ a$, where a is the anti-podal map. If Z is a third manifold, we also use the notation $q_{i,j}$ to denote the (i,j)-th projection. For example q_{13} is the projection from $X \times Y \times Z$ to $X \times Z$.

Proposition 6.3.1. : Let Ω_X be an open conic subset of T^*X, $\underline{K} \in \mathrm{Ob}(D^b(X \times Y))$. Assume :

(6.3.1) The projection : $SS(\underline{K}) \cap p_1^{a-1}(\Omega_X) \longrightarrow \Omega_X$ is proper.

Then we have :

a) For $\underline{G} \in \mathrm{Ob}(D^+(Y))$, set :

$$\underline{F}_1 = \mathbb{R}q_{1!}\,\mathbb{R}\underline{Hom}(\underline{K}, q_2^{-1}\underline{G})$$

$$\underline{F}_2 = \mathbb{R}q_{1*}\,\mathbb{R}\underline{Hom}(\underline{K}, q_2^{-1}\underline{G})$$

Then $\underline{F}_1 \longrightarrow \underline{F}_2$ is an isomorphism in $D^+(X ; \Omega_X)$, and :

$$SS(\underline{F}_1) \cap \Omega_X \subset p_1^a(SS(\underline{K}) \cap p_2^{-1}(SS(\underline{G})))$$

b) If furthermore $\underline{K}$ is cohomologically constructible, then we have the isomorphisms in $D^+(X ; \Omega_X)$:

$$F_1 \cong \mathbb{R}q_{1!}(\mathbb{R}\underline{Hom}(\underline{K}, \underline{A}_{X\times Y}) \overset{\mathbb{L}}{\otimes} q_2^{-1}\underline{G})$$

$$\cong \mathbb{R}q_{1*}(\mathbb{R}\underline{Hom}(\underline{K}, \underline{A}_{X\times Y}) \overset{\mathbb{L}}{\otimes} q_2^{-1}\underline{G})$$

Proof

First we shall show :

$$(6.3.2)\quad \begin{cases} (SS(\underline{K})^a \mathbin{\hat{+}} T^*_X X \times SS(\underline{G})) \cap p_1^{-1}(\Omega_X) = (SS(\underline{K})^a + T^*_X X \times SS(G)) \cap p_1^{-1}(\Omega_X) \\ (SS(\underline{K})^a \underset{\infty}{\hat{+}} T^*_X X \times SS(\underline{G})) \cap p_1^{-1}(\Omega_X) = \emptyset \end{cases}$$

In fact if we choose coordinate systems on X and Y , and if $(x,y ; \xi,\eta)$ are the coordinates on $T^*(X \times Y)$, then :

$$(x,y;\xi,\eta) \in (SS(K)^a \mathbin{\hat{+}} T^*_X X \times SS(\underline{G})) \cap p_1^{-1}(\Omega_X)$$

iff there exist sequences $\{(x_n, y_n; \xi_n, \eta_n)\}$ in $SS(K)^a$ and $\{(y'_n, \eta'_n)\}$ in $SS(\underline{G})$ such that $(x_n, y_n) \underset{n}{\longrightarrow} (x,y)$, $y'_n \underset{n}{\longrightarrow} y$, $\xi_n \underset{n}{\longrightarrow} \xi$, $\eta_n + \eta'_n \underset{n}{\longrightarrow} \eta$, $|\eta_n||y_n - y'_n| \underset{n}{\longrightarrow} 0$. By the assumption (6.3.1), $|\eta_n|$ is bounded, and thus $|\eta'_n|$ is bounded, which proves (6.3.2).

Now we can apply Theorem 4.4.2. to $\mathbb{R}\underline{Hom}(\underline{K}, q_2^{-1}\underline{G})$ which proves a), and b) follows from Corollary 5.6.4. and Theorem 4.4.2. . □

<u>Remark 6.3.2.</u> : By replacing $\mathbb{R}\underline{Hom}(\underline{K}, \cdot)$ with $\underline{K} \overset{\mathbb{L}}{\otimes} \cdot$, $\mathbb{R}\underline{Hom}(\underline{K}, \underline{A}_{X\times Y}) \overset{\mathbb{L}}{\otimes} \cdot$ with $\mathbb{R}\underline{Hom}(\mathbb{R}\underline{Hom}(\underline{K}, \underline{A}_{X,Y}), \cdot)$, and p_1^a with p_1, we have a similar result, that we do not repeat.

Now let Ω_X and Ω_Y be two conic open subsets of T^*X and T^*Y respectively, and let $\underline{K} \in Ob(D^b(X\times Y))$.

Consider the conditions :

$$(6.3.3)_{(X,Y)} \quad \begin{cases} p_1^{a-1}(\Omega_X) \cap SS(\underline{K}) \subset p_2^{-1}(\Omega_Y) \\ p_1^{a-1}(\Omega_X) \cap SS(\underline{K}) \text{ is proper over } \Omega_X \end{cases}$$

and let $\Phi_{\underline{K}}$ be the functor from $D^+(Y)$ to $D^+(X)$ given by :

$$\underline{G} \longmapsto \mathbb{R}q_{1*}\,\mathbb{R}\underline{Hom}(\underline{K}, q_2^!\underline{G})$$

Then if $\underline{K}$ satisfies $(6.3.3)_{(X,Y)}$, $\Phi_{\underline{K}}$ sends $N(\Omega_Y)$ into $N(\Omega_X)$, so that by taking the quotient we may associate to $\Phi_{\underline{K}}$ a well defined functor, that we still denote $\Phi_{\underline{K}}$, from $D^+(Y ; \Omega_Y)$ to $D^+(X ; \Omega_X)$:

$$\Phi_{\underline{K}} : D^+(Y ; \Omega_Y) \longrightarrow D^+(X ; \Omega_X)$$

Remark that $\Phi_{\underline{K}}$ depends only on the image of $\underline{K}$ in $D^b(X \times Y ; \Omega_X^a \times T^*Y)$, that is if $\underline{K}' \cong \underline{K}$ in $D^b(X \times Y ; \Omega_X^a \times T^*Y)$ then $\Phi_{\underline{K}'} \cong \Phi_{\underline{K}}$.

Similarly, let $\underline{L} \in \mathrm{Ob}(D^b(X \times Y))$ satisfying the following condition:

$$(6.3.4)_{(X,Y)} \quad \begin{cases} p_2^{-1}(\Omega_Y) \cap SS(\underline{L}) \subset p_1^{a-1}(\Omega_X) \\ p_2^{-1}(\Omega_Y) \cap SS(\underline{L}) \quad \text{is proper over} \quad \Omega_Y \end{cases}$$

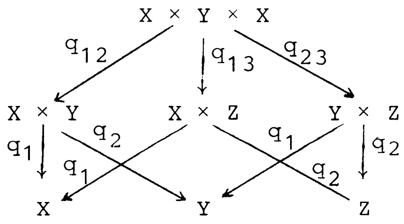
Then we define the functor $\psi_{\underline{L}}$:

$$\psi_{\underline{L}} : D^+(X ; \Omega_X) \longrightarrow D^+(Y ; \Omega_Y)$$

by setting, for $\underline{F} \in \mathrm{Ob}(D^+(X))$:

$$\psi_{\underline{L}}(\underline{F}) = \mathbb{R}q_{2!}(\underline{L} \overset{L}{\otimes} q_1^{-1} \underline{F})$$

Now let Z be another manifold, and Ω_Z a conic open set in T^*Z. Consider the diagram :



Let $\underline{K}_1 \in \mathrm{Ob}(D^b(X \times Y))$, $K_2 \in \mathrm{Ob}(D^b(Y \times Z))$.

<u>Proposition 6.3.3.</u> : <u>Assume</u> $\underline{K}_1$ <u>satisfies</u> $(6.3.3)_{(X,Y)}$ <u>and</u> $\underline{K}_2$ <u>satisfies</u> $(6.3.3)_{(Y,Z)}$. <u>Set</u> :

$$\underline{K}_3 = \mathbb{R}q_{13!}(q_{12}^{-1}\underline{K}_1 \overset{L}{\otimes} q_{23}^{-1}\underline{K}_2) \quad \text{or} \quad \mathbb{R}q_{13*}(q_{12}^{-1}\underline{K}_1 \overset{L}{\otimes} q_{23}^{-1}\underline{K}_2)$$

<u>Then</u> $\underline{K}_3$ <u>satisfies</u> $(6.3.3)_{(X,Z)}$, <u>and</u> :

$$\Phi_{\underline{K}_1} \circ \Phi_{\underline{K}_2} = \Phi_{\underline{K}_3}$$

<u>similarly if we replace condition (6.3.3) by condition (6.3.4), we find that</u> $\underline{K}_3$ <u>satisfies</u> $(6.3.4)_{(X,Z)}$ <u>and</u> :

$$\psi_{\underline{K}_2} \circ \psi_{\underline{K}_1} = \psi_{\underline{K}_3}$$

Proof

The proof that $\underline{K}_3$ satisfies (6.3.3) is a direct application of the preceding results.

Now let $\underline{F} \in Ob(D^+(Z))$

$$\begin{aligned}
\Phi_{\underline{K}_1} \circ \Phi_{\underline{K}_2}(\underline{F}) &= \mathbb{R}q_{1*}\, \mathbb{R}\underline{Hom}(\underline{K}_1, q_2^! \,\mathbb{R}q_{1*}\, \mathbb{R}\underline{Hom}(\underline{K}_2, q_2^!\underline{F}))\\
&= \mathbb{R}q_{1*}\, \mathbb{R}\underline{Hom}(\underline{K}_1, \mathbb{R}q_{12*}q_{23}^! \;\mathbb{R}\underline{Hom}(\underline{K}_2, q_2^!\underline{F}))\\
&= \mathbb{R}q_{1*}\, \mathbb{R}q_{12*}\mathbb{R}\underline{Hom}(q_{12}^{-1}\underline{K}_1, \mathbb{R}\underline{Hom}(q_{23}^{-1}\underline{K}_2, q_{23}^! \; q_2^!\underline{F}))\\
&= \mathbb{R}q_{2*}\, \mathbb{R}q_{13*}\, \mathbb{R}\underline{Hom}(q_{12}^{-1}\underline{K}_1 \overset{\mathbb{L}}{\otimes} q_{23}^{-1}\underline{K}_2, q_{13}^! \; q_2^! \;\underline{F}))\\
&= \mathbb{R}q_{2*}\, \mathbb{R}\underline{Hom}((\mathbb{R}q_{13!}(q_{12}^{-1}\underline{K}_1 \overset{\mathbb{L}}{\otimes} q_{23}^{-1}\underline{K}_2), q_2^! \;\underline{F}))
\end{aligned}$$

Remark that :

$$\mathbb{R}q_{13!}(q_{12}^{-1}\underline{K}_1 \overset{\mathbb{L}}{\otimes} q_{23}^{-1}\,\underline{K}_2) \longrightarrow \mathbb{R}q_{13*}(q_{12}^{-1}\underline{K}_1 \overset{\mathbb{L}}{\otimes} q_{23}^{-1}\,\underline{K}_2)$$

gives an isomorphism in $D^b(X \times Z\ ;\ \Omega_X^a \times \Omega_Z)$.

The proof for $\psi_{\underline{K}_3}$ is similar. $\square$

Theorem 6.3.4. : Let Λ be a closed Lagrangean manifold in $\Omega_X^a \times \Omega_Y$, and let $\underline{K}$ belong to $Ob(D^b(X \times Y))$. Assume :

(6.3.5) $\quad p_1^a : \Lambda \longrightarrow \Omega_X$ and $p_2 : \Lambda \longrightarrow \Omega_Y$

are diffeomorphisms

(6.3.6) $\quad p_1^{a-1}(\Omega_X) \cap SS(\underline{K}) \subset \Lambda$ and $p_2^{-1}(\Omega_Y) \cap SS(\underline{K}) \subset \Lambda$

(6.3.7) $\quad \underline{K}$ is cohomologically constructible

(6.3.8) $\quad \underline{A}_\Lambda \xrightarrow{\sim} \mu hom(\underline{K},\underline{K})\big|_\Lambda$

Then $\Phi_{\underline{K}}$ and $\Psi_{\underline{K}}$ are quasi-inverse to each other and give an equivalence of categories between $D^+(X\,;\,\Omega_X)$ and $D^+(Y\,;\,\Omega_Y)$.

Proof

We set :

(6.3.9) $$D\underline{K} = \mathbb{R}\underline{Hom}(\underline{K}, \underline{A}_{X\times X})$$

Applying Proposition 5.6.2. and 6.3.1. we find that $D\underline{K}$ satisfies condition $(6.3.3)_{(Y,X)}$ and :

$$\Phi_{D\underline{K}} \cong \Psi_{\underline{K}} \quad : \quad D^+(X\,;\,\Omega_X) \longrightarrow D^+(Y\,;\,\Omega_Y) .$$

Let us keep the notations of Proposition 6.3.3. with $Z = X$, and let :

$$\underline{N} = q_{23}^{-1}\, D\underline{K} \overset{\mathbb{L}}{\otimes} q_{12}^{!}\, \underline{K}$$

and

$$\underline{M} = \mathbb{R}q_{13*}\underline{N}$$

Then we have :

$$\Phi_{\underline{K}} \circ \Psi_{\underline{K}} = \Phi_{\underline{M}}$$

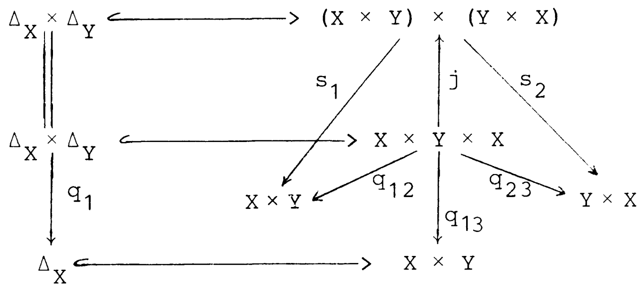
Note that :

$$\underline{N} \cong \underline{N}' \underset{\text{def}}{=\!=} \mathbb{R}\underline{Hom}(q_{23}^{-1}\underline{K}, q_{12}^{!}\underline{K}) \text{ in } D^+(X\times Y\times X\,;\,\Omega_X^a\times T^*Y\times T^*X \cup T^*X\times T^*Y\times\Omega_X)$$

by Corollary 5.6.4. .

Hence $\underline{M}$ is isomorphic to $\underline{M}' = \mathbb{R}q_{13*}\mathbb{R}\underline{Hom}(q_{23}^{-1}\underline{K}, q_{12}^{!}\underline{K})$ in $D^+(X\times X\,;\,\Omega_X^a \times T^*X \cup T^*X\times\Omega_X^a)$ by Theorem 4.4.2. .

Consider the diagram :



Applying Proposition 2.3.4. we get :

$$\mu_{\Delta_X}(\underline{M}') = \mathbb{R}\tilde{q}_{1*}\,\mu_{\Delta_X \times Y}(\underline{N}')$$

where $\tilde{q}_1$ is the natural map from $T^*_{\Delta_X \times Y}(X \times Y \times X)$ to $T^*_{\Delta_X}(X \times X)$.

On the other hand we have :

$$j^! \mathbb{R}\,\underline{Hom}(s_2^{-1}\underline{K}, s_1^!\underline{K}) = \mathbb{R}\,\underline{Hom}(q_{23}^{-1}\underline{K}, q_{12}^!\underline{K})$$

Applying Theorem 5.4.1. we find that $\mu_{\Delta_X}(\underline{M}')$ is the direct image of $\mu_{\Delta_X \times \Delta_Y}(\mathbb{R}\,\underline{Hom}(s_2^{-1}\underline{K}, s_2^!\underline{K})) = \mu hom(\underline{K},\underline{K})$. Thus we obtain $\underline{A}_{\Omega_X^a} \xrightarrow{\sim} \mu_{\Delta_X}(M')\big|_{\Omega_X^a}$. If we denote by i the canonical map $\underline{A}_{\Delta_X} \longrightarrow M'$, then $\mu_{\Delta_X}(i)$ is an isomorphism on Ω_X^a. Since $SS(\underline{M}') \cap (\Omega_X^a \times T^*X) \subset T^*_{\Delta_X}(X \times X)$, i is an isomorphism in $D^+(X \times X\,;\,\Omega_X^a \times T^*X)$. This shows $\Phi_{\underline{M}} \simeq \Phi_{M'} \simeq \Phi_{\underline{A}_{\Delta_X}} = Id$ from $D^+(X\,;\,\Omega_X)$ to $D^+(X\,;\,\Omega_X)$.

The proof for $\psi_{\underline{K}} \circ \Phi_{\underline{K}}$ is similar . $\square$

6.3.2. We can now "extend" contact transformations to sheaves.

Theorem 6.3.5. : Let Ω_X and Ω_Y be two conic open sets in T^*X and T^*Y respectively, ϕ a contact transformation from Ω_Y to Ω_X. For each $\lambda_Y \in \Omega_Y$ there exists a conic open neighborhood Ω'_Y of

λ_Y , and $\underline{K} \in Ob(D^b(X \times Y))$ such that $\underline{K}$ satisfies the conditions $(6.3.1)_{(X,Y)}$ on $\phi(\Omega'_Y)^a \times \Omega'_Y$, and $\Phi_{\underline{K}}$ induces an equivalence of categories between $D^+(Y ; \Omega''_Y)$ and $D^+(X ; \phi(\Omega''_Y))$ for all conic open set $\Omega''_Y \subset \Omega'_Y$.

Proof

It is well-known that ϕ may be locally obtained as the composite $\phi_1 \circ \phi_2$ of two contact transformations such that if Λ_i is the Lagrangean manifold associated to the graph of ϕ_i , then Λ_i is the conormal bundle to a hypersurface $(i = 1,2)$. Applying Proposition 6.3.3. we may assume from the beginning that Λ , the image of the graph of ϕ by the anti-podal map on T^*X , is the conormal bundle to a hypersurface S of $X \times Y$.

By replacing X and Y with small balls, we take $\underline{A}_S$ as $\underline{K}$. Then all the conditions in Theorem 6.3.4. are satisfied . □

Remark 6.3.6. : The extention $\Phi_{\underline{K}}$ of ϕ to the category $D^+(Y ; \Omega_Y)$ is essentially unique. In fact assume ϕ is the identity on a connected open set $\Omega_X \subset T^*X$. Then we find a bounded complex of projective A-modules M such that $\Phi_{\underline{K}} = \Phi_{\underline{M}_\Delta}$. Let ϕ_M be the functor on $D^+(X)$ defined by :

$$\phi_M(\underline{F}) = Hom(M, \underline{F})$$

Then $\Phi_{\underline{M}_\Delta} = \phi_M$, and ϕ_M is an equivalence of categories from $D^+(X ; \Omega_X)$ onto itself.

Let us denote by $D^b(\{p^t\})$ the derived category of the category of bounded complexes of A-modules, and let us denote by $\tilde{\phi}_M$ the functor $Hom(M,\cdot)$ from $D^b(\{p^t\})$ to itself. We shall prove that $\tilde{\phi}_M$ is an equivalence of categories. For that purpose choose a submanifold S of X and $\lambda \in \Omega_X$ such that $\lambda \in T^*_SX$. Set $\Lambda = T^*_SX$, and consider the full subcategory $D^b_\Lambda(X;\lambda)$ consisting of objects $\underline{F}$ of $D^b(X;\lambda)$ with $SS(\underline{F}) \subset \Lambda$. Consider the functor ψ_S from $D^b(\{p^t\})$ to $D^b_\Lambda(X;\lambda)$ which associates the sheaf $\underline{N}_S$ to the A-module N. By Proposition 6.2.2. , this functor is an equivalence of categories. Since $\psi_S \circ \tilde{\phi}_M = \phi_M \circ \psi_S$ and ϕ_M induces an equicalence of categories from $D^b_\Lambda(X;\lambda)$ onto itself, we find that $\tilde{\phi}_M$ is an equivalence of categories.

Definition 6.3.7. : In the situation of Theorem 6.3.5. we say that $\Phi_{\underline{K}}$ is an extended contact transformation above ϕ .

Remark 6.3.8. : Since $\Phi_{\underline{K}}$ is essentially unique, we do not use the terminology "quantized contact transformation" in this context.

In Chapter 11, when ϕ is a complex contact transformation (X and Y are then complex manifolds), we shall construct (non unique) isomorphisms from $\Phi_{\underline{K}}(\mathcal{O}_Y)$ to $\mathcal{O}_X$ (in $D^+(X;\lambda_X)$), and we keep the term of "quantization" to the choice of such an isomorphism.

6.3.3. We shall study the action of extended contact transformations on microlocalization.

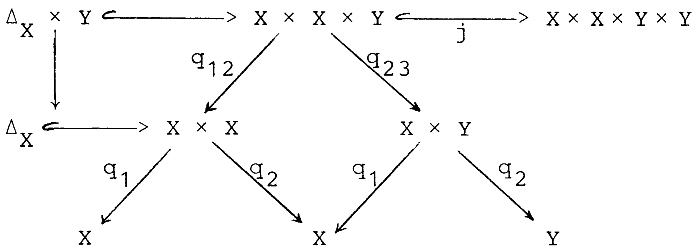
Theorem 6.3.9. : Let Ω_X and Ω_Y be two conic open sets of T^*X and T^*Y respectively, Λ a Lagrangean manifold in $\Omega^a_X \times \Omega_Y$. Let $\underline{K}$ belong to $Ob(D^b(X\times Y))$. We make the assumptions (6.3.5), (6.3.6), (6.3.7) and (6.3.8) of Theorem 6.3.4. . Then for $\underline{F} \in Ob(D^b(X))$ and $\underline{G} \in Ob(D^+(Y))$ we have :

$$\phi_*\mu hom(\psi_{\underline{K}}(\underline{F}),\ \underline{G}) \simeq \mu hom(\underline{F},\ \Phi_{\underline{K}}(\underline{G})) \tag{6.3.10}$$

Here ϕ is the contact transformation $p_1^a \circ (p_2^{-1}|_\Lambda)$ from Ω_Y to Ω_X.

Proof

Consider the diagram :



$$\mu hom(\underline{F}, \Phi_{\underline{K}}(\underline{G})) = \mu_{\Delta_X} \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F}, q_2^! \Phi_{\underline{K}}(\underline{G}))$$

$$= \mu_{\Delta_X} \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F}, \mathbb{R}q_{12*} \mathbb{R}\underline{Hom}(q_{23}^{-1}\underline{K}, q_3^!\underline{G}))$$

$$= \mu_{\Delta_X} \mathbb{R}q_{12*} \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F} \overset{L}{\otimes} q_{23}^{-1}\underline{K}, q_3^!\underline{G})$$

$$= \mathbb{R}p_{1*} \mu_{\Delta_X \times Y}\mathbb{R}\underline{Hom}(q_1^{-1}\underline{F} \overset{L}{\otimes} q_{23}^{-1}\underline{K}, q_3^!\underline{G})$$

Here we have denoted by q_i or q_{ij} the i-th or the (i,j)-th projection from $X \times X \times Y$, or $X \times X$, or $X \times Y$, and by p_1 the natural projection from $T^*_{\Delta_X \times Y}(X \times X \times Y)$ to $T^*_{\Delta_X}(X \times X)$ and we have applied Proposition 2.3.4. .

By the hypotheses on $\underline{K}$, the diagonal embedding j from $X \times X \times Y$ into $X \times X \times Y \times Y$ is non characteristic on $T^*X \times \Omega_X \times T^*Y$ for the sheaf :

$$\underline{E} \underset{\overline{\overline{def}}}{} \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F} \overset{L}{\otimes} q_{23}^{-1}\underline{K}, q_4^!\underline{G})$$

(here we kept the notation q_i , or q_{ij} to denote the projections from $X \times X \times Y \times Y$). Moreover :

$$j^!\underline{E} = \mathbb{R}\underline{Hom}(q_1^{-1}\underline{F} \overset{L}{\otimes} q_{23}^{-1}\underline{K}, q_3^!\underline{G})$$

Thus by applying Corollary 5.4.2. we get that $\mu hom(\underline{F}, \Phi_{\underline{K}}(\underline{G}))\big|_{\Omega_X}$ is the direct image of $\mu_{\Delta_X \times \Delta_Y}(E)\big|_{\Omega_X^a \times \Omega_X \times \Omega_Y^a \times \Omega_Y}$ by the projection from $T^*_{\Delta_X \times \Delta_Y}(X \times X \times Y \times Y)$ to $T^*_{\Delta_X}(X \times X)$. Similarly, we find that $\mu hom(\psi_K(\underline{F}),\underline{G})\big|_{\Omega_Y}$ is the direct image of $\mu_{\Delta_X \times \Delta_Y}(\underline{E})$. Then it is enough to remark that $supp(\mu_{\Delta_X \times \Delta_Y}(\underline{E}))\big|_{\Omega_X^a \times \Omega_X \times \Omega_Y^a \times \Omega_Y}$ is contained in Λ. □

Corollary 6.3.10. : Under the hypotheses of Theorem 6.3.9. let $\lambda \in \Lambda$, $\lambda_X = p_1^a(\lambda)$, $\lambda_Y = p_2(\lambda)$. Then :

$$Hom_{D^+(X;\lambda_X)}(\underline{F}, \Phi_{\underline{K}}(\underline{G})) = Hom_{D^+(Y;\lambda_Y)}(\psi_{\underline{K}}(\underline{F}),\underline{G})$$

That its, $\Phi_{\underline{K}}$ and $\psi_{\underline{K}}$ are adjoint one to each other.

Proof

Apply Theorems 6.3.9. and 6.1.2. . □

Corollary 6.3.11. : In the situation of Theorem 6.3.4., let $\underline{G} \in Ob(D^+(Y))$ and $\underline{G}' \in Ob(D^b(Y))$. Then we have a natural isomorphism of sheaves on Ω_X :

$$\phi_* \mu hom(\underline{G}',\underline{G}) \simeq \mu hom(\Phi_{\underline{K}}(G'),\Phi_{\underline{K}}(G))$$

Similarly for $\underline{F} \in Ob(D^b(X))$ and $\underline{F}' \in Ob(D^+(X))$, we have a natural isomorphism of sheaves on Ω_Y :

$$\mu hom(\psi_{\underline{K}}(F), \psi_{\underline{K}}(F')) \simeq \phi^{-1} \mu hom(\underline{F},\underline{F}')$$

Proof

Apply Theorem 6.3.4. and 6.3.9. . □

Remark 6.3.12. : We may generalize Theorem 6.3.5. by considering the following geometrical situation, already studied by V. Guillemin and S. Sternberg [1] in the framework of Fourier distributions.

Let Λ be a Lagrangean manifold in $T^*(X \times Y)$, $\lambda \in \Lambda$, and assume p_1^a is an immersion on Λ at λ (thus p_2 is a submersion). Let V be the involutive manifold of T^*X, the image of $\Lambda \cap U$ by p_1^a, where U is a sufficiently small neighborhood of λ. We introduce $D_V^+(X ; \lambda_X)$ the full subcategory of $D^+(X ; \lambda_X)$ consisting of those $\underline{F} \in \mathrm{Ob}(D^+(X ; \lambda_X))$ such that $SS(\underline{F}) \subset V$.

Then one can find $\underline{K} \in \mathrm{Ob}(D^b(X \times Y))$ satisfying $(6.3.3)_{(X,Y)}$ such that :

$$\Phi_{\underline{K}} = D^+(Y ; \lambda_Y) \longrightarrow D_V^+(X ; \lambda_X)$$

is an equivalence of categories.

For the proof, assume first V regular involutive. By performing a contact transformation on T^*X (and applying Theorem 6.3.5.), we may assume $X = X' \times Z$, $V = T^*X' \times T_Z^*Z$, $\Lambda = \Lambda' \times T_Z^*Z$, where Λ' is a Lagrangean manifold in $T^*X' \times T^*Y$. Let $\lambda_{X'}$ be the image of λ_X in T^*X'. Then $D_V^+(X ; \lambda_X) \simeq D^+(X' ; \lambda_{X'})$ by Proposition 6.2.3. , and the result follows from Theorem 6.3.5. applied to $\Lambda' \subset T^*(X' \times Y)$.

In the general case we use the "trick of the dummy variable". Let t be a coordinate on $\mathbb{R}$. Replacing Λ by $\Lambda \times T^*_{\Delta_{\mathbb{R}}}(\mathbb{R} \times \mathbb{R})$, we find that $\Phi_{\underline{K}}$ is an equivalence of categories from $D^+(Y \times \mathbb{R} ; (\lambda_Y, (0,dt)))$ to $D^+_{V \times T^*\mathbb{R}}(X \times \mathbb{R} ; (\lambda_X, (0,dt)))$ thus an equivalence of categories from $D^+_{Y \times T^*_{\{0\}}\mathbb{R}}(Y \times \mathbb{R} ; (\lambda_Y, (0,dt)))$ to $D^+_{V \times T^*_{\{0\}}\mathbb{R}}(X \times \mathbb{R}; (\lambda_X, (0,dt)))$. But those two categories are respectively equivalent to $D^+(Y ; \lambda_Y)$ and $D_V^+(X ; \lambda_X)$ by Proposition 6.2.1. .

§6.4. Involutivity of the micro-support

6.4.1. Using contact transformations we are now able to prove the involutivity of micro-supports.

Theorem 6.4.1. : *Let* X *be a manifold of class* C^{α}, $\alpha \geq 2$ *and let* $\underline{F} \in Ob(D^+(X))$. *Then* $SS(\underline{F})$ *is involutive. More precisely, let* f *be a* C^1*-function defined on some open set* U *of* T^*X , *and assume that* $U \cap SS(\underline{F})$ *is contained in* $\{f = 0\}$. *Then* $SS(\underline{F}) \cap U$ *is a union of integral curves of* H_f.

Proof

Let $V = \{f = 0\}$. We may assume V is smooth (there is nothing to prove at points in a neighborhood of which V is not smooth). We know that $SS(\underline{F} \overset{L}{\boxtimes} \underline{\mathbb{Z}}_{\{0\}}) = SS(\underline{F}) \times T^*_{\{0\}}\mathbb{R}$. Thus replacing $\underline{F}$ by $\underline{F} \overset{L}{\boxtimes} \underline{\mathbb{Z}}_{\{0\}}$, we may assume from the beginning V regular involutive. We consider $\underline{F}$ as a complex of sheaves of $\mathbb{Z}$-modules. Then a contact transformation reduces the problem to the case where $X = \mathbb{R}^n$, $n \geq 2$, with coordinates $(x_1,\ldots,x_n)$ and $SS(\underline{F})$ is contained in the set $\{(x,\xi) ; \xi_n = 0\}$ in a neighborhood U of $(0 ; dx_1)$, (i.e. : $f = \xi_n$). Then the result follows from Proposition 6.2.3. □

Remark 6.4.2. : Let Z be a locally closed set in X. Then $SS(\underline{\mathbb{Z}}_Z)$ is involutive. When Z is closed, an interesting set in T^*X is associated to Z by J.M. Bony [1], and a weak form of the involutivity theorem has been proved by Bony, then refined by J. Sjöstrand [1]. But we emphasize that this set defined by Bony is in general strictly smaller than $SS(\underline{\mathbb{Z}}_Z)$, since it may be not closed, and its closure non involutive in the sense of Theorem 6.4.1..

CHAPTER 7 - SIMPLE SHEAVES

§7.1. Index for three Lagrangean planes

We shall recall the definition and main properties of the Maslov index associated to a triplet of Lagrangean planes in a symplectic vector space (cf. Lion-Vergne [1] for proofs and details, and cf. also Maslov [1], Hörmander [2], Leray [2]).

7.1.1. Let (E,σ) be a real symplectic vector space (σ is a non degenerate skew symmetric bilinear form on the finite dimensional vector space E). For a linear subspace ρ of E we set :

$$\rho^{\perp} = \{x \in E \; ; \; \sigma(x,\rho) = 0\}$$

We have $(\rho^{\perp})^{\perp} = \rho$, $(\rho_1 + \rho_2)^{\perp} = \rho_1^{\perp} \cap \rho_2^{\perp}$ and $(\rho_1 \cap \rho_2)^{\perp} = \rho_1^{\perp} + \rho_2^{\perp}$. If $\rho^{\perp} \subset \rho$ (resp. $\rho^{\perp} \supset \rho$, $\rho^{\perp} = \rho$), ρ is called involutive (resp. isotropic, Lagrangean). For an isotropic space ρ, the space $\rho^{\perp}/\rho$ is endowed with a natural structure of symplectic vector space. For $\lambda \subset E$ we set $\lambda^{\rho} = ((\lambda \cap \rho^{\perp}) + \rho)/\rho$. Then :

$$(\lambda^{\perp})^{\rho} = (\lambda^{\rho})^{\perp} .$$

Definition 7.1.1. : Let $\{\lambda_1,\lambda_2,\lambda_3\}$ be a triplet of Lagrangean planes of E. We define $\tau_E(\lambda_1,\lambda_2,\lambda_3)$ as the signature of the quadratic form Q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ defined by $Q(x_1,x_2,x_3) = \sigma(x_1,x_2) + \sigma(x_2,x_3) + \sigma(x_3,x_1)$, for $(x_1,x_2,x_3) \in \lambda_1 \oplus \lambda_2 \oplus \lambda_3$.

Here the signature means the difference of the number of positive eigenvalues and that of negative eigenvalues.

If there is no fear of confusion we write τ for τ_E .

