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DISCRETE LAX EQUATIONS AND DIFFERENTIAL-DIFFERENCE CALCULUS

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PREFACE

Since the discovery of solitons about 15 years ago, the classical theory of completely integrable systems has undergone remarkable transformation. Among many mathematical branches which benefited from this progress, the classical calculus of variations is one of the most conspicuous, being at the same time the most indispensable tool in the study of the structural problems.

For the continuous mechanical systems, the basic developments in both above mentioned theories are by now well known under the name of (differential) Lax equations (see, e.g., Manin's review [10]). Here I take up the case of classical mechanics proper but for the case of an infinite number of particles. It turns out that the appropriate calculus, resulting from an attempt to look at classical mechanics from the point of view of field theories, and not vice versa as is the custom, exists and in its logical structure, resembles very much the classical one though it does not have a geometric model.

The path of the presentation follows, as close as possible, the differential theory of Lax equations. A superficial familiarity with the latter will undoubtedly help the reader to understand the strings in various constructions, although I often supply the necessary motivation. There are no other prerequisites.

A few things have not found their way into the text. Most important among them are the matrix equations and their connections with simple Lie groups. This theory is at present largely unknown, however strange such a state of affairs may appear, especially in contrast with the presumably more complicated differential case, where the beautiful theory has been developed (see [12], [2], [14]). Want of space has led to the exclusion from the notes of the following topics which are of interest:

--Noncommutative calculus of variations which, in its differential part, stands in the same relation to the left invariant calculus of variations on a Lie group G as the Poisson structure on the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ of G stands to the left invariant part of the usual Hamiltonian formalism on the cotangent bundle T^*G.

--The restriction of a family of commuting flows to the stationary manifold of one of them, leading to the theory of the so-called "Finite Depth"-type equations.

--Generalized theorems on splitting and translated invariants for Lie algebras over function rings.

In order not to extend the size of the notes beyond the bounds of reason, I have omitted the most voluminous chapter X with proofs of the Hamiltonian property of a few quadratic and cubic matrices. The reader can reconstruct the proofs using methods of Chapter VIII (see also Chapter 1 in [10]).

These notes are an expanded version of lectures delivered at the Centre de Mathématique de l'École Normale Supérieure in the spring of 1982. I am very grateful to J.-L. Verdier for the invitation to lecture and I am much indebted to him for very stimulating discussions of the problem of deformations. My thanks go to friends and colleagues who read various parts of the manuscript: J. Gibbon, J. Gibbons, A. Greenspoon, D. Holm, S. Omohundro, and especially M. Hazewinkel who suggested numerous improvements.

The material on τ-function in the last section of Chapter IX owes much to the talks with H. Flaschka in August 1982 during my visit to Tucson. The rest of the notes were written in the Spring-Summer of 1982 while I was at the Los Alamos National Laboratory. I am much indebted to the Center for Nonlinear Studies for its hospitality, and to M. Martinez for the speedy typing.

CONTENTS

Chapter 0. Introduction

The subject of these notes is an infinite-particle analog of those integrable systems of classical mechanics which are analogous to the Toda lattice. Let us begin with this lattice in the form first studied by Toda [11].

Consider a classical mechanical system with the Hamiltonian

$$H = \sum_n \left[\frac{1}{2} p_n^2 + \exp(q_{n-1} - q_n)\right] , \tag{0.1}$$

where the summation above takes place either over

$$n \in \mathbb{Z}_N \tag{0.2}$$

or

$$n \in \mathbb{Z} . \tag{0.3}$$

There are other possibilities for the range of n, coupled with alterations of the potential energy at end-points. However, we will not discuss them here, since they lead to different points of view of the Toda lattice.

For the Hamiltonian (0.1) the equations of motion expressed in the form

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} , \quad \dot{q}_n = \frac{\partial H}{\partial p_n} ,$$

become

$$\begin{cases} \dot{p}_n = \exp(q_{n-1} - q_n) - \exp(q_n - q_{n+1}) , \\ \dot{q}_n = p_n . \end{cases} \tag{0.4}$$

If we introduce new variables

$$v_n = \exp(q_{n-1} - q_n) , \quad u_n = p_n , \tag{0.5}$$

then (0.4) implies, but is not equivalent to,

$$\begin{cases} \dot{p}_n = v_n - v_{n+1} , \\ \dot{v}_n = v_n(p_{n-1} - p_n) . \end{cases} \tag{0.6}$$

This system looks algebraic and is open to interpretations. The most important property of this system is the existence of a "Lax representation", i.e., an equation of the form

$$L_t = [P,L] . \tag{0.7}$$

Indeed, if one takes

$$L = \begin{vmatrix} \ddots & & & & \\ & v_{n-2} & p_{n-1} & 1 & \\ & & v_{n-1} & p_n & 1 \\ & & & v_n & p_{n+1} & 1 \\ & & & & & \ddots \end{vmatrix} , \quad P = \begin{vmatrix} \ddots & 0 & & \\ \ddots & v_{n-2} & 0 & \\ & & v_{n-1} & 0 \\ & & & v_n & \ddots \end{vmatrix} , \tag{0.8}$$

then (0.7) turns into (0.6). The immediate consequence of the Lax representation is, usually, a flood of integrals; in our case, from (0.7), we get

$$(\mathrm{Tr}\, L^m)_t = \mathrm{Tr}[P,L^m] , \tag{0.9}$$

and the trace of a commutator is supposed to vanish.

There is a slight problem, however. What is this trace when we have an infinite number of particles (which makes L into an infinite matrix)? The answer can be modelled from the theory of differential Lax equations (see, e.g. [10]). Consider, for instance, the Korteweg-de Vries equation

$$u_t = 6uu_x - u_{xxx} , \tag{0.10}$$

which we write in the form

$$u_t = \partial \frac{\delta}{\delta u} (u^3 + \frac{1}{2} u_x^2) , \quad \partial \equiv \frac{\partial}{\partial x} , \quad \frac{\delta}{\delta u} = \sum_{n \geq 0} (-\partial)^n \frac{\partial}{\partial(\partial^n(u))} . \tag{0.11}$$

One immediately recognizes that since we have an Euler-Lagrange operator $\frac{\delta}{\delta u}$ in (0.11), it must act on differential forms: in the above case, on $(u^3 + \frac{1}{2} u_x^2)dx$. Thus the correct objects of the theory are densities and not integrals $\int_{-\infty}^{\infty} dx(\cdot)$. In consequence, the coordinatized version of the geometric calculus at one point (which was called by Gel'fand and Dikii the "formal calculus of variations"), gives all the machinery necessary to study most of the problems concerning differential Lax equations.

We shall develop an analogous point of view for equations of the type (0.6). Briefly speaking, we should work with densities instead of a global Tr as in (0.9), and for this we need an appropriate calculus.

Here is a clue of how to proceed. First, we need to remove n from our equations. For instance, for (0.6) we consider p(n) and v(n) as functions p and v on $\mathbb{Z}$ with values in some field $\mathbb{k}$ of characteristic zero, say $\mathbb{R}$ or $\mathbb{C}$. (I will not comment anymore on the periodic case $n \in \mathbb{Z}_N$: all results will remain true if we impose the periodicity condition.) The set of all such functions on $\mathbb{Z}$ is a $\mathbb{k}$-algebra, with pointwise multiplication. Let us introduce the shift operators Δ^k acting as

$$(\Delta^k f)(n) = f(n+k) , \tag{0.12}$$

for any function f. Then we can rewrite (0.6) as

$$\begin{cases} p_t = v - \Delta v , \\ v_t = v(\Delta^{-1}p - p) , \end{cases} \tag{0.13}$$

which are equalities between functions. At this stage it becomes clear that the base $\mathbb{Z}$ is not important and we can make sense out of (0.13) in any situation where we have an automorphism Δ acting on a ring C generated by $\Delta^k p$ and $\Delta^s v$.

Now we can find a densities-related version of the matrix form (0.7), (0.8) of our system (0.6). Consider the associative ring $C[\zeta, \zeta^{-1}]$ of operators with coefficients in C and relations

$$\zeta^s f = \Delta^s(f)\zeta^s \ . \tag{0.14}$$

Take

$$L = \zeta + p + v\zeta^{-1} \ , \ P = v\zeta^{-1} \in C[\zeta,\zeta^{-1}] \ . \tag{0.15}$$

Then if we extend the action of $\frac{\partial}{\partial t}$ to $C[\zeta,\zeta^{-1}]$ in the natural way,

$$L_t = p_t + v_t\zeta^{-1} \ ,$$

$$[P,L] = [v\zeta^{-1},\zeta+p+v\zeta^{-1}] = v\zeta^{-1}(\zeta+p) - (\zeta+p)v\zeta^{-1}$$

$$= v + v\Delta^{-1}(p)\zeta^{-1} - \Delta(v) - pv\zeta^{-1} = v - \Delta(v) + [v\Delta^{-1}(p)-vp]\zeta^{-1} \ ,$$

and equating ζ^o- and ζ^{-1}-terms we obtain (0.13).

Thus (0.15) provides us with a Lax representation for the equations (0.13), which strongly suggests that, in all likelihood, there exist many remarkable features associated with Lax equations in the differential case (see, e.g., [2,12-14]). This is indeed the case and we will see this in the subsequent chapters. The breakdown of the chapters is as follows.

In Chapter I we consider an abstract scheme which generates Lax derivations. In Chapter II we develop a calculus which plays for the equations of type (0.13) the same role as that played by the formal calculus of variations over differential rings for differential Lax equations. In Chapter III we specialize constructions of Chapter I to get discrete Lax equations such as (0.13). We use Chapter II to find an infinite number of integrals of those equations and study various Hamiltonian forms of these Lax equations; that is to say, connections between the conservations laws (= integrals) and the equations themselves. Chapter IV is devoted to modified equations, and their morphisms into (unmodified) equations of Chapter III. Using the modified equations, in Chapter V we study certain one-parameter families of discrete equations which contain discrete Lax equations when this parameter vanishes. Considering these families as curves in the space of equations, we use the results of Chapter IV to find contractions of these curves into their basepoints. In Chapter VI we discuss various aspects of passing to a "continuous limit", from discrete to differential equations. In Chapter VII we develop a calculus which incorporates both differential and discrete degrees of freedom. We show that this calculus behaves naturally with respect to continuous limits. In Chapter VIII we begin to study the Hamiltonian formalism and find a one-to-one correspondence between linear Hamiltonian operators and Lie algebras over rings with calculus. In Chapter IX we study formal eigenfunctions of the Lax operators, together with associated constructions of conservation laws. Finally, in Chapter X we provide proofs of the Hamiltonian property of the various operators constructed in Chapters III and IV.

Chapter I. The Construction of Lax Equations

In this chapter we fix the structure of basic equations and discuss their first properties.

1. Abstract Lax Derivations

Before embarking on the construction of Lax equations in our discrete framework, let us briefly review the corresponding construction in the differential case [12].

Consider a differential operator

$$L = \sum_{i=0}^{n} u_i \xi^i \ , \tag{1.1}$$

where ξ can be thought of as "$\frac{d}{dx}$".

The u_i are $\ell\times\ell$ matrices satisfying the following conditions:

(1.2) the leading coefficient u_n is an invertible diagonal matrix, $u_n =$ diag $(c_1,\ldots,c_\ell)$, where the c_α are constants;

(1.3) if $c_\alpha = c_\beta$, then $u_{n-1,\alpha\beta} = 0$.

Let $\bar{B}$ be the differential algebra

$$\bar{B} = k[u^{(j)}_{i,\alpha\beta}] \ , \ 1 \le \alpha,\beta \le \ell \ , \ 0 \le i \le n-1 \ , \ j \ge 0 \ , \tag{1.4}$$

where k is an arbitrary field of characteristic zero to which the constants c_α belong (say, $k = \mathbb{R}$ or $\mathbb{C}$); in accordance with (1.3) we do not introduce any symbols $u^{(j)}_{n-1,\alpha\beta}$ if $c_\alpha = c_\beta$. The derivation on $\bar{B}$ which makes it into a differential algebra is defined as usual by its action on generators:

$$\partial : u^{(j)}_{i,\alpha\beta} \to u^{(j+1)}_{i,\alpha\beta} \ , \ \partial : k \to 0 \ .$$

Let $\mathrm{Mat}_\ell(\bar{B})$ be the ring of $\ell x \ell$ matrices over $\bar{B}$. Now consider the associative ring of formal pseudo-differential operators with coefficients in $\bar{B}$:

$$\mathrm{Mat}_\ell(\bar{B})((\xi^{-1})) = \{ \sum_{i\le N<\infty} v_i \xi^i | v_i \in \mathrm{Mat}_\ell(\bar{B})\} \ , \tag{1.5}$$

with commutation relations

$$\xi^m b = \sum_{j=0}^{m} \binom{m}{j} b^{(j)} \xi^{m-j} \ , \ m \geq 0 \ ,$$

$$b \in Mat_\ell(\bar{B}) \ ,$$

$$\xi^{-m} b = \sum_{j=0}^{\infty} (-1)^j \binom{m+j-1}{j} b^{(j)} \xi^{-m-j} \ , \ m > 0 \ ,$$

where $b^{(j)} = \partial^j(b)$ and ∂ is naturally extended from $\bar{B}$ to $Mat_\ell(\bar{B})$.