Proposition 7.1.2. : i) $\tau(\lambda_1,\lambda_2,\lambda_3)$ is alternating with respect to permutations of the triplet $\{\lambda_1,\lambda_2,\lambda_3\}$.

ii) (Cocycle condition). For a quadruplet $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ of Lagrangean planes we have :

$$\tau(\lambda_2,\lambda_3,\lambda_4) - \tau(\lambda_1,\lambda_3,\lambda_4) + \tau(\lambda_1,\lambda_2,\lambda_4) - \tau(\lambda_1,\lambda_2,\lambda_3) = 0$$

iii) If ρ is a subspace contained in $(\lambda_1 \cap \lambda_2) + (\lambda_2 \cap \lambda_3) + (\lambda_3 \cap \lambda_1)$ we have :

$$\tau_E(\lambda_1,\lambda_2,\lambda_3) = \tau_{E^\rho}(\lambda_1^\rho,\lambda_2^\rho,\lambda_3^\rho)$$

In particular if $\lambda_1 \cap (\lambda_2+\lambda_3) \subset (\lambda_1 \cap \lambda_2) + (\lambda_1 \cap \lambda_3)$, then $\tau(\lambda_1,\lambda_2,\lambda_3) = 0$.

iv) If $\{\lambda_1,\lambda_2,\lambda_3\}$ moves continuously so that $\dim(\lambda_i \cap \lambda_j)$ $(i,j = 1,2,3)$ remains unchanged, then $\tau(\lambda_1,\lambda_2,\lambda_3)$ is constant.

v) $\tau(\lambda_1,\lambda_2,\lambda_3) \equiv \frac{1}{2} \dim E + \dim(\lambda_1 \cap \lambda_2) + \dim(\lambda_2 \cap \lambda_3) + \dim(\lambda_3 \cap \lambda_1) \mod 2\,\mathbb{Z}$.

More generally for a set of Lagrangean planes $\{\lambda_1,\ldots,\lambda_N\}$ $(N \geqslant 4)$ we define :

$$\tau(\lambda_1,\ldots,\lambda_N) = \sum_{i=2}^{N-1} \tau(\lambda_1,\lambda_i,\lambda_{i+1})$$

Then by the cocycle condition we have :

Proposition 7.1.3. : i) $\tau(\lambda_1,\ldots,\lambda_N) = \tau(\lambda_2,\ldots,\lambda_N,\lambda_1) = - \tau(\lambda_N,\lambda_{N-1},\ldots,\lambda_1)$.

ii) $\tau(\lambda_1,\ldots,\lambda_N) = \sum_{i=1}^{N} \tau(\mu,\lambda_i,\lambda_{i+1})$ (where $\lambda_{N+1} = \lambda_1$) , for any Lagrangean plane μ.

iii) If $\{\lambda_1,\ldots,\lambda_N\}$ moves continuously so that $\dim(\lambda_i \cap \lambda_{i+1})$ is unchanged $(1 \leqslant i \leqslant N)$, then $\tau(\lambda_1,\ldots,\lambda_N)$ remains constant.

Example 7.1.4. : Take $E = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\}$ with $\sigma((x,\xi),(x',\xi')) = \langle\xi,x'\rangle - \langle\xi',x\rangle$. Then we have :

$$\tau(\{\xi=0\}, \{x=0\}, \{B\xi=Ax\}) = \operatorname{sgn}(A\,{}^tB)$$

where A and B are $n \times n$ matrices such that $A\,{}^tB$ is symmetric and that (A,B) has rank n. In this case $\{(x,\xi) ; B\xi = Ax\} = \{(x,\xi);$ $x = {}^tBz,\ \xi = {}^tAz$ for some $z \in \mathbb{R}^n\}$.

Proposition 7.1.5. : Assume $\lambda_1 \cap \lambda_2 = \{0\}$. Then E is the direct sum $E = \lambda_1 \oplus \lambda_2$. Let us denote by p_1 (resp. p_2) the projection on λ_1 (resp. λ_2), and consider the quadratic form q on λ_3 :

$$q : x \longmapsto \sigma(p_1(x), p_2(x)) .$$

we have : $\tau(\lambda_1,\lambda_2,\lambda_3) = -\operatorname{sgn} q$.

§.7.2. Pure sheaves

7.2.1. Let X be a differentiable manifold of class C^α, $\alpha \geq 2$. For any $p \in T^*X$ the tangent space T_pT^*X has a canonical structure of symplectic vector space. We denote by $\lambda_0(p)$ the Lagrangean plane $T_p\pi^{-1}\pi(p)$. For a Lagrangean manifold Λ of T^*X we denote by $\lambda_\Lambda(p)$ the tangent space to Λ at p. Then for a Lagrangean plane μ transversal to $\lambda_0(p)$ and $\lambda_\Lambda(p)$, $e^{\pi\sqrt{-1}\,\tau(\lambda_0(p),\lambda_\Lambda(p),\mu)/4}$ is the cocycle used to construct the Maslov bundle over Λ (Maslov [1], Hörmander [2]).

For a real valued function ϕ on X, we define $Y_\phi \subset T^*X$ by :

$$Y_\phi = \{(x; d\phi(x)) ; x \in X\} .$$

For $p \in Y_\phi$ we denote by $\lambda_\phi(p)$ the tangent space to Y_ϕ at p. Remark that $\lambda_\phi(p) \cap \lambda_0(p) = \{0\}$. Notice that Y_ϕ is Lagrangean but not homogeneous.

Definition 7.2.1. : We say that ϕ is transversal to Λ at p if $\phi(\pi(p)) = 0$ and if Y_ϕ and Λ intersect transversally at p.

Lemma 7.2.2. : Let Y be a submanifold of X and let ϕ be transversal to $T^*_Y X$ at $p \in T^*_Y X$. Then there exists a local coordinate system $(x_1, \ldots, x_\ell)$ of Y around $\pi(p)$ such that :

i) $\pi(p) = 0$

ii) $\phi|_Y = \sum_{j=1}^{\ell} a_j x_j^2$ with $a_j \neq 0$ $\forall j$, and

iii) $\tau(\lambda_\Lambda(p), \lambda_0(p), \lambda_\phi(p)) = \# \{j ; a_j > 0\} - \# \{j ; a_j < 0\}$

Proof

Let us take a local coordinate system $(x_1, \ldots, x_n)$ of X such that $\pi(p) = 0$ and $Y = \{x ; x_j = 0 \text{ for } j > \ell\}$. Let $(\xi_1, \ldots, \xi_n)$ be the corresponding fiber coordinates of T^*X. Then Y_ϕ is defined by :

$$\xi_j = \frac{\partial \phi}{\partial x_j}$$

Hence $T_p(Y_\phi) = \{(x,\xi) ; \xi_j = \sum_k \frac{\partial^2 \phi}{\partial x_j \partial x_k}(0) x_k\}$ and $T_p T^*_Y X = \{(x,\xi) ; x_j = 0 \ \forall j > \ell, \ \xi_k = 0 \ \forall k \leqslant \ell\}$.

Then $T_p(Y_\phi) \cap T_p(T^*_Y X) = \{0\}$ is equivalent to the non degeneracy of the matrix $(\partial^2 \phi / \partial x_j \partial x_k(0))_{1 \leqslant j, k \leqslant \ell}$.

By Morse's lemma we may assume, after a change of coordinates $(x_1, \ldots, x_n)$ that $\phi|_Y = \sum_{j \leqslant \ell} a_j x_j^2$, with $a_j \neq 0$.

Assertion iii) is obtained by an immediate calculation . $\square$

Lemma 7.2.3. : Let Y be a submanifold of X, $\Lambda = T^*_Y X$, ϕ a C^α function transversal to Λ at $p \in \Lambda$. Then we have for a complex of A-modules $M^\bullet$:

$$H^k_{\{x;\phi(x)\geq 0\}}(\underline{M}^\bullet_Y)_{\pi(p)} = H^{k-\frac{1}{2}\dim Y - \frac{1}{2}\tau(\lambda_o(p),\lambda_\Lambda(p),\lambda_\phi(p))}(M^\bullet)$$

Proof

We may choose coordinates $(x_1,\dots,x_\ell)$ on Y as in the preceding lemma. Then we have :

$$\begin{aligned} H^k_{\{x;\phi(x)\geq 0\}}(\underline{M}^\bullet_Y)_{\pi(p)} &= H^k_{\{\sum_{j\leq\ell} a_j x_j^2\geq 0\}}(\mathbb{R}^\ell ; \underline{M}^\bullet) \\ &= H^{k-q}(\underline{M}^\bullet) \end{aligned}$$

where $q = \# \{j ; a_j < 0\}$. On the other hand we have :

$$\tau(\lambda_o(p),\ \lambda_\Lambda(p),\ \lambda_\phi(p)) = q - (\ell-q) = 2q - \dim Y . \quad \square$$

Lemma 7.2.4. : Let Λ be a Lagrangean submanifold of T^*X, p a point of Λ, $\underline{F} \in \mathrm{Ob}(D^+(X))$. We assume that $SS(\underline{F}) \subset \Lambda$ on a neighborhood of p. Let ϕ be a function transversal to Λ at p and j be a number such that $j \equiv \frac{1}{2}(\dim X + \dim(\lambda_o(p) \cap \lambda_\Lambda(p))) \bmod \mathbb{Z}$. Then $H^{j+\tau_\phi/2}_{\{\phi\geq 0\}}(\underline{F})_{\pi(p)}$ does not depend on ϕ, where $\tau_\phi = \tau(\lambda_o(p),\ \lambda_\Lambda(p),\ \lambda_\phi(p))$.

Proof

Let S be a manifold and let $\phi_s(x)$ be a C^α-function on $S \times X$ such that ϕ_s is transversal to Λ at $p(s)$ for $s \in S$.

We shall show first that for any integer k :

(7.2.1) $\quad H^k_{\{\phi_s\geq 0\}}(\underline{F})_{\pi(p(s))}$ is locally constant with respect to s.

Let $g : S \times X \longrightarrow X$ be the projection and let :

$$Y = \{(s,x) \; ; \; \phi_s(x) = 0\} \subset S \times X$$

$$Z = \{(s,\pi(p(s)) \; ; \; d\phi_s(\pi(p(s)))) \; ; \; s \in S\} \subset T^*_Y(S \times X)$$

We have :

$$(7.2.2) \qquad H^j_{\{\phi_s \geq 0\}}(\underline{F})_{\pi(p(s))} = H^j(\mu_{Y \cap \{s\}\times X}(g^{-1}\underline{F}|_{\{s\}\times X})_{d\phi_s(\pi(p(s))}$$

and

$$(7.2.3) \qquad SS(\mu_Y(g^{-1}\underline{F})) \subset C_{T^*_Y(S\times X)}(SS(g^{-1}\underline{F})) \subset C_{T^*_Y(S\times X)}(T^*_S S \times \Lambda)$$

Since $T^*_Y(S\times X) \cap (T^*_S S \times \Lambda) = \mathbb{R}^+ Z$, and the intersection is transversal,

$$C_{T^*_Y(S\times X)}(T^*_S S \times \Lambda) = T^*_{\mathbb{R}^+ Z}(T^*_Y(S\times X))$$

and this implies $\mu_Y(g^{-1}(\underline{F}))|_Z$ has locally constant cohomologies. Since the right-hand side of (7.2.2) equals $H^j(\mu_Y(g^{-1}F))_{d\phi_s(\pi(p(s)))}$ by Theorem 5.4.1. we get (7.2.1).

Now let ϕ and ϕ' be two functions transversal to Λ at p, and let us prove :

$$H^{j+\frac{1}{2}\tau_\phi}_{\{\phi\geq 0\}}(\underline{F})_{\pi(p)} = H^{j+\frac{1}{2}\tau_{\phi'}}_{\{\phi'\geq 0\}}(\underline{F})_{\pi(p)}$$

We choose two families ϕ_s and ϕ'_s transversal to Λ at $p(s)$ such that $\phi_0 = \phi$, $\phi'_0 = \phi'$. Let Λ' be the set of points of Λ around which the projection from Λ to X has constant rank. Then Λ' is an open dense subset of Λ and, locally on Λ', it is the conormal bundle to a submanifold Y of X.

Hence by Proposition 6.2.2. $\underline{F}$ is microlocally isomorphic to $\underline{M}^\bullet_Y$. By applying the preceding lemma if $k + \frac{1}{2}\tau_{\phi_s} \equiv 0 \mod \mathbb{Z}$:

$$H^{k+\frac{1}{2}\tau_{\phi_s}}_{\{\phi_s \geqslant 0\}}(\underline{F})_{\pi(p(s))} = H^{k+\frac{1}{2}\tau_{\phi'_s}}_{\{\phi'_s \geqslant 0\}}(\underline{F})_{\pi(p(s))}$$

for $p(s) \in \Lambda'$.

Applying (7.2.1) we obtain :

$$H^{k+\frac{1}{2}\tau_{\phi_s}}_{\{\phi \geqslant 0\}}(F)_{\pi(p)} = H^{k+\frac{1}{2}\tau_{\phi'_s}}_{\{\phi' \geqslant 0\}}(F)_{\pi(p)}$$

Now we have :

$$\tau_{\phi_s} - \tau_{\phi'_s} = \tau(\lambda_o(p(s)),\lambda_\Lambda(p(s)),\lambda_{\Lambda_s}(p(s))) - \tau(\lambda_o(p(s)),\lambda_\Lambda(p(s)),\lambda_{\phi'_s}(p(s)))$$

$$= \tau(\lambda_\Lambda(p(s)),\lambda_{\phi_s}(p(s)),\lambda_o(p(s)),\lambda_{\phi'_s}(p(s)))$$

Since $\lambda_{\phi_s}(p(s))$ and $\lambda_{\phi'_s}(p(s))$ are transversal to $\lambda_o(p(s))$ and $\lambda_\Lambda(p(s))$, this is locally constant with respect to s (Proposition 7.1.3) and we get $\tau_{\phi_s} - \tau_{\phi_{s'}} = \tau_\phi - \tau_{\phi'}$. □

Definition 7.2.5. : Let Λ be a Lagrangean submanifold of T^*X, $p \in \Lambda$, $\underline{F} \in Ob(D^+(X))$. We assume $SS(\underline{F}) \subset \Lambda$ in a neighborhood of p. if for a real function ϕ transversal to Λ at p and an A-module M :

$$H^j_{\{\phi \geqslant 0\}}(\underline{F}^\bullet)_{\pi(p)} = M \text{ for } j = -d + \tfrac{1}{2}\dim X + \tfrac{1}{2}\tau(\lambda_o(p),\lambda_\Lambda(p),\lambda_\phi(p))$$

$$\text{"} \quad = 0 \text{ otherwise}$$

then we say that $\underline{F}$ is pure with shift d of type M at p (along Λ). If moreover M is a free A-module of rank one, we say that $\underline{F}$ is a simple sheaf at p with shift d.

Remark : We have $d \equiv \frac{1}{2}\dim(\lambda_o(p) \cap \lambda_\Lambda(p)) \bmod \mathbb{Z}$.

Examples 7.2.6. : i) For a submanifold Y of X, $\underline{A}_Y$ is simple with shift $\frac{1}{2}$ codim Y (on $T^*_Y X$).

ii) If $\underline{F}$ is with shift d then $\underline{F}[k]$ is with shift $d+k$.

iii) Take $X = \mathbb{R}$, $Z = \{x ; x \geqslant 0\}$, $U = \{x ; x < 0\}$.

Then $\underline{A}_Z$ (resp. $\underline{A}_U$) is simple with shift $\frac{1}{2}$ (resp. $-\frac{1}{2}$) at $(0;dx)$.

iv) Take $X = \mathbb{R}^2$, $Z = \{(x,y) ; x > 0, -x^{3/2} \leqslant y < x^{3/2}\}$

$\Lambda = \{(x,y ; \xi,\eta) ; \eta > 0, y = -(2\xi/3\eta)^3, x = (2\xi/3\eta)^2$, $\underline{F} = \underline{A}_Z$.

Then $SS(\underline{F}) \subset \Lambda \cup T^*_X X$, and $\underline{F}$ is simple with shift $+1/2$ $(\xi > 0)$, 0 $(\xi = 0)$, $-1/2$ $(\xi < 0)$ along Λ.

Remark 7.2.7. : Let $\mu(s)$ be a Lagrangean plane of $T_{p(s)}T^*X$ such that $p(s) \in \Lambda$ and $\mu(s)$ depends continuously on s and that $\mu(s)$ is transversal to $\lambda_0(p(s))$ and $\lambda_\Lambda(p(s))$. If $\underline{F}$ is pure of type M with shift $d(s)$ at $p(s)$, then $\underline{F}$ is pure of type M with shift $d(s')$ at $p(s')$ where

$$(7.2.4) \quad d(s')-d(s) = \frac{1}{2}(\tau(\lambda_0(p(s)), \lambda_\Lambda(p(s)), \mu(s)) - \tau(\lambda_0(p(s'), \lambda_\Lambda(p(s')), \mu(s')))$$

In fact denote by $d_\mu(s,s')$ the right-hand side, and choose an other Lagrangean family $\nu(s)$. Then $d_\mu(s,s') - d_\nu(s,s')$ is locally constant and vanishes for $s = s'$. Thus it is zero, which proves that the right-hand side of (7.2.4) does not depend on the choice of $\mu(s)$, and the remark follows.

7.2.2. The two next statements are proved by similar arguments as for Lemma 7.2.4., by reducing to the case where the Lagrangean manifold is the conormal bundle to a submanifold of X.

Proposition 7.2.8. : Let Λ be a Lagrangean manifold in T^*X, $p \in \Lambda$, $\underline{F} \in Ob(D^b(X))$. Assume $\underline{F}$ pure of type M with shift d

along Λ at p and assume $\mathrm{Ext}^j(M,A) = 0$ for $j \neq 0$. Then $\mathbb{R}\underline{\mathrm{Hom}}(\underline{F},\underline{A})$ is pure of type $\mathrm{Hom}(M,A)$ with shift $-d$ along Λ^a at p^a.

Proposition 7.2.9. : Let Λ_j be a Lagrangean manifold in T^*X_j, $p_j \in \Lambda_j$, and let $\underline{F}_j \in \mathrm{Ob}(D^+(X_j))$ pure of type M_j with shift d_j along Λ_j at p_j $(j = 1,2)$. Denote by q_j the j-th projection on $X_1 \times X_2$.

a) Assume $\mathrm{wg}\ell\mathrm{d}(A)$ finite and $\mathrm{Tor}_j(M_1,M_2) = 0$ for $j \neq 0$. Then $q_1^{-1}\underline{F}_1 \overset{\mathbb{L}}{\otimes} q_2^{-1}\underline{F}_2$ is pure of type $M_1 \otimes M_2$ with shift $d_1 + d_2$ along $\Lambda_1 \times \Lambda_2$ at (p_1,p_2).

b) Assume $\mathrm{Ext}^j(M_1,M_2) = 0$ for $j \neq 0$ and $\underline{F}_1 \in \mathrm{Ob}(D^b(X_1))$. Then $\mathbb{R}\underline{\mathrm{Hom}}(q_1^{-1}\underline{F}_1, q_2^{-1}\underline{F}_2)$ is pure of type $\mathrm{Hom}(M_1,M_2)$ with shift d_2-d_1 along $\Lambda_1^a \times \Lambda_2$ at (p_1^a, p_2).

§7.3. Operations on pure sheaves

7.3.1. Let f be a map from Y to X, ρ and ϖ the associated maps from $Y \underset{X}{\times} T^*X$ to T^*Y and T^*X respectively. First we study direct images of pure sheaves.

Theorem 7.3.1. : Let Λ be a Lagrangean submanifold of T^*Y, $p \in Y \underset{X}{\times} T^*X$, $\underline{G} \in \mathrm{Ob}(D^+(Y))$, and assume :

i) f is proper over $\mathrm{supp}(\underline{G})$

ii) ρ is transversal to Λ at p, and $\rho^{-1}(\Lambda)$ is isomorphic to a manifold Λ_o of T^*X by the map ϖ

iii) $\rho^{-1}(SS(\underline{G})) \cap \varpi^{-1}\varpi(p) \subset \{p\}$

iv) $\underline{G}$ is pure of type M with shift d along Λ at $\rho(p)$. Then $\Lambda_o = \varpi\,\rho^{-1}(\Lambda)$ is Lagrangean and $\mathbb{R}f_*(\underline{G})$ is pure of type M

<u>with shift</u> d' <u>along</u> Λ_o <u>at</u> $\varpi(p)$, <u>where</u> :

$$d'-d = \frac{1}{2}(\dim X - \dim Y) - \frac{1}{2}\,\tau(\lambda_o(\rho(p)), \lambda_\Lambda(\rho(p)), \rho\,\varpi^{-1}(\lambda_o(\varpi(p))))$$

(we have written $\rho\,\varpi^{-1}\,\lambda_o(\varpi(p))$ instead of $d\rho(p)\,d\varpi(p)^{-1}(\varpi(p))(\lambda_o(\varpi(p)))$.

<u>Proof</u>

Remark first that Λ_o is Lagrangean since

$$\begin{aligned} \varpi^{-1}(\omega_X)|\varpi^{-1}(\Lambda_o) &= \varpi^{-1}(\omega_X)|\rho^{-1}(\Lambda) \\ &= \rho^{-1}(\omega_Y)|\rho^{-1}(\Lambda) \\ &= 0 \end{aligned}$$

and Λ_o being isotropic in T^*Y is Lagrangean since $\dim \Lambda_o = \dim \Lambda - (\dim Y - \dim X) = \dim X$.

Let $y_o \in Y$ be the projection of $\rho(p)$ and $x_o = f(y_o)$ the projection of $\varpi(p)$. Let us take a function ϕ on X transversal to Λ_o at $\varpi(p)$. Then $\phi' = \phi \circ f$ is transversal to Λ at $\rho(p)$. Hence we have :

$$\begin{aligned} H^j_{\{\phi'\geq 0\}}(\underline{G})_{y_o} &= M \quad \text{for} \quad j = j_o \\ &= 0 \quad \text{for} \quad j \neq j_o \end{aligned}$$

where $j_o = -d + \frac{1}{2}\dim Y + \frac{1}{2}\,\tau(\lambda_o(\rho(p)), \lambda_\Lambda(\rho(p)), \lambda_\phi(\rho(p)))$.

On the other hand iii) implies :

$$\mathbb{R}\Gamma_{\{\phi'\geq 0\}}(\underline{G})_y = 0 \quad \text{for} \quad y \in f^{-1}(x_o) - \{y_o\}$$

thus :

$$H^j_{\{\phi\geq 0\}}(\mathbb{R}f_*(\underline{G}))_{x_o} = H^j_{\{\phi'\geq 0\}}(\underline{G})_{y_o}$$

and it remains to show :

(7.3.1) $\tau(\lambda_o(\rho(p)),\lambda_\Lambda(\rho(p)),\lambda_{\phi'}(\rho(p)))-\tau(\lambda_o(\rho(p)),\lambda_\Lambda(\rho(p)),\rho\,\varpi^{-1}(\lambda_o(\varpi(\rho))))$

$$= \tau(\lambda_o(\varpi(p)),\lambda_{\Lambda_o}(\varpi(p)),\lambda_\phi(\varpi(p)))$$

We set :

$$V_1 = T_{\varpi(p)} T^*X \ , \quad V_2 = T_{\rho(p)}(T^*Y) \ ,$$

$$W = T_p(Y \underset{X}{\times} T^*X)$$

$$\lambda = \lambda_\phi(\varpi(p))$$

$$\mu_1 = \lambda_o(\rho(p)), \quad \mu_2 = \lambda_\Lambda(\rho(p))$$

$$\lambda' = \lambda_{\phi'}(\rho(p))$$

Then (7.3.1) is a consequence of the following lemma.

Lemma 7.3.2. : Let (E_1, σ_1) and (E_2, σ_2) be two symplectic vector spaces and let ν be a Lagrangean plane of $(E_1 \oplus E_2, \sigma)$, with $\sigma = \sigma_1 \oplus (-\sigma_2)$.

Let λ be a Lagrangean plane of E_1 and let μ_1 and μ_2 be Lagrangean planes of E_2. Let p_j be the projection from ν to E_j $(j = 1,2)$. Set $\lambda' = p_2 p_1^{-1}(\lambda)$, $\mu_i' = p_1 p_2^{-1}(\mu_i)$, $\mu_i'' = p_2 p_1^{-1}(\mu_i')$.

Then we have :

$$\tau_{E_1}(\lambda,\mu_1',\mu_2') = \tau_{E_2}(\lambda',\mu_1,\mu_2) - \tau_{E_2}(\mu_1'',\mu_1,\mu_2)$$

Proof

Remark first that for any subspace α of E_1 we have $p_2p_2^{-1}(\alpha^\perp) = (p_2p_1^{-1}(\alpha))^\perp$. In particular if α is Lagrangean, so is $p_2p_1^{-1}(\alpha)$.

Set $\rho = p_2p_1^{-1}(O)$. Then $\mu_1'' = (\mu_1 \cap \rho^\perp) + \rho$.

We have :

$$\tau_{E_2}(\lambda',\mu_1,\mu_2)-\tau_{E_2}(\mu_1'',\mu_1',\mu_2) = \tau_{E_2}(\lambda',\mu_1,\mu_1'')+\tau_{E_2}(\lambda',\mu_1'',\mu_2).$$

Since λ' and μ_1'' contain ρ and $\mu_1^\rho = \mu_1''^\rho$, $\tau_{E_2}(\lambda',\mu_1,\mu_1'') = 0$ by Proposition 7.1.2. .

Since $p_1p_2^{-1}(\mu_1'') = p_1p_2^{-1}(\mu_1)$ it is enough to show :

$$\tau_{E_1}(\lambda,\mu_1',\mu_2') = \tau_{E_2}(\lambda',\mu_1,\mu_2)$$

if $\mu_1 \subset \rho^\perp = p_2(W)$.

Now by setting $\tau = \tau_{E_1\oplus E_2}$, we have :

$$\tau_{E_1}(\lambda,\mu_1',\mu_2') - \tau_{E_2}(\lambda',\mu_1,\mu_2) = \tau(\lambda\oplus\lambda',\mu_1'\oplus\mu_1,\mu_2'\oplus\mu_2)$$

Thus by the cochain property it will follow from :

$$\text{(7.3.2)} \qquad \tau(\nu,\ \lambda\oplus\lambda',\ \mu_i'\oplus\mu_i) = 0 \qquad (i = 1,2)$$

$$\text{(7.3.3)} \qquad \tau(\nu,\ \mu_1'\oplus\mu_1,\ \mu_2'\oplus\mu_2) = 0$$

Now (7.3.2) follows from $\nu + (\lambda\oplus\lambda') = \nu + \lambda$ and $(\mu_i'\oplus\mu_i)\cap(\nu+\lambda) \subset [(\mu_i'\oplus\mu_i)\cap\nu] + \mu_i'\cap\lambda$ by Proposition 7.1.2. iii).

The second inclusion is proved as follows : if $x \in \lambda$, $w \in \nu$, and $w + x \in \mu_i'\oplus\mu_i$, then $p_2(w) \in \mu_i$ hence $p_1(w) \in \mu_i'$. Therefore $x \in \lambda\cap\mu_i'$ and $w \in \mu_i'\oplus\mu_i$.

(7.3.3.) follows from (7.3.2) and $p_2p_1^{-1}\mu_1' = \mu_1$. □

<u>7.3.2</u>. Now we study inverse images of pure sheaves.

<u>Theorem 7.3.3.</u> : <u>Let</u> Λ <u>be a Lagrangean submanifold of</u> T^*X , $p \in Y \underset{X}{\times} \Lambda$, <u>F</u> $\in Ob(D^+(X))$, <u>and assume</u> :

i) f <u>is non characteristic for</u> <u>F</u> (<u>cf. Definition 5.3.1.</u>).

ii) ϖ <u>is transversal to</u> Λ <u>at</u> p <u>and</u> $\varpi^{-1}(\Lambda)$ <u>is isomorphic to a manifold</u> Λ_o <u>of</u> T^*Y <u>by the map</u> ρ.

iii) $\varpi^{-1}(SS(\underline{F})) \cap \rho^{-1}\rho(p) \subset \{p\}$.

iv) $\underline{F}$ <u>is pure of type</u> M <u>with shift</u> d <u>along</u> Λ <u>at</u> $\varpi(p)$.

<u>Then</u> $\Lambda_o = \rho\,\varpi^{-1}(\Lambda)$ <u>is Lagrangean and</u> $f^{-1}(\underline{F})$ <u>is pure of type</u> M <u>with shift</u> d <u>along</u> Λ_o <u>at</u> $\rho(p)$.

<u>Proof</u>

By the induction on the codimension of Y , we may assume from the beginning that Y is a hypersurface of X . When p belongs to the zero section, the proposition is immediate, and hence, we may assume $p \in \dot{T}^*X$. Now, we shall take a local coordinate system $(t,x) = (t,x_1,\ldots,x_n)$ of X such that Y is given by $t = 0$ and p is given by $(t,x\,;\,\tau,\xi) = (0,0\,;\,0,\xi_o)$, where $(t,x\,;\,\tau,\xi)$ is the coordinates of T^*X associated with (t,x). We shall take a function $\phi(x)$ transversal to Λ_o at $\rho(p)$ and a function $\psi(x)$ such that $\psi(0) = d_x\,\psi(0) = 0$ and $\mathrm{Hess}\,\psi(0) >> 0$.

For $0 < a$, δ, we set :

$$\phi_{a,\delta}(t,x) = a(\phi(x) + \delta\psi(x)) - t^2$$

We shall show first :

$$(7.3.4) \qquad (Y_{\phi_{a,\delta}} + \mathbb{R}^- d|t|) \cap SS(\underline{F}) \cap \{t \neq 0\} = \emptyset$$

on $0 < |t| << 1$, $|x| << 1$, for $0 < a << 1$, $|\delta| << 1$.

If this were false, there would exist sequences $\{(t_n,x_n)\}$, $\{a_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$, $a_n \geqslant 0$, $\delta_n \geqslant 0$, $\sigma_n \geqslant 0$ such that :

$$p_n = (t_n,x_n\,;\,-2a_n^{-1}t_n - (\mathrm{sgn}\,t_n)\sigma_n\,,\,d\phi(x_n) + \delta_n d\psi(x_n)) \in SS(\underline{F})$$

and $t_n,x_n \xrightarrow[n]{} 0$, $a_n,\delta_n \xrightarrow[n]{} 0$, $t_n \neq 0$.

Since $t_n > 0$ or $t_n < 0$, we may assume $t_n > 0$.
Since $d\phi(x_n) + \delta_n d\psi(x_n)$ converges to ξ_o, iii) and i) imply $\tau_n = -2a_n^{-1}t_n - \sigma_n \xrightarrow[n]{} 0$. On the other hand, we have $|t_n| / |\tau_n| \leqslant |t_n| / 2|a_n^{-1}t_n|$, which tends to 0. Hence if we take $c_n > 0$ such that $c_n(p_n - p)$ converges to a non zero vector $(\tilde{t},\tilde{x} ; \tilde{\tau},\tilde{\xi})$ which belongs to $T_p\Lambda$, then $\tilde{t} = 0$. By ii), $(\tilde{x};\tilde{\xi})$ is a non zero vector belonging to $T_{\rho(p)} \Lambda_o$.
On the other hand $c_n(x_n;d\phi(x_n)-\xi_o)$ converges to $(\tilde{x};\tilde{\xi})$, which contradicts the transversality of ϕ.

This shows (7.3.4).

Set $Z_a = \{(t,x) ; \phi_{a,o} = a\phi(x) - t^2 \geqslant 0\}$ and $Z_{a,\delta,\varepsilon} = \{(t,x) ; \phi_{a,\delta} \geqslant \varepsilon\}$ for $0 < \varepsilon << a$, $\delta << 1$.

Then $\{Z_a \setminus Z_{a,\delta,\varepsilon}\}_{\varepsilon>0}$ forms a neighborhood system of $(0,0)$ in Z_a.
For $\alpha > 0$, set $Z(\alpha) = \{(t,x) ; |t| \leqslant \alpha\}$.

Then for $0 < \varepsilon << a$, $\delta << 1$ and $0 < \alpha$, $N^*(Z_a \cap Z(\alpha))$, $N^*(Z_a \cap Z_{a,\delta,\varepsilon} \cap Z(\alpha))$ and $N^*(Z_{a,\delta,\varepsilon} \cap Z(\alpha))$ are disjoint from $SS(\underline{F}) \cap \dot{T}^*X$ on their boundary sets. Thus :

$$\mathbb{R}\Gamma_{(Z_a \cap Z(\alpha)) \setminus (Z_a \cap Z_{a,\delta,\varepsilon} \cap Z(\alpha))}(X ; \underline{F})$$

does not depend on α.

Hence by taking the projective limit with respect to α, we obtain :

$$\mathbb{R}\Gamma_{Z_a \setminus Z_{a,\delta,\varepsilon}}(X ; \underline{F}) \xleftarrow{\sim} \mathbb{R}\Gamma_{Z_a \setminus Z_{a,\delta,\varepsilon}}(Y ; \mathbb{R}\Gamma_Y(\underline{F}))$$

By taking the inductive limit with respect to ε we obtain :

$$\mathbb{R}\Gamma_{Z_a}(\underline{F})_o \xleftarrow{\sim} \mathbb{R}\Gamma_{Z_a \cap Y}(\mathbb{R}\Gamma_Y(\underline{F}))_o \quad \text{for } 0 < a \ll 1 .$$

On the other hand, Corollary 5.3.3. implies $\mathbb{R}\Gamma_Y(\underline{F}) = \underline{F}_Y[-1]$.
Thus, by setting $\phi_a = a\phi - t^2$

$$H^{j}_{\{x;\phi(x)\geq 0\}}(\underline{F}|_Y)_o = H^{j+1}_{\{(t,x);\phi_a(t,x)\geq 0\}}(\underline{F})_o$$

The last term vanishes except for $j+1 = -d + \frac{1}{2}\dim X + \frac{1}{2}\tau(\lambda_o(p), \lambda_\Lambda(p), \lambda_{\phi_a}(p))$ and the remaining term is M.

Hence it is sufficient to show, by setting $p_o = \rho(p)$

(7.3.5) $\quad \tau(\lambda_o(p_o), \lambda_{\Lambda_o}(p_o), \lambda_\phi(p_o)) = \tau(\lambda_o(p), \lambda_{\Lambda_o}(p), \lambda_{\phi_a}(p)) - 1$

Now, we have $\lambda_{\phi_a}(p) = \{(t,x\ ;\ \tau,\xi)\ ;\ -2t = a\tau\ ,\ (x;\xi)\ \varepsilon\ \lambda_\phi(p_o)\}$.

We have :

(7.3.6) $\quad \tau(\lambda_o(p_o), \lambda_{\Lambda_o}(p_o), \lambda_\phi(p_o)) = \tau(\lambda_o(p), \lambda_\Lambda(p), \lambda_\phi(p) \times \{t=o\})$

$$= \tau(\lambda_o(p), \lambda_\Lambda(p), \lambda_{\phi_a}(p)) + \tau(\lambda_\Lambda(p), \lambda_\phi(p) \times \{t=o\},\ \lambda_{\phi_a}(p))$$

$$+ \tau(\lambda_\phi(p) \times \{t=o\},\ \lambda_o(p),\ \lambda_{\phi_a}(p)).$$

In (7.3.6), the last term vanishes because $\mu = \{t = x = \xi = 0\} \subset \lambda_o(p)$ and $\lambda^{\mu}_{\phi_a}(p) = (\lambda_\phi(p) \times \{t=o\})^{\mu}$.

Now, set $\nu = \{t = \tau = 0,\ (x,\xi)\ \varepsilon\ \lambda_{\phi_a}(p)\}$.

Then by identifying $\nu^{\perp}/\nu$ with the (t,x)-space ,

$$\tau(\lambda_\Lambda(p),\ \lambda_\phi(p) \times \{t=o\},\ \lambda_{\phi_a}(p)) = \tau(\lambda_\Lambda(p)^{\nu},\ \{t=o\},\ \{-2t = a\tau\}).$$

By the assumption, $\lambda_\Lambda(p)^{\nu}$ is different from $\{t = 0\}$, and we can write it $\tau = ct$ for some c.

Then $\tau((\tau = ct),\ (t = 0), (-2t = a\tau)) = -1$ for $0 < a << 1$.

Thus we obtain (7.3.5). □

<u>7.3.3</u>. Finally we study the functors <u>Hom</u>$(\cdot,\cdot)$ and $\cdot \overset{\mathbb{L}}{\otimes} \cdot$.

<u>Corollary 7.3.4</u>. : <u>We assume</u> A <u>commutative. Let</u> q_1 <u>and</u> q_2 <u>be the projections from</u> $X \times Y$ <u>to</u> X <u>and</u> Y <u>respectively</u>, p_1

and p_2 the projections from $T^*(X\times Y) = T^*X\times T^*Y$ to T^*X and T^*Y respectively, and set $p_j^a = p_j \circ a$. Let Λ be a Lagrangean submanifold of $T^*(X\times Y)$, Λ_Y a Lagrangean submanifold of T^*Y , $p \in \Lambda$, and set $p_Y = p_2(p)$, $p_X = p_1^a(p)$. Let $\underline{K} \in Ob(D^b(X \times Y))$, $\underline{F} \in Ob(D^+(Y))$, and assume :

i) $p_2|_\Lambda$ is transversal to Λ_Y at p and $p_2^{-1}(\Lambda_Y) \cap \Lambda$ is isomorphic to a submanifold Λ_X of T^*X by p_1^a.

ii) $\underline{K}$ is pure of type M with shift d along Λ at p.

iii) $\underline{F}$ is pure of type N with shift d' along Λ_Y at p_Y.

iv) The projection q_1 is proper over $\mathrm{supp}\, \underline{K} \cap q_2^{-1}(\mathrm{supp}\, \underline{F})$.

v) $(p_1^a)^{-1}(p_X) \cap SS(\underline{K}) \subset \{p\}$.

vi) $(SS(\underline{K}) \underset{T^*Y}{\times} SS(\underline{F})) \cap (T_X^*X \times T^*Y) \subset T_X^*X \times T_Y^*Y$ on a neighborhood of $\pi_X(p_X)$.

vii) $\mathrm{Ext}^j(M,N) = 0$ for $j \neq 0$.

Then we have :

$$\mathbb{R}q_{1*}\, \mathbb{R}\underline{Hom}(\underline{K}, q_2^{-1}\underline{F}) \text{ is pure of type } Hom(M,N)$$

with shift $d'-d - \frac{1}{2}\dim Y + \frac{1}{2}\tau$ along Λ_X at p_X where :

$$\tau = \tau(\lambda_o(p), \lambda_\Lambda(p), \lambda_o(p_X^a) \times \lambda_{\Lambda_Y}(p_Y))$$

$$= \tau(\lambda_o(p_Y), p_2(\lambda_\Lambda(p) \cap p_1^{a-1}\lambda_o(p_X)), \lambda_{\Lambda_Y}(p_Y))$$

Proof

Let us denote by r_1 and r_2 the projections $(X \times Y) \times Y \to X \times Y$ and $(X \times Y) \times Y \to Y$ and let Δ be the diagonal $X \times Y \underset{Y}{\times} Y$ of $X \times Y \times Y$.

We have :

$$\mathbb{R}\underline{Hom}(\underline{K},\ q_2^{-1}\underline{F}) = \mathbb{R}\Gamma_\Delta\ \mathbb{R}\underline{Hom}(r_1^{-1}\underline{K},\ r_2^{-1}\underline{F}) \otimes \underline{\omega}_Y[\dim Y]$$

The condition vi) permits us to apply Proposition 5.3.2. and we get :

$$\mathbb{R}\underline{Hom}(\underline{K},\ q_2^{-1}\underline{F}) = \mathbb{R}\underline{Hom}(r_1^{-1}\underline{K},\ r_2^{-1}\underline{F})\Big|_\Delta$$

Applying Proposition 7.2.9. and 7.3.3. we find that $\mathbb{R}\underline{Hom}(\underline{K},\ q_2^{-1}\underline{F})$ is pure of type $Hom(M,N)$ with shift $d'-d$ along $\tilde{\Lambda}$ at p_o, where :

$$p_o = (p_X, 0),\quad \tilde{\Lambda} = \{(x,y\,;\,\xi,\eta+\eta')\,;\,(x,y\,;\,\xi,\eta) \in \Lambda^a,\ (y,\eta') \in \Lambda_Y\}.$$

Therefore one can apply Proposition 7.3.1. to obtain that $\mathbb{R}q_{1*}\,\mathbb{R}\underline{Hom}(\underline{K},\ q_2^{-1}\underline{F})$ is pure of type $Hom(M,N)$ with shift $d' - d - \frac{1}{2}\dim Y - \frac{1}{2}\tau$, where :

$$\tau = \tau(\lambda_o(p_o),\ \lambda_{\tilde{\Lambda}}(p_o),\ \lambda_o(p_X) \times TT_Y^*Y)$$

Set $\lambda_\Lambda(p) = \lambda_\Lambda$, $\lambda_{\Lambda^a}(p^a) = \lambda_{\Lambda^a}$, $\lambda_{\Lambda_Y}(p_Y) = \lambda_{\Lambda_Y}$; we get by applying Proposition 7.1.2. with $\rho^\perp = T(T^*(X \times Y \times Y) \underset{Y\times Y}{\times} (Y \underset{Y}{\times} Y)$

$$\tau = \tau(\lambda_o(p^a) \times \lambda_o(p_Y),\ \lambda_{\Lambda^a} \times \lambda_{\Lambda_Y},\ \lambda_o(p_X) \times T(T_Y^*Y \times Y))$$

Hence we have :

$$\begin{aligned}\tau &= \tau(\lambda_o(p^a) \times \lambda_{\Lambda_Y},\ \lambda_o(p^a) \times \lambda_o(p_Y)\ ,\ \lambda_{\Lambda^a} \times \lambda_{\Lambda_Y})\\ &+ \tau(\lambda_o(p^a) \times \lambda_{\Lambda_Y},\ \lambda_{\Lambda^a} \times \lambda_{\Lambda_Y}\ ,\ \lambda_o(p_X) \times T(T_Y^*(Y \times Y)))\\ &+ \tau(\lambda_o(p^a) \times \lambda_{\Lambda_Y},\ \lambda_o(p_X) \times T(T_Y^*(Y \times Y)),\ \lambda_o(p^a) \times \lambda_o(p_Y))\end{aligned}$$

Here the first and the last terms vanish by i) and iii) of Proposition 7.1.2. The middle term is equal, by applying the same proposition with $\rho = \{0\} \times \{0\} \times \lambda_{\Lambda_Y}$ to

$$\tau(\lambda_o(p^a),\ \lambda_{\Lambda^a}(p),\ \lambda_o(p_X) \times \lambda_{\Lambda_Y^a}(p_Y^a))$$

$$= -\tau(\lambda_o(p),\ \lambda_\Lambda(p),\ \lambda_o(p_X^a) \times \lambda_{\Lambda_Y}(p_Y)) \ . \ \square$$

Corollary 7.3.5. : We assume A commutative and $wg\ell d(A)$ finite. Let q_1, q_2, p_1, p_2, Λ, Λ_Y, p, p_Y, p_X and $\underline{F}$ be as in Corollary 7.3.4. and let $\underline{K} \in Ob(D^+(X \times Y))$. We make the assumptions i), iii), iv), vi) of Corollary 7.3.4. and also :

ii)' $\underline{K}$ is pure of type M with shift $-d$ along Λ^a at p^a.

v)' $(p_1)^{-1}(p_X) \cap SS(\underline{K}) \subset \{p^a\}$.

vii)' $Tor_j(M,N) = 0$ for $j \neq 0$.

Then we have :

$\mathbb{R}q_{1*}(\underline{K} \overset{\mathbb{L}}{\otimes} q_2^{-1}\underline{F})$ is pure of type $M \otimes N$ with shift $d' - d - \frac{1}{2}\dim Y + \frac{1}{2}\tau$ along Λ_X at p_X, where τ is the same as in Corollary 7.3.4. .

§7.4. Contact transformations

7.4.1. We can use pure sheaves in order to perform contact transformations for sheaves.

Let ϕ be a contact transformation between two conic open sets $\Omega_X \subset T^*X$ and $\Omega_Y \subset T^*Y$, and let Λ be the image of the graph of ϕ by the anti-podal map in T^*X. Let $p \in \Lambda$, $p_X = p_1^a(p) \in \Omega_X$, $p_Y = p_2(p) \in \Omega_Y$.

Let $\underline{K}$ belong to $Ob(D^b(X \times Y))$, satisfying hypothesis (6.3.6) of Theorem 6.3.4. .

Theorem 7.4.1. : In the preceding situation, assume $\underline{K}$ is a simple sheaf along Λ . Then $\Phi_{\underline{K}}$ is an equivalence of categories.

Proof

This is an immediate application of Proposition 6.3.3. and the results of §7.3., since if $\underline{M}$ is a simple sheaf along $T^*_{\Delta_X}(X\times X)$, then $\underline{M}$ is microlocally equivalent to $\underline{A}_{\Delta_X}[d]$, for some shift d . □

Corollary 7.4.2. : In the situation of Theorem 6.3.4. let N (resp. M) be a submanifold of Y (resp. X) and assume that ϕ interchanges $T^*_N Y \cap \Omega_Y$ and $T^*_M X \cap \Omega_X$.

Let $p \in \Lambda$, $p_X = p_1^a(p)$, $p_Y = p_2(p)$, and assume $\underline{K}$ is a simple sheaf on Λ at p. Then for $\underline{G} \in Ob(D^+(Y))$ we have in a neighborhood of p_X :

$$\mu_M(\Phi_{\underline{K}}(\underline{G})) \simeq \phi_*(\mu_N(G))[d]$$

where the shift d is calculated as follows :

$$d = \frac{1}{2}(\dim M - \dim N + \dim X + \tau) - d'$$

where d' is the shift of $\underline{K}$ along Λ at p and τ is given by :

$$\tau = \tau(\lambda_o(p_Y) , \phi^*(\lambda_o(p_X), \lambda_{\Lambda_Y}(p_Y)) .$$

Here $\phi^*(\lambda_o(p_X))$ is the image of $\lambda_o(p_X)$ by $\phi^* : T_{p_X}T^*X \xrightarrow{\sim} T_{p_Y}T^*Y$, and $\Lambda_Y = T^*_N Y$.

Proof

First we remark that we have an isomorphism of sheaves on Ω_X :

$$\mu hom(\Phi_{\underline{K}}(\underline{A}_N) , \Phi_{\underline{K}}(\underline{G})) \simeq \phi_* \mu hom(\underline{A}_N, \underline{G})$$

In fact this follows from Theorem 6.3.9. . Now we apply Proposition 5.5.3. and remark that $\Phi_{\underline{K}}(\underline{A}_N)$ is a simple sheaf along $T^*_M X$ of shift $\frac{1}{2}$ codim $M + d$ by Corollary 7.3.4. . Hence $\phi_{\underline{K}}(\underline{A}_N)$ is microlocally isomorphic to $\underline{A}_M[d]$. □

CHAPTER 8 - APPLICATION 1 : CONSTRUCTIBLE SHEAVES

§8.1. Stratifications and Lagrangean sets

8.1.1. Let X be a manifold of class C^α, $\alpha \geqslant 2$. Recall that a stratification $(X_\alpha)_\alpha$ of X is a partition $X = \bigsqcup_\alpha X_\alpha$ such that :

i) the family (X_α) is locally finite ,

ii) each X_α is a smooth (locally closed) manifold ,

iii) for each pair (α,β) such that $X_\alpha \cap \overline{X_\beta}$ is non empty, X_α is contained in $\overline{X_\beta}$. (One says that X_β dominates X_α and writes $X_\alpha \prec X_\beta$).

Such a stratification is a Whitney stratification if moreover :

iv) for all pairs (α,β) such that $X_\alpha \subset \overline{X_\beta}$, (X_α, X_β) satisfies the conditions a) and b) of Witney (cf. Whitney [1],[2]).

8.1.2. Let us recall somme results of Kashiwara [2][5] and Kashiwara-Schapira [2]. For the reader's convenience we repeat some proofs.

Definition 8.1.1. : Let Λ be a conic subset of T^*X. A locally closed subset Y of X is called flat at $y \in Y$ with respect to Λ if for any $p \in \pi^{-1}(Y)$:

$$C(\Lambda , \pi^{-1}(Y))_p \subset \{v \in T_pT^*X \,;\, \langle v,\omega(p)\rangle \leqslant 0\}$$

Lemma 8.1.2. : If a submanifold Y is flat with respect to Λ then $\pi^{-1}(Y) \cap \Lambda \subset T^*_YX$.

Proof

Take a point p in $\Lambda \cap \pi^{-1}(Y)$. Then $C(\Lambda, \pi^{-1}(Y))$ contains $T_p(\pi^{-1}(Y))$. Hence $\omega(p) = 0$ on $T_p(\pi^{-1}(Y))$. This is equivalent to saying that p belongs to $T^*_Y X$. □

Proposition 8.1.3. : Suppose that X is an open set in $\mathbb{R}^N$ and that a subset Y is flat with respect to a conic set Λ in T^*X at a point x_o. Then there exists $\varepsilon > 0$ such that $(x ; x-y)$ does not belong to Λ for $x \in X$, $y \in Y$ satisfying $|x-x_o|, |y-x_o| < \varepsilon$, $x \neq y$.

Proof

We shall prove the Proposition by contradiction.

If the Proposition is false, then there are sequences $\{x_n\}$ and $\{y_n\}$, $x_n \in X$, $y_n \in Y$ which converge to x_o, such that $\{(x_n ; x_n-y_n)\}$ is contained in Λ and $x_n \neq y_n$. Let $\{c_n\}$ be a sequence with $c_n > 0$, such that $c_n(x_n-y_n)$ tends to $v \neq 0$. Then $\{(x_n ; c_n(x_n-y_n))\}$ is a sequence in Λ which converges to $p = (x_o ; v)$ and $\{(y_n ; c_n(x_n-y_n))\}$ is a sequence in $\pi^{-1}(Y)$ which converges to p. Since $c_n((x_n ; c_n(x_n-y_n)) - (y_n ; c_n(x_n-y_n)))$ converges to $(v,0)$, $(v,0)$ belongs to $C(\Lambda, \pi^{-1}(Y))$. Thus :

$$\langle (v,0), \omega(p) \rangle = \langle v,v \rangle$$

which is a contradiction. □

Proposition 8.1.4. : Let $X = \bigsqcup_\alpha X_\alpha$ be a stratification of Whitney. Then $\Lambda = \bigsqcup_\alpha T^*_{X_\alpha} X$ is a closed subset and each stratum X_α is flat with respect to Λ.

Proof

Let $\{(x_n;\xi_n)\}$ be a sequence in $T^*_{X_\alpha}X$ which converges to $(x;\xi)$. We shall prove that $(x;\xi)$ belongs to $T^*_{X_\beta}X$ for β such that X_β contains x. By the condition of Whitney, if $T_{x_n}X_\alpha$ converges to a plane $\tau \subset T_xX$, then τ contains T_xX_β. Therefore the orthogonal $(T^*_{X_\alpha}X)_{x_n}$ converges to $\tau^\perp$ which is contained in $(T^*_{X_\beta}X)_x$. This implies $(x;\xi) \in T^*_{X_\beta}X$. Let us show th at X_β is flat with respect to $T^*_{X_\alpha}X$. Let x be a point in X_β, $p = (x;\xi)$ a point in $\pi^{-1}(x)$ and q a point in $C_p(T^*_{X_\alpha}X, \pi^{-1}(X_\beta))$. Then there are sequences $\{(x_n;\xi_n)\}$ in $T^*_{X_\alpha}X$, $\{(y_n;\eta_n)\}$ in $\pi^{-1}(X_\beta)$, $\{c_n\}$ in $\mathbb{R}^+$ such that $c_n(x_n-y_n;\ \xi_n-\eta_n)$ converges to $q = (v\,;w)$ and that $(x_n;\xi_n)$ and $(y_n;\eta_n)$ converge to p. Suppose that $T_{x_n}X_\alpha$ converges to a plane τ in T_xX. Then by the condition of Whitney, τ contains v and T_xX_β. Since ξ is contained in $\tau^\perp$ we have

$$\langle q\,,\omega(p)\rangle = \langle v\,,\xi\rangle = 0\,. \quad \square$$

Remark 8.1.5. : Conversely if Λ is closed and if each X_α is flat with respect to Λ, then $X = \bigcup_\alpha X_\alpha$ is a Whitney stratification.

§8.2. $\mathbb{R}$-constructible sheaves

8.2.1. We assume X real analytic, and we shall use the theory of subanalytic sets of H. Hironaka [1]. Here subanalytic sets are always locally closed.

Recall that a stratification $X = \bigcup_\alpha X_\alpha$ is said to be $\mathbb{R}$-analytic if each X_α is subanalytic in X.

Definition 8.2.1. : Let Λ be a conic subanalytic set in T^*X. We say that Λ is isotropic if there is a dense open smooth submanifold $\Lambda' \subset \Lambda$ such that $\omega|_{\Lambda'} = 0$. We say that Λ is Lagrangean if it is both isotropic and involutive (in the sense of Theorem 6.4.1.).

Proposition 8.2.2. : Let Λ be a conic subanalytic isotropic set in T^*X, V a conic subanalytic subset of Λ. Then V is isotropic.

Proof

We may find a Whitney stratification $\Lambda = \bigcup_\alpha \Lambda_\alpha$ such that V is a union of strata. We have to prove that for each α, $\omega|_{\Lambda_\alpha} = 0$, and we may argue by induction on the codimension of the strata since $\Lambda_\alpha \subset \overline{\Lambda_\beta}$ implies $\overline{T^*_{\Lambda_\beta} T^*X} \underset{T^*X}{\times} \Lambda_\alpha \subset T^*_{\Lambda_\alpha} T^*X$ by the Whitney conditions. $\square$

Proposition 8.2.3. (cf. Kashiwara-Schapira [2]) : Let Λ be a closed conic subanalytic isotropic set in T^*X. Then there exists a Whitney $\mathbb{R}$-analytic stratification $X = \bigcup_\alpha X_\alpha$ such that Λ is contained in $\bigcup_\alpha T^*_{X_\alpha} X$.

Proof

Let $S = \pi(\Lambda)$. Then S is subanalytic since Λ is conic. There exists a Whitney stratification $\Lambda = \bigsqcup_{\alpha \in I} \Lambda_\alpha$ of Λ, $S = \bigsqcup_{\beta \in J} S_\beta$ and a map $\tau : I \longrightarrow J$ such that $\pi(\Lambda_\alpha) \subset S_{\tau(\alpha)}$ and $\Lambda_\alpha \longrightarrow S_{\tau(\alpha)}$ is smooth (i.e. of maximal rank).

We have the inclusion $\Lambda_\alpha \subset T^*_{S_{\tau(\alpha)}} X$.

In fact we may choose coordinates $(x_1,\dots,x_n) = (x',x'')$ (where $x' = (x_1,\dots,x_p)$) such that $S_{\tau(\alpha)} = \{(x',x'') ; x' = 0\}$. Then $\omega|_{\pi^{-1}(S_{\tau(\alpha)})} = \xi''dx''$ is zero on Λ_α and the linear forms $dx_{p+1},\dots,dx_n$ being linearly independant on $S_{\tau(\alpha)}$ are also linearly independant on Λ_α since $\pi|_{\Lambda_\alpha}$ is smooth. Thus $\xi'' = 0$ on Λ_α

which gives the inclusion and achieves the proof. $\square$

Proposition 8.2.4. : Let V be a conic closed involutive subset in T^*X, and $\Lambda = \bigcup_{\alpha \in I} \Lambda_\alpha$ a locally finite union of locally closed smooth connected Lagrangean subanalytic manifolds of T^*X. Assume $V \subset \Lambda$. Then there exists a subset $J \subset I$ such that $V = \overline{\bigcup_{\beta \in J} \Lambda_\beta}$.

Proof

By the involutivity of V, $V \cap \Lambda_\alpha$ is open and closed in Λ_α. Let $J = \{\beta \in I ; \Lambda_\beta \subset V\}$, $\Lambda' = \overline{\bigcup_{\beta \in J} \Lambda_\beta}$. Then $V \setminus \Lambda'$ is contained in $\bigcup_\alpha \partial \Lambda_\alpha$. Take a filtration $\{W_i\}_{-1 \leq i < \dim X}$ such that $\bigcup_i W_i = \bigcup_\alpha \partial \Lambda_\alpha$ and $W_i \setminus W_{i-1}$ is an i-dimensional manifold ($W_{-1} = \emptyset$). We shall show $V \setminus \Lambda' \subset W_{-1}$ by induction on i. Assume $W_i \supset (V \setminus \Lambda')$. Then for $p \in W_i \setminus W_{i-1}$ there exist two functions f and g such that $\{f,g\}(p) \neq 0$ and $f|_{W_i} = g|_{W_i} = 0$. If $p \in V \setminus \Lambda'$, then $V \setminus \Lambda'$ contains the integral path through p of H_f, which contradicts the vanishing of g on $V \setminus \Lambda'$. Thus we may proceed by induction and conclude that $V \setminus \Lambda' = \emptyset$. $\square$

8.2.2. Now let $\underline{F} \in Ob(D^+(X))$.

Definition 8.2.5. : We say that $\underline{F}$ is weakly $\mathbb{R}$-constructible if :

i) there exists a Whitney $\mathbb{R}$-analytic stratification $X = \bigcup_\alpha X_\alpha$ such that for all j, all α, the sheaves $H^j(\underline{F})|_{X_\alpha}$ are locally constant (on X_α).

One says that $\underline{F}$ is $\mathbb{R}$-constructible if $\underline{F}$ is weakly $\mathbb{R}$-constructible and moreover :

ii) $\underline{F} \in Ob(D^b(X))$, and for all $x \in X$, $\underline{F}_x$ is quasi-isomorphic to a bounded complex of finitely generated projective A-modules.

Remark : The category of $\mathbb{R}$-constructible sheaves is studied in Kashiwara [7].

Theorem 8.2.6. : Let $\underline{F} \in Ob(D^+(X))$. The following conditions are equivalent :

a) $\underline{F}$ is weakly $\mathbb{R}$-constructible

b) $SS(\underline{F})$ is contained in a closed conic subanalytic isotropic set of T^*X

c) $SS(\underline{F})$ is a closed conic subanalytic Lagrangean set in T^*X.

Proof

a) $\Longrightarrow$ c). Let $X = \bigsqcup_\alpha X_\alpha$ be a Whitney $\mathbb{R}$-analytic stratification such that $H^j(\underline{F})|_{X_\alpha}$ is locally constant, and let us prove that $SS(\underline{F})$ is contained in $\bigsqcup_\alpha T^*_{X_\alpha} X$. By the definition of $SS(\underline{F})$ it is sufficient to show that for any $x \in X_\alpha$ and a C^ω-function f defined in a neighborhood of x such that $d(f|_{X_\alpha})(x) \neq 0$, and $f(x) = 0$, we have :

$$(\mathbb{R}\Gamma_{\{f \geq 0\}}(\underline{F}))_x = 0$$

If we take a submanifold Y of $f^{-1}(0)$ transversal to X_α at x, with $\dim Y + \dim X_\alpha = \dim X$, we have a topological isomorphism (Thom [1]) :

$$(X, \{X_\beta\}, X_\alpha, x) \simeq (Y \times X_\alpha, \{(Y \cap X_\beta) \times X_\alpha\}, X_\alpha, x)$$

such that ϕ and ψ denoting the projections from $X \simeq Y \times X_\alpha$ to Y and X_α respectively, then $f = (f|_{X_\alpha}) \circ \psi$ and $\underline{F} = \phi^{-1}(\underline{F}')$ for $\underline{F}' \in Ob(D^+(Y))$. Thus we have for any j :

$$(H^j_{\{f \geq 0\}}(\underline{F}))_x = H^j_{\{f \geq 0\} \cap X_\alpha}(\mathbb{R}\psi_* \phi^{-1} \underline{F}')_x$$
$$= 0$$

Thus $SS(\underline{F}) \subset \bigcup_\alpha T^*_{X_\alpha} X$, and we apply Proposition 8.2.4. and Theorem 6.4.1. to get c).

<u>b) ⟹ a).</u> Assume $SS(\underline{F}) \subset \Lambda$, where Λ is a closed conic subanalytic isotropic set in T^*X . Applying Proposition 8.2.3. we find a Whitney $\mathbb{R}$-analytic stratification $X = \bigcup_\alpha X_\alpha$ such that $SS(\underline{F})$ is contained in $\bigcup_\alpha T^*_{X_\alpha} X$. Then for all j and α , $H^j(\underline{F})|_{X_\alpha}$ is a locally constant sheaf by the following lemma.

<u>Lemma 8.2.7.</u> : <u>Let</u> Y <u>be a submanifold of</u> X, <u>flat with respect to</u> $SS(\underline{F})$. <u>Then</u> $H^j(\underline{F})|_Y$ <u>is locally constant</u>.

<u>Proof</u>

Let $(x_1,\ldots,x_n)$ be a system of local coordinates on X such that Y is linear, and let $y_o \in Y$.

We have seen (Proposition 8.1.3.) that there exists $\varepsilon > 0$ such that :

(8.2.1) $\quad (x;x-y) \notin SS(\underline{F})$ for $x \in X$, $y \in Y$, $|x-y_o| < \varepsilon$, $|y-y_o| < \varepsilon$, $x \neq y$

Let $U_r(y) = \{x \,;\, |x-y| < r\}$. In order to prove that $\underline{F}|_{X_\alpha}$ is locally constant it is enough to show the isomorphism :

(8.2.2) $$\mathbb{R}\Gamma(U_\varepsilon(y_o)\,;\,\underline{F}) \xrightarrow{\sim} \mathbb{R}\Gamma(U_\rho(y)\,;\,\underline{F})$$

for $y \in Y$, $\rho > 0$, $|y-y_o| + \rho < \varepsilon$. In fact then we get :

$$H^j(U_\varepsilon(y_o)\,;\,\underline{F}) \simeq \varinjlim_{\rho>0} H^j(U_\rho(y)\,;\,\underline{F}) = H^j(\underline{F})_y$$

for $y \in Y \cap U_\varepsilon(y_o)$, any $j \in \mathbb{Z}$.

Set $\Omega_t = U_{t\varepsilon+(1-t)\rho}(ty_o + (1-t)y)$. Then $\Omega_1 = U_\varepsilon(y_o)$, $\Omega_o = U_\rho(y)$, and it is easy to check that $\{\Omega_t\}_{0\leq t\leq 1}$ is an increasing sequence and that :

$$\Omega_{t_o} = \bigcup_{t<t_o} \Omega_t \qquad 0 < t < < 1$$

$$\overline{\Omega}_{t_o} = \bigcap_{t>t_o} \Omega_t \qquad 1 > t_o \geqslant 0$$

Moreover :

$$\mathbb{R}\Gamma_{X \setminus \Omega_t}(\underline{F})\Big|_{\partial\Omega_t} = 0$$

by (8.2.1) and the definition of the micro-support. Then (8.2.2) follows from Theorem 1.4.3. . □

Remark 8.2.8. : Assume $\underline{F}$ is $\mathbb{R}$-constructible on X. Let K be a compact subanalytic set in X. Then $\mathbb{R}\Gamma(K,\underline{F})$ is isomorphic to a bounded complex of finitely generated projective A-modules.

Remark 8.2.9. : $\mathbb{R}$-constructible complexes are cohomologically constructible in the sense of §5.6. .

§8.3. Functorial properties of $\mathbb{R}$-constructible sheaves

8.3.1. We shall study in this section the functorial properties of constructible sheaves, using Theorem 8.2.6. . Of course many results are already wellknown (cf. Goreski-Mac Pherson [1] for a review on this subject).

All manifolds we consider here are real analytic.

We denote by $D^b_{\mathbb{R}-c}(X)$ the subcategory of $D^b(X)$ consisting of $\mathbb{R}$-constructible complexes.

8.3.2. Let f be a map from Y to X, ρ and ϖ the natural associated maps from $Y \times_X T^*X$ to T^*Y and T^*X, respectively.

Proposition 8.3.1. : Let $\underline{G} \in \mathrm{Ob}(D^b_{\mathbb{R}-c}(Y))$ and assume f is proper on $\mathrm{supp}\,\underline{G}$. Then $\mathbb{R}f_*(\underline{G}) \in \mathrm{Ob}(D^b_{\mathbb{R}-c}(X))$.

Proof

We know by Proposition 4.1.1. that $SS(\mathbb{R}f_*(\underline{G}))$ is contained in $\varpi\rho^{-1}(SS(\underline{G}))$ and this last set is subanalytic since ϖ is proper, and it is isotropic in T^*X since $\varpi^{-1}(\omega_X) = \rho^{-1}(\omega_Y)$ and $\rho^{-1}(\omega_Y)$ vanishes on $\rho^{-1}(SS(\underline{G}))$. Thus $\mathbb{R}f_*(\underline{G})$ is weakly constructible and the finiteness properties are proved using Remark 8.2.8. by standard arguments. □

Remarl 8.3.2. : In the same line if we consider Theorem 4.4.1. and assume $\underline{G}$ is $\mathbb{R}$-constructible, we find that $\mathbb{R}f_*(\underline{G})$ (resp. $\mathbb{R}f_!(\underline{G})$) is $\mathbb{R}$-constructible.

Proposition 8.3.3. : Let $\underline{F} \in Ob(D^b_{\mathbb{R}-c}(X))$. Then $f^{-1}(\underline{F})$ and $f^!(\underline{F})$ belong to $Ob(D^b_{\mathbb{R}-c}(Y))$.

Proof

We may reduce the problem to the case where f is an immersion. Then it is a consequence of Remark 8.3.2. and the following :

Proposition 8.3.4. : Assume $Y \subset X$. Let $\underline{F} \in Ob(D^b_{\mathbb{R}-c}(X))$. Then $\nu_Y(\underline{F}) \in Ob(D^b_{\mathbb{R}-c}(T_YX))$ and $\mu_Y(\underline{F}) \in Ob(D^b_{\mathbb{R}-c}(T^*_YX))$.

Proof

We know by (Kashiwara-Schapira [2 Theorem 10.5.2.]) that $C_{T^*_YX}(SS(\underline{F}))$ is a closed subanalytic isotropic set in $T^*T^*_YX$. Thus $\mu_Y(F)$ is weakly constructible, and the finiteness properties follow from Remark 8.2.8. . The proof for $\nu_Y(F)$ is the same. □

Proposition 8.3.5. : Let E be a vector bundle, $\underline{F} \in Ob(D^b_{\mathbb{R}-c}(E))$ and assume $\underline{F}$ is conic. Then $\underline{F}^\wedge$ the Fourier-Sato transform of $\underline{F}$ belongs to $Ob^b_{\mathbb{R}-c}(E^*))$.