If $P = \sum_{i=-\infty}^{N} p_i \xi^i$ is any element of $Mat_\ell(\bar{B})((\xi^{-1}))$, we denote

$$P_+ = \sum_{i=0}^{N} p_i \xi^i \ , \ P_- = P - P_+ = \sum_{i<0} p_i \xi^i \ . \tag{1.6}$$

Now let Z(L) be the centralizer of L (1.1) in $Mat_\ell(\bar{B})((\xi^{-1}))$:

$$Z(L) = \{P \in Mat_\ell(\bar{B})((\xi^{-1})) \mid PL = LP\} \ . \tag{1.7}$$

Definition 1.8. An evolutionary derivation of $\bar{B}$ is a derivation that commutes with ∂ and k.

Obviously, an evolutionary derivation is uniquely defined by its values on the generators $u_{i,\alpha\beta}$, and it can be naturally extended to act coefficient-wise on $Mat_\ell(\bar{B})((\xi^{-1}))$.

Definition 1.9. For L given by (1.1), a Lax equation is an equation of the form

$$\partial_t(L) = [Q,L] \tag{1.10}$$

with some $Q \in Mat_\ell(\bar{B})((\xi^{-1}))$, and an evolutionary derivation $\frac{\partial}{\partial t}$ where $\partial_t(L) := \sum_{i=0}^{n} \partial_t(u_i)\xi^i$, provided (1.10) makes sense: that is, [Q,L] is a differential operator of order $\leq n-1$ with its ξ^{n-1} - coefficient satisfying condition (1.3).

For a given L, the full description of all possible Lax equations is not known outside the scalar case $\ell = 1$ [10]. In the matrix case, the current theory proceeds as follows [12].

For any $P \in Z(L)$, consider the evolutionary derivation ∂_P of $\bar{B}$ defined by the Lax equation

$$\partial_P(L) = [P_+, L] = [-P_-, L] \ . \tag{1.11}$$

(The first of these equalities shows that $\partial_P(L)$ is a differential operator while the second implies that $\partial_P(L)$ has order $\leq n-1$ and satisfies (1.3). Also, $[P_+, L] = [-P_-, L]$ because $0 = [P_+ + P_-, L]$.)

Thus each element of Z(L) determines the corresponding Lax equation, and hence the evolutionary derivation of $\bar{B}$. The main property of these derivations is that <u>they mutually commute</u>:

<u>Proposition 1.12</u>. If $P, Q \in Z(L)$, then

$$[\partial_P, \partial_Q] = 0 \ . \tag{1.13}$$

As Wilson explains, this in turn follows from the two facts: 1) that P and Q commute:

<u>Proposition 1.14</u>. Z(L) is an abelian subalgebra in $\mathrm{Mat}_\ell(\bar{B})((\xi^{-1}))$; and 2) Z(L) can be described explicitly:

<u>Proposition 1.15</u>. Every element of Z(L) is a sum of elements with highest terms of the form $p\xi^r$ where p is a constant matrix belonging to the center of the centralizer of u_n in $\mathrm{Mat}_\ell(k)$.

The interested reader can consult [12] for the proofs of (1.13)-(1.15). The message one can extract from the propositions above is this: whenever one has a reasonably detailed description of an abelian centralizer Z(L), then, whatever the situation, one can hope to show that all related Lax derivations mutually commute.

The path indicated above is the one which we shall follow in this chapter, but first we need to describe a formal framework for our study (alluded to above as "situation").

Let $k[\bar{x}]$ denote the associative algebra over k

$$k[\bar{x}]: = k[x_o, x_1, \dots] \tag{1.16}$$

with generators $x_o, x_1, \dots$, whose number can be finite or infinite. We make $k[\bar{x}]$ into a graded algebra over k giving variables x_j the weights

$$w(x_j) = \beta - \alpha j \ , \tag{1.17}$$

with some $\beta, \alpha \in \mathbb{N}$. Thus we may consider the completion of $k[\bar{x}]$ with respect to the above grading (allowing infinite sums). We denote this completion by $\hat{k}[x]$. Consider the following element in $\hat{k}[\bar{x}]$:

$$L = x_o + x_1 + \dots \tag{1.18}$$

Proposition 1.19. The centralizer Z(L) of L in $\hat{k}[\bar{x}]$ is generated over k by the elements $\{L^n, n \in \mathbb{Z}_+\}$.

Proof. Obvious. Let $Q \in Z(L)$ and let q, say, be its homogeneous component of highest weight. Then q must commute with the highest weight component of L, viz. x_o. But in our set-up nothing commutes with a given element, save for the constants from k and the powers of itself. Thus $q = \text{const}\cdot x_o^n$, then we take $Q - \text{const}\cdot L^n$, etc. □

Notation. If $P = \sum_k p_k$ is an element of $\hat{k}[x]$, then

$$P_+ = \sum_{w(p_k) \geq 0} p_k \ , \ P_- = P - P_+ = \sum_{w(p_k) < 0} p_k \ . \tag{1.20}$$

Let (β, α) denote the greatest common divisor of β and α, and let

$$\gamma = \frac{\alpha}{(\beta, \alpha)} \ . \tag{1.21}$$

Let us take a look at $P = L^{k\gamma}$, $k \in \mathbb{N}$. Writing it in long hand, with subscripts standing for weights, we have

$$L^{k\gamma} = p_{k\gamma\beta} + p_{k\gamma\beta-\alpha} + p_{k\gamma\beta-2\alpha} + \ldots + p_{k\gamma\beta-\bar{k}\alpha=0} + p_{-\alpha} + \cdots , \tag{1.22}$$

where

$$\bar{k} = k \frac{\beta}{(\beta,\alpha)} .$$

Thus $P = L^{k\gamma}$ has an element of weight zero.

Consider now, for this $P = L^{k\gamma}$, the following expression

$$[P_+,L] = [-P_-,L] . \tag{1.23}$$

since

$$P_- = p_{-\alpha} + p_{-2\alpha} + \cdots , \tag{1.24}$$

the weights of the elements in (1.23) take the values $\beta-\alpha$, $\beta-2\alpha, \ldots$, and therefore we can afford the following definition:

Definition 1.25. For any $k \in \mathbb{N}$, the derivation ∂_P of $\hat{k}[\bar{x}]$ is defined by

$$\begin{cases} w(\partial_P) = 0 \text{ (that is, } \partial_P \text{ is homogeneous of degree zero)} , \\ \partial_P(L) = [P_+,L] = [-P_-,L] , \quad P = L^{k\gamma} . \end{cases} \tag{1.26}$$

In other words

$$\partial_P(x_n) = \{\text{component of weight } \beta-n\alpha \text{ in } [-P_-,L]\}, n \in \mathbb{Z}_+ . \tag{1.27}$$

In particular,

$$\partial_P(x_0) = 0 . \tag{1.28}$$

Let us denote $(x_n, x_{n+1}, \ldots)$, for a given $n \in \mathbb{N}$, an ideal in $\hat{k}[\bar{x}]$ generated by those words which contain at least one of x_j, with $j \geq n$, among their letters.

Proposition 1.29. For $P = L^{k\gamma}$,

$$\partial_P(x_n, x_{n+1}, \dots) \subset (x_n, x_{n+1}, \dots) .$$

Proof. Since P_+ has elements of non-negative weights only, we have from (1.27):

$$\partial_P(x_j) = \{\text{component of weight } \beta - j\alpha \text{ in } [P_+, L]\}$$

$$= \sum_{r=0}^{k\beta/(\beta,\alpha)} [p_{r\alpha}, x_{j+r}] \subset (x_j, x_{j+1}, \dots) . \qquad \square$$

Remark 1.30. I hope that the importance of the proposition (1.29) is clear to the reader: it says that allowing for an infinite number of x's, we treat the universal case which we can specialize to our liking by putting $x_n = x_{n+1} = \dots = 0$ whenever we please.

Remark 1.31. The reader may wonder what had happened to the elements of Z(L) other than those we looked at above. The answer is clear from a glance at (1.24): for the weights in $\partial_P(L)$ to form an arithmetic progression $\beta - j\alpha$, $j > 0$, $P = L^n$ must have weights belonging to $\mathbb{Z}\alpha$. Thus n must be proportional to $\gamma = \alpha/(\beta,\alpha)$. This fact can be explained also from a different point of view which is to consider another ring $\hat{\mathcal{R}}[\bar{y}]$ with variables y_o, $y_1, \dots$, and weights $w(y_j) = \beta - j$. Thus $\alpha = \gamma = 1$, and the full centralizer of $\bar{L} = y_o + y_1 + \dots$ is important. If we now want to specialize to the case

$$\{y_j = 0 \ , \ j \not\equiv 0 \ (\mathrm{mod}\ \alpha)\} , \tag{1.32}$$

and denote the remaining variables $y_{\alpha j}$ as x_j, the only derivations (of $\hat{\mathcal{R}}[\bar{y}]$) which survive the specialization (1.32) are exactly those which correspond to elements $\bar{L}^n$ of $Z(\bar{L})$ with $n \equiv 0 \ (\mathrm{mod}\ \gamma)$.

2. Commutativity of Lax Derivations

In this section we prove an analog of (1.13).

Theorem 2.1. Let $P = L^{k\gamma}$, $Q = L^{r\gamma}$, where $k, r \in \mathbb{N}$, $\gamma = \alpha/(\beta,\alpha)$. Then the derivations ∂_P and ∂_Q defined by (1.26), commute.

Proof. Applying ∂_P to the equality $[Q,L] = 0$ and using (1.26), we obtain

$$0 = [\partial_P(Q),L] + [Q,[-P_-,L]] = [\partial_P(Q)+[P_-,Q],L] ,$$

and so $\partial_P(Q) + [P_-,Q]$ commutes with L. I contend that it is zero:

$$\partial_P(Q) = [-P_-,Q] . \tag{2.2}$$

Indeed $(Q-x_o^{r\gamma}) \in (x_1,x_2,\ldots)$ and by (1.28) and (1.29), $\partial_P(Q) \in (x_1,x_2,\ldots)$. Also $P_- \in (x_1,x_2,\ldots)$, and thus $[P_-,Q] \in (x_1,x_2,\ldots)$. Altogether we have $(\partial_P(Q)+[P_-,Q]) \in (x_1,x_2,\ldots)$ and it follows from (1.19) that $\partial_P(Q)+[P_-,Q]$ must be a constant, which is, of course, zero, since $\partial_P(Q)+[P_-,Q]$ is a homogeneous polynomial of degree $(k+r)\gamma$ in variables x_j. Thus we have proved (2.2) which can be rewritten as

$$\partial_P(Q) = [P_+,Q] = [-P_-,Q] , \tag{2.3}$$

thanks to the relation $[P,Q] = 0$. Since $\partial_P(Q_+) = (\partial_P(Q))_+$, we find that

$$\partial_P(Q_+) = [-P_-,Q]_+ = [-P_-,Q_+]_+ . \tag{2.4}$$

We can now deduce that $[\partial_P,\partial_Q](L) = 0$, and this will be enough since both ∂_P and ∂_Q have weight zero. We now have

$$\partial_P\partial_Q(L) = \partial_P([Q_+,L]) = [[-P_-,Q_+]_+,L] + [Q_+,[P_+,L]] ,$$

$$\partial_Q\partial_P(L) = \partial_Q([P_+,L]) = [[-Q_-,P_+]_+,L] + [P_+,[Q_+,L]] .$$

Subtracting and using the Jacobi identity, we find that $[\partial_P,\partial_Q](L)$ is equal to the bracket of L with

$$[P_-,Q_+]_+ + [P_+,Q_-]_+ + [P_+,Q_+] ,$$

which is zero as we can see at once by taking the positive part of

$$0 = [P,Q] = [P_++P_-,Q_++Q_-] . \qquad \square$$

Remark 2.5. The proof above, based upon Wilson's treatment of the differential case [12], shows also that a large part of the differential theory is due to general algebraic principles and not to the specifics of differential algebras.