This follows from Theorem 5.1.4. . □

Remark : In the complex case, Proposition 8.3.5. has already been proved by B. Malgrange [1] and J.L. Brylinski [2].

Proposition 8.3.6. : Assume A commutative and let $\underline{F}$ and $\underline{G}$ belong to $\mathrm{Ob}(D^b_{\mathbb{R}-c}(X))$.

a) $\mathbb{R}\underline{Hom}(\underline{F},\underline{G}) \in \mathrm{Ob}(D^b_{\mathbb{R}-c}(X))$.

b) Assume moreover $wg\ell d(A)$ finite .

Then $\underline{F} \overset{\mathbb{L}}{\otimes} \underline{G} \in \mathrm{Ob}(D^b_{\mathbb{R}-c}(X))$.

§8.4. Contact transformations

8.4.1. Assume X real analytic. Let $p \in T^*X$. We construct the triangulated category $D^b_{\mathbb{R}-c}(X;p)$ exactly as for $D^+(X;p)$, starting with $D^b_{\mathbb{R}-c}(X)$ instead of $D^+(X)$.

Let ϕ be a real analytic contact transformation between an open set U of T^*X and an open set V of T^*Y. Let $p \in U$, $q = \phi(p)$.

Proposition 8.4.1. : Let $\Phi_{\underline{K}}$ be an extended contact transformation above ϕ (cf. §6). Then $\Phi_{\underline{K}}$ defines an equivalence of categories from $D^+_{\mathbb{R}-c}(X;p)$ to $D^+_{\mathbb{R},c}(Y;q)$.

Proof

In the construction of contact transformations in §6.3. we may take for $\underline{K}'$ an $\mathbb{R}$-constructible sheaf. Then the Proposition follows from the results of Chapter 6 and Chapter 8, §3. . □

Remark 8.4.2. : In the complex case a similar result has been obtained by J.L. Brylinski [2] assuming X projective, and extended by G. Laumon [1].

§8.5. $\mathbb{C}$-constructible sheaves

8.5.1. In this section X will denote a complex manifold. We shall often confuse X and $X^{\mathbb{R}}$, the real underlying manifold. To specify that a set Z is complex analytic we shall say that Z is $\mathbb{C}$-analytic. A $\mathbb{C}$-stratification of X is a stratification by complex manifolds. A $\mathbb{C}$-constructible (resp. weakly $\mathbb{C}$-constructible) sheaf is an $\mathbb{R}$-constructible (resp. weakly $\mathbb{R}$-constructible) sheaf along a $\mathbb{C}$-analytic stratification. We define the category $D^b_{\mathbb{C}-c}(X)$ similarly to $D^b_{\mathbb{R}-c}(X)$. We denote by ω the canonical 1-form on T^*X, and by $2\,\mathrm{Re}\,\omega = \omega^{\mathbb{R}}$ the canonical 1-form on $T^*X^{\mathbb{R}}$. The meaning of "$\mathbb{R}$-isotropic" or "$\mathbb{C}$-isotropic", etc ... is the obvious one.

8.5.2. Let $\underline{F} \in \mathrm{Ob}(D^+(X))$.

Definition 8.5.1. : *We say that $\underline{F}$ is monodromic if $SS(\underline{F})$ is stable for the action of $\mathbb{C}^\times$ on T^*X.*

Theorem 8.5.2. : *The following conditions are equivalent.*

a) $\underline{F}$ *is weakly $\mathbb{C}$-constructible.*

b) $\underline{F}$ *is weakly $\mathbb{R}$-constructible and monodromic.*

c) $SS(\underline{F})$ *is contained in a closed conic $\mathbb{R}$-isotropic subanalytic subset of T^*X stable by the action of $\mathbb{C}^\times$.*

d) $SS(\underline{F})$ *is a closed conic $\mathbb{C}$-analytic Lagrangean subset of T^*X.*

Proof

a) $\Longrightarrow$ c) : as in the proof of Theorem 8.2.6.

d) $\Longrightarrow$ a) : as in the proof of Theorem 8.2.6. since the analogous to Proposition 8.2.3. is true if we replace in its statement "subanalytic" by "$\mathbb{C}$-analytic" (cf. Kashiwara [2]).

d) $\Longrightarrow$ b) : is obvious .

b) $\Longrightarrow$ c) by Theorem 8.2.6. .

It remains to prove c) $\Longrightarrow$ d), and it will follow from Theorem 8.2.6. and the next Proposition.

Proposition 8.5.3. : Let Λ_o be a closed $\mathbb{R}$-isotropic subanalytic subset of T^*X , stable by the action of $\mathbb{C}^\times$ and let Λ be a closed $\mathbb{R}$-involutive subanalytic subset contained in Λ_o. Then Λ is complex analytic.

Proof

Let Λ' be the non-singular locus of Λ . Then Λ' is a real analytic manifold of dimension $2n$ and is open in Λ_o . Hence for any $p \in \Lambda'$, $T_p\Lambda' \supset \mathbb{C}\, H_\omega$. For $v \in T_pT^*X$, we have $\langle \mathrm{Re}\, d\omega, H_\omega \wedge v\rangle = \langle \omega, v\rangle$. Therefore $\omega|_{T_p\Lambda'} = 0$, thus $d\omega|_{T_p\Lambda' + \sqrt{-1}\, T_p\Lambda'} = 0$, and hence $T_p\Lambda' = \sqrt{-1}\, T_p\Lambda'$.

This shows that Λ' is complex analytic. Now, we shall show that $\Lambda = \mathrm{clos}\, \Lambda'$ is complex analytic. Let S' be the set of points p in $S = \Lambda \setminus \Lambda'$ such that S is a real analytic manifold of real dimension $2n-1$ on a neighborhood of p and (Λ', S) satisfies the condition of Whitney on a neighborhood of p. We shall show first that Λ is complex analytic on a neighborhood of S'.

For $p \in S'$ and a sequence $\{p_n\}$ in Λ' which tends to p such that $T_{p_n}\Lambda'$ tends to a plane $\tau \in T_p(T^*X)$, we have $T_pS' \subset \tau$. Since T_pS' is not a complex vector space we have

$T_pS' + \sqrt{-1}\, T_pS' = \tau$. Therefore :

(8.5.1) $$\dim_{\mathbb{C}}(T_pS' + \sqrt{-1}\, T_pS') = n$$

for any point p in S'.

This implies that, for any $p \in S'$, there exists a complex manifold $S^{\mathbb{C}}$ with complex dimension n such that $S^{\mathbb{C}} \supset S'$ on a neighborhood of p. Hence S' is a real hypersurface of $S^{\mathbb{C}}$.

Now we shall show that Λ' is contained in $S^{\mathbb{C}}$ on a neighborhood of p. We take a local coordinate system $(z_1,\dots,z_{2n}) = (z',z'')$, where $z' = (z_1,\dots,z_n)$, such that :

$$S^{\mathbb{C}} = \{(z',z'')\,;\, z'=0\} \;;\; S = \{(z',z'')\,;\, z'=0,\ \psi(z'')=0\}$$

Since $\overline{T^*_{\Lambda'}\, T^*X \underset{T^*X}{\times} S} \subset T^*_S\, T^*X$, the projection $\phi : (z',z'') \longmapsto (z'')$ is finite on Λ on a neighborhood of p, and is a local isomorphism on Λ'.

Since $\phi(S) = \{z \in \mathbb{C}^n\,;\, \psi(z'') = 0\}$, in order to see that $\Lambda' \subset S^{\mathbb{C}}$, we may assume :

$$\Lambda' \subset \{z \in \mathbb{C}^{2n}\,;\, \psi(z'') > 0\}$$

without loss of generality. Then setting $U = \{z \in \mathbb{C}^{2n}\,;\, \psi(z'') > 0\}$, $\phi : \Lambda' \cap \phi^{-1}(U) \longrightarrow U$ is an unramified covering and hence for any holomorphic function u defined on a neighborhood of p :

$$u_m(z') = \sum_{\phi(q)=z'} u(q)^m$$

is a holomorphic function in $z' \in \mathbb{C}^n$ defined on U.

If $u|_{S^{\mathbb{C}}} = 0$, then u_m can be continued to $\overline{U}$, so that $u_m|_{\phi(S)} = 0$, because $\overline{\Lambda}' \cap \phi^{-1}\phi(S) \subset S^{\mathbb{C}}$. Hence we have $u_m = 0$. Since this holds for any m , $u|_{\Lambda'} = 0$. Therefore Λ' is an open subset of $S^{\mathbb{C}}$.

Since $S^{\mathbb{C}}$ is Lagrangean, the involutivity of Λ implies $\Lambda \supset S^{\mathbb{C}}$ on a neighborhood of p.

Thus we have $S' = \emptyset$, and the real dimension of S is less than or equal to $2(n-1)$. By the extension theorem for complex analytic subsets, $\overline{\Lambda}'$ is complex analytic. □

§8.6. Direct images of $\mathbb{C}$-constructible sheaves for non proper maps

8.6.1. Let Y be another complex manifold, f a holomorphic map from Y to X. As an immediate application of Theorem 8.5.2. and Remark 8.3.2. we get :

Proposition 8.6.1. : Let $(Y_s)_{s>0}$ be a family of subanalytic open subsets in $Y^{\mathbb{R}}$, let $\underline{G} \in \mathrm{Ob}(D^b_{\mathbb{C}-c}(Y))$ and assume :

i) $Y = \bigcup_s Y_s$, $\bigcup_{r<s} Y_r = Y_s$, $\bigcap_{t>s} Y_t \subset \overline{Y}_s$, $N^*_y(Y_s) \neq T^*_y Y$ for any $y \in Y$, any s.

ii) f is proper over $\overline{Y}_s \cap \mathrm{supp}(\underline{G})$ for all s.

iii) $\overline{N^*(Y_s) \cap (SS(\underline{G}) + \rho(Y \times_X T^*X))} \subset T^*_Y Y$

Then $\mathbb{R}f_*(\underline{G})$ and $\mathbb{R}f_!(\underline{G})$ are $\mathbb{C}$-constructible and moreover :

$$SS(\mathbb{R}f_*(\underline{G})) \subset \varpi\, \rho^{-1}(SS(\underline{G}))$$

$$SS(\mathbb{R}f_!(\underline{G})) \subset \varpi\, \rho^{-1}(SS(\underline{G}))$$

8.6.2. The hypotheses of Proposition 8.6.1. are "locally" always satisfied when $\dim X = 1$. More precisely :

Proposition 8.6.2. : Let $f : Y \longrightarrow X$ be a holomorphic map, with $\dim X = 1$, and let $\underline{G} \in \mathrm{Ob}(D^b_{\mathbb{C}-c}(Y))$. Let $x \in X$ and let K be a compact subset of $f^{-1}(x)$. Then there exist open neighborhoods U of x, V of K, with $V \subset f^{-1}(U)$, such that, denoting by f_V the restriction of f to V, $f_V : V \longrightarrow U$, $\mathbb{R}f_{V*}(\underline{G})$ and $\mathbb{R}f_{V!}(\underline{G})$ are $\mathbb{C}$-constructible and moreover :

$$SS(\mathbb{R}f_{V*}(\underline{G})) \subset \varpi\, \rho^{-1}(SS(\underline{G}))$$

$$SS(\mathbb{R}f_{V!}(\underline{G})) \subset \varpi\, \rho^{-1}(SS(\underline{G}))$$

Proof

By decomposing f into $Y \longrightarrow Y \times X \longrightarrow X$, we may assume from the beginning that $Y = Z \times X$, and f is the projection.

We set $S = SS(\underline{G})$, $Z_o = f^{-1}(x)$.

Let $\Lambda = T^*Z_o \cap C_{T^*_{Z_o}Y}(S)$. Since we know that $C_{T^*_{Z_o}Y}(S)$ is isotropic in $T^*T^*_{Z_o}X$ (Kashiwara-Schapira [2, §10]), we obtain by Lemma 8.2.2. that Λ is isotropic in T^*Z_o.

Lemma 8.6.3. : The image of $Z_o \underset{Y}{\times} \overline{(S + \rho(Y \underset{X}{\times} T^*X))}$ by the projection $Z_o \underset{Y}{\times} T^*Y \longrightarrow T^*Z_o$ coïncides with Λ.

Proof

We shall take a local coordinate system (z,t) of $Z \times X = Y$ and $(z,t\,;\,\zeta,\tau)$ of T^*Y. Then it is enough to show that if a sequence $\{(z_n,t_n\,;\,\zeta_n,\tau_n)\}$ in S satisfies $(z_n,t_n,\zeta_n) \underset{n}{\longrightarrow} (z_o,t_o,\zeta_o)$, then $|\tau_n||t_n-t_o| \underset{n}{\longrightarrow} 0$.

If this is not true, there is a holomorphic map $(z(\lambda),\ t(\lambda),\ \zeta(\lambda),\ \tau(\lambda)) \in S$ defined on $\{\lambda\,;\,0 < |\lambda| < 1\}$ such that when λ tends to 0 we have :

$$z(\lambda) \longrightarrow z_o,\ \zeta(\lambda) \longrightarrow \zeta_o,\ t(\lambda)-t_o \sim \lambda^s,\ \tau(\lambda) \sim \lambda^{-r},\ \text{with } s \leqslant r.$$

Since S is Lagrangean

$$<\zeta(\lambda)\ ,\ \frac{d}{d\lambda}\, z(\lambda)> + \tau(\lambda)\ \frac{dt(\lambda)}{d\lambda} = 0\ .$$

Therefore $\tau(\lambda)\frac{dt(\lambda)}{d\lambda}$ is bounded, which contradicts $s \leqslant r$. □

Lemma 8.6.4. : Let Z be a real analytic manifold and let $\phi : Z \longrightarrow \mathbb{R}$ be a proper map. Then, for any closed conic subanalytic isotropic subset Λ in T^*Z, $\phi(\{x ; d\phi(x) \in \Lambda\})$ is a discrete set of $\mathbb{R}$.

Proof

Otherwise, there exists a real analytic path $x(t)$ such that $d\phi(x(t)) \in \Lambda$ and $\phi(x(t))$ is not constant. Since Λ is isotropic, we have $\frac{d}{dt}\phi(x(t)) = 0$. This is a contradiction. □

We resume the proof of Proposition 8.6.2. .

Let us take a positive valued real analytic function ϕ on $Z_o \setminus K$ such that $Z_o^s = \{z \in Z ; \phi(z) < s\} \cup K$ is relatively compact in Z_o for any $s > 0$. By the preceding lemma there exists $0<s_1<s_2<s_3<s_4$ such that $d\phi(z) \notin \Lambda$ if $s_1 \leqslant \phi(z) \leqslant s_4$.

We shall show that $V = f^{-1}(U) \cap Z_o^{s_3} \times X$ satisfies the conditions of Proposition 8.6.1. for a sufficient small neighborhood U of x.

We argue by contradiction.

If this is not true, there exists a sequence $\{(z_n, x_n)\} \subset Y = Z \times X$ such that $(z_n, x_n ; d\phi(z_n), 0) \in S + \rho(Y \underset{X}{\times} T^*X)$ and $z_n \underset{n}{\longrightarrow} z_o$, $x_n \underset{n}{\longrightarrow} x$ with $s_2 \leqslant \phi(z_o) \leqslant s_3$.

Hence by Lemma 8.6.3. $(z_o ; d\phi(z_o))$ belongs to Λ, which is a contradiction. □

Remark : This proposition is not true if $\dim X \neq 1$. The above proof breaks because Lemma 8.6.3. does not hold if $\dim X \neq 1$.

Remark : In fact, Proposition 8.6.2. is already known (cf. Lê [1], Sabbah [2] who prove it using results of Hironaka in [2]).

CHAPTER 9 - APPLICATION 2 : REGULAR HOLONOMIC MODULES

§9.1. Preliminaries

9.1.1. In this chapter, using the Riemann - Hilbert correspondence (cf. Kashiwara [6], [7], Mebkhout [2], [3]) and more precisely, the functor TH of Kashiwara (loc. cit.), we shall translate our results on $\mathbb{C}$-constructible sheaves to results on regular holonomic Modules (cf. Kashiwara-Kawaï [6]).

9.1.2. Let X be a complex manifold of dimension n, $\mathcal{O}_X$ (resp. Ω_X) the sheaf on X of holomorphic functions (resp. holomorphic n-forms).

The following sheaves have been defined by Sato-Kashiwara-Kawaï [1]. Let Z be a complex submanifold of X of codimension d. One sets :

$$(9.1.1) \qquad C^{\mathbb{R}}_{Z|X} = \mu_Z(\mathcal{O}_X)\ [d]$$

Remark that this complex is concentrated in degree zero. Its restriction to the zero section , $\mathbb{R}\Gamma_Z(\mathcal{O}_X)\ [d]$ is also denoted by $B^{\infty}_{Z|X}$.

Let M be a real analytic manifold such that X is a complexification of M. The sheaf of Sato's microfunctions is defined by :

$$(9.1.2) \qquad C_M = \mu_M(\mathcal{O}_X) \otimes \omega_M\ [n]$$

It is concentrated in degree zero, and its restriction to the zero section, $\mathbb{R}\Gamma_M(\mathcal{O}_X) \otimes \omega_M[n]$ is the sheaf of Sato's hyperfunctions, and is denoted by B_M.

Now let Y be another complex manifold, $f : Y \longrightarrow X$ a holomorphic map. We identify Y with the graph of f in $Y \times X$ and $Y \underset{X}{\times} T^*X$ with $T^*_Y(Y \times X)$. Hence the natural maps ρ and ϖ from $Y \underset{X}{\times} T^*X$ to T^*Y and T^*X respectively are associated to the first and the second projection on $T^*(Y \times X) = T^*Y \times T^*X$ respectively. We denote by p_1 and p_2 these projections. We denote by q_j the j-th projection $(j = 1,2)$ on $Y \times X$.

One sets :

$$(9.1.3) \qquad \begin{cases} \mathcal{E}^{\mathbb{R}}_{Y \longrightarrow X} = C^{\mathbb{R}}_{Y|Y\times X} \underset{q_2^{-1}\mathcal{O}_X}{\otimes} q_2^{-1}\Omega_X \\ \mathcal{E}^{\mathbb{R}}_{X \longleftarrow Y} = (C^{\mathbb{R}}_{Y|Y\times X} \underset{q_1^{-1}\mathcal{O}_X}{\otimes} q_1^{-1}\Omega_X)^a \end{cases}$$

When f is the identity one sets :

$$(9.1.4) \qquad \mathcal{E}^{\mathbb{R}}_X = \mathcal{E}^{\mathbb{R}}_{X \longrightarrow X}$$

The sheaf $\mathcal{E}^{\mathbb{R}}_X$ on T^*X is naturally endowed with a structure of a unitary Ring, and the sheaves C_M and $C^{\mathbb{R}}_{Z|X}$ (Z a complex submanifold of X) are naturally left $\mathcal{E}^{\mathbb{R}}_X$-modules. Moreover the sheaf $\mathcal{E}^{\mathbb{R}}_{Y \longrightarrow X}$ has a natural structure of $(\rho^{-1}\mathcal{E}^{\mathbb{R}}_Y, \varpi^{-1}\mathcal{E}^{\mathbb{R}}_X)$ bimodule, and the sheaf $\mathcal{E}^{\mathbb{R}}_{X \longleftarrow Y}$ a structure of $(\varpi^{-1}\mathcal{E}^{\mathbb{R}}_X, \rho^{-1}\mathcal{E}^{\mathbb{R}}_Y)$-bimodule.

Let γ be the map : $T^*X \longrightarrow T^*X/\mathbb{C}^\times$. Then one sets :

$$(9.1.5) \qquad \mathcal{E}^{\infty}_X = \gamma^{-1} \mathbb{R}\gamma_*(\mathcal{E}^{\mathbb{R}}_X)$$

This is the Ring of "infinite order microdifferential operators". It contains the important subring $\mathcal{E}_X$ of "finite order microdifferential operators", but we do not recall the construction of $\mathcal{E}_X$ here. The sheaves $C^{\infty}_{Z|X}$, $\mathcal{E}^{\infty}_{Y \longrightarrow X}$, $C_{Z|X}$, $\mathcal{E}_{Y \longrightarrow X}$, are similarly defined.

Identifying X with $T^*_X X$, we have :

(9.1.6) $$\mathcal{E}^{\mathbb{R}}_X \Big|_{T^*_X X} = \mathcal{E}^{\infty}_X \Big|_{T^*_X X} = \mathcal{D}^{\infty}_X$$

(9.1.7) $$\mathcal{E}_X \Big|_{T^*_X X} = \mathcal{D}_X$$

where $\mathcal{D}^{\infty}_X$ (resp. $\mathcal{D}_X$) is the Ring of infinite (resp. finite) order differential operators. Moreover :

(9.1.8) $$\mathcal{D}_{Y \longrightarrow X} = \mathcal{E}_{Y \longrightarrow X}\Big|_{Y \times_X T^*_X X} \simeq \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathcal{D}_X$$

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, its characteristic variety, denoted char($\mathcal{M}$), satisfies :

(9.1.9) $$\begin{aligned} \mathrm{char}(\mathcal{M}) &= \mathrm{supp}(\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M}) \\ &= \mathrm{supp}(\mathcal{E}^{\mathbb{R}}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M}) \end{aligned}$$

The last equality follows from the fact that $\mathcal{E}^{\mathbb{R}}_X$ is faithfully flat over $\mathcal{E}_X$ (cf. Chapter 10 below). Recall that $\mathcal{E}_X$ is flat over $\pi^{-1}\mathcal{D}_X$.

Although we shall note use it here, it may be useful to notice that the functor $\mu hom(\cdot, \mathcal{O}_X)$ allows us to construct new sheaves of microfunctions.

As an example consider the sheaf $C_{M^+|X}$ of Kataoka [1] [2]. One may recover it as follows.

Let M be a real analytic manifold of dimension n such that X is a complexification of M, and let ϕ be a real analytic function on M , with $d\phi \neq 0$ on the set $\{\phi = 0\}$. Let $M^+ = \{x \in M ; \phi(x) \geqslant 0\}$. Then :

$$C_{M^+|X} \simeq \mu hom(\mathbb{C}_{M^+}, \mathcal{O}_X) \otimes \underline{\omega}_M [n]$$

(use Proposition 5.5.8. to compare with Kataoka's definition). For further developments, cf. also Kaneko [1], Ôaku [1].

9.1.3. Let Diff(X) be the category of left $\mathcal{D}_X$-modules. We denote by $D(\mathcal{D}_X)$ its derived category, by $D^+(\mathcal{D}_X)$ the full subcategory consisting of complexes with cohomology bounded from below, by $D^b_{coh}(X)$ the full subcategory consisting of complexes with bounded and coherent cohomologies and finally by $D^b_{rh}(X)$ the full subcategory consisting of complexes with bounded and regular holonomic cohomologies.

We shall keep the same notations for the categories constructed with right $\mathcal{D}_X$-modules, since there will have no risk of confusion.

9.1.4. Now assume X is a real analytic manifold. Let $\tilde{X}$ be a complexification of X. We denote by $\mathcal{A}_X = (\mathcal{O}_{\tilde{X}})|_X$ the sheaf of real analytic functions on X, by $\Omega_X = (\Omega_{\tilde{X}})|_X \otimes \underline{\omega}_X$ the sheaf of real analytic densities, $\mathcal{D}_X = (\mathcal{D}_{\tilde{X}})|_X$ the sheaf of finite order real analytic differential operators, etc ... . We also use the sheaf $\mathcal{D}'_X$ of L. Schwartz's distributions.

If X is a complex manifold, we denote by $X^{\mathbb{R}}$ the real underlying manifold, and $\overline{X}$ the anti-holomorphic manifold associated to X. The diagonal embedding $X^{\mathbb{R}} \hookrightarrow X \times \overline{X}$ identify $X \times \overline{X}$ with a complexification of $X^{\mathbb{R}}$.

Of course when X is complex, one shall not confuse the sheaf $\mathcal{D}_X$ with the sheaf $\mathcal{D}_{X^{\mathbb{R}}}$, or Ω_X with $\Omega_{X^{\mathbb{R}}}$ for example.

9.1.5. On a real analytic manifold X, we denote by $\mathbb{R}$-C(X) the category of $\mathbb{R}$-constructible sheaves on X, with base ring $\mathbb{C}$.

Then the derived category $D^b(\mathbb{R}\text{-}C(X))$ is equivalent to $D^b_{\mathbb{R}\text{-}c}(X)$, (cf. Kashiwara [7]).

§9.2. Review on the Riemann-Hilbert correspondence

9.2.1. First we recall the main properties of the functor of "cohomology with moderate growth" constructed in Kashiwara [7].

Theorem 9.2.1. : Let X be a real analytic manifold. There exists a contravariant functor, denoted TH, from $\mathbb{R}\text{-}C(X)$ to $Diff(X)$ such that :

i) TH is exact

ii) $TH(\underline{\mathbb{C}}_X) = \mathcal{D}'_X$ the sheaf of Schwartz distributions on X

iii) if Z is a closed subanalytic subset of X and $\underline{F} \in Ob(\mathbb{R}\text{-}C(X))$,

$$TH(\underline{F}_Z) = \Gamma_Z(TH(\underline{F}))$$

iv) let $f : Y \longrightarrow X$ be a real analytic map, and let $\underline{G} \in Ob(D^b_{\mathbb{R}\text{-}c}(Y))$.

Assume f is proper on $\operatorname{supp} \underline{G}$. Then still denoting by TH the functor from $D^b_{\mathbb{R}\text{-}c}(X)$ to $D^+(\mathcal{D}_X)$ obtained by passing to the derived categories, we have :

$$\mathbb{R}f_!(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\underline{G})) = TH(\mathbb{R}f_* \underline{G})$$

Corollary 9.2.2. : With the same hypotheses and notations as in Theorem 4.4.1., assuming moreover Y_s subanalytic, we have :

$$\mathbb{R}f_!(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\underline{G})) = TH(\mathbb{R}f_* \underline{G}) ,$$

(resp.

$$\mathbb{R}f_*(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\underline{G})) = TH(\mathbb{R}f_! \underline{G})).$$

Proof

We keep the notations of Theorem 4.4.1. . Thus i_s denotes the injection $Y_s \hookrightarrow Y$, and $f_s = i_s \circ f$. Applying Theorem 9.2.1. we get :

$$\mathbb{R} f_!(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\mathbb{R} i_{s*}\, i_s^{-1}\, \underline{G})) = TH(\mathbb{R} f_* \,\mathbb{R} i_{s*}\, i_s^{-1}\, \underline{G})$$

By Theorem 4.4.1. the right-hand side is isomorphic to $TH(\mathbb{R} f_* \underline{G})$.

On the other hand we have :

$$H^k \mathbb{R} f_!(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\underline{G})) = \varinjlim_s H^k \mathbb{R} f_* \mathbb{R}\Gamma_{\overline{Y}_s}(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(G))$$

$$= \varinjlim_s H^k \mathbb{R} f_*(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(G_{\overline{Y}_s}))$$

In fact there are natural morphisms :

$$\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\underline{G}_{\overline{Y}_s}) \longrightarrow \mathbb{R}\Gamma_{\overline{Y}_s}(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\underline{G})) \longrightarrow \mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\underline{G}_{\overline{Y}_{s'}})$$

for $s' > s$ such that $Y_{s'} \supset \overline{Y}_s$. Similarly we have :

$$\underline{G}_{\overline{Y}_{s''}} \longrightarrow \mathbb{R} i_{s'*}\, i_{s'}^{-1}\, \underline{G} \longrightarrow \underline{G}_{\overline{Y}_s}$$

for $s'' > s' > s$ such that $Y_{s''} \supset \overline{Y}_s$ and $Y_{s'} \supset \overline{Y}_s$.

Therefore we obtain :

$$H^k \mathbb{R} f_!(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\underline{G})) = \varinjlim_s H^k \mathbb{R} f_!(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} TH(\mathbb{R} i_{s*}\, i_s^{-1}\, \underline{G}))$$

The other case is similarly proved. □

9.2.2. Recall the "reconstruction theorem" of Kashiwara [6],[7]. (cf. also Mebkhout [1],[2],[3] for another approach of the Riemann-Hilbert correspondence).

Theorem 9.2.3. : Let X be a complex manifold. Then the functors

$$\mathrm{sol} \;:\; D^b_{rh}(\mathcal{D}_X) \longrightarrow D^b_{\mathbb{C}-c}(X)$$

$$\mathcal{M} \longmapsto \mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

and

$$\mathrm{R\text{-}H} \;:\; D^b_{\mathbb{C}-c}(X) \longrightarrow D^b_{rh}(\mathcal{D}_X)$$

$$\underline{F} \longmapsto \mathbb{R}\underline{Hom}_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathrm{TH}(\underline{F}))$$

are well defined, and inverse of each other .

§9.3. Microlocal Riemann-Hilbert correspondence

9.3.1. We may microlocalize the construction in Theorem 9.2.3. by using the functor $\mu hom(\cdot,\cdot)$.

Proposition 9.3.1. : Let $\underline{F} \in D^b_{\mathbb{C}-c}(X)$ and $\mathcal{M} = \mathrm{R\text{-}H}(\underline{F})$ (and hence $\mathcal{M}^\infty = \mathcal{D}^\infty_X \underset{\mathcal{D}_X}{\otimes} \mathcal{M} = \mathbb{R}\underline{Hom}(\underline{F}, \mathcal{O}_X))$. Then we have :

$$\mu hom(\underline{F}, \mathcal{O}_X) = \mathcal{E}^{\mathbb{R}}_X \underset{\pi^{-1}\mathcal{D}^\infty_X}{\otimes} \pi^{-1}\mathcal{M}^\infty = \mathcal{E}^{\mathbb{R}}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M} \quad .$$

Proof

We shall not write π^{-1} for short.

Let q_j be the j-th projection from $X \times X$ to X $(j = 1,2)$.
By the definition of $\mathcal{E}^{\mathbb{R}}_X$ we get :

$$\mathcal{E}^{\mathbb{R}}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{M} = \mu_\Delta(\mathcal{O}^{(o,n)}_{X\times X}[n] \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}) \;.$$

On the other hand (Kashiwara-Kawaï [6], Mebkhout [4]) :

$$\mathcal{O}^{(o,n)}_{X\times X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M} = q_1^{-1}\mathcal{O}_X \otimes q_2^{-1}(\Omega_X \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M})$$

$$= q_1^{-1}\mathcal{O}_X \otimes q_2^{-1}\mathbb{R}\underline{Hom}(\underline{F}, \mathbb{C}_X)\ [n]$$

Therefore we have, by Proposition 5.6.2.

$$\mathscr{E}_X^{\mathbb{R}} \underset{\mathscr{D}_X}{\otimes} \mathscr{M} = \mu_\Delta(q_2^{-1}\mathbb{R}\underline{Hom}(\underline{F},\underline{\mathbb{C}}_X) \otimes q_1^{-1}\mathscr{O}_X)\ [2n]$$
$$= \mu_\Delta(\mathbb{R}\underline{Hom}(q_2^{-1}\underline{F},\ q_1^{!}\mathscr{O}_X))$$

The last term equals, by the definition , $\mu hom(\underline{F}, \mathscr{O}_X)$. □

Remark 9.3.2. : When restricting the isomorphism of Proposition 9.3.1. to the zero section of T^*X we recover a result of Z. Mebkhout [4].

Corollary 9.3.3. : *In the situation of Theorem 9.2.3. we have :*

$$SS(\underline{F}) = char(\Phi(\underline{F})) \ .$$

Proof

Let $\mathscr{M} = R\text{-}H(\underline{F})$. We know by Kashiwara [5] that $SS(\mathbb{R}\underline{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X))$ is contained in $char(\mathscr{M})$. The converse inclusion follows from Proposition 9.3.1. . □

Remark 9.3.4. : We shall generalize this Corollary in Chapter 10. Let us notice that a related result to Corollary 9.3.3. with the notion of "vanishing cycle" replacing that of micro-support, was already known, (cf. Deligne [1], Brylinski [1], Kashiwara [5]).

9.3.2. Let Y and X be complex manifolds, f a holomorphic map from Y to X.

Proposition 9.3.3. : *For* $\underline{F} \in D^b_{\mathbb{R}-c}(X^{\mathbb{R}})$ *and* $\underline{G} \in D^b_{\mathbb{R}-c}(Y^{\mathbb{R}})$, *we have the following isomorphisms.*

(i) *If* f *is an immersion, and* f *is non characteristic for* $\underline{F}$ *on an open subset* U *of* T^*X :

$$\mu hom(f^{-1}\underline{F}, \mathcal{O}_Y)|_U = \mathbb{R}\rho_*(\mathcal{D}_{Y \to X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \varpi^{-1} \mu hom(\underline{F}, \mathcal{O}_X))|_U$$

(ii) <u>If</u> f <u>is smooth and</u> supp $\underline{G}$ <u>is proper over</u> X , <u>then</u> :

$$\mu hom(\mathbb{R}f_!\underline{G}, \mathcal{O}_X) = \mathbb{R}\varpi_*(\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \rho^{-1} \mu hom(\underline{G}, \mathcal{O}_Y)) \quad [\dim Y - \dim X]$$

<u>Proof</u>

(i) We have

$$\mathcal{D}_{Y \to X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \varpi^{-1} \mu hom(\underline{F}, \mathcal{O}_X) = \varpi^{-1} \mu hom(\underline{F}, \mathcal{D}_{Y \to X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X)$$

Since $\mathcal{D}_{Y \longrightarrow X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X = \mathbb{R}f_*\mathcal{O}_Y$, it is enough to apply Corollary 5.5.7.

(ii) In this case, we have :

$$\mathcal{D}_{X \leftarrow Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{O}_Y = f^! \mathcal{O}_X [\dim X - \dim Y] .$$

Hence (ii) follows immediately from Corollary 5.5.6. . □

<u>§9.4. Direct images of regular holonomic Modules for non proper maps</u>

<u>9.4.1</u>. Let Y and X be complex manifolds, f a holomorphic map from Y to X. Recall that for a right $\mathcal{D}_Y$-module $\mathcal{M}$ the direct image of $\mathcal{M}$ (or the proper direct image of $\mathcal{M}$) is the complexe of right $\mathcal{D}_X$-modules given by :

$$\int_f \mathcal{M} \overline{\overline{\text{def}}} \mathbb{R}f_*(\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{D}_{Y \to X})$$

$$\int_f^{pr} \mathcal{M} \overline{\overline{\text{def}}} \mathbb{R}f_!(\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{D}_{Y \to X})$$

These functors extend naturally to $D^b_{coh}(\mathcal{D}_Y)$.

9.4.2. We shall "translate" the results of §8.6. to regular holonomic Modules.

Theorem 9.4.1. : Let $\mathcal{M}$ be a right regular holonomic $\mathcal{D}_Y$-module (or more generally an object of $D^b_{rh}(\mathcal{D}_Y)$). We make the assumptions of Proposition 8.6.1. with $\mathrm{supp}(\underline{G})$ replaced by $\mathrm{supp}(\mathcal{M})$ and $SS(\underline{G})$ by $\mathrm{char}(\mathcal{M})$. Then :

i) $$\int_f \mathcal{M} = \int_{f_s} (\mathcal{M}|_{Y_s}), \quad \int_f^{pr} \mathcal{M} = \int_{f_s}^{pr} (\mathcal{M}|_{Y_s})$$

and these complexes belong to $\mathrm{Ob}(D^b_{rh}(\mathcal{D}_X))$.

ii) $$\mathrm{char}(\int_f \mathcal{M}) \subset \varpi\rho^{-1}(\mathrm{char}\,\mathcal{M})$$

$$\mathrm{char}(\int_f^{pr} \mathcal{M}) \subset \varpi\rho^{-1}(\mathrm{char}\,\mathcal{M})$$

iii) $$\mathbb{R}f_*(\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{O}_Y) = (\int_f \mathcal{M}) \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X$$

$$\mathbb{R}f_!(\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{O}_Y) = (\int_f^{pr} \mathcal{M}) \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{O}_X$$

Proof

Set :

$$\underline{G} = \mathbb{R}\underline{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \Omega_Y)$$

Then $\underline{G} \in \mathrm{Ob}(D^b_{\mathbb{C}-c}(Y))$ and

$$\mathcal{M} \cong (\Omega_{\overline{Y}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\overline{Y}}}{\otimes}} TH(\underline{G})) \underset{\mathcal{O}_Y}{\otimes} \Omega_Y\,[\dim Y]$$

$$\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{O}_Y = \mathbb{R}\underline{Hom}(\underline{G}, \mathbb{C}_Y)\quad [\dim Y]$$

Moreover we know by Corollary 9.3.3. that :

$$\mathrm{char}(\mathcal{M}) = SS(\underline{G}) .$$

Now we get :

$$\int_f \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \Omega_X^{\otimes -1} = \mathbb{R}f_*(\mathcal{D}_{X<-Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_Y}{\otimes}} \Omega_{\overline{Y}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\overline{Y}}}{\otimes}} \mathrm{TH}(\underline{G})) \; [\dim Y]$$

$$= \Omega_{\overline{X}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\overline{X}}}{\otimes}} \mathbb{R}f_*(\mathcal{D}_{X^{\mathbb{R}}<-Y^{\mathbb{R}}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{Y^{\mathbb{R}}}}{\otimes}} \mathrm{TH}(\underline{G})) \; [\dim Y]$$

$$= \Omega_{\overline{X}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\overline{X}}}{\otimes}} \mathrm{TH}(\mathbb{R}f_! \underline{G}) \; [\dim Y]$$

by applying Corllary 9.2.2. (and similarly for $\int_f^{pr} \mathcal{M}$) .

By Proposition 8.6.1., $\mathbb{R}f_!(\underline{G})$ is $\mathbb{C}$-constructible. Thus i) , ii) and iii) follow from the reconstruction Theorem 9.2.3. and :

$$\mathbb{R}f_* \mathbb{R}\underline{Hom}(\underline{G},\underline{\mathbb{C}}_Y) \; [2\dim Y] = \mathbb{R}\underline{Hom}(\mathbb{R}f_! \; \underline{G},\underline{\mathbb{C}}_X) \; [2\dim X]$$

$$\mathbb{R}f_! \mathbb{R}\underline{Hom}(\underline{G},\underline{\mathbb{C}}_Y) \; [2\dim Y] = \mathbb{R}\underline{Hom}(\mathbb{R}f_* \; \underline{G},\underline{\mathbb{C}}_X) \; [2\dim X] \; . \; \square$$

By the proof of Proposition 8.6.2. we get :

<u>Corollary 9.4.2.</u> : <u>Let</u> $f : Y \longrightarrow X$ <u>be a holomorphic map, with</u> $\dim X = 1$, <u>and</u> $\mathcal{M}$ <u>a regular holonomic</u> $\mathcal{D}_Y$<u>-module. Let</u> $x \in X$ <u>and</u> K <u>be a compact subset of</u> $f^{-1}(x)$. <u>Then there exist open neighborhoods</u> U <u>of</u> x, V <u>of</u> K, <u>with</u> $V \subset f^{-1}(U)$, <u>such that denoting by</u> $f_V : V \longrightarrow U$ <u>the restriction of</u> f <u>to</u> V, $\int_{f_V} \mathcal{M}$ <u>and</u> $\int_{f_V}^{pr} \mathcal{M}$ <u>belong to</u> $\mathrm{Ob}(D^b_{rh}(\mathcal{D}_U))$, <u>and conclusions ii) and iii) of Theorem 9.4.1. are satisfied, with</u> f <u>replaced by</u> f_V.

<u>Remark 9.4.3.</u> : The assumptions of Theorem 9.4.1. are clearly satisfied when f is proper on $\mathrm{supp}\,\mathcal{M}$. This case was first treaded by Kashiwara [3] (assuming moreover that f is projective) then by Laumon [1]. Another case where the assumptions are satisfied, is the case where Y is a vector bundle on X, and $\mathcal{M}$ is

"conic", that is when char($\mathcal{M}$) is contained in the hypersurface $S_Y \subset T^*Y$ canonically associated to the vector bundle structure, (S_Y is defined in Chapter 5, §1.).

Remark 9.4.4. : The relation between direct images of holonomic Modules and integration of ramified holomorphic functions is explained by Pham [1], where some examples are discussed.

Remark 9.4.5. : A non proper direct image theorem for general coherent differential Modules is proved by Houzel-Schapira [1].

§9.5. Perverse sheaves and pure sheaves

9.5.1. Let X be a complex manifold, $\underline{F} \in \mathrm{Ob}(D^b_{\mathbb{C}-c}(X))$. We do not recall here the definition of perversity, and refer to Goreski-Mac Pherson [1], but we use the fact (which may be taken as a definition here) that $\underline{F}$ is perverse if and only if, in the Riemann-Hilbert correspondence, $R\text{-}H(\underline{F})$ is a complex concentrated in degree 0 in $\mathrm{Ob}(D^b_{rh}(\mathcal{D}_X))$.

Lemma 9.5.1. : *Let Λ be a complex Lagrangean manifold in T^*X, $\underline{F} \in \mathrm{Ob}(D^+(X))$. Assume $\underline{F}$ is pure at any point $p \in \Lambda$. Then the shift of $\underline{F}$ is locally constant on Λ.*

Proof

With the same notations as in Remark 7.2.7. it is enough to show :

$$\tau(\lambda_o(p(s)),\lambda_\Lambda(p(s)),\mu(s))=\tau(\lambda_o(p(s'),\lambda_\Lambda(p(s'),\mu(s'))$$

We may assume that $\mu(s)$ is a complex Lagrangean plane. Then the lemma follows from the following remark : let (E,σ) be a complex symplectic vector space, $(E^{\mathbb{R}},\sigma^{\mathbb{R}})$ the real underlying symplectic vector space (i.e. : $\sigma^{\mathbb{R}}(x,y) = \sigma(x,y) + \overline{\sigma(x,y)}$). Then for a triplet of complex Lagrangean planes λ_1, λ_2, λ_3, we have $\tau_{E^{\mathbb{R}}}(\lambda_1,\lambda_2,\lambda_3) = 0$, since denoting par I the multiplication by $\sqrt{-1}$ on $E^{\mathbb{R}}$, we have $\sigma^{\mathbb{R}} \circ I = -\sigma^{\mathbb{R}}$. □

Theorem 9.5.2. : Let X be a complex manifold, $\underline{F} \in \mathrm{Ob}(D^b_{\mathbb{C}-c}(X))$ and let $\Lambda = SS(\underline{F})$. Then the following conditions are equivalent :

a) $\underline{F}$ is a perverse sheaf.

b) At any point of the non singular locus Λ_{reg} of Λ, $\underline{F}$ is pure with shift zero.

Proof

We set $\mathcal{M} = R\text{-}H(\underline{F})$. Consider :

a') $H^j(\mathcal{M}) = 0 \quad j \neq 0$

It is equivalent to :

a") $H^j(\mathcal{E}_X^{\mathbb{R}} \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}(\mathcal{M}))|_{\Lambda'} = 0 \qquad j \neq 0$

where Λ' is the set of points of Λ_{reg} where the projection π on X has constant rank.

Now we have $\mathcal{E}_X^{\mathbb{R}} \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M} = \mu hom(\underline{F}, \mathcal{O}_X)$, and in a neighborhood of $p \in \Lambda'$, we have $\Lambda = T^*_Y X$, for a submanifold $Y \subset X$.

Hence $\underline{F} \cong \underline{M}^\bullet_Y$, microlocally at p, for a complex $M^\bullet$ of vector spaces.

Let d be the complex codimension of Y. We have in a neighborhood of p :

$$\mu hom(\underline{F}, \mathcal{O}_X) = \bigoplus_j \mathrm{Hom}(H^j(M^\bullet), \mu_Y(\mathcal{O}_X)) \quad [j]$$

$$= \bigoplus_j \mathrm{Hom}(H^j(M^\bullet), C^{\mathbb{R}}_{Y|X}[-d]) \quad [j]$$

(since by the definition $C^{\mathbb{R}}_{Y|X} = \mu_Y(\mathcal{O}_X)[d]$) .

Thus a") is equivalent to

b') $H^j(M^\bullet) = 0$ for $j \neq d$ and this is clearly equivalent to b). $\square$

CHAPTER 10 - APPLICATION 3 - MICRODIFFERENTIAL SYSTEMS

In all this chapter, X will denote a complex manifold. We keep the notations of §9.1. concerning the sheaves $\mathcal{O}_X$, $\mathcal{D}_X$, $\mathcal{E}_X$, $\mathcal{E}_X^{\mathbb{R}}$, etc ... .

§10.1. Characteristic variety of $\mathcal{D}_X$-modules

10.1.1. Some results of this section will be generalized later on (§10.4), but the proofs concerning $\mathcal{D}_X$-modules are more elementary than that concerning $\mathcal{E}_X^{\mathbb{R}}$-modules.

10.1.2. Let $\mathcal{M} \in \mathrm{Ob}(D^b_{coh}(\mathcal{D}_X))$. Locally on X, $\mathcal{M}$ is quasi-isomorphic to a bounded complex $\mathcal{M}^\cdot$:

$$0 \longleftarrow \mathcal{D}_X^{N_o} \xleftarrow[P_o]{} \cdots \longleftarrow \mathcal{D}_X^{N_p} \longleftarrow 0 \tag{10.1.1}$$

where the P_j's are matrices of differential operators acting on the right.

We set :

$$\mathcal{M}^* = \mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \tag{10.1.2}$$

$$\mathcal{M}^{\mathbb{R}} = \mathcal{E}_X^{\mathbb{R}} \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M} \tag{10.1.3}$$

We denote by char($\mathcal{M}$) the characteristic variety of a coherent $\mathcal{D}_X$-module, and for $\mathcal{M} \in \mathrm{Ob}(D^b_{coh}(\mathcal{D}_X))$ we set :

$$\mathrm{char}(\mathcal{M}) = \bigcup_j \mathrm{char}(H^j(\mathcal{M}))$$

(Recall that the characteristic variety of a coherent $\mathcal{D}_X$-module

$\mathcal{M}$ is the support in T^*X of $gr(\mathcal{M})$, where $gr(\mathcal{M})$ is the graded Module obtained by endowing $\mathcal{M}$ with a good filtration and taking the associated graded Module).

We have :

(10.1.4) $$\operatorname{char}(\mathcal{M}) = \operatorname{char}(\mathcal{M}^*)$$

and

$$\operatorname{char}(\mathcal{M}) = \operatorname{supp}(\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M})$$

(cf. Björk [1] for **example**).

Moreover the Ring $\mathcal{E}_X^{\mathbb{R}}$ is faithfully flat over $\mathcal{E}_X$: this is proved exactly as for $\mathcal{E}_X^{\infty}$ (cf. Sato-Kashiwara-Kawaï [1,Chapter II, §3.4.]) by the division theorem in $\mathcal{E}_X^{\mathbb{R}}$ (cf. Kashiwara-Schapira [2,§6] or Aoki-Kashiwara-Kawaï [1]). Hence we get :

(10.1.5) $$\operatorname{char}(\mathcal{M}) = \operatorname{supp}(\mathcal{M}^{\mathbb{R}})$$

<u>10.1.3</u>. We shall prove :

<u>Theorem 10.1.1</u>. : <u>Let</u> $\mathcal{M} \in Ob(D^b_{coh}(\mathcal{D}_X))$. <u>Then</u> :

$$\operatorname{char}(\mathcal{M}) = SS(\mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))$$

<u>Proof</u>

i) First we prove the inclusion of the micro-support in $\operatorname{char}(\mathcal{M})$. Of course one may reduce the proof to the case where $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module.

Let $(x_o;\xi_o) \notin \operatorname{char}(\mathcal{M})$. Let ϕ be a real C^1-function with $\phi(x_o) = 0$, $d\phi(x_o) = \xi_o$. Using classical results on spectral sequences, it is enough to prove :

(10.1.6) $$\underline{Ext}^j_{\mathcal{D}_X}(\mathcal{M}, H^k_{\{\phi\geq 0\}}(\mathcal{O}_X))_{x_o} = 0 \qquad \forall j \geq 0,\ \forall k \geq 0.$$

Then, using a standard argument, we may reduce the proof of (10.1.6) to the case where $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$, for a differential operator P with $\sigma(P)(x_o;\xi_o) \neq 0$, $\sigma(P)$ denoting the principal symbol of P.

If $\xi_o = 0$, the result is clear. Assume $\xi_o \neq 0$. By the Cauchy-Kowalewski theorem the sequence :

(10.1.7) $$0 \longrightarrow \mathcal{O}_X^P \longrightarrow \mathcal{O}_X \xrightarrow[P]{} \mathcal{O}_X \longrightarrow 0$$

is exact, $\mathcal{O}_X^P$ denoting the sheaf of holomorphic solutions of the equation Pu = 0. Hence $\mathcal{O}_X^P$ is quasi-isomorphic to $\mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$.

We shall prove :

(10.1.8) $$(x_o;\xi_o) \notin SS(\mathcal{O}_X^P)$$

Applying the functor $\mathbb{R}\Gamma_{\{\phi \geq 0\}}(\cdot)$ to the exact sequence (10.1.7) it is clear that (10.1.8) will imply (10.1.6).

To prove (10.1.8) we may assume X is open in $\mathbb{C}^n$, $x_o = 0$, $\xi_o = (1,0,\dots,0)$. Let us denote by $z = (z_1,\dots,z_n)$ the coordinates on $\mathbb{C}^n$, with $z = (z_1,z')$ and set :

$$H = \{z \in \mathbb{C}^n ; \quad \operatorname{Re} z_1 \geq -\varepsilon\}$$
$$L = \{z \in \mathbb{C}^n ; \quad \operatorname{Re} z_1 = -\varepsilon\}$$
$$G = \{z \in \mathbb{C}^n ; \quad \operatorname{Im} z_1 = 0, -\operatorname{Re} z_1 \geq \delta|z'|\}$$

By the refined Cauchy-Kowalewski theorem (Leray [1]), there exists $\varepsilon > 0$, $\delta > 0$ and an open neighborhood V of 0 such that for any $x \in V$, the restriction morphism induces a quasi-isomorphism from the complex :

$$0 \longrightarrow \mathcal{O}_X((x+G) \cap H) \xrightarrow[P]{} \mathcal{O}_X((x+G) \cap H) \longrightarrow 0$$

to the complex :

$$0 \longrightarrow \mathcal{O}_X((x+G) \cap L) \xrightarrow[P]{} \mathcal{O}_X((x+G) \cap L) \longrightarrow 0$$

Since $\mathcal{O}_X(K) \cong \mathbb{R}\Gamma(K, \mathcal{O}_X)$ for any convex compact set K, we obtain :

$$\mathbb{R}\Gamma((x+G) \cap H, \mathcal{O}_X^P) \cong \mathbb{R}\Gamma((x+G) \cap L, \mathcal{O}_X^P)$$

Applying Theorem 3.1.1. we get (10.1.6).

ii) Let us prove the converse inclusion.

Let Z be another complex manifold, and let p_1 be the projection $T^*(X \times Z) \longrightarrow T^*X$. First we prove :

<u>Lemma 10.1.2.</u> : <u>We have</u> :

$$p_1(SS\, \mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\times Z})) \subset SS(\mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))$$

<u>Proof of Lemma 10.1.2.</u>

We represent $\mathcal{M}$ by a bounded complex of free $\mathcal{D}_X$-modules as in (10.1.1).

Let K and L be convex compact subsets of X and Z respectively, and let $\alpha^\bullet(K)$ and $\alpha^\bullet(K \times L)$ denote the complexes :

$$0 \longrightarrow \mathcal{O}_X(K)^{N_0} \xrightarrow[(P_0)]{} \cdots \longrightarrow \mathcal{O}_X(K)^{N_p} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{X\times Z}(K\times L)^{N_0} \xrightarrow[(P_0)]{} \cdots \longrightarrow \mathcal{O}_{X\times Z}(K\times L)^{N_p} \longrightarrow 0$$

whose cohomology groups calculate $H^\bullet(\mathbb{R}\Gamma(K; \mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)))$ and $H^\bullet(\mathbb{R}\Gamma(K\times L; \mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\times Z})))$ respectively.

Let K_1 and K_2 be two convex compact subsets of X with $K_1 \subset K_2$ and assume that the restriction mapping $\mathcal{O}(K_2) \longrightarrow \mathcal{O}(K_1)$ induces a quasi-isomorphism $\alpha^\bullet(K_2) \xrightarrow[q.is]{\sim} \alpha^\bullet(K_1)$.

Let L be a closed ball in Z. Then $\mathcal{O}_{X\times Z}(K_j \times L) = \mathcal{O}_X(K_j) \hat{\otimes} \mathcal{O}_Z(L)$, $(j = 1,2)$, where $\cdot \hat{\otimes} \cdot$ denotes the topological tensor product of the nuclear spaces (Grothendieck , [1]). By Lemma 10.1.3. below we obtain a quasi-isomorphism $\alpha^{\bullet}(K_2 \times L) \cong \alpha^{\bullet}(K_1 \times L)$ and it remains to use criteria (3) of Theorem 3.1.1. . □

Lemma 10.1.3. : Let $E^{\bullet}$ and $F^{\bullet}$ be two bounded complexes of nuclear D-F-spaces, u a linear continuous morphism from $E^{\bullet}$ to $F^{\bullet}$ which induces for all $j \in \mathbb{Z}$ an isomorphism $H^j(E^{\bullet}) \cong H^j(F^{\bullet})$. Let G be a nuclear D-F-space and let $u \otimes 1$ be the morphism from $E^{\bullet} \hat{\otimes} G$ to $F^{\bullet} \hat{\otimes} G$ associated to u. Then for all $j \in \mathbb{Z}$, $u \otimes 1$ induces an isomorphism $H^j(E^{\bullet} \hat{\otimes} G) \cong H^j(F^{\bullet} \hat{\otimes} G)$.

Proof of Lemma 10.1.3.

By considering the mapping cone of u we may reduce the lemma to the case where $F^{\bullet} = 0$. Then it is well known that in this situation, the functor $\cdot \hat{\otimes} G$ is exact (Grothendieck [1]) . □

End of the proof of Theorem 10.1.1.

We have :

$$\mathrm{char}(\mathcal{M}) = \mathrm{char}(\mathcal{M}^{*}) = \mathrm{supp}(\mathcal{M}^{*\mathbb{R}})$$

$$= \mathrm{supp}\ \mu_{\Delta}\ \mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X\times X})$$

where Δ denotes the diagonal of $X \times X$. Then one applies Theorem 5.2.1. and Lemma 10.1.2. . □

Remark 10.1.4. : i) The inclusion of the micro-support in the characteristic variety may also be obtained by noticing that if $p = (x_o;\xi_o)$, then $\mathcal{E}^{\mathbb{R}}_{X,p}$ operates on $(\mathbb{R}\Gamma_{\{\phi \geq o\}}(\mathcal{O}_X))$ where $\phi(x_o) = 0$, $d\phi(x_o) = \xi_o$, (cf. Kashiwara [5]).

ii) When replacing $\mathcal{O}_X$ by various sheaves of $\mathcal{D}_X$-modules or $\mathcal{E}_X$-modules, and the characteristic variety by suitable "micro-characteristic varieties" one can extend the inclusion $* \supset *$ of Theorem 10.1.1. (cf. Kashiwara-Schapira [1],[2], Laurent [1], Monteiro-Fernandes [1], Schapira [2] and Theorem 10.5.1. below).

§10.2. Characteristic variety of the induced systems

10.2.1. Let Y be a complex submanifold of X . Recall that the sheaf $\mathcal{D}_{Y \to X} = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is naturally endowed with a structure of a $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, or more generally an object of $D^b_{coh}(\mathcal{D}_X)$.

We set :

$$\mathcal{M}_Y^{\mathbb{L}} = \mathcal{D}_{Y \to X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M} = \mathcal{O}_Y \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{M} \tag{10.2.1}$$

In general the cohomology groups $H^j(\mathcal{M}_Y^{\mathbb{L}})$ are no more coherent over $\mathcal{D}_Y$, but locally they are a union of an increasing sequence of coherent $\mathcal{D}_Y$-modules.

We shall also consider the algebraic cohomology groups $\mathbb{R}\Gamma_{[Y]}(\mathcal{M})$ with their structure of left $\mathcal{D}_X$-modules (Kashiwara [4]).

Finally we set :

$$\mathcal{M}^{\infty} = \mathcal{D}_X^{\infty} \otimes_{\mathcal{D}_X} \mathcal{M} \tag{10.2.2}$$

<u>Theorem 10.2.1.</u> : <u>Let</u> $\mathcal{M} \in Ob(D^b_{coh}(\mathcal{D}_X))$ <u>and assume</u> :

a) $\mathcal{M}_Y^{\mathbb{L}} \in Ob(D^b_{coh}(\mathcal{D}_Y))$

b) $\mathcal{D}_X^{\infty} \otimes_{\mathcal{D}_X} (\mathbb{R}\Gamma_{[Y]}(\mathcal{M})) = \mathbb{R}\Gamma_Y(\mathcal{D}_X^{\infty} \otimes_{\mathcal{D}_X} \mathcal{M})$

<u>Then</u> :

$$\operatorname{char}(\mathcal{M}_Y^{\mathbb{L}}) \subset T^*Y \cap C_{T_Y^*X}(\operatorname{char}(\mathcal{M}))$$

Proof

Since $\mathrm{char}(\mathcal{M}) = \mathrm{char}(\mathcal{M}^*)$ we have by Theorem 10.1.1. :

$$\mathrm{char}(\mathcal{M}) = SS(\Omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}) \tag{10.2.3}$$

Set :

$$\underline{F} = \Omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}$$

Then :

$$\underline{F} = \Omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes_\infty}} \mathcal{M}^\infty$$

Let us calculate $\mathbb{R}\Gamma_Y(\underline{F})$:

$$\begin{aligned}\mathbb{R}\Gamma_Y(\underline{F}) &= \Omega_X \overset{L}{\underset{\mathcal{D}^\infty_X}{\otimes_\infty}} \mathbb{R}\Gamma_Y(\mathcal{M}^\infty) \\ &= \Omega_X \overset{L}{\underset{\mathcal{D}^\infty_X}{\otimes_\infty}} \mathcal{D}^\infty_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathbb{R}\Gamma_{[Y]}(\mathcal{M}) \\ &= \Omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathbb{R}\Gamma_{[Y]}(\mathcal{M})\end{aligned}$$

Since

$$\mathbb{R}\Gamma_{[Y]}(\mathcal{M}) = \mathcal{D}_{X \longleftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{D}_{Y \longrightarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}\ [-\ell]$$

with $\ell = \mathrm{codim}_{\mathbb{C}}(Y)$, we obtain :

$$\begin{aligned}\mathbb{R}\Gamma_Y(\underline{F}) &= \Omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{D}_{X \longleftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{M}^{\mathbb{L}}_Y\ [-\ell] \\ &= \Omega_Y \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{M}^{\mathbb{L}}_Y\ [-2\ell]\end{aligned}$$

Thus :

$$\mathrm{char}(\mathcal{M}^{\mathbb{L}}_Y) = (SS(\mathbb{R}\Gamma_Y(\underline{F})))$$

and it remains to apply Theorem 5.2.1. . □

Remark : The proof of Theorem 10.2.1. replaces the proof of Theorem 3 of Kashiwara-Schapira [4] which was not correct.

10.2.2. The hypotheses of Theorem 10.2.1. are satisfied for regular holonomic Modules (Kashiwara-Kawaï [6]). Hence we get :

Corollary 10.2.2. : i) *Let $\mathcal{M}$ be a regular holonomic system on* X. *Then* :

$$\operatorname{char}(\mathcal{M}_Y^{\mathbb{L}}) \subset T^*Y \cap C_{T_Y^*X}(\operatorname{char}(\mathcal{M}))$$

ii) *Let $\mathcal{M}$ and $\mathcal{N}$ be two regular holonomic systems on* X. *Then* :

$$\operatorname{char}(\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{N}) \subset \operatorname{char}(\mathcal{M}) \hat{+} \operatorname{char}(\mathcal{N})$$

(The structure of left $\mathcal{D}_X$-module of $\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{N}$ is explained in Kashiwara [4] ; cf. also Bernstein [1]).

Remark 10.2.3. : An exact formula for the restriction of the characteristic cycle of regular holonomic Modules is obtained by Sabbah [1].

10.2.3. : Let $(f_j)_{j=1}^{\ell}$ be holomorphic functions on X, $\lambda_j \in \mathbb{C}$ and let $\mathcal{M}$ be the left $\mathcal{D}_X$-module :

$$\mathcal{M} = \mathcal{D}_X\left(\prod_{j=1}^{\ell} f_j^{\lambda_j}\right) \tag{10.2.4}$$

that is, $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ where $\mathcal{I}$ is the left Ideal of sections P of $\mathcal{D}_X$ which satisfy $P(\prod_j f_j^{\lambda_j}) = 0$ generically.

Let us recall the result of Kashiwara-Schapira [3].

Define the subset Z of T^*X as follows :

$$(10.2.5)\begin{cases} (x;\xi) \in Z \iff \text{there exists a sequence } \{(x_p, a_{j,p})_{1\leq j\leq \ell}\}_p \\ \text{in } X \times \mathbb{C}^{\ell} \text{ such that :} \\ x_p \xrightarrow[p]{} x,\ \sum_{j=1}^{\ell} a_{j,p}\, df_j(x_p) \xrightarrow[p]{} \xi\ ;\ a_{j,p} f_k(x_p) \xrightarrow[p]{} 0 \quad \forall j,k \end{cases}$$

Corollary 10.2.4. : We have :

$$\mathrm{char}(\mathcal{M}) \subset Z$$

For the proof one introduces the left $\mathcal{D}_{X\times\mathbb{C}^\ell}$-module $\mathcal{N}$ generated by $\prod_{j=1}^{\ell} (t_j + f_j(x))^{\lambda_j}$. Then :

$\mathrm{char}(\mathcal{N}) \subset \{(x,t;\sum_j a_j(df_j(x)+dt_j));\ a_j(t_j+f_j(x))=0$ for any $j, a_j \in \mathbb{C}\}$

and we apply Corollary 10.2.2., since :

$$\mathcal{M} = \underline{\mathrm{Tor}}_0^{\mathcal{O}_{X\times\mathbb{C}^\ell}}(\mathcal{O}_{X\times\{0\}}, \mathcal{N})$$

Remark 10.2.5. : The result we obtain in Corollary 10.2.4. has to be compared to that of Kashiwara-Kawaï [5] ; cf. also Iagolnitzer [1] for another approach.

Remark 10.2.6. : When Y is a (singular) hypersurface and $\mathcal{M} = H^1_{[Y]}(\mathcal{O}_X)$, an exact formule for $\mathrm{char}(\mathcal{M})$ has been obtained (cf. Lê-Mebkhout [1], Sato-Kashiwara-Kimura-Oshima [1]).

§10.3. Singular support of $\prod_j (f_j+io)^{\lambda_j}$

10.3.1. We shall calculate the singular support (i.e. : the "analytic wave front set") of hyperfunctions of the type $\prod_j(f_j+io)^{\lambda_j}$. We refer to Kashiwara-Kawaï [5] or Lebeau [2] for another approach to this problem.

Let M be a real analytic manifold, X its complexification, n = dim M.

The sheaf C_M of Sato's microfunctions is defined by :

$$C_M = \mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M[n]$$

and its restriction to M, the zero section of T_M^*X is the sheaf B_M of Sato's hyperfunctions (cf. Chapter 9, §1).

If u is a hyperfunction on M, we denote by $sp(u)$ the section of C_M it defines all over T_M^*X, and by $SS(u)$ the support of $sp(u)$.

<u>10.3.2</u>. Let N be a submanifold of M, of codimension d, $Y \subset X$ its complexification. Let $\Lambda = T_Y^*X$ and $\Lambda^{\mathbb{R}} = T_Y^*X \cap T_M^*X$. Then Λ is a complex Lagrangean manifold in T^*X, and it is the complexification of the purely imaginary Lagrangean manifold $\Lambda^{\mathbb{R}}$ of $T_M^*X \cong \sqrt{-1}\, T^*M$. The manifold Λ is endowed with the sheaf (Chapter 9, §1) :

$$C_{Y|X}^{\mathbb{R}} = \mu_Y(\mathcal{O}_X)[d]$$

and one may define the "second microlocalization" by replacing the sheaf $\mathcal{O}_X$ on X by the sheaf $C_{Y|X}^{\mathbb{R}}$ on Λ (cf. Laurent [1], Kashiwara-Kawaï [3], Kashiwara-Laurent [1]).

One defines :

$$C_\Lambda^2 = \mu_{\Lambda^{\mathbb{R}}}(C_{Y|X}^{\mathbb{R}}) \otimes \underline{\omega}_N[n]$$

$$B_\Lambda^2 = \mathbb{R}\Gamma_{\Lambda^{\mathbb{R}}}(C_{Y|X}^{\mathbb{R}}) \otimes \underline{\omega}_N[n] = \mathbb{R}\tilde{\pi}_* C_\Lambda^2$$

where $\tilde{\pi}$ is the projection $T^*_{\Lambda^{\mathbb{R}}}(T^*X) \longrightarrow T^*X$.

There exists an injective natural homomorphism :

$$\tau : \tilde{\pi}^{-1} C_M\Big|_{\Lambda^{\mathbb{R}}} \rightarrowtail C_\Lambda^2$$

The morphism τ is obtained as follows. There are natural morphisms $\underline{\mathbb{C}}_Y \longrightarrow \underline{\mathbb{C}}_N$ and $\underline{\omega}_N \longrightarrow \underline{\omega}_M[d]$, which define :

$$\mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M \longrightarrow \mu_N(\mathcal{O}_X) \otimes \underline{\omega}_N[d] \longrightarrow \mu_Y(\mathcal{O}_X) \otimes \underline{\omega}_N[d]$$

then :

$$\mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M[n] \longrightarrow \mathbb{R}\Gamma_{T_M^*X}\, \mu_Y(\mathcal{O}_X) \otimes \underline{\omega}_N[n+d] .$$

Let $f_j(y)$ $(1 \leq j \leq r)$ be real analytic functions on N, with $\operatorname{Im} f_j \geq 0$ on N. Assume $M = \mathbb{R} \times N$, and let t be a coordinate on $\mathbb{R}$ (we also write t for its complexification on $\mathbb{C}$).

Let u be the hyperfunction on M defined as the boundary value of the holomorphic function $\prod_{j=1}^{r} (t + f_j(y))^{\lambda_j}$ from $\operatorname{Im}(t + f_j(x)) > 0$ and let v be the hyperfunction on N given by $v = \prod_j (f_j(v) + io)^{\lambda_j}$.