Finally, we prepare the grounds for appearance of conservation laws, which will be abbreviated as c.l.'s.

Definition 2.6. For $P = \Sigma p_k \in \hat{\mathcal{k}}[\bar{x}]$, where $w(p_k) = k$,

$$\mathrm{Res}\, P := p_o \ .$$

Taking the residue of both sides of (2.3), we obtain

Proposition 2.7. Let P,Q be as in (2.1). Then

$$\partial_P(\mathrm{Res}\, Q) = \mathrm{Res}([P_+,Q]) \ . \tag{2.8}$$

Chapter II. Discrete Calculus of Variations

In this chapter we develop a discrete version of the calculus, which is the foundation of the Hamiltonian interpretation of the Lax equations. This interpretation will be given in subsequent chapters. Before reading on, the reader might wish to review the differential case, e.g. from [5], [10].

Again, k is a field of characteristic zero. Let K be a commutative algebra over k, and let $\Delta_1, \ldots, \Delta_r$: K → K be r mutually commuting automorphisms of K over k. For any $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathbb{Z}^r$, denote $\Delta^\sigma = \Delta_1^{\sigma_1} \cdots \Delta_r^{\sigma_r}$.

Let C denote the ring of polynomials

$$C = K[q_j^{(\nu_j)}] \ , \ j \in J, \ \nu_j \in \mathbb{Z}^r \ , \tag{1}$$

with independent commuting variables $q_j^{(\nu_j)}$. We extend the action of Δ's to C defining

$$\Delta^\sigma(q_j^{(\nu)}) = q_j^{(\sigma+\nu)} \ , \tag{2}$$

where $\sigma+\nu$ is defined naturally by the additive structure of $\mathbb{Z}^r$.

We also denote

$$q_j = q_j^{(0)} \ . \tag{3}$$

Definition 4. A derivation $\hat{X}$ or C is called evolutionary if it commutes with $\Delta_1, \ldots, \Delta_r$ and is trivial on K.

Thus an evolutionary derivation, sometimes also called on evolutionary (vector) field, is uniquely determined by its values on q_j's which are, of course, arbitrary:

$$\hat{X} = \sum_{j \in J, \sigma \in \mathbb{Z}^r} \Delta^\sigma(\hat{X}(q_j)) \cdot \frac{\partial}{\partial q_j^{(\sigma)}} \ . \tag{5}$$

Notice that evolutionary derivations form a Lie algebra.

Definition 6. $\Omega^1(C)$, called the module of 1-forms over C, is a C-bimodule

$$\{\Sigma f_j^\sigma dq_j^{(\sigma)} \mid f_j^\sigma \in C, \text{ finite sums}\} \ . \tag{7}$$

The usual universal derivation $d: C \to \Omega^1(C)$ (over K) is defined by its values on generators:

$$d: q_j^{(\sigma)} \to dq_j^{(\sigma)} \ . \tag{8}$$

We extend the Δ's from C to $\Omega^1(C)$ by requiring the following diagram to be commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Delta^\sigma} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \Omega^1(C) & \xrightarrow{\Delta^\sigma} & \Omega^1(C) \end{array} \ , \ \forall \sigma \in \mathbb{Z}^r \ , \tag{9}$$

which amounts to

$$\Delta^\sigma(f dq_j^{(\nu)}) = \Delta^\sigma(f) dq_j^{(\sigma+\nu)} \ , \ \forall f \in C \ . \tag{10}$$

Again, as usual there is the standard pairing between $\Omega^1(C)$ and the C-module Der(C) of derivations of C over K: If $Z \in \mathrm{Der}(C)$, then

$$\begin{aligned} (f dq_j^{(\nu)})(Z) &= f Z(q_j^{(\nu)}) \ , \\ Z(H) &= (dH)(Z) \ , \ \forall H \in C \ . \end{aligned} \tag{11}$$

Denote:

$$\mathcal{D}_i = \Delta_i - 1 \ , \ i = 1, \dots, r \ ; \tag{12}$$

$$\mathrm{Im}\, \mathcal{D} = \sum_{i=1}^{r} \mathrm{Im}\, \mathcal{D}_i \ , \tag{13}$$

wherever we consider C or $\Omega^1(C)$. Elements in $\mathrm{Im}\,\mathcal{D}$ will be called trivial.

Denote by $\Omega_o^1(C)$ the C-bimodule of special 1-forms:

$$\Omega_o^1(C) = \{\Sigma f_j dq_j \mid f_j \in C, \text{ finite sums}\} . \tag{14}$$

The most important property of $\Omega_o^1(C)$ is the following analog of the classical du Bois-Reymond lemma:

Theorem 15. If $\omega \in \Omega_o^1(C)$ and $\omega \in \operatorname{Im}\mathfrak{D}$ then $\omega = 0$.

We break the proof into a few lemmas.

Lemma 16. If $\omega \in \Omega^1(C)$ and $\omega \in \operatorname{Im}\mathfrak{D}$ then $\omega(\hat{X}) \in \operatorname{Im}\mathfrak{D}$, for any evolutionary derivation $\hat{X}$.

Proof. We have, $[(\Delta_i - 1)(f dq_j^{(\nu)})](\hat{X}) =$

$$[\Delta_i(f) dq_j^{(\nu+1_i)} - f dq_j^{(\nu)}](\hat{X}) = \Delta_i(f)\hat{X}(q_j^{(\nu+1_i)}) - f\hat{X}(q_j^{(\nu)}) =$$

$$= [\text{by } (5)] = \Delta_i(f)\Delta^{\nu+1_i}(\hat{X}(q_j)) - f\Delta^{\nu}(\hat{X}(q_j)) =$$

$$= (\Delta_i - 1)[f\Delta^{\nu}(\hat{X}(q_j))] .$$ □

Let us write $a \sim b$ to mean: $(a-b) \in \operatorname{Im}\mathfrak{D}$.

Lemma 17. If $g \in C$ is such that $gC \sim 0$, then $g = 0$.

Proof. Let us introduce operators $\frac{\delta}{\delta q_j} : C \to C$ by

$$\frac{\delta}{\delta q_j} = \sum_{\sigma \in \mathbb{Z}^r} \Delta^{-\sigma} \frac{\partial}{\partial q_j^{(\sigma)}} . \tag{18}$$

We have

$$\frac{\delta}{\delta q_j} (\operatorname{Im} \mathfrak{D}) = 0 . \tag{19}$$

[Remark. In the differential case, $\frac{\delta}{\delta q_j} = \sum_{\sigma \in \mathbb{Z}_+^r} (-\partial)^{\sigma} \frac{\partial}{\partial q_j^{(\sigma)}}$ in obvious notations. Also, $\frac{\delta}{\delta q_j}(\mathrm{Im}\partial_i) = 0.]$ Indeed,

$$\frac{\delta}{\delta q_j}(\Delta_i - 1) = \sum_\sigma \{\Delta^{-\sigma} \frac{\partial}{\partial q_j^{(\sigma)}} \Delta_i - \Delta^{-\sigma} \frac{\partial}{\partial q_j^{(\sigma)}}\}$$

$$= \sum_\sigma \{\Delta^{-\sigma} \Delta_i \frac{\partial}{\partial q_j^{(\sigma - 1_i)}} - \Delta^{-\sigma} \frac{\partial}{\partial q_j^{(\sigma)}}\}$$

$$= \sum_{\nu = \sigma - 1_i} \Delta^{-\nu} \frac{\partial}{\partial q_j^{(\nu)}} - \sum_\sigma \Delta^{-\sigma} \frac{\partial}{\partial q_j^{(\sigma)}} = 0 ,$$

where 1_i stands for the element of $\mathbb{Z}^r$ with 1 in the i-th place and zeros everywhere else, and I used the obvious commutation rule

$$\frac{\partial}{\partial q_j^{(\sigma)}} \Delta^\nu = \Delta^\nu \frac{\partial}{\partial q_j^{(\sigma-\nu)}} . \tag{20}$$

Now choose one of the q_j's present in g, and call it q. Denote by V the minimal convex hull in $\mathbb{Z}^r$ containing all points ν for which $\frac{\partial g}{\partial q^{(\nu)}} \neq 0$. Notice that the assumptions on g imply that $\Delta^\sigma(g)$ has the same property for any $\sigma \in \mathbb{Z}^r$, since $\Delta^\sigma(\mathrm{Im}\,\mathfrak{D}) = \mathrm{Im}\,\mathfrak{D}$. Thus we can assume that $0 \in \mathbb{Z}^r$ is one of the vertices of V. Let us imbed $\mathbb{Z}^r$ into $\mathbb{R}^r$. Let h be a hyperplane through 0 in $\mathbb{R}^r$ which is not parallel to any face of V and which leaves V in one of the halfspaces in which h divides $\mathbb{R}^r$. Then there exists a unique vertex ν_o of V such that V lies between h and ν_o+h. In other words, $V \cap \{V - \nu_o\} = \{0\}$.

Now take any $f(q) \in C$. If $fg \sim 0$ then $\frac{\delta}{\delta q}(fg) = 0$. So

$$0 = \frac{\partial}{\partial q^{(-\nu_o)}} \frac{\delta}{\delta q}(fg) = \frac{\partial}{\partial q^{(-\nu_o)}} [\sum_{\sigma \in V} \Delta^{-\sigma} f \frac{\partial g}{\partial q^{(\sigma)}} + \frac{\partial f}{\partial q} g] = \quad [\text{if } V \neq \{0\}]$$

$$= \sum_{\sigma\in V} \Delta^{-\sigma} \frac{\partial}{\partial q^{(-\nu_o+\sigma)}} (f \frac{\partial g}{\partial q^{(\sigma)}}) = \text{[only } \sigma = \nu_o \text{ yields something]}$$

$$= \Delta^{-\nu_o} \frac{\partial}{\partial q} (f \frac{\partial g}{\partial q^{(\nu_o)}}) .$$

Thus

$$\frac{\partial}{\partial q} (f \frac{\partial g}{\partial q^{(\nu_o)}}) = 0$$

which is a contradiction. Finally, if $V = \{0\}$, i.e. $g = g(q)$ then

$$\frac{\delta}{\delta q} (fg) = \frac{\partial}{\partial q} (fg) = 0 \Rightarrow g = 0 .$$ □

Proof of theorem 15. For an evolutionary field $\hat{X}$, let $\widehat{fX}$ denote another evolutionary field satisfying $\widehat{fX}(q_j) = f\hat{X}(q_j)$, $j\in J$. In other words, $\omega(\widehat{fX}) = f\omega(\hat{X})$, $\forall \omega\in \Omega^1_o(C)$. Now suppose there exists $\omega\in\Omega^1_o(C)$ such that $\omega\sim 0$ and $\omega\neq 0$, then we can find an evolutionary field $\hat{X}$ such that $\omega(\hat{X}) \neq 0$. Denote $g = \omega(\hat{X})$. By lemma 16, $g\sim 0$, and therefore $gf = \omega(\widehat{fX}) \sim 0$, $\forall f\in C$, which is a contradiction to the assumption $g \neq 0$. Thus $\omega = 0$. □

Corollary 21. There exists a unique projection

$$\hat{\delta} : \Omega^1(C) \to \Omega^1_o(C) \tag{22}$$

such that

$$(\hat{\delta}-1)(\Omega^1(C)) \sim 0 . \tag{23}$$

Proof. Uniqueness follows from, and is equivalent to, theorem 15. To prove existence notice that

$$f dq_j^{(\sigma)} = \Delta^{\sigma}[\Delta^{-\sigma}(f)dq_j] \sim \Delta^{-\sigma}(f)dq_j , \tag{24}$$

since $\mathrm{Im}(\Delta^{\sigma}-1) \in \mathrm{Im}\,\mathfrak{D}$. □

Let us define now the map

$$\delta : C \to \Omega_o^1(C) \tag{25}$$

by

$$\delta = \hat{\delta} d \ . \tag{26}$$

Proposition 27. For $H \in C$,

$$\delta H = \Sigma \frac{\delta H}{\delta q_j} dq_j \ , \tag{28}$$

where $\frac{\delta H}{\delta q_j} := \frac{\delta}{\delta q_j}(H)$ is defined by (18).