Then $\tau(sp(u))$ admits an asymptotic expansion (Kashiwara-Kawaï [3])

$$\tau(sp(u)) = \sum_{k,\mu} (t + io)^{\mu} (\log t + io)^{k}\, v_{k,\mu}$$

and $sp(v)$ is the coefficient of $(t + io)^{o}(\log t + io)^{o}$. Thus :

(10.3.1) $$\operatorname{supp}(sp\ v) \subset T_N^*Y \cap \operatorname{supp}(\tau(sp(u)))$$

We shall estimate this set.

Let $\Omega = \{(t,y) \in \mathbb{C} \times Y = X ;\ \operatorname{Im}(t + f_j(y)) > 0 \text{ for } j = 1,\dots,r\}$.

Then $\prod_j (t + f_j(y))^{\lambda_j}$ defines a homomorphism :

$$\underline{\mathbb{C}}_\Omega \longrightarrow \mathcal{O}_X$$

Since Ω is a tube over M, we have a canonical morphism :

$$\underline{\mathbb{C}} \longrightarrow \mu_M(\underline{\mathbb{C}}_\Omega) \otimes \underline{\omega}_M[n]$$

and $sp(u)$ is the image of $1 \in \mathbb{C}$ by :

$$\underline{\mathbb{C}} \longrightarrow \mu_M(\underline{\mathbb{C}}_\Omega) \otimes \underline{\omega}_M[n] \longrightarrow \mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M[n]$$

Consider the commutative diagram :

$$\begin{array}{ccc}
\underline{\mathbb{C}} & & \\
\downarrow & & \\
\mu_M(\underline{\mathbb{C}}_\Omega) \otimes \underline{\omega}_M[n] \,\big|_{\Lambda \mathbb{R}} & \longrightarrow & \mathbb{R}\tilde{\pi}_* \mu_{\Lambda \mathbb{R}}\, \mu_Y(\underline{\mathbb{C}}_\Omega) \otimes \underline{\omega}_N\,[n+1] \\
\downarrow & & \downarrow \\
\mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M[n] \,\big|_{\Lambda \mathbb{R}} & \longrightarrow & \mathbb{R}\tilde{\pi}_* \mu_{\Lambda \mathbb{R}}\, \mu_Y(\mathcal{O}_X) \otimes \underline{\omega}_N[n+1]
\end{array}$$

Hence we have :

$$\text{(10.3.2)} \qquad \operatorname{supp}(\tau(sp(u))) \subset \operatorname{supp} \mu_{\Lambda \mathbb{R}}\, \mu_Y(\underline{\mathbb{C}}_\Omega)$$

and we obtain by Theorem 5.2.1. :

$$\text{(10.3.3)} \qquad SS(v) \subset T^*_N Y \cap C_{T^*_Y X}(SS(\underline{\mathbb{C}}_\Omega))$$

If we set $\Omega_j = \{(t,y) \; ; \; \operatorname{Im}(t + f_j(y)) > 0\}$, then $\underline{\mathbb{C}}_\Omega = \underline{\mathbb{C}}_{\Omega_1} \overset{L}{\otimes} \cdots \overset{L}{\otimes} \underline{\mathbb{C}}_{\Omega_r}$.

Hence by Theorem 5.2.2. we have :

$$SS(\underline{\mathbb{C}}_\Omega) \subset \{(t,y \; ; \; i \sum_j a_j(dt + df_j(y)) \; ; \; a_j \geqslant 0$$
$$\operatorname{Im}(t + f_j(y)) \geqslant 0, \; a_j \operatorname{Im}(t + f_j(y)) = 0\}$$

then we can state, replacing N by M and Y by X for convenience :

<u>Theorem 10.3.1.</u> : <u>Let</u> f_j $(1 \leqslant j \leqslant r)$ <u>be real analytic functions on the real manifold</u> M, <u>such that</u> $\operatorname{Im} f_j \geqslant 0$. <u>Let</u> X <u>be a complexification of</u> M . <u>We have</u> :

$$SS(\prod_j (f_j(x) + io)^{\lambda_j}) \subset \bigcup_J Z_J$$

<u>where</u> J <u>ranges over subsets of</u> $\{1,\ldots,r\}$ <u>and</u>

$Z_J = \{(x \; ; \; i\xi) \in \sqrt{-1}\; T^*M \cong T^*_M X$; <u>there exists a sequence</u> $\{(x_p, a^j_p)_{j \in J}\}_p$ <u>in</u> $X \times \mathbb{R}^J$ <u>such that</u> $a^j_p \geqslant 0$, $x_p \xrightarrow[p]{} x$,

$\sum_{j \in J} a_p^j \, df_j(x_p) \xrightarrow[p]{} \xi$, $f_j(x) = 0$, $\operatorname{Im} f_j(x_p) = \operatorname{Im} f_{j'}(x_p)$ <u>and</u> $a_p^j \operatorname{Im} f_j(x_p) \xrightarrow[p]{} 0$ <u>for any</u> j , $j' \in J\}$.

§10.4. Characteristic variety of $\mathcal{E}_X^{\mathbb{R}}$-modules

<u>10.4.1</u>. We briefly recall, following Kashiwara-Schapira [2 , §3, especially Corollary 3.2.5.], the action of $\mathcal{E}_X^{\mathbb{R}}$ over $\mathcal{O}_X$. We take $X = \mathbb{C}^n$ and consider a closed convex proper cone G containing 0. An open set D is said to be G-round if $(D+G) \cap (D+G^a) = D$. Then for a G-round open set D, the ring $\mathcal{E}(G ; D)$ is defined by :

$$\mathcal{E}(G ; D) = H_Z^n(D \times D ; \mathcal{O}_{X \times X}^{(0,n)})$$

with $Z = \{(x,y) \in X \times X ; y-x \in G\}$.

Then there exists a natural ring homomorphism :

$$\mathcal{E}(G ; D) \longrightarrow \Gamma(D \times \operatorname{Int} G^{ao} ; \mathcal{E}_X^{\mathbb{R}}) \tag{10.4.1}$$

and for $p = (x_o ; \xi_o) \in T^*X$ we have :

$$\mathcal{E}_{X,p}^{\mathbb{R}} = \varinjlim_{G,D} \mathcal{E}(G ; D) \tag{10.4.2}$$

where G ranges over the set of closed convex proper cones such that $G \subset \{\gamma ; \langle\gamma,\xi_o\rangle < 0\} \cup \{0\}$ and D ranges over the set of G-round open neighborhoods of x_o .

The action of $\mathcal{E}_X^{\mathbb{R}}$ on $\mathcal{O}_X$ is described by the following :

<u>Theorem 10.4.1</u>. : <u>Let</u> x_o <u>belong to the G-round open set</u> D. <u>Then there exists a G-round open neighborhood</u> U <u>of</u> x_o <u>such that for any G-open sets</u> Ω_o, Ω_1 <u>with</u> $\Omega_o \subset \Omega_1$, $\Omega_1 \setminus \Omega_o \subset U$, <u>we can define naturally</u> $\mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_o}(\mathbb{R}\phi_{G^*}(\mathcal{O}_X))$ <u>in the derived category of the category of sheaves of left</u> $\mathcal{E}(G ; D)$-<u>Modules defined on</u> $\Omega_{1,G}$.

10.4.2. Now we consider a bounded complex M of left free $\mathcal{E}(G\,;\,D)$-modules of finite rank :

$$(10.4.3) \qquad M : 0 \longrightarrow \mathcal{E}(G;D)^{N_q} \longrightarrow \dots \longrightarrow \mathcal{E}(G;D)^{N_p} \longrightarrow 0$$

and we denote by $\mathcal{M}$ the complex $\mathcal{E}_X^{\mathbb{R}} \underset{\mathcal{E}(G;D)}{\otimes} M$ over $D \times G^{oa}$:

$$(10.4.4) \qquad \mathcal{M} : 0 \longrightarrow (\mathcal{E}_X^{\mathbb{R}})^{N_q} \longrightarrow \dots \longrightarrow (\mathcal{E}_X^{\mathbb{R}})^{N_p} \longrightarrow 0$$

Recall that $\mathrm{supp}(\mathcal{M})$ is the complementary in $D \times G^{oa}$ of the set of points where the germ of $\mathcal{M}$ is exact.

Let Z be another manifold. We denote by π_Z the projection $X \times Z \longrightarrow X$ or $T^*(X \times Z) \longrightarrow T^*X$.

Let ϕ_Z be the continuous map $X \times Z \longrightarrow X_G \times Z$.

Then $\mathbb{R}\Gamma_{(\Omega_1 \setminus \Omega_o) \times Z}(\mathbb{R}\phi_{Z*}\, \mathcal{O}_{X \times Z})$ is also in the derived category of the category of $\mathcal{E}(G,D)$-Modules over $\Omega_{1,G} \times Z$.

<u>Theorem 10.4.2.</u> : <u>Let</u> Ω_o <u>and</u> Ω_1 <u>be two G-open sets in</u> X <u>with</u> $\Omega_o \subset \Omega_1$, $\Omega_1 \setminus \Omega_o \subset U$. Let $V = \mathrm{Int}(\Omega_1 \setminus \Omega_o) \times \mathrm{Int}(G^{oa})$. <u>Then</u> :

$$\mathrm{supp}\, \mathcal{M} \cap V = \bigcup_Z \pi_Z(SS(\mathbb{R}\underline{Hom}_{\mathcal{E}(G;D)}(M, \phi_Z^{-1}\mathbb{R}\Gamma_{(\Omega_1 \setminus \Omega_o)\times Z}\, \mathbb{R}\phi_{Z*}\, \mathcal{O}_{X\times Z}))) \cap V$$

<u>where</u> Z <u>runs over the set of complex manifolds</u>.

<u>Proof</u>

i) Assume $\mathcal{M}$ exact on a neighborhood W of $(x_o;\xi_o) \in V$. We first prove :

$$(10.4.5) \qquad \pi_Z^{-1}(W) \cap SS(\mathbb{R}\underline{Hom}_{\mathcal{E}(G;D)}(M, \phi_Z^{-1}\mathbb{R}\Gamma_{(\Omega_1 \setminus \Omega_o)\times Z}\mathbb{R}\phi_{Z*}\, \mathcal{O}_{X\times Z})) = 0$$

Since M is exact on $\pi_Z^{-1}(W)$ we may replace $X \times Z$ by X .

Let ψ be a real C^1-function with $d\psi(x_o) = \xi_o$. We shall assume $\xi_o \neq 0$, otherwise the proof is trivial.

If we set $S = \{x \,;\, \psi(x) \geq \psi(x_o)\}$, $\mathbb{R}\Gamma_S(\mathcal{O}_X)_{x_o}$ is an $\mathcal{E}^{\mathbb{R}}_{(x_o,\xi_o)}$-module. Moreover :

$$\mathbb{R}\Gamma_S(\mathcal{O}_X)_{x_o} = \mathbb{R}\Gamma_S(\phi^{-1}\,\mathbb{R}\Gamma_{\Omega_1\setminus\Omega_o}\,\mathbb{R}\phi_*\,\mathcal{O}_X)_{x_o}$$

and this isomorphism is compatible to the map $\mathcal{E}(G;D) \longrightarrow \mathcal{E}^{\mathbb{R}}_{(x_o,\xi_o)}$.

Therefore we have :

$$\begin{aligned}&\mathbb{R}\Gamma_S\,\mathbb{R}\underline{Hom}_{\mathcal{E}(G;D)}(M,\phi^{-1}\,\mathbb{R}\Gamma_{\Omega_1\setminus\Omega_o}\,\mathbb{R}\phi_*\,\mathcal{O}_X)_{x_o}\\ &= \mathbb{R}\underline{Hom}_{\mathcal{E}^{\mathbb{R}}_{(x_o,\xi_o)}}(\mathcal{E}^{\mathbb{R}}_{(x_o,\xi_o)} \underset{\mathcal{E}(G;D)}{\otimes} M,\ \mathbb{R}\Gamma_S(\mathcal{O}_X)_{x_o})\end{aligned}$$

which vanishes by the assumption.

Since this holds at any point $(x;\xi)$ in W , (10.4.3) follows.

ii) To prove the converse inclusion, set :

$$X' = \mathrm{Int}(\Omega_1\setminus\Omega_o)$$

ϕ the continuous map $X \times X' \longrightarrow X_G \times X'$

$$\underline{F} = \phi^{-1}\,\mathbb{R}\Gamma_{(\Omega_1\setminus\Omega_o)\times X'}\,\mathbb{R}\phi_*(\mathcal{O}_{X\times X'})$$

and remark :

$$\mu_{X'}(\mathcal{O}_{X'\times X'})|_V = \mu_{X'}(\underline{F})|_V$$

(where X' is identified with the diagonal of $X' \times X'$).

Thus on V :

$$\begin{aligned}\mathrm{supp}(\mathbb{R}\underline{Hom}_{\mathcal{E}(G;D)}(M,\mathcal{E}^{\mathbb{R}}_X)) &\subset SS(\mathbb{R}\underline{Hom}_{\mathcal{E}(G,D)}(M,\underline{F})) \cap T^*_{X'}(X'\times X')\\ &\subset p_1\ SS(\mathbb{R}\underline{Hom}_{\mathcal{E}(G,D)}(M,\underline{F}))\end{aligned}$$

(where p_1 denotes the projection from $T^*(X'\times X') = T^*X' \times T^*X'$ on the first factor).

Finally we remark that :

(10.4.6) $$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{E}(G;D)}(M, \mathcal{E}_X^{\mathbb{R}}) = \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X^{\mathbb{R}}}(\mathcal{M}, \mathcal{E}_X^{\mathbb{R}})$$

and the support of the cohomology of this last complex is the same as the support of the cohomology of $\mathcal{M}$, because

(10.4.4) $$\mathcal{M} = \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X^{\mathbb{R}}}(\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X^{\mathbb{R}}}(\mathcal{M}, \mathcal{E}_X^{\mathbb{R}}), \mathcal{E}_X^{\mathbb{R}}) . \square$$

Corollary 10.4.3. : i) Let $\mathcal{M}$ be a bounded complex of free left $\mathcal{E}_X^{\mathbb{R}}$-modules of finite rank on an open set U of T^*X . Then $\mathrm{supp}(\mathcal{M})$ is involutive in $T^*(X^{\mathbb{R}})$ (in the sense of Theorem 6.4.1.).

ii) Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module on U. Then $\mathrm{supp}(\mathcal{M})$ is involutive in T^*X .

Proof

The second assertion follows from the first one since $\mathcal{E}_X^{\mathbb{R}}$ is faithfully flat over $\mathcal{E}_X$ and any coherent $\mathcal{E}_X$-module is locally quasi-isomorphic to a bounded complex of free $\mathcal{E}_X$-modules of finite rank.

The first assertion follows from Theorem 6.4.1. since the projection $\pi_Z(S)$ of an involutive set in $T^*((X \times Z)^{\mathbb{R}})$ is involutive in $T^*(X^{\mathbb{R}})$. $\square$

Remark 10.4.4. : i) In Kawaï [1] we see an example of a bounded complex $\mathcal{M}$ of free $\mathcal{D}_X^{\infty}$-Modules of finite rank such that $\mathrm{char}(\mathcal{M}) = \mathrm{supp}(\mathcal{E}_X^{\mathbb{R}} \otimes_{\mathcal{D}_X^{\infty}} \mathcal{M})$ is a real Lagrangean subset.
The same article implies the following : let M be a real analytic manifold and X its complexification. Then the sheaf of microfunctions admits locally a finite resolution by locally free $\mathcal{E}_X^{\mathbb{R}}$-Modules of finite rank.

ii) Of course, assertion ii) of Corollary 10.4.3. is well-known (cf. Sato-Kashiwara-Kawaï [1]).

§10.5. Propagation theorem and Cauchy problem

10.5.1. Let X be a complex manifold of dimension n, Y a real submanifold of class C^{α} $(\alpha \geq 2)$.

The complex $\mu_Y(\mathcal{O}_X)$ is naturally endowed with a structure of a left $\mathcal{E}_X^{\mathbb{R}}$-module. In fact we may assume X is open in $\mathbb{C}^n$, and let G, D, Ω_o, Ω_1, be as in Theorem 10.4.1. . Set $V = \mathrm{Int}(\Omega_1 \setminus \Omega_o) \times \mathrm{Int}\, G^{oa}$. We have :

$$(10.5.1) \qquad \mu_Y(\mathcal{O}_X)|_V \simeq (\mu_Y(\phi_G^{-1}\, \mathbb{R}\Gamma_{(\Omega_1\setminus\Omega_o)}\, \mathbb{R}\phi_{G*}(\mathcal{O}_X)))|_V$$

and we may apply Theorem 10.4.1. .

10.5.2. Now let M and $\mathcal{M}$ be defined as in (10.4.3) and (10.4.4) respectively. We know by Theorem 10.4.2. that $SS(\mathbb{R}\underline{Hom}_{\mathcal{E}(G;D)}(M, \phi_G^{-1}\mathbb{R}\Gamma_{(\Omega_o\setminus\Omega_1)}\mathbb{R}\phi_{G*}(\mathcal{O}_X)))$ is contained in $\mathrm{supp}\,\mathcal{M} \cap V$. Hence we get by Theorem 5.2.1. :

<u>Theorem 10.5.1.</u> : <u>Let $\mathcal{M}$ be a bounded complex of free $\mathcal{E}_X^{\mathbb{R}}$-modules of finite rank on an open set U of T^*X. Then</u> :

$$SS(\mathbb{R}\underline{Hom}_{\mathcal{E}_X^{\mathbb{R}}}(\mathcal{M}, \mu_Y(\mathcal{O}_X))) \subset C_{T_Y^*X}(\mathrm{supp}(\mathcal{M})) \; .$$

<u>Remark 10.5.2.</u> : This theorem generalizes Theorem 6.3.1. of Kashiwara-Schapira [2] which made some assumptions on the submanifold Y. When Y is real analytic and X is a complexification of Y, we recover the theorem of propagation of micro-analyticity for solutions of micro-hyperbolic systems (Kashiwara-Schapira, loc. cit.). Let us recall that this result was first obtained for single differential operators (for hyperfunction solutions) by Bony-Schapira [2] then extended to single microdifferential operators by Kashiwara-Kawaï [2].

Remark 10.5.3. : As a particular case of Theorem 10.5.1. we find :

$$(10.5.2) \qquad \operatorname{supp} \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X^{\mathbb{R}}}(\mathcal{M}, \mu_Y(\mathcal{O}_X)) \subset T_Y^*X \cap \operatorname{supp}(\mathcal{M}) .$$

Assume now that $\mathcal{M}$ is defined all over T^*X ($\mathcal{M}$ is a bounded complex of free $\mathcal{D}_X^\infty$-modules of finite rank), and assume Y is non characteristic for $\mathcal{M}$, that is, $T_Y^*X \cap \operatorname{supp} \mathcal{M} \subset T_X^*X$. Applying Proposition 5.3.2. we obtain the isomorphism :

$$(10.5.3) \qquad \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X^\infty}(\mathcal{M}, \mathcal{O}_X)|_Y \simeq \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X^\infty}(\mathcal{M}, \mathbb{R}\Gamma_Y(\mathcal{O}_X) \otimes \omega_{Y,X}[d])$$

where $d = \operatorname{codim} Y$.

Remark that if Y is real analytic and X is a complexification of Y, the hypothesis that Y is non characteristic is often translated as "$\mathcal{M}$ is elliptic" , and (10.5.3) asserts that the hyerfunction solutions of the system are real analytic functions.

10.5.3. One can also recover Theorem 2.3.1. of Kashiwara-Schapira [2] on the Cauchy problem for microfunction solutions of microhyperbolic systems. For sake of simplicity we shall only give the proof for differential systems. Let $f : N \longrightarrow M$ be a map of real analytic manifolds, and let us denote by $f : Y \longrightarrow X$ a complexification of f. We denote as usual by ρ and ϖ the associated maps from $Y \times_X T^*X$ to T^*Y and T^*X respectively.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, $\mathcal{M}_Y = \mathcal{D}_{Y \to X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}$ the induced system on Y. Assume first f is non characteristic for $\mathcal{M}$ (i.e. : $T_Y^*X \cap \varpi^{-1}(\operatorname{char}(\mathcal{M})) \subset Y \times_X T_X^*X$). Then one knows (Kashiwara [1]) that one has a natural isomorphism :

$$(10.5.4) \qquad f^{-1}\, \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$$

Theorem 10.5.4. : Let V be an open subset of $T^*_N Y$. Assume :

i) f is non characteristic for $\mathcal{M}$.

ii) f is micro-hyperbolic for $\mathcal{M}$ on V, that is the map $\varpi_M : \rho^{-1}(V) \cap N \underset{M}{\times} T^*_M X \longrightarrow T^*_M X$ is non characteristic for $C_{T^*_M X}(\mathrm{char}(\mathcal{M}))$.

iii) $\rho^{-1}(V) \cap \varpi^{-1}(\mathrm{char}(\mathcal{M})) \subset Y \underset{X}{\times} T^*_M X$.

Then the natural morphism :

$$\rho_* \varpi^{-1} \mathbb{R}\underline{Hom}_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, C_M) \longrightarrow \mathbb{R}\underline{Hom}_{\pi^{-1}\mathcal{D}_Y}(\pi^{-1}\mathcal{M}_Y, C_N)$$

is an ismorphism on V.

The sheaves C_M and C_N of Sato's microfunctions on M and N respectively, are defined in Chapter 9, §1.

Proof

Let us apply Theorem 5.4.1. to the complex $\underline{F} \underset{\mathrm{def}}{=} \mathbb{R}\underline{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. We get :

$$\mu_N(f^{-1}\underline{F}) \overset{L}{\otimes} f^! \underline{\mathbb{Z}}_X \Big|_V \simeq \mathbb{R}\rho_* \varpi^{-1} \mu_M(\underline{F}) \overset{L}{\otimes} \varpi^{-1} \underline{\mathbb{Z}}_{N \underset{M}{\times} T^*_M X}\Big|_V$$

We have :

$$f^! \underline{\mathbb{Z}}_X \simeq \underline{\mathbb{Z}}_Y \,[\dim_{\mathbb{R}} Y - \dim_{\mathbb{R}} X]$$

$$\varpi^{-1}\underline{\mathbb{Z}}_{N \underset{M}{\times} T^*_M X} \simeq \underline{\mathbb{Z}}_N \,[\dim N - \dim M] \otimes \omega_{N/M}$$

Hence :

$$\mu_N(f^{-1}\underline{F})\Big|_V \otimes \omega_N \,[\dim N] \simeq \mathbb{R}\rho_* \varpi^{-1} \mu_M(F)\Big|_V \otimes \omega_M \,[\dim M] .$$

Applying the isomorphism (10.5.4) we get the result. □

Remark 10.5.5. : In case $\mathcal{M}$ is an $\mathcal{E}_X$-module, one has to use

Theorem 3.1.2. of Bony-Schapira [3] instead of (10.5.4).

Remark 10.5.6. : Here again, this theorem was first obtained by Bony-Schapira [2] for single differential operators, then by Kashiwara-Kawaï [2] for single microdifferential operators.

§10.6. Microlocal action of $\mathcal{E}_X^{\mathbb{R}}$ on $\mathcal{O}_X$

10.6.1. Let X be an n-dimensional complex manifold, and let q_j be the j-th projection from $X \times X$ to X $(j = 1,2)$.

Lemma 10.6.1. : We have :

$$\underline{\mathrm{Ext}}^j(q_2^{-1}\mathcal{O}_X , q_1^! \mathcal{O}_X) = 0 \quad \text{for} \quad j \neq -n$$

and we have canonical homomorphisms :

$$\mathcal{O}_{X\times X}^{(o,n)} \longrightarrow \underline{\mathrm{Ext}}^{-n}(q_2^{-1}\mathcal{O}_X , q_1^! \mathcal{O}_X)$$

$$\mathcal{O}_{X\times X}^{(o,n)} \longrightarrow \underline{\mathrm{Ext}}^{-n}(q_1^{-1}\Omega_X , q_2^! \Omega_X)$$

Here, $\underline{\mathrm{Hom}}(\cdot,\cdot)$ or $\underline{\mathrm{Ext}}(\cdot,\cdot)$ are taken over the ring $\mathbb{C}$.

Proof

For pseudo-convex open sets U and V of X, we have, by the Poincaré duality (Corollary 1.3.2.) :

$$\mathbb{R}\Gamma(U\times V, \mathbb{R}\underline{\mathrm{Hom}}(q_2^{-1}\mathcal{O}_X, q_1^! \mathcal{O}_X) \simeq \mathrm{Hom}(\mathbb{R}\Gamma_c(V,\mathcal{O}_X), \mathbb{R}\Gamma(U,\mathcal{O}_X))$$

$$\simeq \mathrm{Hom}(H_c^n(V,\mathcal{O}_X), H^o(U,\mathcal{O}_X)) \; [n]$$

Now we have morphisms :

$$(10.6.1) \qquad \mathcal{O}_{X\times X}^{(o,n)}(U\times V) \times H_c^n(V,\mathcal{O}_X) \longrightarrow H^n(U;\mathbb{R}\tilde{q}_{1!}(\mathcal{O}_{X\times X}^{(o,n)}))$$
$$\longrightarrow \mathcal{O}_X(U)$$

where the last arrow comes from the Serre duality :
$\mathbb{R}\tilde{q}_{1!}\mathcal{O}^{(o,n)}_{X\times X}[n] \longrightarrow \mathcal{O}_X$.

The map (10.6.1) being functorial in U and V , we obtain the morphism :

$$\mathcal{O}^{(o,n)}_{X\times X} \longrightarrow \underline{\mathrm{Ext}}^{-n}(q_2^{-1}\mathcal{O}_X, q_1^{!}\mathcal{O}_X)$$

The second morphism is similarly obtained. □

By Lemma 10.6.1. , we obtain :

(10.6.2) $$\mathcal{O}^{(o,n)}_{X\times X}[n] \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}(q_2^{-1}\mathcal{O}_X, q_1^{!}\mathcal{O}_X)$$

Applying the functor $\mu_\Delta(\cdot)$ to both terms in (10.6.2) we have :

<u>Proposition 10.6.2.</u> : <u>There exist canonical morphisms</u>

$$\mathcal{E}^{\mathbb{R}}_X \longrightarrow \mu\hom(\mathcal{O}_X, \mathcal{O}_X)$$
$$(\mathcal{E}^{\mathbb{R}}_X)^a \longrightarrow \mu\hom(\Omega_X, \Omega_X)$$

Thus we again obtain the microlocal action of $\mathcal{E}^{\mathbb{R}}_X$ on $\mathcal{O}_X$:

<u>Corollary 10.6.3.</u> : <u>Let</u> $p \in T^*X$. <u>There exists a natural morphism of rings</u> :

$$\mathcal{E}^{\mathbb{R}}_{X,p} \longrightarrow \mathrm{Hom}_{D^+(X;p)}(\mathcal{O}_X, \mathcal{O}_X) .$$

<u>Proof</u>

The morphism is defined by Propositions 10.1.2. and 6.1.2. . The verification that this morphism is a morphism of rings is left to the reader . □

<u>Remark 10.6.4.</u> : We have got another interpretation of Theorem 10.4.1.

<u>Remark 10.6.5.</u> : Cf. Mebkhout [4] for a construction similar to that of Lemma 10.6.1.

CHAPTER 11 - QUANTIZED CONTACT TRANSFORMATIONS FOR THE SHEAF $\mathcal{O}_X$

§11.1. Construction of quantized contact transformations

11.1.1. Let X and Y be two complex manifolds with the same dimension n. We denote by q_1 and q_2 the first and the second projection from $X \times Y$ respectively, and by p_1 and p_2 the first and the second projection from $T^*(X\times Y) = T^*X \times T^*Y$ respectively. We set $p_1^a = p_1 \circ a$, where a is the anti-podal map on T^*X.

Let ϕ be a holomorphic contact transformation between two conic open sets $\Omega_X \subset T^*X$ and $\Omega_Y \subset T^*Y$, and let Λ be the complex conic Lagrangean manifold associated to ϕ, obtained by taking the image of the graph of ϕ by the anti-podal map on T^*X.

Let $p \in \Lambda$, $p_X = p_1^a(p) \in \Omega_X$, $p_Y = p_2(p) \in \Omega_Y$. Let $\underline{K} \in Ob(D^b_{\mathbb{R}-c}(X\times Y))$ which satisfies :

(11.1.1) $\underline{K}$ is a simple sheaf along Λ with shift 0

(11.1.2)
$$\begin{cases} (p_1^a)^{-1}(\Omega_X) \cap SS(\underline{K}) \subset \Lambda \\ p_2^{-1}(\Omega_Y) \cap SS(\underline{K}) \subset \Lambda \end{cases}$$

First we shall construct a morphism :

$$\psi_{\underline{K}[n]}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y \quad \text{in} \quad D^+(Y;p_Y)$$

We have by the definition :

$$\psi_{\underline{K}[n]}(\mathcal{O}_X) \cong \mathbb{R}q_{2!}(\underline{K}[n] \overset{\mathbb{L}}{\otimes} q_1^{-1}\mathcal{O}_X)$$

Let us take an element :

$$s \in H^0(\mu hom(\underline{K}, \mathcal{O}^{(n,0)}_{X\times Y}))_p$$

Applying Proposition 6.1.2., we find that, shrinking Ω_X and Ω_Y if necessary, s gives a morphism in $D^+(X \times Y ; \Omega_X^a \times \Omega_Y)$:

$$s : \underline{K} \longrightarrow \mathcal{O}_{X\times Y}^{(n,o)}$$

Thus we get a morphism, in $D^+(Y;\Omega_Y)$:

$$\begin{aligned}\psi_{\underline{K}[n]}(\mathcal{O}_X) &\longrightarrow \mathbb{R}q_{2!}(\mathcal{O}_{X\times Y}^{(n,o)}[n] \overset{\mathbb{L}}{\otimes} q_1^{-1}\mathcal{O}_X) \\ &\longrightarrow \mathbb{R}q_{2!}(\mathcal{O}_{X\times Y}^{(n,o)}[n]) \\ &\longrightarrow \mathcal{O}_Y ,\end{aligned}$$

where the last arrow comes from the Serre duality.

We shall denote the morphism from $\psi_{\underline{K}[n]}(\mathcal{O}_X)$ to $\mathcal{O}_Y$ so constructed, by $\alpha(s)$:

(11.1.3) $\qquad \alpha(s) = \psi_{\underline{K}[n]}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y$ in $D^+(Y;\Omega_Y)$.

Remark that $H^o(\mu hom(\underline{K}, \mathcal{O}_{X\times Y}^{(n,o)}))_p$ has a structure of $(\mathcal{E}_{X,p_X}^{\mathbb{R}}, \mathcal{E}_{Y,p_Y}^{\mathbb{R}})$-bi-module by Proposition 10.6.2. .

<u>Theorem 11.1.1.</u> : <u>There exists an</u> $s \in H^o(\mu hom(\underline{K}, \mathcal{O}_{X\times Y}^{(n,o)}))_p$ <u>which satisfies the following conditions</u> :

(1) $\alpha(s) : \psi_{\underline{K}[n]}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y$ <u>is an isomorphism in</u> $D^+(Y;p_Y)$.

(2) $Ps = sQ$ $(P \in \mathcal{E}_{X,p_X}^{\mathbb{R}}, Q \in \mathcal{E}_{Y,p_Y}^{\mathbb{R}})$ <u>gives a ring isomorphism</u>

$$\phi_* : \mathcal{E}_{X,p_X}^{\mathbb{R}} \xrightarrow{\sim} \mathcal{E}_{Y,p_Y}^{\mathbb{R}}$$

(3) $\alpha(s)$ <u>is compatible with the action of</u> $\mathcal{E}_{X,p_X}^{\mathbb{R}}$ <u>and</u> $\mathcal{E}_{Y,p_Y}^{\mathbb{R}}$ <u>on</u> $\mathcal{O}_X$ <u>and</u> $\mathcal{O}_Y$.

We shall prove this result at the end of this section.

Remark 11.1.2. : The isomorphism $\alpha(s)$ is not unique, but unique up to the multiplication by an invertible microdifferential operator. This is the same as the situation encountered in the theory of microdifferential equations (cf. Sato-Kashiwara-Kawaï [1]). We call $\alpha(s)$ a quantized contact transformation above ϕ .

11.1.2. In order to prove the theorem, we shall study the composition of such morphisms.

Let X_j $(j = 1,2,3)$ be three complex manifolds of the same dimension n, $X_{ij} = X_i \times X_j$, $X_{123} = X_1 \times X_2 \times X_3$. We denote by q_{ij} the projection from X_{123} to X_{ij}, by p_{ij} the projection from T^*X_{123} to T^*X_{ij} , by p_i any of the projections from T^*X_{ij} to T^*X_i. We denote by p^a_{ij} the map obtained by composing p_{ij} with the anti-podal map on T^*X_j.

Let $\underline{K}_{12}$ and $\underline{K}_{23}$ be objects of $D^b_{\mathbb{C}-c}(X_{12})$ and $D^b_{\mathbb{C}-c}(\underline{K}_{23})$ respectively both with shift 0.

Let $\mathcal{M}_{ij}$ be the regular holonomic $\mathcal{D}_{X_{ij}}$-module associated to $\underline{K}_{ij}$ by the Riemann-Hilbert correspondence (cf. Theorem 9.2.3.). We set :

$$\mathcal{M}_{ij} = \text{R-H}(\underline{K}_{ij})$$

$$\mathcal{M}^{\mathbb{R}}_{ij} = \mathcal{E}^{\mathbb{R}}_{X_{ij}} \underset{\pi^{-1}\mathcal{D}_{X_{ij}}}{\otimes} \pi^{-1}\mathcal{M}_{ij}$$

Thus :

$$\underline{K}_{ij} = \mathbb{R}\underline{Hom}_{\mathcal{D}_{X_{ij}}}(\mathcal{M}_{ij}, \mathcal{O}_{X_{ij}})$$

$$\mu hom(\underline{K}_{ij}, \mathcal{O}_{X_{ij}}) = \mathcal{M}^{\mathbb{R}}_{ij}$$

Now let Ω_{13} be an open subset of T^*X_{13}, let U be a relatively compact subanalytic open subset of X_2 and Ω_2 an open subset of T^*U. Assume :

$$(11.1.4)\quad \begin{cases} p_{13}^{a-1}(\Omega_{13}) \cap p_{12}^{a-1}(SS(\underline{K}_{12})) \cap p_{23}^{a-1}(SS(\underline{K}_{23})) \subset T^*X_1 \times \Omega_2 \times T^*X_3 \\ \text{and the set on the left-hand side is isomorphic to } \Omega_{13} \\ \text{by } p_{13}^a . \end{cases}$$

Assume moreover that we have isomorphisms :

$$(11.1.5)\quad \begin{cases} SS(\underline{K}_{12}) \cap T^*X_1 \times \Omega_2^a \xrightarrow{\sim} \Omega_2^a \\ SS(\underline{K}_{23}) \cap \Omega_2 \times T^*X_3 \xrightarrow{\sim} \Omega_2 \end{cases}$$

We set $U' = X_1 \times U \times X_2$.

Proposition 11.1.3. : Under the assumptions (11.1.4), (11.1.5) we have a commutative diagram in $D^+(\Omega_{13})$, and an isomorphism γ :

$$\begin{array}{ccc} A & \xrightarrow{\sim} & B \\ \lambda\downarrow & & \downarrow\mu \\ C & \xrightarrow[\gamma]{\sim} & D \end{array}$$

$$A : \mathbb{R}p_{13*}(p_{12}^{a-1}\,\mu hom(\underline{K}_{12}, \mathcal{O}_{X_{12}}^{(o,n)}) \underset{\mathbb{C}}{\otimes} p_{23}^{-1}\,\mu hom(\underline{K}_{23}, \mathcal{O}_{X_{23}}^{(o,n)}))\Big|_{\Omega_{13}}$$

$$B : \mathbb{R}p_{13*}^a(p_{12}^{a-1}\,\mathcal{M}_{12}^{\mathbb{R}(o,n)} \underset{\mathbb{C}}{\otimes} p_{23}^{a-1}\,\mathcal{M}_{23}^{\mathbb{R}(o,n)})\Big|_{\Omega_{13}}$$

$$C : \mu hom(\mathbb{R}q_{13*}(q_{12}^{-1}\underline{K}_{12} \underset{\mathbb{C}}{\otimes} q_{23}^{-1}\,\underline{K}_{23})_{U'}[n], \mathcal{O}_{X_{13}}^{(o,n)})\Big|_{\Omega_{13}}$$

$$D : \mathbb{R}p_{13*}^a(p_{12}^{a-1}\,\mathcal{M}_{12}^{\mathbb{R}(o,n)} \underset{\mathcal{E}_{X_2}^{\mathbb{R}}}{\otimes} p_{23}^{a-1}\,\mathcal{M}_{23}^{\mathbb{R}(o,n)})\Big|_{\Omega_{13}}$$

Here $\mathcal{O}_{X_{ij}}^{(o,n)} = \mathcal{O}_{X_{ij}} \underset{q_j^{-1}\mathcal{O}_{X_j}}{\otimes} \Omega_{X_j}^{(n)}$ and $\mathcal{M}_{ij}^{\mathbb{R}(o,n)} = \mathcal{O}_{X_{ij}}^{(o,n)} \underset{\mathcal{O}_{X_{ij}}}{\otimes} \mathcal{M}_{ij}^{\mathbb{R}}$

Proof

We set $X_{1223} = X_{12} \times X_{23}$ and denote by $\tilde{q}_1$ and $\tilde{q}_2$ the projections from X_{1223} to X_{12} and X_{23} respectively, and by $\tilde{p}_1$ and $\tilde{p}_2$ the projections from T^*X_{1223} to T^*X_{12} and T^*X_{23} respectively.

Let $\mathcal{M}_{12} \hat{\otimes} \mathcal{M}_{23}$ be the coherent $\mathcal{D}_{X_{1223}}$-module:

$$\mathcal{M}_{12} \hat{\otimes} \mathcal{M}_{23} = \mathcal{D}_{X_{1223}} \otimes_{(\tilde{q}_1^{-1} \mathcal{D}_{X_{12}} \otimes \tilde{q}_2^{-1} \mathcal{D}_{X_{23}})} (\tilde{q}_1^{-1} \mathcal{M}_{12} \otimes \tilde{q}_2^{-1} \mathcal{M}_{23})$$

We have :

$$\text{(11.1.6)} \quad \mathbb{R}\underline{Hom}_{\mathcal{D}_{X_{1223}}} (\mathcal{M}_{12} \hat{\otimes} \mathcal{M}_{23} , \mathcal{O}_{X_{1223}}) \cong \tilde{q}_1^{-1} \underline{K}_{12} \otimes \tilde{q}_2^{-1} \underline{K}_{23}$$

Hence we have a commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{\sim} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{\sim} & D \end{array}$$

A : $\tilde{p}_1^{-1}\ \mu hom(\underline{K}_{12}, \mathcal{O}_{X_{12}}) \otimes \tilde{p}_2^{-1}\ \mu hom(\underline{K}_{23}, \mathcal{O}_{X_{23}})$

B : $\tilde{p}_1^{-1} \mathcal{M}_{12}^{\mathbb{R}} \otimes \tilde{p}_2^{-1} \mathcal{M}_{23}^{\mathbb{R}}$

C : $\mu hom(\tilde{q}_1^{-1}\ \underline{K}_{12} \otimes \tilde{q}_2^{-1}\ \underline{K}_{23}, \mathcal{O}_{X_{1223}})$

D : $\mathcal{M}_{12}^{\mathbb{R}}\ \hat{\otimes}\ \mathcal{M}_{23}^{\mathbb{R}}$

$$\text{Here} \quad \mathcal{M}_{12}^{\mathbb{R}}\ \hat{\otimes}\ \mathcal{M}_{23}^{\mathbb{R}} = \mathcal{E}_{X_{1223}}^{\mathbb{R}} \otimes_{\tilde{p}_1^{-1} \mathcal{E}_{X_{12}}^{\mathbb{R}} \otimes \tilde{p}_2^{-1} \mathcal{E}_{X_{23}}^{\mathbb{R}}} (\tilde{p}_1^{-1}\mathcal{M}_{12}^{\mathbb{R}} \otimes \tilde{p}_2^{-1}\mathcal{M}_{23}^{\mathbb{R}})$$

$$= \mathcal{E}_{X_{1223}}^{\mathbb{R}} \otimes_{\pi^{-1}\mathcal{D}_{X_{1223}}} \pi^{-1}(\mathcal{M}_{12} \hat{\otimes} \mathcal{M}_{23})$$

Let j be the diagonal embedding :

$$j : X_{123} \hookrightarrow X_{1223}$$

and let ϖ and ρ be the associated maps from $X_{123} \times_{X_{1223}} T^*X_{1223}$ to T^*X_{1223} and T^*X_{123} respectively. We have by Corollary 5.5.7. :

$$(11.1.7)\begin{cases} \mu hom(j^{-1}(\tilde{q}_1^{-1}\underline{K}_{12} \otimes \tilde{q}_2^{-1}\underline{K}_{23}), \mathcal{O}_{X_{123}}) \\ \qquad \cong \mathbb{R}\rho_*\varpi^{-1}\ \mu hom(\tilde{q}_1^{-1}\underline{K}_{12} \otimes \tilde{q}_2^{-1}\underline{K}_{23}, j_*\mathcal{O}_{X_{123}}) \end{cases}$$

Set :

$$\underline{K} = j^{-1}(\tilde{q}_1^{-1}\ \underline{K}_{12} \otimes \tilde{q}_2^{-1}\ \underline{K}_{23}) = q_{12}^{-1}\ \underline{K}_{12} \overset{\mathbb{L}}{\otimes} q_{23}^{-1}\ \underline{K}_{23}$$

The restriction homomorphism $\mathcal{O}_{X_{1223}} \longrightarrow j_*\mathcal{O}_{X_{123}}$ induces :

$$(11.1.8)\quad \mathbb{R}\rho_*\varpi^{-1}\ \mu hom(\tilde{q}_1^{-1}\underline{K}_{12} \otimes \tilde{q}_2^{-1}\underline{K}_{23}, \mathcal{O}_{X_{1223}}) \longrightarrow \mu hom(\underline{K}, \mathcal{O}_{X_{123}})$$

Set :

$$\mathcal{M} = \mathcal{D}_{X_{123}\to X_{1223}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{X_{1223}}}{\otimes}} (\mathcal{M}_{12}\ \hat{\otimes}\ \mathcal{M}_{23})$$

Then we have :

$$\mathcal{M}^{\mathbb{R}} = \mathbb{R}\rho_*\varpi^{-1}(\mathcal{E}^{\mathbb{R}}_{X_{123}\to X_{1223}} \overset{\mathbb{L}}{\underset{\mathcal{E}^{\mathbb{R}}_{X_{1223}}}{\otimes}} (\mathcal{M}^{\mathbb{R}}_{12}\ \hat{\otimes}\ \mathcal{M}^{\mathbb{R}}_{23}))$$

and Proposition 9.3.3. implies :

$$(11.1.9)\qquad \mu hom(\underline{K}, \mathcal{O}_{X_{123}}) = \mathcal{M}^{\mathbb{R}}$$

Thus by (11.1.7) and (11.1.8) we obtain a commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{\sim} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{\sim} & D \\ \downarrow & & \downarrow \\ E & \xrightarrow{\sim} & F \end{array}$$

$$A : \mathbb{R}\rho_*\varpi^{-1}\ (\tilde{p}_1^{-1}\ \mu hom(\underline{K}_{12}, \mathcal{O}_{X_{12}}) \otimes \tilde{p}_2^{-1}\ \mu hom(\underline{K}_{23}, \mathcal{O}_{X_{23}}))$$

$$B : \mathbb{R}\rho_*\varpi^{-1}\ (\tilde{p}_1^{-1}\ \mathcal{M}^{\mathbb{R}}_{12} \otimes \tilde{p}_2^{-1}\ \mathcal{M}^{\mathbb{R}}_{23})$$

$$C : \mathbb{R}\rho_*\varpi^{-1}\ (\mu hom(\tilde{q}_1^{-1}\underline{K}_{12} \otimes \tilde{q}_2^{-1}\underline{K}_{23}, \mathcal{O}_{X_{1223}}))$$

D : $\mathbb{R}\rho_* \varpi^{-1}(\mathcal{M}_{12}^{\mathbb{R}} \hat{\otimes} \mathcal{M}_{23}^{\mathbb{R}})$

E : $\mu hom(\underline{K}, \mathcal{O}_{X_{123}})$

F : $\mathcal{M}^{\mathbb{R}}$

Remark that the morphism $\mathbb{R}\rho_* \varpi^{-1}(\mathcal{M}_{12}^{\mathbb{R}} \hat{\otimes} \mathcal{M}_{23}^{\mathbb{R}}) \longrightarrow \mathcal{M}^{\mathbb{R}}$ is obtained by the canonical section of $\mathcal{E}^{\mathbb{R}}_{X_{123} \to X_{1223}}$.

Now we shall study the direct image of $\mathcal{M}^{\mathbb{R}}$ by p_{13}.

Let ϖ' and ρ' be the maps from $X_{123} \times_{X_{13}} T^*X_{13}$ to T^*X_{13} and T^*X_{123} respectively, associated to the projection q_{13}.

Set $\underline{K}' = \underline{K}_{U'}$ (recall that $U' = X_1 \times U \times X_3$). By (11.1.4) we have an isomorphism :

(11.1.10) $$\underline{K}' \cong \underline{K} \text{ in } D^+(X_{123} ; p_{13}^{-1}(\Omega_{13}))$$

Moreover $\underline{K}' \in Ob(D^b_{\mathbb{R}-c}(X_{123}))$ and Proposition 9.3.3. implies :

$$\mathbb{R}\varpi'_* (\mathcal{D}_{X_{13} \leftarrow X_{123}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{X_{123}}}{\otimes}} \rho'^{-1} \mu hom(\underline{K}', \mathcal{O}_{X_{123}}))$$

$$\cong \mu hom(\mathbb{R}q_{13*}\underline{K}'[n], \mathcal{O}_{X_{13}})$$

Thus :

(11.1.11) $$\begin{cases} \mathbb{R}\varpi'_* (\mathcal{D}_{X_{13} \leftarrow X_{123}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{X_{123}}}{\otimes}} \rho'^{-1} \mu hom(\underline{K}, \mathcal{O}_{X_{123}}))|_{\Omega_{13}} \\ \cong \mu hom(\mathbb{R}q_{13*}\underline{K}'[n], \mathcal{O}_{X_{13}})|_{\Omega_{13}} \end{cases}$$

Now remark that for a left $\mathcal{D}_{X_{123}}$-module F we have :

$$\mathcal{D}_{X_{13} \leftarrow X_{123}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{X_{123}}}{\otimes}} F = (F \underset{\mathcal{O}_{X_2}}{\otimes} \mathcal{O}_{X_2}^{(n)}) \overset{\mathbb{L}}{\underset{\mathcal{D}_{X_2}}{\otimes}} \mathcal{O}_{X_2}$$

Thus we obtain :

$$(11.1.12)\begin{cases} \mathbb{R}\varpi^!_* \rho'^{-1}(\mu hom(\underline{K}, \mathcal{O}^{(o,n,o)}_{X_{123}}) \overset{\mathbb{L}}{\underset{\mathcal{D}_{X_2}}{\otimes}} \mathcal{O}_{X_2})|_{\Omega_{13}} \\ \qquad \simeq \mu hom(\mathbb{R}q_{13*}\underline{K}'[n], \mathcal{O}_{X_{13}})|_{\Omega_{13}} \end{cases}$$

Here $\mathcal{O}^{(o,n,o)}_{X_{123}} = \mathcal{O}_{X_{123}} \underset{\mathcal{O}_{X_2}}{\otimes} \Omega^{(n)}_{X_2}$.

Set :

$$\underline{K}_{13} = \mathbb{R}q_{13*}\underline{K}' \quad .$$

We obtain the commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{\sim} & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \\ \downarrow & & \downarrow \\ E & \xrightarrow{\sim} & F \end{array}$$

$A : \mathbb{R}\varpi^!_* \rho'^{-1} \mathbb{R}\rho_* \varpi^{-1} (\tilde{p}_1^{-1} \mu hom(\underline{K}_{12}, \mathcal{O}^{(o,n)}_{X_{12}}) \otimes \tilde{p}_2^{-1} \mu hom(\underline{K}_{23}, \mathcal{O}_{X_{23}}))|_{\Omega_{13}}$

$B : \mathbb{R}\varpi^!_* \rho'^{-1} \mathbb{R}\rho_* \varpi^{-1} (\tilde{p}_1^{-1} \mathcal{M}^{\mathbb{R}(o,n)}_{12} \otimes \tilde{p}_2^{-1} \mathcal{M}^{\mathbb{R}}_{23})|_{\Omega_{13}}$

$C : \mathbb{R}\varpi^!_* \rho'^{-1} \mu hom(\underline{K}, \mathcal{O}^{(o,n,o)}_{X_{123}})|_{\Omega_{13}}$

$D : \mathbb{R}\varpi^!_* \rho'^{-1} (\mathcal{M}^{\mathbb{R}(o,n,o)})|_{\Omega_{13}}$

$E : \mu hom(\mathbb{R}q_{13*}K'[n], \mathcal{O}_{X_{13}})|_{\Omega_{13}}$

$F : \mathbb{R}\varpi^!_* \rho'^{-1} (\mathcal{M}^{\mathbb{R}(o,n,o)} \underset{\mathcal{D}_{X_2}}{\otimes} \mathcal{O}_{X_2})|_{\Omega_{13}}$

Now we consider the commutative diagram :

$$\begin{array}{ccccc}
\rho^{-1}\rho'(X_{123}\times_{X_{13}} T^*X_{13}) & \overset{a}{\hookrightarrow} & X_{123}\times_{X_{1223}} T^*X_{1223} & \overset{\varpi}{\hookrightarrow} & T^*X_{1223} \\
\downarrow b & & \downarrow \rho & & \\
X_{123}\times_{X_{13}} T^*X_{13} & \overset{\rho'}{\hookrightarrow} & T^*X_{123} & & \\
\downarrow \varpi' & & & & \\
T^*X_{13} & & & &
\end{array}$$

Then $\rho^{-1}\rho'(X_{123}\times_{X_{13}} T^*X_{13})$ is isomorphic to T^*X_{123}, and by this isomorphism, $b\circ\varpi'$, $\tilde{p}_1\circ\varpi\circ a$, $\tilde{p}_2\circ\varpi\circ a$ are identified with p^a_{13}, p^a_{12} and p^a_{23} respectively.

Hence we obtain the diagram on Ω_{13} :

$$\text{(11.1.13)}\qquad \begin{array}{ccc}
A & \xrightarrow{\sim} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{\sim} & D
\end{array}$$

A : $\mathbb{R}p^a_{13*}(p^{a-1}_{12}\,\mu hom(\underline{K}_{12}, \mathcal{O}^{(o,n)}_{X_{12}}) \otimes p^{a-1}_{23}\,\mu hom(\underline{K}_{23}, \mathcal{O}_{X_{23}}))$

B : $\mathbb{R}p^a_{13*}(p^{a-1}_{12}\,\mathcal{M}^{\mathbb{R}(o,n)}_{12} \otimes p^{a-1}_{23}\,\mathcal{M}^{\mathbb{R}}_{23})$

C : $\mu hom(\mathbb{R}q_{13*}\underline{K}'[n], \mathcal{O}_{X_{13}})$

D : $\mathbb{R}\varpi^!_*\rho'^{-1}(\mathcal{D}_{X_{13}\leftarrow X_{123}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{X_{123}}}{\otimes}} \mathcal{M}^{\mathbb{R}})$

Finally, we remark that (11.1.5) implies :

$$\mathbb{R}\varpi^!_*\rho'^{-1}(\mathcal{D}_{X_{13}\leftarrow X_{123}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{X_{123}}}{\otimes}} \mathcal{M}^{\mathbb{R}}) \cong \mathbb{R}p^a_{13*}(p^{a-1}_{12}\mathcal{M}^{\mathbb{R}(o,n)}_{12} \overset{\mathbb{L}}{\underset{\mathcal{E}^{\mathbb{R}}_{X_2}}{\otimes}} p^{a-1}_{23}\mathcal{M}^{\mathbb{R}}_{23})$$

This completes the proof of Proposition 11.1.3. . □

Now let Ω_{ij} be a conic open subset of T^*X_{ij} ((ij) = (12) or (23)), such that $\Omega_{12} \subset T^*X_1 \times \Omega_2$ and $\Omega_{23} \subset \Omega_2 \times T^*X_3$.

Let s_{ij} be a section of $H^0(\mu hom(\underline{K}_{ij}, \mathcal{O}_{X_{ij}}^{(o,n)}))$ on Ω_{ij}. We set :

$$s_{13} = \lambda(s_{12} \otimes s_{23})$$

where λ is defined in Proposition 11.1.3. .

By Proposition 6.3.3. we have, with the same notations as in the proof of Proposition 11.1.3. :

(11.1.14) $$\psi_{\underline{K}_{13}[n]}(\mathcal{O}_{X_3}) \cong \psi_{\underline{K}_{12}[n]} \circ \psi_{\underline{K}_{23}[n]}(\mathcal{O}_{X_3})$$

and we have morphisms :

(11.1.5) $$\psi_{\underline{K}_{12}[n]} \circ \psi_{\underline{K}_{23}[n]}(\mathcal{O}_{X_3}) \xrightarrow[\psi_{\underline{K}_{12}[n]}(\alpha(s_{23}))]{} \psi_{\underline{K}_{12}[n]}(\mathcal{O}_{X_2}) \xrightarrow[\alpha(s_{12})]{} \mathcal{O}_{X_1}$$

By Proposition 6.3.3. we get :

<u>Lemma 11.1.4.</u> : <u>We have</u> :

$$\alpha(s_{13}) = \alpha(s_{12}) \circ \psi_{\underline{K}_{12}[n]}(\alpha(s_{23}))$$

<u>11.1.3</u>. Now we prove Theorem 11.1.1. . Since any contact transformation is the product of two contact transformations whose associated Lagrangean manifold is the conormal bundle to a hypersurface, we may assume $\underline{K}$ is $\mathbb{C}$-constructible with shift 0.

We take $X_1 = Y$, $X_2 = X$, $X_3 = Y$, $\underline{K}_{12} = \underline{K}$, $\underline{K}_{23} = \underline{K}$.

Then we choose a neighborhood Ω_{12} of (p_Y, p_X^a), a neighborhood Ω_{23} of (p_X, p_Y^a) and an open set U in X , such that (11.1.4) and (11.1.5) are satisfied.

Let s_{ij} be a section of $\mathcal{M}_{ij}^{\mathbb{R}(o,n)}$ which satisfies ((ij) = (12) or (23)) :

$$(11.1.16)\begin{cases}\text{The morphisms} \\ \mathcal{E}^{\mathbb{R}}_{X_i} \longrightarrow p_{i*}(\mathcal{M}^{\mathbb{R}(o,n)}_{ij}),\ P \longmapsto Ps_{ij} \\ \text{and} \\ \mathcal{E}^{\mathbb{R}}_{X_j} \longrightarrow p^a_{j*}(\mathcal{M}^{\mathbb{R}(o,n)}_{ij}),\ Q \longmapsto s_{ij}Q \\ \text{are isomorphisms.}\end{cases}$$

We know that such sections locally exist, (cf. Sato-Kashiwara-Kawaï [1], Kashiwara [5]).

Then by Proposition 11.1.3., $\mu(s_{12} \otimes s_{23}) = \gamma(s_{13})$ gives an isomorphism between $\mathbb{R}p_{13*}(p^{a-1}_{12}\mathcal{M}^{\mathbb{R}(o,n)}_{12} \otimes_{\mathcal{E}^{\mathbb{R}}_{X_2}} p^{-1}_{23}\mathcal{M}^{\mathbb{R}(o,n)}_{23}$ and $\mathcal{E}^{\mathbb{R}}_{X_1}$ (resp. $\mathcal{E}^{\mathbb{R}}_{X_3}$) .

Hence s_{13} gives an isomorphism between $H^o(\mu hom(\underline{K}_{13}, \mathcal{O}^{(o,n)}_{X_{13}}))$ and $\mathcal{E}^{\mathbb{R}}_{X_1}$ (resp. $\mathcal{E}^{\mathbb{R}}_{X_3}$) .

Since $\underline{K}_{13}$ is a simple sheaf with shift 0 along $T^*_Y(Y\times Y)$, $\underline{K}_{13}$ is microlocally isomorphic to $\underline{\mathbb{C}}_Y[-n]$. By this identification, $H^o(\mu hom(\underline{K}_{13}, \mathcal{O}^{(o,n)}_{X_{13}}))$ is isomorphic to $C^{\mathbb{R}(o,n)}_{Y|X_{13}} \cong \mathcal{E}^{\mathbb{R}}_Y$. Then the multiplication on the right, and the multiplication on the left by s_{13}, are isomorphisms on $\mathcal{E}^{\mathbb{R}}_Y$. Hence s_{13} is invertible in $\mathcal{E}^{\mathbb{R}}_Y$, in a neighborhood of p_Y.

As $\alpha(s_{13})$ is nothing but s_{13} acting on $\mathcal{O}_Y$ through the morphism :

$$\mathcal{E}^{\mathbb{R}}_Y \longrightarrow Hom_{D^+(Y;p_Y)}(\mathcal{O}_Y, \mathcal{O}_Y)$$

(Proposition 10.6.2.), $\alpha(s_{13})$ is an isomorphism. Therefore $\alpha(s_{12})$ has a right inverse. Taking s_{12} as s , $\alpha(s)$ has a right inverse. Similar argument replacing X and Y, shows that $\alpha(s)$ has also a left inverse. This completes the proof of Theorem 11.1.1. . $\square$

§11.2. Quantized contact transformations and microlocalization for the sheaf $\mathcal{O}_X$

11.2.1. Let (E,σ) be a complex symplectic vector space, λ a real linear subspace. One says that λ is $\mathbb{R}$-Lagrangean if λ is Lagrangean in the real underlying symplectic vector space $(E^{\mathbb{R}}, 2\mathrm{Re}\,\sigma)$.

If ρ is a complex linear subspace of E, we define as in §7.1.,

$$\rho^{\perp} = \{x \in E\ ;\ \sigma(x,\rho) = 0\}$$

and if ρ is isotropic, we set for a real linear subspace λ :

$$\lambda^{\rho} = ((\lambda \cap \rho^{\perp}) + \rho)/\rho$$

Remark that if λ is $\mathbb{R}$-Lagrangean, then λ^{ρ} is $\mathbb{R}$-Lagrangean in the complex symplectic space $(\rho^{\perp}/\rho, \sigma_{\rho})$, σ_{ρ} denoting the image of σ in $\rho^{\perp}/\rho$.

Let λ be an $\mathbb{R}$-Lagrangean subspace. One says that λ is I-symplectic if $\mathrm{Im}\,\sigma$ is non degenerate on λ. This is equivalent to saying that $(E, \frac{1}{i}\sigma)$ is a complexification of the real symplectic space $(\lambda, \mathrm{Im}\,\sigma|_{\lambda})$, or that $\lambda \cap i\lambda = \{0\}$. Now let λ be an $\mathbb{R}$-Lagrangean subspace and consider :

$$\rho = \lambda \cap i\lambda\ .$$

Then ρ is a (complex) isotropic subspace, and λ^{ρ} is $\mathbb{R}$-Lagrangean and I-symplectic in $\rho^{\perp}/\rho$.

Let λ_0 be a complex Lagrangean subspace of E. We define the bilinear form γ_{λ} on λ_0^{ρ} , by setting for $(u,v) \in \lambda_0^{\rho} \times \lambda_0^{\rho}$:

$$\gamma_{\lambda}(u,v) = \sigma_{\rho}(u,\bar{v}) \tag{11.2.1}$$

where $\bar{v}$ is the complex conjugate of v with respect to the isomorphism :

$$\mathbb{C} \otimes_{\mathbb{R}} \lambda^{\rho} \cong \rho^{\perp}/\rho$$

Lemma 11.2.1. : The bilinear form γ_λ on λ_o^ρ is hermitian. Moreover $\gamma_\lambda(u,v) = 0$ if u or v belongs to $(\lambda^\rho \cap \lambda_o^\rho)^{\mathbb{C}}$, and the hermitian form defined by γ_λ on $\lambda_o^\rho / (\lambda^\rho \cap \lambda_o^\rho)^{\mathbb{C}}$ is non degenerate.

The proof is immediate (cf. Schapira [1 , Proposition 1.3.]). □

Let s^+ (resp. s^-) be the number of positive (resp. negative) eigenvalues of γ_λ on λ_o^ρ. Set :

$$2n = \dim_{\mathbb{C}} E$$

$$c = \dim_{\mathbb{R}}(\lambda \cap \lambda_o)$$

$$d = \dim_{\mathbb{C}}(\lambda \cap i\lambda)$$

$$\delta = \dim_{\mathbb{C}}(\lambda \cap i\lambda \cap \lambda_o)$$

By Lemma 11.2.1. we have :

$$s^+ + s^- + \dim_{\mathbb{R}}(\lambda^\rho \cap \lambda_o^\rho) = \dim_{\mathbb{C}} \lambda_o^\rho$$

$$= n - d$$

Since $\lambda^o \cap \lambda_o^\rho = (\lambda \cap \lambda_o) / (\lambda \cap i\lambda \cap \lambda_o)$ we obtain :

(11.2.2) $$s^+ + s^- = n - d - c + 2\delta$$

We shall denote by $\operatorname{sgn} \gamma_\lambda$ the signature of γ_λ on λ_o^ρ, that is ,

$$\operatorname{sgn} \gamma_\lambda = s^+ - s^- .$$

Proposition 11.2.2. : Let λ be an $\mathbb{R}$-Lagrangean subspace of (E,σ) , $\rho = \lambda \cap i\lambda$ and let λ_o be a complex Lagrangean subspace of (E,σ). We have :

(11.2.3) $$\operatorname{sgn} \gamma_\lambda = \frac{1}{2} \tau(\lambda, i\lambda, \lambda_o)$$

Here τ is the index associated to the symplectic form $2\operatorname{Re}\sigma$

(cf. Chapter 7, §1. for the definition of τ) .

Proof

Since $\gamma_\lambda = \gamma_{\lambda^\rho}$, and $\tau(\lambda, i\lambda, \lambda_o) = \tau(\lambda^\rho, i\lambda^\rho, \lambda_o^\rho)$, we may assume $\lambda \cap i\lambda = \{0\}$.

Let q be the real quadratic form on λ_o :

$$q(z) = \sigma(z,\bar{z}) .$$

Then $\operatorname{sgn} \gamma_\lambda = \frac{1}{2} \operatorname{sgn} q$. If we write $z = x + iy$, $x \in \lambda$, $iy \in i\lambda$, we get :

$$q(z) = -2\sigma(x, iy)$$

and by Proposition 7.1.5., the signature of q is $\tau(\lambda, i\lambda, \lambda_o)$. □

11.2.2. Now let X be a complex manifold, M a real submanifold of class C^2. The conormal bundle T_M^*X is clearly $\mathbb{R}$-Lagrangean (i.e. : $T_pT_M^*X$ is $\mathbb{R}$-Lagrangean in T_pT^*X at each $p \in T_M^*X$). We set for $p \in T_M^*X$:

$$(11.2.4) \quad \begin{cases} E(p) &= T_pT^*X \\ \lambda_M(p) &= T_p(T_M^*X) \\ \lambda_o(p) &= T_p\pi^{-1}\pi(p) \\ 2n &= \dim_{\mathbb{C}} E(p) \\ c &= \dim_{\mathbb{R}}(\lambda_M(p) \cap \lambda_o(p)) = \operatorname{codim} M \\ d(p) &= \dim_{\mathbb{C}}(\lambda_M(p) \cap i\lambda_M(p)) \\ \delta(p) &= \dim_{\mathbb{C}}(\lambda_M(p) \cap i\lambda_M(p) \cap \lambda_o(p)) \end{cases}$$

Definition 11.2.3. : Let $p \in T_M^*X$. We define the integer :

$$s(M,p) = \frac{1}{2}\tau(\lambda_M(p), i\lambda_M(p), \lambda_o(p))$$

(where τ is the Maslov index on $E^{\mathbb{R}}(p)$).

We also define the integers $s^{\pm}(M,p)$ by the relations :

$$s^{+}(M,p) - s^{-}(M,p) = s(M,p)$$

$$s^{+}(M,p) + s^{-}(M,p) = n-c-d(p)+2\delta(p)$$

Remark that it follows from Proposition 11.2.2. that $s^{+}(M,p)$ (resp. $s^{-}(M,p)$) is the number of positive (resp. negative) eigenvalues of the hermitian form $\gamma_{\lambda_M(p)}$ on $\lambda_o^{\rho(p)}(p)$, where $\rho(p) = \lambda_M(p) \cap i\lambda_M(p)$.

Example 11.2.4. : Let $x \in M$, and assume $T_xX = (T_xM)^{\mathbb{C}}$. Then at each $p \in T_M^*X$, $\lambda_M(p)$ is $\mathbb{R}$-Lagrangean and I-symplectic in $E(p)$, and $s^{+}(M,p) = s^{-}(M,p) = 0$. A particular case is obtained when M is real analytic and X is a complexification of M.

Example 11.2.5. : Assume M is a complex submanifold of X . Then in that case , $\lambda_M(p) = i\lambda_M(p)$, and $s^{+}(M,p) = s^{-}(M,p) = 0$ at each $p \in T_M^*X$.

Example 11.2.6. : Let ϕ be a real function of class C^2 , $M = \{x ; \phi(x) = 0\}$, and assume $d\phi \neq 0$ on M. Let :

$$T_{x_o}^{\mathbb{C}}M = \{v \in T_{x_o}X ; \langle v, \partial_x \phi(x_o)\rangle = 0\}$$

where $\partial_x \phi$ is the differential of ϕ with respect to the holomorphic variables.

Let L_ϕ be the Levi form of ϕ. Recall that if $(x_1,\dots,x_n)$ is a system of holomorphic coordinates on X, L_ϕ is represented by the matrix :

$$\left(\frac{\partial^2\phi}{\partial x_i\,\partial \bar{x}_j}\right)_{(1\leq i,j\leq n)}$$

Proposition 11.2.7. : In the preceding situation, we have :

$$s(M, \partial\phi(x_o)) = \operatorname{sgn}(L_\phi \big|_{T^{\mathbb{C}}_{x_o} M})$$

Proof

We shall follow an argument similar to that of (Schapira [1 , Proposition 1.6.]).