Proof. We have, by (24),

$$dH = \Sigma \frac{\partial H}{\partial q_j^{(\sigma)}} dq_j^{(\sigma)} \sim \sum_j \left(\sum_\sigma \Delta^{-\sigma} \left(\frac{\partial H}{\partial q_j^{(\sigma)}}\right)\right) \cdot dq_j \ . \qquad \square$$

We will call $\frac{\delta H}{\delta q_j}$ the functional derivative of H with respect to q_j.

The name comes, of course, from the formula for the first variation:

Proposition 29. For any evolutionary field $\hat{X}$, denote by $\bar{X} = \{X_j\}$ the vector $\{\hat{X}(q_j)\}_{j \in J}$. For any $H \in C$, denote by $\frac{\delta H}{\delta \bar{q}}$ the vector $\{\frac{\delta H}{\delta q_j}\}_{j \in J}$.

Then

$$\hat{X}(H) \sim \bar{X}^t \frac{\delta H}{\delta \bar{q}} \ , \tag{30}$$

where "t" stands for "transpose".

Proof.

$$\hat{X}(H) = (\Sigma\, \Delta^{\sigma}(\hat{X}(q_j)) \cdot \frac{\partial}{\partial q_j^{(\sigma)}})(H) = \Sigma\, \Delta^{\sigma}(X_j)\, \frac{\partial H}{\partial q_j^{(\sigma)}} \sim$$

$$\sim \sum_j X_j \sum_\sigma \Delta^{-\sigma} \frac{\partial H}{\partial q_j^{(\sigma)}} = \sum_j X_j \frac{\delta H}{\delta q_j} . \qquad \square$$

Now we can describe the Kernel of the operator δ.

Theorem 31.

$$\text{Ker } \delta = \text{Im}\, \mathfrak{D} + K .$$

Proof. Let $H \in C$ be such that $\delta H = 0$. Then $\hat{\delta}(dH) = 0$ and so by corollary 21, $dH \sim 0$ in $\Omega^1(C)$. But this is not enough since we don't know (yet) that

$$\{(\text{Im}\, \mathfrak{D} \cap \text{ Ker } d) \text{in } \Omega^1(C)\} = d(\text{Im}\, \mathfrak{D} \text{ in } C) . \tag{32}$$

To prove the theorem we choose the standard way of converting (30) into the homotopy formula.

Let $\omega(t)$ be a real smooth monotonically decreasing function on the interval $[0,1]$ satisfying properties $\omega(0) = 1$; $\omega(1) = 0$.

Let us extend our basic field k to $k \otimes_{\mathbb{Q}} \mathbb{R}$ but leave the notations unchanged, allowing Δ's to act on $\mathbb{R}$ as identical transformations.

Let $\rho_t : C \to C$ be the automorphism over K which takes $q_j^{(\sigma)}$ into $\omega q_j^{(\sigma)}$. Thus $\rho_t^{-1} : q_j^{(\sigma)} \to q_j^{(\sigma)}\omega^{-1}$. Consider an evolutionary field

$$\hat{X}_t = \mu\, \Sigma\, q_j^{(\sigma)} \frac{\partial}{\partial q_j^{(\sigma)}} , \text{ where } \mu = \omega^{-1} \frac{d\omega}{dt} .$$

Obviously we have

$$\frac{d}{dt} \rho_t = \rho_t \hat{X}_t . \tag{33}$$

Now let $H \in C$ and $\delta H = 0$. Then

$$\hat{X}_t(H) = \Sigma\, (\Delta^{\sigma}-1)[\hat{X}_t(q_j)\Delta^{-\sigma}(\frac{\partial H}{\partial q_j^{(\sigma)}})] . \tag{34}$$

Applying ρ_t from the left to (34), using (33) and commutativity of ρ_t with Δ^σ, we obtain

$$\frac{d}{dt}\rho_t(H) = \Sigma(\Delta^\sigma - 1)\rho_t[\mu q_j \, \Delta^{-\sigma}(\frac{\partial H}{\partial q_j^{(\sigma)}})] \ . \tag{35}$$

Integrating (35) with respect to t from t = 0 to t = 1, we find that

$$\rho_1(H) - \rho_o(H) = \Sigma(\Delta^\sigma - 1) \int_o^1 dt\{\mu(t) \ \rho_t[q_j \ \Delta^{-\sigma}(\frac{\partial H}{\partial q_j^{(\sigma)}})]\} \ . \tag{36}$$

But $\rho_1(H) = [H \text{ (all } q_j^{(\sigma)} = 0)] \in K$, and $\rho_o(H) = H$. On the other hand, the right-hand side of (36) belongs to $\mathrm{Im}\mathfrak{D} \cap C$. Indeed, take any monomial from the expression $q_j \ \Delta^{-\sigma}(\frac{\partial H}{\partial q_j^{(\sigma)}})$ and let it be

$$\alpha q_{j_1}^{(\nu_1)} \cdots q_{j_n}^{(\nu_n)} \ , \ \alpha \in K \ , \ n \geq 1 \ .$$

Then ρ_t multiplies it by ω^n. Therefore the integration produces a multiplier

$$\int_0^1 dt[\mu(t)\omega(t)^n] = \int_0^1 dt[\ \omega'(t)\omega^{-1}\omega^n] = \frac{1}{n}\,\omega(t)|_0^1 = -\frac{1}{n} \qquad .$$

Thus

$$H \sim H(0) \ . \qquad \square$$

Having found the Kernel of the operator δ, the next step is to describe its Image. This is usually done by constructing a resolvent

$$K + \mathrm{Im}\,\mathfrak{D} \to C \overset{\delta}{\to} \Omega_o^1\,(C) \overset{?}{\to} \tag{37}$$

which is exact. However, one can sidestep the problem of exactness in the term $\Omega_o^1(C)$ if one is able to find an appropriate operator which makes (37) into a complex. This is enough for most questions of the Hamiltonian formalism.

Let $A : C^n \to C^m$ be a linear operator over k.

Definition 38. An operator $A^* : C^m \to C^n$ is called adjoint to A, if

$$u^t Av \sim (A^* u)^t v \ , \ \forall u \in C^m \ , \forall v \in C^n \ ,$$

where "t" stands for "transpose". If A^* exists, then it is unique, which follows from lemma 17.

The following properties of adjoint operators are standard:

$$(A+B)^* = A^* + B^* \ , \ (AB)^* = B^* A^* \ .$$

If A is represented by the matrix $A = (A_{ij})$ then

$$(A^*)_{ij} = (A_{ji})^*$$

where A_{ij} acts on C. For such an action we record the following formula:

Proposition 39. Let $A : C \to C$ be given as $A = f\Delta^{\sigma}$, $f \in C$. Then

$$(f\Delta^{\sigma})^* = \Delta^{-\sigma} f \ .$$

Proof. $uf\Delta^{\sigma}(v) = \Delta^{\sigma}[\Delta^{-\sigma}(uf) \cdot v] \sim \Delta^{-\sigma}(fu) \cdot v$. □

The important notion is that of Fréchet derivative. Let $H \in C$ and denote

$$D_j(H) = \sum_{\sigma} \frac{\partial H}{\partial q_j^{(\sigma)}} \Delta^{\sigma} \ . \tag{40}$$

Let D(H), called the Fréchet derivative of H, be the row vector with components $D_j(H)$. By $D(H)^t$ we denote the corresponding column. Again, for any evolutionary derivation $\hat{X}$ we denote by $\bar{X}$ the vector with components $(\bar{X})_j = \hat{X}(q_j)$. We can write this fact as

$$\bar{X} = \hat{X}(\bar{q}) \ , \tag{41}$$

where $\bar{q}$ is a vector with components q_j.

Lemma 42.

$$\hat{X}(H) = D(H)\bar{X} = D(H)\hat{X}\bar{q} .$$

Proof.

$$\hat{X}(H) = \sum_{j,\sigma} \frac{\partial H}{\partial q_j^{(\sigma)}} \Delta^\sigma(\hat{X}(q_j)) = \sum_j D_j(H)\hat{X}(q_j) = D(H)\bar{X} . \qquad \square$$

Definition 43. Let $\bar{R}$ be a vector. $D(\bar{R})$, called the Fréchet derivative of $\bar{R}$ is the matrix with matrix elements $D(\bar{R})_{ij} = D_j(R_i)$, and $\hat{X}(\bar{R})$ is a vector with components $\hat{X}(R_i)$.

Lemma 44.

$$\hat{X}(\bar{R}) = D(\bar{R})\bar{X} .$$

Proof.

$$\hat{X}(\bar{R})_i = \hat{X}(R_i) = \sum_j D_j(R_i)\hat{X}(q_j) = \sum_j D(\bar{R})_{ij}X_j . \qquad \square$$

Lemma 45.

$$D\Delta^\sigma = \Delta^\sigma D .$$

Proof. For any vector $\bar{R}$, and any $\hat{X}$, we have from lemma 44:

$$\Delta^\sigma D(\bar{R})\bar{X}) = \Delta^\sigma(\hat{X}(\bar{R})) = (\text{since } \hat{X} \text{ is evolutionary})$$

$$= \hat{X}(\Delta^\sigma\bar{R}) = D(\Delta^\sigma\bar{R})\bar{X} .$$

If two operators produce the same result acting on any $\bar{X}$, they coincide. Thus

$$\Delta^\sigma D(\bar{R}) = D\Delta^\sigma(\bar{R}) \text{ , whatever } \bar{R}. \qquad \square$$

Definition 46. If $A : C^n \to C^n$ is an operator, it is called symmetric if $A^* = A$, and skew-symmetric, or skew, if $A^* = -A$.

Theorem 47. For any $H \in C$, the operator $D(\frac{\delta H}{\delta \bar{q}})$ is symmetric:

$$D\left(\frac{\delta H}{\delta \bar{q}}\right)^* = D\left(\frac{\delta H}{\delta \bar{q}}\right) , \tag{48}$$

where $\frac{\delta H}{\delta \bar{q}}$ is the vector with components $\frac{\delta H}{\delta q_j}$.

Proof. Taking matrix elements from both sides of (48), we obtain by summing on repeated indices, the result

$$\left[D\left(\frac{\delta H}{\delta \bar{q}}\right)^*\right]_{ji} = \left[D\left(\frac{\delta H}{\delta \bar{q}}\right)_{ij}\right]^* = D_j\left(\frac{\delta H}{\delta q_i}\right)^*$$

$$= \left[\frac{\partial}{\partial q_j^{(\sigma)}}\left(\frac{\delta H}{\delta q_i}\right) \cdot \Delta^{\sigma}\right]^* = \Delta^{-\sigma} \frac{\partial}{\partial q_j^{(\sigma)}}\left(\frac{\delta H}{\delta q_i}\right) . \tag{49}$$

Now

$$\frac{\partial}{\partial q_j^{(\sigma)}}\left(\frac{\delta H}{\delta q_i}\right) = \left(\frac{\partial}{\partial q_j^{(\sigma)}} \circ \frac{\delta H}{\delta q_i}\right)(1)$$

$$= \left(\frac{\partial}{\partial q_j^{(\sigma)}} \Delta^{-\nu} \frac{\partial H}{\partial q_i^{(\nu)}}\right)(1) = \left(\Delta^{-\nu} \frac{\partial}{\partial q_j^{(\sigma+\nu)}} \frac{\partial H}{\partial q_i^{(\nu)}}\right)(1)$$

$$= \Delta^{-\nu} \frac{\partial^2 H}{\partial q_j^{(\sigma+\nu)} \partial q_i^{(\nu)}} \Delta^{\nu} ,$$

and thus

$$\left[D\left(\frac{\delta H}{\delta \bar{q}}\right)^*\right]_{ji} = \Delta^{-\sigma}\Delta^{-\nu} \frac{\partial^2 H}{\partial q_j^{(\sigma+\nu)} \partial q_i^{(\nu)}} \Delta^{\nu} = \Delta^{-\mu} \frac{\partial^2 H}{\partial q_j^{(\mu)} \partial q_i^{(\nu)}} \Delta^{\nu} , \tag{50}$$

where $\mu = \sigma+\nu$. For the right-hand side of (48), we similarly get

$$\left[D\left(\frac{\delta H}{\delta \bar{q}}\right)\right]_{ji} = D_i\left(\frac{\delta H}{\delta q_j}\right) = \frac{\partial}{\partial q_i^{(\sigma)}}\left(\frac{\delta H}{\delta q_j}\right)\cdot \Delta^{\sigma} \tag{51}$$

$$= \text{(using the computation above)} = \Delta^{-\mu}\,\frac{\partial^2 H}{\partial q_i^{(\sigma+\mu)}\,\partial q_j^{(\mu)}}\,\Delta^{\mu}\Delta^{\sigma} =$$

$$= \Delta^{-\mu}\,\frac{\partial^2 H}{\partial q_i^{(\nu)}\,\partial q_j^{(\mu)}}\,\Delta^{\nu}\ ,$$

for $\nu = \mu+\sigma$, which is the same as (50). □

The theorem 47 shows that one can take the operator $D(\cdot)-D(\cdot)^*$ to form a complex in (37). To prove exactness, one then will have to construct an analog of "the higher Lagrangian formalism" ([5], Ch. II, §7,8) and use its homotopy formula. Instead of doing this, I will show how the continuous calculus comes into the picture, through an analog of "the first complex" for the operator δ ([5], ch. II, §5; [10], ch. I).

So let $\partial : K \to K$ be a derivation over k, commuting with Δ's. Let $\bar{C}$ now be $K[q_j^{(\nu_j;k_j)}]$, $\nu_j \in \mathbb{Z}^r$, $k_j \in \mathbb{Z}_+$. Δ's and ∂ act on $\bar{C}$ as

$$\Delta^{\sigma}(q_j^{(\nu;k)}) = q_j^{(\nu+\sigma;k)}\ ;\ \partial(q_j^{(\nu;k)}) = q_j^{(\nu;k+1)}\ .$$

All definitions of evolutionary fields, $\Omega^1(\bar{C})$, etc., are practically the same, the operator $\delta : \bar{C} \to \Omega^1_o(\bar{C})$ now being defined as

$$\delta(G) = \sum_j dq_j\Big[\sum_{k,\sigma} (-\partial)^k\,\Delta^{-\sigma}\,\frac{\partial G}{\partial q_j^{(\sigma;k)}}\Big]\ . \tag{52}$$

Denote by $\bar{\tau}$ the homomorphic imbedding of C and $\Omega^1(C)$ into $\bar{C}$ over K:

$$\begin{aligned}\bar{\tau}(q_j^{(\nu)}) &= q_j^{(\nu;0)}\ ,\\ \bar{\tau}(dq_j^{(\nu)}) &= q_j^{(\nu;1)}\ .\end{aligned} \tag{53}$$

Theorem 54. (First complex for the operator δ).

$\delta\bar{\tau}\delta = 0$ on C .

Proof. For $H \in C$,

$$\bar{\tau}\delta(H) = \bar{\tau}\Big(\sum_i \frac{\delta H}{\delta q_i}\, dq_i\Big) = \sum_i \bar{\tau}\Big(\frac{\delta H}{\delta q_i}\Big) q_i^{(0;1)} \; .$$

From now on let us identify $q_j^{(\nu)}$ with $q_j^{(\nu;0)}$ and thus drop the sign $\bar{\tau}$ from $\frac{\delta H}{\delta q_i}$.

Then

$$\frac{\delta}{\delta q_j}(\bar{\tau}\delta H) = \frac{\delta}{\delta q_j}\Big[\sum_i \frac{\delta H}{\delta q_i}\, q_i^{(0;1)}\Big] = \sum_{\sigma,i} \Delta^{-\sigma}\Big[\frac{\partial}{\partial q_j^{(\sigma)}}\, \frac{\delta H}{\delta q_i}\, q_i^{(0;1)}\Big]$$

$$+ (-\partial)\Big(\frac{\delta H}{\delta q_j}\Big) = \sum_{\sigma,i} \Delta^{-\sigma}\Big[\frac{\partial}{\partial q_j^{(\sigma)}}\, \frac{\delta H}{\delta q_i}\Big]\cdot q_i^{(-\sigma;1)} - \sum_{\sigma,i} \frac{\partial}{\partial q_i^{(-\sigma)}}\Big(\frac{\delta H}{\delta q_j}\Big)\cdot q_i^{(-\sigma;1)}$$

$$= \sum_{\sigma,i} q_i^{(-\sigma;1)}\Big\{\Delta^{-\sigma}\Big[\frac{\partial}{\partial q_j^{(\sigma)}}\Big(\frac{\delta H}{\delta q_i}\Big)\Big] - \frac{\partial}{\partial q_i^{(-\sigma)}}\Big(\frac{\delta H}{\delta q_j}\Big)\Big\} = 0 \; .$$

The final expression is zero since all expressions in the curly brackets vanish by (48), (49), (51).

Chapter III. Hamiltonian Form of Lax Equations

In this chapter we consider various types of discrete Lax equations and analyze different approaches for deriving their Hamiltonian forms.

1. Discrete Lax Equations

First we describe the equations with which we shall be concerned from now on. They are specializations of those considered in Chap. I.

Let $C = k[q_j^{(n_j)}]$, $j \in \mathbb{Z}_+$, $n_j \in \mathbb{Z}$, so that $K = k$ and <u>$r = 1$</u> in the notations of Chap. II. We shall write Δ instead of Δ_1.

Consider the associative algebra $C((\zeta^{-1}))$ over k with commutation relations

$$\zeta^k b = \Delta^k(b)\zeta^k \ , \ \forall b \in C \ , \ \forall k \in \mathbb{Z}, \tag{1.1}$$

which is an analog of the ring of pseudo-differential operators of Chap. I. We make $C((\zeta^{-1}))$ into a graded algebra by giving the following weights:

$$w(C) = 0 \ , \ w(\zeta^k) = k \ , \tag{1.2}$$

which is compatible with (1.1).

Denote

$$x_o = \zeta^\beta \ , \ x_{j+1} = \zeta^{\beta-\alpha(j+1)} q_j \ , \ j \in \mathbb{Z}_+ \ , \tag{1.3}$$

$$L = x_o + x_1 + \cdots = \zeta^\beta + \zeta^{\beta-\alpha} q_o + \cdots \tag{1.4}$$

By (1.2), $w(x_j) = \beta - \alpha j$, thus we can read off the results of Chap. I for the Lax equations with the operator L given by (1.4). By (I 1.28), for every appropriate P,

$$0 = \partial_P(x_o) = \partial_P(\zeta^\beta) \ ,$$

thus we can put

$$\partial_P(\zeta) = 0 \ . \tag{1.5}$$

which allows us to consider ∂_P as coming from and equivalent to an evolutionary derivation of C which we shall continue to denote by ∂_P.

Also, for $R \in C((\zeta^{-1}))$, $R = \sum_k r_k \zeta^k$, let us denote

$$R_+ = \sum_{k \geq 0} r_k \zeta^k \ , \ R_- = \sum_{k<0} r_k \zeta^k \ , \ \mathrm{Res}\, R = r_o \ , \tag{1.6}$$

which agrees with (I 1.20) and (I 2.6), thanks to (1.2).

The properties of the Lax equations can now be summarized as follows:

Proposition 1.7. Let α, $\beta \in \mathbb{N}$, $\gamma = \alpha/(\alpha,\beta)$ and let L be given by

$$L = \zeta^\beta (1 + \sum_{j \geq 0} \zeta^{-\alpha(j+1)} q_j) \ . \tag{1.8}$$

Then for every $k \in \mathbb{N}$, the evolutionary derivations ∂_P: $C \to C$, defined for $P = L^{\gamma k}$ by the formulae

$$\partial_P(L) = [P_+, L] = [-P_-, L] \ , \ w(\partial_P) = 0 \ , \ \partial_P(\zeta) = 0 \ , \tag{1.9}$$

all commute. Further, for $Q = L^{k'\gamma}$, $k' \in \mathbb{N}$,

$$\partial_P(\mathrm{Res}\, Q) = \mathrm{Res}[P_+, Q] \ . \tag{1.10}$$

Remark 1.11. The formula (1.10) can be interpreted to assert that all Lax equations (1.9) have an infinite common set of conservation laws Res $L^{k'\gamma}$, $k' \in \mathbb{N}$. This follows from the following observation:

Lemma 1.12. If $R,S \in \mathrm{Mat}_\ell(C)((\zeta^{-1}))$, then

$$\mathrm{Tr}\, \mathrm{Res}[R,S] \sim 0 \ .$$

Proof. If $R = \Sigma R_j \zeta^j$, $S = \Sigma S_j \zeta^j$ where $R_j, S_j \in \mathrm{Mat}_\ell(C)$, then

$$\begin{aligned} \mathrm{Tr}\, \mathrm{Res}[R,S] &= \mathrm{Tr}\, \mathrm{Res}\, \Sigma (R_j \zeta^j S_{-j} \zeta^{-j} - S_{-j} \zeta^{-j} R_j \zeta^j) \\ &= \Sigma\, \mathrm{Tr}[R_j \Delta^j (S_{-j}) - S_{-j} \Delta^{-j}(R_j)] \sim \Sigma\, \mathrm{Tr}[\Delta^{-j}(R_j) S_{-j} - S_{-j} \Delta^{-j}(R_j)] \\ &= \Sigma\, \mathrm{Tr}[\Delta^{-j}(R_j), S_{-j}] = 0 \ . \end{aligned}$$

□

Naturally, one would like to know that the c.l.'s (= conservation laws) Res $L^{k\gamma}$ are not trivial. This is indeed the case.

Lemma 1.13. For $k \in \mathbb{N}$, Res $L^{k\gamma} \neq 0$.

Proof. H = Res $L^{k\gamma}$ is a nonzero polynomial in variables $q_j^{(\cdot)}$ having homogeneous components of degree ≥ 1 (with respect to the usual degree) with positive integer coefficients. Its functional derivatives $\frac{\delta H}{\delta q_j}$, preserving this property of positivity, therefore do not vanish. Thus $H \neq 0$. □

Now let us show that the derivations ∂_P are not trivial:

Lemma 1.14. $\partial_P \neq 0$.

Proof. Let L be given as

$$L = \zeta^{\beta}(1+\zeta^{-\alpha}q_o+\cdots+\zeta^{-\alpha(r+1)}q_r) \ , \ \beta < \alpha(r+1) \ . \tag{1.15}$$

Since $\partial_P(L) = [P_+,L]$, we get for $\partial_P(q_r)$:

$$\zeta^{\beta-\alpha(r+1)} \ \partial_P(q_r) = \text{Res } P\cdot\zeta^{\beta-\alpha(r+1)}q_r - \zeta^{\beta-\alpha(r+1)}q_r\text{Res } P \ ,$$

so

$$\partial_P(q_r) = q_r(\Delta^{\alpha(r+1)-\beta}-1)\text{Res } P \neq 0 \ ,$$

since Res $P \notin k$. Now for the "general L" (1.8), with infinite number of q's, ∂_P couldn't vanish, for otherwise its specialization (1.15) $\bigcap_{j>r} \{q_j=0\}$ would vanish, and we have just seen this not to be the case. □

Remark 1.16. The arguments above show a little more. Let $P = \sum_{k\leq N} c_k L^{k\gamma}$, $c_k \in k$, $c_N \neq 0$. Then: a) Res $P \neq 0$, and b) $\partial_P \neq 0$. Indeed, the property b) follows from a). On the other hand, the homogeneous components of highest degree in Res P come from Res $L^{N\gamma}$, and they are not trivial.

Remark 1.17. One could obviously take q's above being $\ell\times\ell$ matrices over a ring with an automorphism as lemma 1.12 suggests. Everything we do would still be correct, but notations become more cumbersome, and we are going to have enough trouble with infinite matrices later on. I therefore avoid any mentioning of the matrix versions, leaving this to the interested reader.

2. Variational Derivatives of Conservation Laws

The main goal of the Hamiltonian description of Lax equations, is to express the derivations ∂_P's in terms of Res P's. The method, which is standard by now (see [10]), is to extend the calculus to the ring $C((\zeta^{-1}))$. The details follow.

Again, $C = k[q_j^{(n_j)}]$ and we let C' denote $C((\zeta^{-1}))$ with Δ acting on C' commuting with ζ. Denote $\Omega^1(C)((\zeta^{-1})) = \{ \sum_{i<\infty} \omega_i \zeta^i | \omega_i \in \Omega^1(C)\}$. We make $\Omega^1(C)((\zeta^{-1}))$ into a C'-bimodule by putting

$$c\zeta^i \omega \zeta^j = c\Delta^i(\omega)\zeta^{i+j}, \quad \omega\zeta^j c\zeta^i = \Delta^j(c)\omega\zeta^{i+j} \quad , \ c\in C \ , \ \omega\in\Omega^1(C) \ .$$

We also extend Δ to $\Omega^1(C)((\zeta^{-1}))$ by requiring $\Delta\zeta = \zeta\Delta$.