Let ψ be the morphism $M \times \mathbb{R} \longrightarrow T^*_M X$ given by :

$$(x,k) \longmapsto (x, k\, \partial\phi(x))$$

Setting $p = \partial\phi(x_o) \in T^*_M X \subset T^* X$, ψ induces the $\mathbb{C}$-linear homomorphism :

$$\psi_* : \mathbb{C} \otimes_{\mathbb{R}} T_{(x_o,1)}(M \times \mathbb{R}) \longrightarrow T_p\, T^* X$$

Set $\lambda_M = \lambda_M(p) = T_p\, T^*_M X$, and $\lambda_o = \lambda_o(p) = T_p\, \pi^{-1}\pi(p)$. Then :

$$\operatorname{Im} \psi_* = \lambda_M + i\lambda_M$$

Now we identify $T_{(x_o,1)}(M \times \mathbb{R})$ with $(T_{x_o} M) \times \mathbb{R}$, and we embedd $T_{x_o} M$ into $T_{x_o} X \oplus T_{x_o} \overline{X}$. We extend this embedding to the $\mathbb{C}$-linear homomorphism :

$$\mathbb{C} \otimes_{\mathbb{R}} T_{x_o} M \hookrightarrow T_{x_o} X \oplus T_{x_o} \overline{X}$$

by which $\mathbb{C} \otimes_{\mathbb{R}} T_{x_o} M$ is identified with the space :

$$\{(\alpha,\beta) \in T_{x_o} X \oplus T_{x_o} \overline{X} \ ; \ \langle \alpha, \partial\phi(x_o)\rangle + \langle \beta, \overline{\partial}\phi(x_o)\rangle = 0\}$$

We have :

$$\psi_*^{-1}(\lambda_o) = \{(\alpha,\beta,k) \in T_{x_o} X \oplus T_{x_o} \overline{X} \oplus \mathbb{C} \ ; \ \alpha = 0, \ \langle \beta, \overline{\partial}\phi(x_o)\rangle = 0\}$$

$$\psi_*^{-1}(\lambda_M) = \{(\alpha,\beta,k) \in T_{x_o} X \oplus T_{x_o} \overline{X} \oplus \mathbb{C} \ ; \ \alpha = \overline{\beta}, \ k \in \mathbb{R}, \operatorname{Re} \langle \alpha, \partial\phi(x_o)\rangle = 0\}$$

Hence we have an embedding :

$$\overline{T^{\mathbb{C}}_{x_o} M} \subset \psi_*^{-1}(\lambda_o)$$

given by $\beta \longmapsto (0,\beta,0)$.

By this embedding :

$$\psi_*(\overline{T^{\mathbb{C}}_{x_o} M}) \subset \lambda_o \cap (\lambda_M + i\lambda_M)$$

$$\psi_*(\overline{T_{x_o} M}) + (\lambda_o \cap \lambda_M)^{\mathbb{C}} = \lambda_o$$

Hence $\operatorname{sgn} \gamma_{\lambda_M} = \operatorname{sgn}(\gamma_{\lambda_M} \circ \psi_*)$.

For $\beta \in \overline{T^{\mathbb{C}}_x M}$, $\psi_*((0,\beta) + (\overline{\beta},0)) \in \lambda_M$, and $\psi_*((0,\beta) - (\overline{\beta},0)) \in i\lambda_M$.

Hence :

(11.2.5) $$(\gamma_{\lambda_M} \circ \psi_*)(\beta,\beta) = (\psi^*(d\omega))((0,\beta), (\overline{\beta},0))$$

Since $\psi^*(\omega) = k\, \partial\phi$ and $\psi^*(d\omega) = kd\, \partial\phi + dk\, \partial\phi$, the right hand side of (11.2.5) equals $\langle d\, \partial\phi, (0,\beta) \wedge (\overline{\beta},0)\rangle$ that is, $\langle \overline{\partial}\partial\phi, (0,\beta) \wedge (\overline{\beta},0)\rangle$. This completes the proof. $\square$

<u>Theorem 11.2.8.</u> : <u>Let</u> M <u>and</u> N <u>be two real submanifolds of class</u> C^2 <u>of</u> X , U <u>and</u> V <u>two conic open subsets of</u> T^*X, <u>and let</u> ϕ <u>be a complex contact transformation from</u> U <u>to</u> V. <u>Assume that</u> ϕ <u>sends</u> $U \cap T^*_M X$ <u>to</u> $V \cap T^*_N X$. <u>Then</u> :

i) The function $s(M,p) - s(N,\phi(p))$ <u>is locally constant on</u> $T^*_M X \cap U$.

ii) <u>Locally on</u> U , <u>we may quantize</u> ϕ as an isomorphism :

$$\tilde{\phi} : \phi_*\mu_M(\mathcal{O}_X) \cong \mu_N(\mathcal{O}_X) \quad [\tfrac{1}{2}(\dim M + s(M,p)) - \tfrac{1}{2}(\dim N + s(N,\phi(p))]$$

<u>where</u> $p \in U \cap T^*_M X$.

<u>Proof</u>

Let $p \in T^*_M X \cap U$, $q = \phi(p)$. We set $\lambda_o(p) = T_p(\pi^{-1}\pi(p))$,

$\lambda_o(q) = T_q(\pi^{-1}\ \pi(q))$, $\lambda_M = \lambda_M(p) = T_p\ T^*_M X$, $\lambda_N = \lambda_N(q) = T_q\ T^*_N X$ and we denote by ϕ^* the differential of ϕ at p.

Let Λ be the Lagrangean manifold of $T^*(X \times X)$ associated to the graph of ϕ , $\underline{K}$ a simple sheaf on Λ with shift 0 . Applying Theorem 11.1.1. and Corollary 7.4.2. we find, locally on U, an isomorphism :

$$\phi_* \mu_M(\mathcal{O}_X) = \mu_N(\mathcal{O}_X) \quad [d]$$

where :

$$d = \frac{1}{2}(\dim M - \dim N) + \frac{1}{2}\tau$$

$$\tau = \tau(\lambda_o(q),\ \phi^*(\lambda_o(p)),\ \lambda_N(q)) \ .$$

First let us prove that τ is locally constant.

Let $\nu(q)$ be a complex Lagrangean plane of $T_q\ T^*X$, transversal to each of $\lambda_o(q)$, $\phi^*(\lambda_o(p))$, $\lambda_N(q)$. Then :

$$\tau = \tau_1 + \tau_2 + \tau_3$$

where :

$$\tau_1 = \tau(\lambda_o(q),\ \phi^*(\lambda_o(p)),\ \nu(q))$$
$$\tau_2 = \tau(\phi^*(\lambda_o(p)),\ \lambda_N(q),\ \nu(q))$$
$$\tau_3 = \tau(\lambda_N(q),\ \lambda_o(q),\ \nu(q))$$

Since $\lambda_o(q)$, $\phi^*(\lambda_o(p))$, $\nu(q)$ are complex Lagrangean, $\tau_1 \equiv 0$. Since $\dim(\phi^*(\lambda_o(p)) \cap \lambda_N(q))$ and $\dim(\lambda_o(q) \cap \lambda_N(q))$ are locally constant, τ_2 and τ_3 are locally constant by Proposition 7.1.3. .

Now we have :

$$\begin{aligned} 2(s(M,p)-s(N,q)) &= \tau(\lambda_M, i\lambda_M, \lambda_o(p)) - \tau(\lambda_N, i\lambda_N, \lambda_o(q)) \\ &= \tau(\lambda_N, i\lambda_N, \phi^*(\lambda_o(p))) - \tau(\lambda_N, i\lambda_N, \lambda_o(q)) \\ &= \tau(i\lambda_N, \phi^*(\lambda_o(p)), \lambda_o(q)) - \tau(\lambda_N, \phi^*(\lambda_o(p)), \lambda_o(q)) \\ &= 2\tau(\lambda_o(q),\ \phi^*(\lambda_o(p)), \lambda_N) \ . \end{aligned}$$

The last equality follows from the fact that the isomorphism of the multiplication by i transforms σ to $-\sigma$. Hence we have found :

(11.2.6) $$s(M,p) - s(N,q) = \tau$$

which completes the proof. □

§11.3. Applications

11.3.1. We shall give some applications to Theorem 11.2.8. .

Let M be a real submanifold of class C^2. We keep the notations (11.2.4) . We also introduce :

(11.3.1) $\nu(p)$: the complex line of $T_p T^*X$ generated by the Euler vector field.

(Recall that the Euler vector field is the image by the Hamiltonian isomorphism of the canonical 1-form ω of T^*X).

Proposition 11.3.1. : Assume :

(11.3.2) $$\dim_{\mathbb{R}}(\lambda_M(p) \cap \nu(p)) = 1$$

Then we have $H^j(\mu_M(\mathcal{O}_X))_p = 0$ for $j \notin [c + s^-(M,p) - \delta(p),\ n - s^+(M,p) + \delta(p)]$

Proof

We may find a complex Lagrangean plane λ'_o in $T_p T^*X$ such that :

(11.3.3) $\lambda'_o \supset \nu(p)$, $\dim_{\mathbb{R}}(\lambda_M(p) \cap \lambda'_o) = 1$, and the hermitian form $\gamma_{\lambda_M(p)}$ on $(\lambda'_o)^{\rho(p)}$ has no positive eigenvalues

(where $\rho(p) = \lambda_M(p) \cap i\lambda_M(p)$). In fact let $\rho' = (\lambda_M(p) \cap i\lambda_M(p)) + \nu(p)$. It is enough to find a Lagrangean plane $\overline{\lambda}'_o$ in $\rho'^{\perp}/\rho'$ which does not intersect $(\lambda_M(p))^{\rho'}$ and such that the hermitian form $\gamma_{\lambda_M(p)}$

has no positive eigenvalues on $\overline{\lambda}'_o$, then to take the pull-back of $\overline{\lambda}'_o$ in $T_p T^*X$. Similarly we may find a complex Lagrangean plane λ''_o such that :

$$(11.3.4)\quad \begin{cases} \lambda''_o \supset \nu(p),\ \dim_{\mathbb{R}}(\lambda_M(p) \cap \lambda''_o) = 1, \text{ and the hermitian form} \\ \gamma_{\lambda_M(p)} \text{ on } (\lambda''_o)^{\rho(p)} \text{ has no negative eigenvalues} \end{cases}$$

Now we choose a contact transformation ϕ which interchanges λ'_o (resp. λ''_o) with $T_q\,\pi^{-1}\,\pi(q)$. Thus ϕ interchanges (T^*X, T^*_MX, p) and (T^*X, T^*_NX, q), for a real hypersurface N of X, with $s^+(N,q) = 0$ (resp. $s^-(N,q) = 0$).

Assume we know that $H^j(\mu_N(\mathcal{O}_X))_q = 0$ for $j \notin [1 + s^-(N,q) + \alpha,\ n - s^+(N,q) + \beta]$.

Applying Theorem 11.2.8. we find :

$$H^j(\mu_M(\mathcal{O}_X))_p = 0 \quad \text{for} \quad j \notin [1 + s^-(N,q) + \alpha - \gamma,\ n - s^+(N,q) + \beta - \gamma]$$

where :

$$\gamma = \frac{1}{2}(s(M,p) - s(N,q) - \operatorname{codim} M + 1)$$

Let us write for short $s(M)$, $s^{\pm}(M)$, λ_M, $s(N)$, ... instead of $s(M,p)$, $s^{\pm}(M,p)$, $\lambda_M(p)$, $s(N,q)$, ... We have :

$$1 + \alpha + s^-(N) - \gamma = \alpha + \frac{1}{2}(1 + s^+(N) + s^-(N) - s(M) + c) .$$

$$\text{Since}\quad s^+(N) + s^-(N) = n - 1 - \dim_{\mathbb{C}}(\lambda_N \cap i\lambda_N)$$
$$= n - 1 - d(p)$$

we get by Definition 11.2.3. :

$$1 + \alpha + s^-(N) - \gamma = \alpha + \frac{1}{2}(n + c - s(M) - d(p))$$
$$= \alpha + s^-(M) + c - \delta(p)$$

Similarly :

$$n + \beta - s^+(N) - \gamma = n + \beta + \frac{1}{2}(c - s(M) - 1 - s^+(N) - s^-(N))$$
$$= \beta + \frac{1}{2}(n - d(p) + c - s(M))$$
$$= n + \beta + \delta(p) - s^+(M)$$

Thus :

$$H^j(\mu_M(\mathcal{O}_X))_p = 0 \quad \text{for} \quad j \notin [c + s^-(M) - \delta(p) + \alpha, n - s^+(M) + \delta(p) + \beta].$$

We know that for a locally closed subset Z of X, $H^j_Z(\mathcal{O}_X)_x = 0$ for $j \notin [1,n]$ if $x \notin \operatorname{Int} Z$ (for $j = 0$ this is the "analytic extension principle" and for $j > n$ this is a theorem of Malgrange [2]). In particular $H^j(\mu_N(\mathcal{O}_X))_q = 0$ for $j \notin [1,n]$.

Thus if we choose N such that $s^-(N,q) = 0$ (this is possible by (11.3.4)) we may take $\alpha = 0$. Similarly if we choose $s^+(N,q) = 0$ we may take $\beta = 0$, which completes the proof. $\square$

Remark 11.3.2. : Since

$$\lambda_M(p) \cap \lambda_o(p) = T_p(T^*_M X \cap \pi^{-1}\pi(p)) ,$$

we have :

$$\delta(p) = \operatorname{codim}_{\mathbb{C}}(T_{\pi(p)} M + i\, T_{\pi(p)} M) \tag{11.3.5}$$

Hence $\delta(p) = 0$ is equivalent to saying that M is a so-called "generic" submanifold. This is of course the case when M is a real hypersurface.

When $\delta(p) = 0$, similar results to that of Proposition 11.3.1. are well-known, and there exists an extensive literature on this subject. Let us only quote Andreotti-Grauert [1], Naruki [1], Baouendi-Chang-Trèves [1], Nacinovich [1], Sjöstrand [2]. When M is real analytic, and $\delta(p) = 0$, $\mu_M(\mathcal{O}_X) \otimes \omega_M$ [codim M] is isomorphic to the complex of solutions of the induced Cauchy-Riemann

system with values in the sheaf of Sato's microfunctions on $\sqrt{-1}\ T^*M$
This follows easily from the Cauchy-Kowalewski theorem (formula 10.5.4), (cf. Kashiwara-Kawaï [1]). In this situation, Proposition 11.3.1. follows from Sato-Kashiwara-Kawaï [1, Chapter III, Theorem 2.3.10.]. We refer to Tajima [1], and the bibliography of this paper, for more details.

Remark 11.3.3. : Let us say that M is non degenerate at p if :

(11.3.6) $\lambda_M(p) \cap i\lambda_M(p) = \{0\}$

In this case the cohomology of $\mu_M(\mathcal{O}_X)_p$ is concentrated in degree $m = n - s^+(M,p) = \operatorname{codim} M + s^-(M,p)$ (at $p \in \dot{T}^*_M X$). Let γ be the projection from $\dot{T}^*_M X$ to the spherical cotangent bundle $S^*_M X = \dot{T}^*_M X/\mathbb{R}^+$.
Let us put :

$$\tilde{C}_M = \gamma_* \mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M\ [n]$$

Since we may find a complex contact transformation which interchanges $T^*_M X$ with the conormal bundle to the boundary of a strictly pseudo-convex open set (in a neighborhood of p), we find that the sheaf $\tilde{C}_M$ on $S^*_M X$ is flabby in a neighborhood of p. Using Proposition 2.3.2. it would be possible to formulate results of the type "edge of the wedge theorem", but we do not develop this here (cf. Martineau [1], Bros-Iagolnitzer [1], Bengel-Schapira [1]). Remark that when M is real analytic (and (11.3.6) is satisfied) quantized contact transformations for $\mu_M(\mathcal{O}_X)$ where already performed by Kashiwara-Kawaï [4] (cf. also Boutet de Monvel [1], Lebeau [1], Hörmander [3], Sjöstrand [1]).

11.3.2. We can also recover Proposition 1.1.2. of Sato-Kashiwara-Kawaï [1 Chapter II].

Proposition 11.3.4. : Let M be a complex submanifold of complex codimension d . Then :

$$H^j(\mu_M(\mathcal{O}_X)) = 0 \quad \text{for} \quad j \neq d \ .$$

Proof

First let us choose $p \in \dot{T}^*_M X$. By a complex contact transformation we may interchange $(\dot{T}^*_M X, p)$ with $(\dot{T}^*_N X, q)$, where N is any other complex submanifold of X. By Theorem 11.2.8. we have :

$$\mu_M(\mathcal{O}_X)_p \ [\operatorname{codim}_{\mathbb{C}} M] \simeq \mu_N(\mathcal{O}_X)_q \ [\operatorname{codim}_{\mathbb{C}} N]$$

Since $\mu_N(\mathcal{O}_X)_q \ [\operatorname{codim}_{\mathbb{C}} N]$ is concentrated in degree $\leqslant 0$ (resp. $\geqslant 0$) for N a submanifold of dimension 0 (resp. of codimension 1), the results follows. The case where $p \in T^*_X X$ is easily deduced. □

11.3.3. We can also study real submanifolds which are "microlocally weakly pseudo-convex". More precisely :

Proposition 11.3.5. : Let M be a real submanifold of class C^2, $p_o \in T^*_M X$. Assume (11.3.2) at p_o and also :

(11.3.7) $s^-(M,p) - \delta(p)$ is locally constant in a neighborhood of p_o.

Set $j_o = \operatorname{codim} M + s^-(M,p) - \delta(p)$.

Then $H^j(\mu_M(\mathcal{O}_X))_{p_o} = 0$ for $j \neq j_o$, and for $j = j_o$, this space is infinite dimensional.

Proof

Let ϕ be a complex contact transformation defined in a neighborhood of p_o, such that ϕ interchanges $(T^*_M X, p_o)$ and $(T^*_N X, q_o)$, where N is a real submanifold. We have already noticed that

$s(M,p) - s(N,\phi(p))$ is locally constant.

On the other hand we have :

$$s^+(M,p)+s^-(M,p)-(s^+(N,\phi(p))+s^-(N,\phi(p)))$$

$$= \operatorname{codim} N - \operatorname{codim} M + \dim_{\mathbb{C}}(\lambda_M(p) \cap i\lambda_M(p))$$

$$- \dim_{\mathbb{C}}(\lambda_N(\phi(p)) \cap i\lambda_N(\phi(p)) + 2\delta(p) - 2\delta(\phi(p)).$$

Since $\dim_{\mathbb{C}}(\lambda_M(p) \cap i\lambda_M(p)) = \dim_{\mathbb{C}}(\lambda_N(\phi(p)) \cap i\lambda_N(\phi(p)))$, we find that $(s^-(M,p) - \delta(p)) - (s^-(N,\phi(p)) - \delta(\phi(p)))$ is locally constant.

We choose the contact transformation ϕ such that N is a real hypersurface, and $s^-(N,\phi(p_o)) = 0$. Since $\delta(\phi(p)) = 0$, we find by the hypothesis that :

$$s^-(N,q) = 0$$

for $q \in T^*_N X$, in a neighborhood of $q_o = \phi(p_o)$.

Then applying Proposition 11.2.7. and (Hörmander [1 , Theorem 2.6.12.]), we see that N is the boundary of a pseudo-convex open set . Therefore $H^j(\mu_N(\mathcal{O}_X))_{q_o} = 0$ for $j \neq 1$, and $H^j(\mu_M(\mathcal{O}_X))_{q_o} = 0$ for $j \neq j_o$, where $j_o = 1 - \frac{1}{2}[s(M,p)-s(N,q)- \operatorname{codim} M + 1]$ and the same calculation as for Proposition 11.3.1. shows that $j_o = \operatorname{codim} M + s^-(M,p) - \delta(p)$. Finally $H^{j_o}(\mu_M(\mathcal{O}_X))_{p_o} \simeq \varinjlim_{U \ni x_o} \mathcal{O}(U \cap \Omega)/\mathcal{O}(U)$, where $x_o = \pi(q_o)$, Ω is a pseudo-convex open set with N as boundary, and U ranges over the family of neighborhoods of x_o in X . □

<u>Remark 11.3.6.</u> : We shall extend Proposition 11.3.5. to general systems of microdifferential equations with constant multiplicities in our forthcoming paper [6] .

Masaki KASHIWARA
R.I.M.S.
Kyoto University
KYOTO
Japon

Pierre SCHAPIRA
Math. C.S.P.
Université Paris-Nord
93430 VILLETANEUSE
France

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INDEX

Special notations and conventions

General notations

$Int(A)$: interior of A

$\overline{A}$: closure of A

$\{x_n\}_n$ or $\{x_n\}$: a sequence indexed by $n \in \mathbb{N}$

$x_n \xrightarrow[n]{} x$: the sequence $\{x_n\}$ has a limit x

G^o : the polar cone to a cone G in a real vector space (§1.1)

TX : the tangent bundle to X

τ : the natural projection $TX \longrightarrow X$

T^*X : the cotangent bundle to X

π : the natural projection $T^*X \longrightarrow X$

$T_Y X$: the normal bundle to Y in X

T_Y^*X : the conormal bundle to Y in X

T_X^*X : the zero section of T^*X, identified to X

$\dot{T}^*X = T^*X \setminus T_X^*X$

$\dot{\pi}$: the restriction of π to $\dot{T}^*X$

a : the antipodal map on a vector bundle

ρ_f, ϖ_f or simply ρ, ϖ : the natural maps from $Y \times T^*X$ to T^*Y and T^*X respectively, naturally associated to a map $f : Y \longrightarrow X$

ω_X or ω : the canonical 1-form on T^*X (§1.1)

H : the Hamiltonian isomorphism from T^*T^*X to TT^*X (§1.1)

$X^{\mathbb{R}}$: the real underlying manifold to a complex manifold X

$\overline{X}$: the anti-holomorphic manifold associated to a complex manifold X

X_G : the space X endowed with the G-topology (X contained in a real vector space, G a closed convexe cone) (§1.5)

ϕ_G or ϕ_X or ϕ : the natural map $X \longrightarrow X_G$ (§1.5)

$C(S,V)$: the normal cone of S along V in TX (§1.2)

$C_V(S)$: the normal cone of S along V in T_VX , (V smooth), (§1.2)

$N(S) = TX \setminus C(X \setminus S, S)$, the strict normal cone to S (§1.2)

$N^*(S) = N^o(S)$: the conormal cone to S (§1.2)

$S_1 \hat{+} S_2$, $S_1 \hat{+}_\infty S_2$: cf. Definition 1.2.3.

$f^{\#}(A)$, $f^{\#}_\infty(A)$: cf. definition 1.2.5.

$\underline{\omega}_X$: the orientation sheaf on X

$\underline{\omega}_{Y/X} = \underline{\omega}_Y \otimes f^{-1}\underline{\omega}_X$: the relative orientation sheaf associated to $f : Y \longrightarrow X$

$\varinjlim$: inductive limit

$\varprojlim$: projective limit

"$\varinjlim$" : ind-object (§5.6)

"$\varprojlim$" : pro-object (§5.6)

$\tau(\cdot,\cdot,\cdot)$: Maslov index (§7.1)

$s(M,p)$: index associated to a real submanifold M in a complex manifold X (§11.2)

Categories

$D(X)$: the derived category of the category of complexes of sheaves of A-modules over X, (A is a fixed unitary ring)

$D^+(X)$ (resp. $D^b(X)$) : the full subcategory of $D(X)$ consisting of complexes with cohomology bounded from below (resp. bounded cohomology)

$D^+(X;\Omega)$: the localization of $D^+(X)$ with respect to $N(\Omega) = \{\underline{F} \in Ob(D^+(X)); SS(\underline{F}) \cap \Omega = \emptyset\}$ (§6.1)

$D^+(X;p) = D^+(X;\{p\})$

$D^+_V(X)$: the full subcategory of $D^+(X)$ consisting of $\underline{F} \in Ob(D^+(X))$ with $SS(\underline{F}) \subset V$

$D^+_{w-\mathbb{R}-c}(X)$ (resp. $D^b_{\mathbb{R}-c}(X)$) : the full subcategory of $D^+(X)$ (resp. $D^b(X)$) consisting of complexes with weakly $\mathbb{R}$-constructible (resp. $\mathbb{R}$-constructible) cohomology (X is a real analytic manifold), (§8.2)

$D^{+}_{w\text{-}\mathbb{C}\text{-}c}(X)$ (resp. $D^{b}_{\mathbb{C}\text{-}c}(X)$) : the full subcategory of $D^{+}(X)$ (resp. $D^{b}(X)$) consisting of complexes with weakly $\mathbb{C}$-constructible (resp. $\mathbb{C}$-constructible) cohomology (X is a real analytic manifold), (§8.5).

Diff(X) : the category of left $\mathcal{D}_X$-modules

$D(\mathcal{D}_X)$: the derived category of Diff(X)

$D^{b}_{coh}(X)$: the full subcategory of $D(\mathcal{D}_X)$ consisting of complexes with bounded and coherent cohomology, (§9.1)

$D^{b}_{rh}(X)$: the full subcategory of $D^{b}_{coh}(X)$ consisting of complexes with regular holonomic cohomology

Sheaves

$\underline{F}$, $\underline{G}$, ... : objects of $D^{+}(X)$

$\mathrm{supp}(\underline{F}) = \overline{\bigcup_j \mathrm{supp}\, H^j(\underline{F})}$, where $\mathrm{supp}\, H^j(\underline{F})$ is the support of the sheaf $H^j(\underline{F})$

$SS(\underline{F})$: the micro-support of $\underline{F}$ (Definition 3.1.2)

[d] : shift in $D^{+}(X)$ (§1.3)

wgℓd(A) : weak global dimension of A (§1.3)

$\Gamma_Z(X,\underline{F})$: global sections of $\underline{F}$ supported by Z

$\Gamma_Z(\underline{F})$: sheaf of sections of $\underline{F}$ supported by Z

$\underline{F} \otimes \underline{G}$: tensor product sheaf

$\underline{Hom}(\underline{F},\underline{G})$: sheaf of homomorphisms from $\underline{F}$ to $\underline{G}$

$Hom(\underline{F},\underline{G})$: group of homomorphisms from $\underline{F}$ to $\underline{G}$ (= $\Gamma(X,\underline{Hom}(\underline{F},\underline{G}))$)

$\underline{F}_x$: the stalk of $\underline{F}$ at x

$\underline{F}_Z$: naturally associated to $\underline{F}$, a sheaf such that $(\underline{F}_Z)_x = \underline{F}_x$ for $x \in \underline{F}$, $(\underline{F}_Z)_x = 0$, $x \notin Z$

$f_*(\cdot)$: direct image

$f_!(\cdot)$: direct image with proper supports

$f^{-1}(\cdot)$: inverse image

$H^j(\cdot)$: j-th cohomology object

$\mathbb{R}F(\cdot)$: right derived functor of $F(\cdot)$

$\mathbb{L}F(\cdot)$: left derived functor of $F(\cdot)$

$f^!(\cdot)$: adjoint of $\mathbb{R}f_!(\cdot)$

$H^j_Z(X,\cdot) = H^j\ \mathbb{R}\Gamma_Z(X,\cdot)$

$H^j_Z(\cdot) = H^j\ \mathbb{R}\Gamma_Z(\cdot)$

$\mathrm{Ext}^j(\cdot,\cdot) = H^j\ \mathbb{R}\mathrm{Hom}(\cdot,\cdot)$

$\underline{\mathrm{Ext}}^j(\cdot,\cdot) = H^j\ \mathbb{R}\underline{\mathrm{Hom}}(\cdot,\cdot)$

$\mathrm{Tor}_j(\cdot,\cdot) = H^{-j}(\cdot \overset{\mathbb{L}}{\otimes} \cdot)$

$\underline{F}^{\wedge}$: Fourier-Sato transform of $\underline{F}$ (§2.1)

$\underline{F}^{\vee}$: inverse Fourier Sato transform of $\underline{F}$ (§2.1)

$\nu_M(\cdot)$: specialization along M (§2.2)

$\mu_M(\cdot)$: microlocalization along M (§2.3)

$\mu hom(\underline{F},\underline{G})$: microlocalization of $\underline{G}$ along $\underline{F}$ (§5.5)

$\Psi_{\underline{K}}(\cdot), \Phi_{\underline{K}}(\cdot)$: extended contact transformations (§6.3)

Special sheaves and functors

$\mathcal{O}_X$: sheaf of holomorphic functions on a complex manifold X

$\mathcal{D}_X$ (resp. $\mathcal{D}^\infty_X$) : sheaf of finite (resp. infinite) order differential operators on X

$\Omega^{(p)}_X$: sheaf of holomorphic p-forms on X

$\Omega_X = \Omega^{(\dim X)}_X$

$B_M = \mathbb{R}\Gamma_M(\mathcal{O}_X) \otimes \underline{\omega}_M[n]$: the sheaf of Sato's hyperfunctions on the real analytic manifold M of dimension n

$C_M = \mu_M(\mathcal{O}_X) \otimes \underline{\omega}_M[n]$: the sheaf of Sato's microfunctions

$\mathcal{E}^{\mathbb{R}}_X = \mu_\Delta(\mathcal{O}_{X\times X} \otimes_{q_2^{-1}\mathcal{O}_X} q_2^{-1}\Omega_X)$: the Ring of holomorphic microlocal operators (§9 .1)

$\mathcal{E}^\infty_X$: Ring of infinite order microdifferential operators (§9.1)

$\mathcal{E}_X$: Ring of finite order microdifferential operators

$\mathcal{E}^{\mathbb{R}}_{Y\to X}$, $\mathcal{E}^{\infty}_{Y\to X}$, $\mathcal{E}_{Y\to X}$, $\mathcal{D}^{\infty}_{Y\to X}$, $\mathcal{D}_{Y\to X}$: rings of microlocal or (micro)differential infinite or finite order operators from Y to X (§9.1)

$\mathcal{E}(G;D)$: cf. §10.4.

$\mathcal{A}_X$: sheaf of real analytic functions on a real analytic manifold X

$\mathcal{V}_X$: sheaf of real analytic densities on a real analytic manifold X

$\mathcal{D}'_X$: sheaf of Schwartz's distributions on a real manifold X

$\int_f \cdot$: functor of direct images for $\mathcal{D}_X$-modules (§9 .4)

$\int_f^{pr} \cdot$: functor of direct images with proper supports for $\mathcal{D}_X$-modules (§9.4)

TH(·) : functor of temperate homomorphisms, (§9.2)

RH(·) : Riemann-Hilbert functor (§9.2)

char($\mathcal{M}$) : characteristic variety of $\mathcal{M}$ (§10.1)

$\mathcal{M}^{\infty} = \mathcal{D}^{\infty}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{M}$

$\mathcal{M}^{*} = \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$

$\mathcal{M}^{\mathbb{R}} = \mathcal{E}^{\mathbb{R}}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M}$

ABSTRACT

To a sheaf $\underline{F}$ on a real manifold X, we associate its micro-support, $SS(\underline{F})$. This is a closed conic subset of the cotangent bundle T^*X, which shall appear as being involutive, and which, roughly speaking, describes the set of codirections on X where $\underline{F}$ and its cohomology do not propagate.

We study the behaviour of the micro-support under the usual operations on sheaves, and localize the derived category of sheaves on X with respect to the micro-support. This allows us to work with sheaves "microlocally" and in particular to give a meaning to the action of contact transformations on sheaves. For that purpose we make a detailed study of "simple sheaves" along a smooth Lagrangean manifold, calculating their shift with the help of the Maslov index. Next, we apply this theory to the study of real or complex analytic constructible sheaves (these are the sheaves whose micro-support is Lagrangean), to regular holonomic Modules (including a direct image theorem in the non proper case), and to microdifferential systems (estimation of the characteristic variety, wave front set of $\prod_j (f_j + io)^{\lambda_j}$, propagation theorems, including micro-hyperbolic systems, etc ...) .

Finally we show that one can locally "quantize" complex contact transformations for the sheaf $\mathcal{O}_X$ of holomorphic functions on a complex manifold X, and we derive some applications of it to the calculation of Sato's microlocalization of $\mathcal{O}_X$ along real submanifolds of X .

RÉSUMÉ

Sur une variété réelle X , nous associons à un faisceau $\underline{F}$ son micro-support, $SS(\underline{F})$. C'est un fermé conique du fibré cotangent T^*X, qui se révèlera être involutif, et qui décrit l'ensemble des codirections de X où $\underline{F}$ et sa cohomologie "ne se propagent pas". Nous étudions le comportement du micro-support vis à vis des opérations usuelles sur les faisceaux et localisons la catégorie dérivée des faisceaux sur X par rapport au micro-support. Cela nous permet de travailler "microlocalement" avec les faisceaux, et en particulier de donner un sens à l'action des transformations canoniques sur les faisceaux. Pour cela nous faisons une étude détaillée des "faisceaux simples" le long d'une variété Lagrangienne lisse, calculant leur décallage à l'aide de l'indice de Maslov.

On applique ensuite cette théorie à l'étude des faisceaux constructibles analytiques réels ou complexes (ce sont les faisceaux dont le micro-support est Lagrangien), aux systèmes holonomes réguliers (avec un théorème d'images directes dans le cas non propre), et aux systèmes microdifférentiels (estimation de la variété caractéristique, front d'onde de $\prod_j (f_j + io)^{\lambda_j}$, théorèmes de propagation, en particulier pour les systèmes micro-hyperboliques, etc ...).

Finalement nous montrons que l'on peut localement "quantifier" les transformations canoniques complexes sur le faisceau $\mathcal{O}_X$ des fonctions holomorphes sur une variété complexe X , et en déduisons quelques applications au calcul du microlocalisé de Sato de $\mathcal{O}_X$ le long de sous-variétés réelles de X .