For $\omega\in\Omega^1(C)((\zeta^{-1}))$, $\omega = \Sigma\omega_i\zeta^i$, we define

$$\text{Res } \omega = \omega_o \ .$$

Finally, let us extend the map $d: C \to \Omega^1(C)$ to

$$d: C' \to \Omega^1(C)((\zeta^{-1})) \ , \ \text{by } d(c\zeta^i) = d(c)\cdot\zeta^i \ .$$

The maps introduced above obviously commute:

Lemma 2.1. The maps Res, Δ and d all commute.

Lemma 2.2. If $c_1, c_2 \in C'$, then

$$d(c_1c_2) = dc_1\cdot c_2 + c_1dc_2 \ .$$

The proofs are obvious.

Lemma 2.3. Let $\omega\in\Omega^1(C)((\zeta^{-1}))$, $c\in C'$. Then

$$\text{Res}(\omega c - c\omega) \sim 0 \ .$$

Proof. Using the usual summation convention, $\mathrm{Res}(\omega_j\zeta^j c_k\zeta^k - c_k\zeta^k\omega_j\zeta^j) = \omega_j\Delta^j(c_{-j}) - c_{-j}\Delta^{-j}(\omega_j) \sim 0$, since c_{-j} commutes with $\Delta^{-j}(\omega_j)$: no ζ's are involved. □

Lemma 2.4. Let $L \in C'$, $n \in \mathbb{N}$. Then

$$\mathrm{Res}\, dL^n \sim n\, \mathrm{Res}(L^{n-1}dL) \sim n\, \mathrm{Res}(dL\cdot L^{n-1}) .$$

Proof. $\mathrm{Res}\, dL^n = \mathrm{Res}(dL\cdot L^{n-1} + L\cdot dL\cdot L^{n-2} + \cdots + L^{n-1}\, dL) \sim$

$\sim \mathrm{Res}(ndL\cdot L^{n-1}) \sim \mathrm{Res}(nL^{n-1}dL)$, by lemma 2.3. □

Now we give first application of lemma 2.4. Let L be as in (1.8) or (1.15). We define

$$H_n = \frac{1}{n}\mathrm{Res}\, L^n , \tag{2.5}$$

$$L^n = \sum^{n\beta} p_s(n)\zeta^s . \tag{2.6}$$

(Of course, $H_n = 0$ for $n \not\equiv 0 \pmod{\gamma}$, but this shouldn't worry us for the moment).

Theorem 2.7.

$$p_{\alpha(j+1)-\beta}(n) = \frac{\delta H_{n+1}}{\delta q_j} .$$

Proof. Applying lemma 2.4 to our L with n+1 substituted for n, we have

$$\mathrm{Res}\, dL^{n+1} = (n+1)dH_{n+1} \sim (n+1)\, \mathrm{Res}(L^n dL)$$

$$= (n+1)\,\mathrm{Res}\Big(\sum_{s,j} p_s(n)\zeta^s\zeta^{\beta-\alpha(j+1)}dq_j\Big) = (n+1)\sum_j p_{\alpha(j+1)-\beta}(n)dq_j .$$ □

3. First Hamiltonian Structure, $\alpha = 1$

There are four different types of operators L, depending upon whether $\beta = 1$ or $\beta > 1$, and whether $\gamma = 1$ or $\gamma > 1$ (recall that $\gamma = \alpha/(\alpha,\beta)$). The difficulties of the Hamiltonian description steadily increase in the direction $(\beta=\gamma=1) \to (\beta > 1,\ \gamma = 1) \to (\beta = 1,\ \gamma > 1) \to (\beta,\ \gamma > 1)$. The case $(\beta = \gamma = 1)$ is the most transparent and the richest. We begin our study with this case.

To have one derivation simultaneously for both cases ($\gamma = 1$, $\beta = 1$) and ($\gamma = 1$, $\beta > 1$), let us take L as

$$L = \zeta^{\beta}(1 + \sum_{j\geq 0} \zeta^{-(j+1)} q_j) \tag{3.1}$$

This L is indeed the general one, for we can always reduce the case $(\beta,\alpha) > 1$ for the operator L in (1.8), to one with $(\beta,\alpha) = 1$ simply by introducing a new variable $\bar{\zeta} = \zeta^{(\beta,\alpha)}$ and replacing Δ with $\Delta^{(\alpha,\beta)}$. Then, the condition $\gamma = 1$ is equivalent to $\alpha = 1$, as in (3.1).

Now let $P = L^n = \Sigma\, p_s(n)\zeta^s$. Then

$$\partial_P(L) = [P_+,L] = [\sum_{s\geq 0} p_s(n)\zeta^s\ ,\ \zeta^{\beta}(1 + \Sigma\zeta^{-(j+1)}q_j)]$$

$$= \zeta^{\beta}[\sum_{s\geq 0} \Delta^{-\beta}(p_s(n))(\zeta^s + \Sigma\zeta^{s-j-1}q_j)]$$

$$- \zeta^{\beta}[\sum_{s\geq 0} p_s(n)\zeta^s + \sum_{s,j\geq 0} \zeta^{-j-1}q_j\, p_s(n)\zeta^s]\ .$$

Picking out the $\zeta^{\beta-r-1}$-terms from both sides, we get

$$\partial_P(q_r) = \sum_{s\geq 0} [\Delta^{r+1-\beta}(p_s(n))\cdot q_{s+r} - \Delta^{-s}(q_{s+r}p_s(n))]$$

$$= \sum_{s\geq 0} [q_{s+r}\Delta^{r+1-\beta} - \Delta^{-s}q_{s+r}]p_s(n)\ . \tag{3.2}$$

Now consider the case $\beta = 1$. Substituting (2.7) into (3.2) we arrive at

<u>Theorem 3.3</u>. (First Hamiltonian structure for the $\alpha=\beta=1$-case). The equations $\partial_P(L) = [P_+,L]$ with $P = L^n$ can be written as

$$\partial_P(q_r) = \sum_s B_{rs} \frac{\delta H}{\delta q_s}\ ,\ B_{rs} = q_{s+r}\Delta^r - \Delta^{-s}q_{s+r}\ , \tag{3.4}$$

with $H = H_{n+1} = \frac{1}{n+1}\,\mathrm{Res}\, L^{n+1}$.

A few comments are in order. The system (3.4) is Hamiltonian since the matrix $B = (B_{rs})$ is skew and the usual axioms of the Hamiltonian formalism (see sec. 2, Ch. VIII) are satisfied: the proofs of such satisfaction, for various matrices appearing from now on, are all relegated to Chapter X. Let us just see that B is skew:

$$(B_{rs})^* = \Delta^{-r} q_{s+r} - q_{s+r}\Delta^{s} = -B_{sr} ,$$

as stated.

Notice also the adjective "first" referring to the Hamiltonian structure (3.4): it means, that the derivation ∂_P with $P = L^n$ is expressed through $H_{n+1} = (n+1)^{-1}$ Res L^{n+1}. If it were expressed through $H_{n-k} = (n-k)^{-1}$ Res L^{n-k}, $k \in \mathbb{Z}$, it would be called the $(k+2)^{nd}$ Hamiltonian structure, etc.

Now let us see a first instance of the troubles ahead: suppose $\beta > 1$. Substituting (2.7) into (3.2), we get

$$\partial_P(q_r) = \sum_{s\geq 0} [q_{s+r}\, \Delta^{r+1-\beta} - \Delta^{-s}\, q_{s+r}] \frac{\delta H_{n+1}}{\delta q_{\beta-1+s}} . \tag{3.5}$$

Thus, if we write

$$\partial_P(\bar{q}) = B \frac{\delta H}{\delta \bar{q}}, \tag{3.6}$$

to mean

$$\partial_P(q_r) = \sum_{p\geq 0} B_{rp} \frac{\delta H}{\delta q_p} , \tag{3.7}$$

then the matrix B which corresponds to (3.5) is not even skewsymmetric, since its first β-1 columns are zeros, while the same is not true for the first β-1 rows. Thus the representation (3.5) is of no use and we need another one. Before looking for a remedy though, one can try to save as much as possible from (3.5). Denote $Q_s = q_{\beta-1+s}$, $s \geq 0$; $R_j = q_{\beta-2-j}$, $0 \leq j \leq \beta-2$. Then (3.5) implies

$$\partial_P(Q_r) = \sum_{s\geq 0} [Q_{s+r}\Delta^r - \Delta^{-s}Q_{s+r}] \frac{\delta H_{n+1}}{\delta Q_s} , \tag{3.8}$$

which is the same as (3.4), up to a change in notation. Thus the Q-variables split from the rest.

To find an appropriate form for the evolution of the R's, we use the second representation for the derivation ∂_P: $\partial_P(L) = [-P_-,L]$. Writing in long hand, we obtain

$$\partial_P(L) = [\zeta^\beta(1+\Sigma\zeta^{-j-1}q_j) , \sum_{s<0} p_s(n)\zeta^s]$$

$$= \zeta^\beta\{ \sum_{s<0} p_s(n)\zeta^s + \sum_{s<0,j\geq 0} \zeta^{-j-1} q_j p_s(n)\zeta^s - \sum_{s<0} \Delta^{-\beta}(p_s(n))\zeta^s$$

$$- \sum_{s<0,j\geq 0} \Delta^{-\beta}(p_s(n))\zeta^{s-j-1}q_j\}$$

$$= \zeta^\beta\{ \sum_{s<0} (1-\Delta^{-\beta})(p_s(n))\zeta^s + \sum_{s<0,j\geq 0} \zeta^{s-j-1}[\Delta^{-s}(q_j p_s(n))$$

$$- q_j\Delta^{j+1-s-\beta}(p_s(n))]\} .$$

Picking out the $\zeta^{\beta-r-1}$-terms, we get

$$\begin{aligned} \partial_P(q_o) &= (1-\Delta^{-\beta})\Delta p_{-1}(n) , \\ \partial_P(q_{r+1}) &= (1-\Delta^{-\beta})\Delta^{r+2}p_{-r-2}(n) + \sum_{s=-1}^{-r-1} [\Delta^{-s}q_{s+r+1} - q_{s+r+1}\Delta^{r+2-\beta}]p_s(n) . \end{aligned} \tag{3.9}$$

We can combine the two formulae in (3.9) into

$$\partial_P(q_r) = (1-\Delta^{-\beta})\Delta^{r+1}p_{-r-1}(n) + \sum_{-r\leq s<0} [\Delta^{-s}q_{s+r} - q_{s+r}\Delta^{r+1-\beta}]p_s(n) , \tag{3.10}$$

agreeing to drop the sum when it is empty for r = 0. Now consider r in (3.10) running from 0 to β-2. Then

$$\partial_P(R_j) = \partial_P(q_{\beta-2-j}) = (1-\Delta^{-\beta})\Delta^{\beta-1-j}p_{j+1-\beta}(n)$$

$$+ \sum_{j+2-\beta\le s<0} [\Delta^{-s}R_{j-s} - R_{j-s}\Delta^{-1-j}]p_s(n) \ , \ 0 \le j \le \beta-2 \ .$$

Substituting $p_s(n) = \frac{\delta H_{n+1}}{\delta q_{s+\beta-1}} = \frac{\delta H_{n+1}}{\delta R_{-s-1}}$, $s < 0$; $p_{j+1-\beta}(n) = \frac{\delta H_{n+1}}{\delta R_{\beta-2-j}}$,

$0 \le j \le \beta-2$, we get

$$\partial_P(R_j) = (1-\Delta^{-\beta})\Delta^{\beta-1-j} \frac{\delta H_{n+1}}{\delta R_{\beta-2-j}}$$
$$+ \sum_{0<s\le\beta-2-j} [\Delta^{s}R_{j+s} - R_{j+s}\Delta^{-1-j}] \frac{\delta H_{n+1}}{\delta R_{s-1}} \ , \ 0 \le j \le \beta-2 \ . \tag{3.11}$$

Thus we see that the R-variables also split from the rest. Changing s into s+1, we once again rewrite (3.11) in the form

$$\partial_P(R_j) = (1-\Delta^{-\beta})\Delta^{\beta-1-j} \frac{\delta H_{n+1}}{\delta R_{\beta-2-j}}$$
$$+ \sum_{0\le s<\beta-2-j} [\Delta^{1+s}R_{s+j+1} - R_{j+s+1}\Delta^{-1-j}] \frac{\delta H_{n+1}}{\delta R_s} \ . \tag{3.12}$$

The matrix B which corresponds to (3.12) via

$\partial_P(R_j) = \sum_s B_{js} \frac{\delta H_{n+1}}{\delta R_s}$, is now clearly skew-symmetric. Its Hamiltonian property will be proven in Chap. VIII (Theorem VIII 5.38).

Remark 3.13. Although an infinite number of q's in (1.8) can be painlessly cut out to reduce (1.8) to (1.15), it might not be the case for the matrices B's which result from the manipulation of formal identities. So far, for the matrices in (3.4) and (3.8), everything is fine: B_{rs}, for all s and fixed r, involves only those q_j (or Q_j) for which $j \ge r$.

Remark 3.14. It should be clear by now that the first Hamiltonian structure could not possibly exist for the case $\alpha > 1$: If, as in remark I 1.31, we try to treat the case $\alpha > 1$ by putting equal to zero some of the original variables q_j (except when $j \equiv -1 (\mathrm{mod}\ \alpha)$), then H_{n+1} will vanish since there are no weight-zero terms in Res L^n for $n \not\equiv 0 (\mathrm{mod}\ \gamma)$. Thus, the most one could hope for in the case $\gamma > 1$, is the second Hamiltonian structure.

4. Second Hamiltonian Structure, $\beta = 1$

Let L be given as

$$L = \zeta(1 + \sum_{j\geq 0} \zeta^{-\gamma(j+1)} q_j) . \tag{4.1}$$

We write, in a notation analogous to that of (2.6),

$$L^{\gamma n} = \sum_s p_s(\gamma n)\zeta^{\gamma s} , \quad L^{\gamma n-1} = \sum_s p_s(\gamma n-1)\zeta^{\gamma s-1} . \tag{4.2}$$

Using (2.4), we have

$$\mathrm{Res}\ dL^{\gamma n} \sim \gamma n\ \mathrm{Res}(L^{\gamma n-1} dL) = \gamma n\ \mathrm{Res}(L^{\gamma n-1} \sum_j \zeta^{1-\gamma(j+1)} dq_j)$$

$$= \gamma n \sum_j p_{j+1}(\gamma n-1) dq_j .$$

Denoting

$$H_{\gamma n} = \frac{1}{\gamma n} \mathrm{Res}\ L^{\gamma n} , \tag{4.3}$$

we thus obtain (as in the case of theorem 2.7)

$$p_{s+1}(\gamma n-1) = \frac{\delta H_{\gamma n}}{\delta q_s} , \quad s \geq 0 . \tag{4.4}$$

Now let us write down the Lax equations. We have

$$\partial_p(L) = [(L^{\gamma n})_+, L] = [\sum_{k\geq 0} p_k(\gamma n)\zeta^{k\gamma}, \zeta(1 + \sum_j \zeta^{-\gamma(j+1)} q_j)]$$

$$= \zeta\{(\Delta^{-1}-1)p_k(\gamma n)\zeta^{k\gamma} + \zeta^{\gamma(k-j-1)}\ \Delta^{-\gamma(k-j-1)-1}(p_k(\gamma n))q_j$$

$$- \zeta^{\gamma(k-j-1)}\ \Delta^{-k\gamma}(q_j\ p_k(\gamma n))\}\ ,$$

and therefore

$$\partial_p(q_r) = \sum_{k\geq 0} (q_{k+r}\ \Delta^{\gamma(r+1)-1} - \Delta^{-k\gamma} q_{k+r}) p_k(\gamma n)\ . \tag{4.5}$$

Now we need to express $p_k(\gamma n)$ through $H_{\gamma n}$. For this, we expand in the powers of ζ the two identities: $L^{\gamma n} = L^{\gamma n-1}L$ and $L^{\gamma n} = LL^{\gamma n-1}$. We have then, from (4.2):

$$L^{\gamma n} = \sum_s p_s(\gamma n)\zeta^{\gamma s} = \sum_s p_s(\gamma n-1)\zeta^{\gamma s-1}\ \zeta(1 + \sum_j \zeta^{-\gamma(j+1)} q_j)$$

$$= \sum_s p_s(\gamma n-1)\zeta^{\gamma s} + \sum_{s,j} p_s(\gamma n-1)\Delta^{\gamma(s-j-1)} q_j\ \zeta^{\gamma(s-j-1)}\ ,$$

therefore

$$p_s(\gamma n) = p_s(\gamma n-1) + \sum_j p_{s+j+1}(\gamma n-1)\Delta^{\gamma s}(q_j)\ . \tag{4.6}$$

Also,

$$L^{\gamma n} = \sum_s p_s(\gamma n)\zeta^{s} = \zeta(1 + \sum_j \zeta^{-\gamma(j+1)} q_j) \sum_s p_s(\gamma n-1)\zeta^{\gamma s-1}$$

$$= \Delta p_s(\gamma n-1)\zeta^{\gamma s} + \sum_{j,s} \Delta^{1-\gamma(j+1)} q_j p_s(\gamma n-1)\zeta^{\gamma(s-j-1)}\ ,$$

and thus

$$p_s(\gamma n) = \Delta p_s(\gamma n-1) + \sum_j \Delta^{1-\gamma(j+1)} q_j p_{s+j+1}(\gamma n-1)\ . \tag{4.7}$$

At this point we are faced with two problems very typical for the subject. Firstly, from (4.4) we can't get $p_o(\gamma n-1)$ which we need in (4.6) and (4.7).

Secondly, which one of the two expressions for $p_s(\gamma n)$, (4.6) of (4.7), should we substitute into (4.5) in order to end up with at least a skew-symmetric matrix B? We begin with the first question.

Let us subtract (4.7) from (4.6) with s = 0 in both equations. We get

$$(\Delta-1)p_o(\gamma n-1) = \sum_j (1 - \Delta^{1-\gamma(j+1)})q_j p_{j+1}(\gamma n-1) , \tag{4.8}$$

and therefore

$$p_o(\gamma n-1) = - \sum_j \frac{\Delta^{1-\gamma(j+1)}-1}{\Delta-1} q_j p_{j+1}(\gamma n-1) , \tag{4.9}$$

where, of course,

$$\frac{1-\Delta^{1-\gamma(j+1)}}{1-\Delta} = \Delta^{1-\gamma(j+1)}(1 + \Delta + \ldots + \Delta^{\gamma(j+1)-2}) \quad \text{for} \quad \gamma(j+1)-1>1 ,$$

$$\frac{1-\Delta^{-1}}{\Delta-1} = \Delta^{-1} .$$

We need a few words about going from (4.8) to (4.9). We effectively divided by Δ-1 both sides of (4.8). The result, naturally, might have been defined modulo Ker(Δ-1) = k. To see that the arbitrary constant does not appear in (4.9), let us introduce another (the usual) grading "deg" in $k[q_j^{(n_j)}]((\xi^{-1}))$ by putting

$$\deg(\zeta^i) = i , \deg(q_j^{(n)}) = \gamma(j+1) , \deg(k) = 0 . \tag{4.10}$$

Thus deg(L) = 1, $\deg(L^m) = m$, $\deg(p_s(\gamma n)) = \gamma(n-s)$, $\deg(p_s(\gamma n-1)) = \gamma(n-s)$, and both sides of (4.9) are homogeneous of degree $\gamma n \neq 0$, so that (4.9) does follow from (4.8).

Now we substitute (4.9) into (4.6) with s = 0, resulting in

$$p_o(\gamma n) = \sum_j \left(\frac{\Delta^{1-\gamma(j+1)}-1}{\Delta-1} + 1 \right) q_j p_{j+1}(\gamma n-1) \ . \tag{4.11}$$

This solves our first problem. There is no obvious answer for the second problem. Since there are two summands in the right-hand side of (4.5), it could very well be that we should use both (4.6) and (4.7). This is indeed what we will do.

So, let us rewrite (4.6) and (4.7) with s+1 substituted for s. Using (4.4), we find that

$$p_{s+1}(\gamma n) = \begin{cases} \dfrac{\delta H}{\delta q_s} + \sum_j q_j^{(\gamma(s+1))} \dfrac{\delta H}{\delta q_{j+s+1}} & (4.12a) \\ \underline{\text{or}} \\ \Delta \dfrac{\delta H}{\delta q_s} + \sum_j \Delta^{1-\gamma(j+1)} q_j \dfrac{\delta H}{\delta q_{j+s+1}} \ , & (4.12b) \end{cases}$$

where from now on I write H for $H_{\gamma n}$. Now let us substitute (4.11) and (4.12) into (4.5), separating the terms with k=0 from those with k > 0:

$$\partial_p(q_r) = q_r(\Delta^{\gamma(r+1)-1}-1) \sum_j \left(\frac{\Delta^{1-\gamma(j+1)}-1}{1-\Delta} + 1 \right) q_j \frac{\delta H}{\delta q_j} + \tag{4.13a}$$

$$+ \sum_{k\geq 0} q_{k+r+1} \Delta^{\gamma(r+1)-1} \begin{cases} \dfrac{\delta H}{\delta q_k} + \sum_j q_j^{(\gamma(k+1))} \dfrac{\delta H}{\delta q_{j+k+1}} & (4.13b) \\ \underline{\text{or}} \\ \Delta \dfrac{\delta H}{\delta q_k} + \sum_j \Delta^{1-\gamma(j+1)} q_j \dfrac{\delta H}{\delta q_{j+k+1}} & (4.13c) \end{cases} -$$

$$- \sum_{k\geq 0} \Delta^{-\gamma(k+1)} q_{k+r+1} \begin{cases} \dfrac{\delta H}{\delta q_k} + \sum_j q_j^{(\gamma(k+1))} \dfrac{\delta H}{\delta q_{j+k+1}} & (4.13d) \end{cases}$$

or

$$\Delta \frac{\delta H}{\delta q_k} + \sum_j \Delta^{1-\gamma(j+1)} q_j \frac{\delta H}{\delta q_{j+k+1}} . \tag{4.13e}$$

To decide which expressions in curly brackets to prefer, let us begin with those linear in q's. It is now immediately clear that (4.13e) is not correct since its contribution to the matrix element B_{rk} is $-\Delta^{-\gamma(k+1)} q_{k+r+1}\Delta$, and its minus adjoint, $\Delta^{-1} q_{k+r+1}\Delta^{\gamma(r+1)}$ could be matched by no terms in (4.13b) or (4.13c). Thus the correct choice is (4.13d). Taking the minus adjoint of its linear in q part, namely $-\Delta^{-\gamma(k+1)} q_{k+r+1}$, we arrive at $q_{k+r+1}\Delta^{\gamma(r+1)}$ which directs us to (4.13c). There is no other choice left and the result is

$$\partial_P(q_r) = q_r(\Delta^{\gamma(r+1)-1}-1) \sum_j \frac{\Delta^{-\gamma(j+1)}-1}{\Delta^{-1}-1} q_j \frac{\delta H}{\delta q_j}$$

$$+ \sum_{k\geq 0} [q_{k+r+1} \Delta^{\gamma(r+1)} - \Delta^{-\gamma(k+1)} q_{k+r+1}] \frac{\delta H}{\delta q_k} \tag{4.14}$$

$$+ \sum_{j,k\geq 0} [q_{k+r+1} \Delta^{\gamma(r-j)} q_j - q_j \Delta^{-\gamma(k+1)} q_{k+r+1}] \frac{\delta H}{\delta q_{j+k+1}}$$

We rewrite this formula in the form $\partial_P(q_r) = \sum_{s\geq 0} B_{rs} \frac{\delta H}{\delta q_s}$, with

$$B_{rs} = q_{r+s+1}\Delta^{\gamma(r+1)} - \Delta^{-\gamma(s+1)} q_{r+s+1} + \tag{4.15a}$$

$$+ q_r(\Delta^{\gamma(r+1)-1}-1) \frac{1-\Delta^{-\gamma(s+1)}}{1-\Delta^{-1}} q_s + \tag{4.15b}$$

$$+ \sum_{j+k+1=s} (q_{k+r+1}\Delta^{\gamma(r-j)} q_j - q_j \Delta^{-\gamma(k+1)} q_{k+r+1}) . \tag{4.15c}$$

It is not immediately clear that the matrix B in (4.15) is skew, so let us check this out.

Proposition 4.16. The matrix B defined by (4.15) is skew.

Proof. The part of B which is linear in the q's (4.15a), is obviously skew. Let us ignore it. Let us rewrite the rest as

$$B_{rs} = q_r(\Delta^{\gamma(r+1)-1}-1)\,\frac{1-\Delta^{-\gamma(s+1)}}{1-\Delta^{-1}}\,q_s$$

$$+\sum_{j<s}[q_{s-j+r}\,\Delta^{\gamma(r-j)}q_j - q_j\Delta^{-\gamma(s-j)}q_{s-j+r}]\ , \tag{4.17}$$

and let us write down $B^*_{rs} + B_{sr}$:

$$q_s(\Delta^{1-\gamma(r+1)}-1)\,\frac{1-\Delta^{\gamma(s+1)}}{1-\Delta}\,q_r + q_s(\Delta^{\gamma(s+1)-1}-1)\,\frac{1-\Delta^{-\gamma(r+1)}}{1-\Delta^{-1}}\,q_r \tag{4.18a}$$

$$+\sum_{j<s}[q_j\,\Delta^{\gamma(j-r)}q_{s-j+r}-q_{s-j+r}\,\Delta^{\gamma(s-j)}q_j] \tag{4.18b}$$

$$+\sum_{j<r}[q_{r-j+s}\,\Delta^{\gamma(s-j)}q_j-q_j\Delta^{-\gamma(r-j)}q_{r-j+s}]\ . \tag{4.18c}$$

Simplifying the Δ-part between q_s and q_r in (4.18a), we obtain

$$\frac{1}{1-\Delta}\,\{\Delta^{1-\gamma(r+1)}-1-\Delta^{1-\gamma(r-s)}+\Delta^{\gamma(s+1)}-\Delta[\Delta^{\gamma(s+1)-1}-1-\Delta^{-1+\gamma(s-r)}-\Delta^{-\gamma(r+1)}]\}$$

$$=\frac{1}{1-\Delta}\,\{\Delta^{\gamma(s-r)}(1-\Delta)+\Delta-1\} = \Delta^{\gamma(s-r)}-1\ ,$$

thus (4.18a) is equal to

$$q_s[\Delta^{\gamma(s-r)}-1]q_r\ . \tag{4.19a}$$

Combining the first term in (4.18b) with the second one in (4.18c), and the first term in (4.18c) with the second one in (4.18b), we get

$$\Big(\sum_{j<s}-\sum_{j<r}\Big)[q_j\Delta^{\gamma(j-r)}q_{s-j+r}-q_{s-j+r}\Delta^{\gamma(s-j)}q_j]\ , \tag{4.19b}$$

which, combined with (4.19a), finally produces

$$\Big(\sum_{j\le s} - \sum_{j<r} \Big) [q_j \Delta^{\gamma(j-r)} q_{s-j+r} - q_{s-j+r} \Delta^{\gamma(s-j)} q_j] \ , \tag{4.20}$$

which vanishes. Indeed, for r=s, both (4.19a) and (4.19b) vanish. So let r<s, say. Then (4.20) reduces to

$$\sum_{r\le j\le s} q_j \Delta^{\gamma(j-r)} q_{s+r-j} - \sum_{r\le j\le s} q_{s+r-j} \Delta^{\gamma(s-j)} q_j \ ,$$

and the second sum turns into the first if we change j into s+r-j. □

The Hamiltonian property of the matrix (4.15) will be given in Chapter X.

Finally a few words about the remaining case $\beta > 1$, $\gamma > 1$: I couldn't find the Hamiltonian form for this case, and it seems probable that this form doesn't exist, - an occurrence which is so far unknown in the domain of Lax equations. Needless to say, to prove the nonexistence is very difficult.

Remark 4.21 For $\gamma = 1$, the matrix (4.15) provides the second Hamiltonian structure for $L = \zeta + \sum_{j\ge 0} \zeta^{-j} q_j$. We also have the first Hamiltonian structure for the same L, given by theorem 3.3. Usually, different Hamiltonian structures for Lax equations are connected. To see what connections we can find here, let us denote the matrix (4.15) by $B^2(q_o)$ and that of (3.4) by B^1: B^1 does not depend upon q_o.

Lemma 4.22.

$$B^2(q_o+\lambda) = B^2(q_o) + \lambda B^1 \ , \forall \lambda \in k \ .$$

Proof. Since the linear in q terms of the matrix B^2 in (4.15) do not involve q_o, we can work with (4.17). If both r,s>0, then the only q_o-term occurs as $q_j|_{j=0}$ inside the sum. Thus $B^2_{rs}(q_o+\lambda) = B^2_{rs}(q_o) + \lambda[q_{s+r}\Delta^r - \Delta^{-s} q_{s+r}]$, which agrees with (3.4). Since B^2 is skew, it's enough to consider B^2_{ro}, in order to verify the lemma. Then the sum in (4.17) drops out and we have

$$B^2_{ro}(q_o+\lambda) = q_r(\Delta^r - 1) \frac{1-\Delta^{-1}}{1-\Delta^{-1}} (q_o+\lambda) = q_r(\Delta^r-1)q_o + \lambda q_r(\Delta^r - 1) \ ,$$

which again agrees with (3.4). □

5. Third Hamiltonian Structure for the Toda Lattices

As we have seen in the preceding section, for the operator L given by

$$L = \zeta + \sum_j \zeta^{-j} q_j \ , \tag{5.1}$$

its Lax equations with $P = L^n$ can be cast into the form (4.5):

$$\partial_P(q_r) = \sum_{k \geq 0} [q_{k+r}\Delta^r - \Delta^{-k} q_{k+r}] p_k(n) \ . \tag{5.2}$$

Then manipulation of the p(n)'s into the p(n-1)'s gives the second Hamiltonian form for the Lax equations. One might be able to make another step and find an appropriate expression of (5.2) in terms of the p(n-2)'s but I could not do it in general. Instead, I propose another derivation of the second Hamiltonian structure which can be repeated to provide the third structure for the operator $L = \zeta + q_o + \zeta^{-1} q$.

So, let us take a <u>finite</u> L,

$$L = \zeta + \sum_{j=0}^{N} \zeta^{-j} q_j \ , \tag{5.3}$$

and let

$$L^m = \sum_s p_s(m)\zeta^s \ , \ H_m = \frac{1}{m} \text{Res } L^m \ .$$

Then, as usual,

$$dH_m \sim \text{Res}(L^{m-1} dL) = \sum_{j=0}^{N} p_j(m-1) dq_j \ ,$$

and so

$$p_j(m-1) = \frac{\delta H_m}{\delta q_j} \ , \ 0 \leq j \leq N \ . \tag{5.4}$$

Writing $L^n = L^{n-1}L$, $L^n = LL^{n-1}$ in terms of p's, we get

$$p_s(n) = p_{s-1}(n-1) + \sum_{k=0}^{N} q_k^{(s)} p_{k+s}(n-1) , \tag{5.5a}$$

$$p_s(n) = \Delta p_{s-1}(n-1) + \sum_{k=0}^{N} \Delta^{-k} q_k p_{k+s}(n-1) . \tag{5.5b}$$

Applying Δ to (5.5a), subtracting from it (5.5b) and putting s=0, we get

$$p_o(n) = \sum_{k=0}^{N} \frac{\Delta^{k+1}-1}{\Delta-1} \Delta^{-k} q_k p_k(n-1) . \tag{5.6}$$

Before proceeding further, we record what is left of (5.2) in our case; that is

$$\partial_P(q_r) = \sum_{k=0}^{N-r} [q_{k+r}\Delta^r - \Delta^{-k} q_{k+r}] p_k(n) . \tag{5.7}$$

Thus we find that in addition to the problem of which one of the expressions in (5.5) we should substitute into (5.7) - a problem we have met before - we now have to take into account the fact that only N+1 among the p's can be expressed as functional derivatives by (5.4), and we have quite a few other p's in (5.5). To separate these other p's, let us first rewrite (5.5) with s+1 substituted for s:

$$p_{s+1}(n) = p_s(n-1) + \sum_{k=0}^{N-s-1} q_k^{(s+1)} p_{k+s+1}(n-1) + \sum_{m=0}^{s} q_{N-s+m}^{(s+1)} p_{N+m+1}(n-1) , \tag{5.8a}$$

$$p_{s+1}(n) = \Delta p_s(n-1) + \sum_{k=0}^{N-s-1} \Delta^{-k} q_k p_{k+s+1}(n-1) + \sum_{m=0}^{s} \Delta^{s-N-m} q_{N-s+m} p_{N+m+1}(n-1) . \tag{5.8b}$$

Now let us rewrite (5.7), using (5.6):

$$\partial_P(q_N) = q_N(\Delta^N - 1)\sum_{k=0}^{N} \frac{\Delta^{k+1}-1}{\Delta-1}\,\Delta^{-k} q_k p_k(n-1) \ , \tag{5.9}$$

$$\partial_P(q_i) = q_i(\Delta^i - 1)p_o(n) + \sum_{s=0}^{N-i-1} [q_{i+s+1}\Delta^i - \Delta^{-s-1} q_{i+s+1}]p_{s+1}(n) \ ,$$

$$0 \le i < N \ . \tag{5.10}$$

The first term on the right, $q_i(\Delta^i - 1)p_o(n)$ presents no problems. We therefore will concentrate on the sum, rewriting it with the help of (5.8) as

$$\sum_{s=0}^{N-i-q} q_{i+s+1}\Delta^i \begin{cases} p_s(n-1) + \sum\limits_{k=0}^{N-s-1} q_k^{(s+1)} p_{k+s+1}(n-1) + \sum\limits_{m=0}^{s} q_{N-s+m}^{(s+1)} p_{N+m+1}(n-1) & (5.11a) \\ \text{or} & - \\ \Delta p_s(n-1) + \sum\limits_{k=0}^{N-s-1} \Delta^{-k} q_k p_{k+s+1}(n-1) + \sum\limits_{m=0}^{s} \Delta^{s-N-m} q_{N-s+m} p_{N+m+1}(n-1) & (5.11b) \end{cases}$$

$$-\sum_{s=0}^{N-i-1} \Delta^{-s-1} q_{i+s+1} \begin{cases} p_s(n-1) + \sum\limits_{k=0}^{N-s-1} q_k^{(s+1)} p_{k+s+1}(n-1) + \sum\limits_{m=0}^{s} q_{N-s+m}^{(s+1)} p_{N+m+1}(n-1) & (5.11c) \\ \text{or} & \\ \Delta p_s(n-1) + \sum\limits_{k=0}^{N-s-1} \Delta^{-k} q_k p_{k+s+1}(n-1) + \sum\limits_{m=0}^{s} \Delta^{s-N-m} q_{N-s+m} p_{N+m+1}(n-1) & (5.11d) \end{cases}$$

Thus we again have the problem of what to choose but this time the purpose is different since we have to eliminate p_j's with $j > N$; that is, all $\sum_m$'s in (5.11). To do this, let us examine only the highest numbered p_j's which occur for m=s=N-i-1:

$$q_N\Delta^i \begin{cases} q_N^{(N-i)} p_{2N-i}(n-1) \\ \text{or} \\ \Delta^{-N} q_n p_{2N-i}(n-1) \end{cases} \qquad - \Delta^{i-N} q_N \begin{cases} q_N^{(N-i)} p_{2n-i}(n-1) \\ \text{or} \\ \Delta^{-N} q_N p_{2N-i}(n-1) \end{cases}$$

Since the first bracket has $q_N = q_N^{(0)}$, thus the second (minus) term must contribute its first row, and hence the first term has to compensate by its second row. Thus, the only choice is (5.11b) with (5.11c). Let us check out that then all unwanted p's in this case disappear. Denoting $\bar{p}_m = p_{N+m+1}(n-1)$, we have

$$\sum_{s=0}^{N-i-1} \sum_{m=0}^{s} [q_{i+s+1}\Delta^i \Delta^{s-N-m} q_{N-s+m}\bar{p}_m - \Delta^{-s-1} q_{i+s+1} q_{N-s+m}^{(s+1)}\bar{p}_m] =$$

$$= \sum_{m=0}^{N-i-1} \sum_{m\le s\le N-i-1} \{q_{i+s+1}\Delta^{i+s-N-m} q_{N-s+m} - q_{N-s+m}\Delta^{-s-1} q_{i+s+1}]\bar{p}_m = 0 ,$$

which can be seen at once by changing s into N-i-1+m-s in the second term. Thus there are no dangerous terms left and we can sum up the result:

$$\partial_P(q_i) = q_i(\Delta^i - 1) \sum_{k=0}^{N} \frac{\Delta^{k+1}-1}{\Delta-1}\Delta^{-k} q_k p_k(n-1) +$$

$$+ \sum_{s+i<N} \{q_{i+s+1}\Delta^i[\Delta p_s(n-1) + \sum_{k=0}^{N-s-1} \Delta^{-k} q_k p_{k+s+1}(n-1)] - \tag{5.12}$$

$$-\Delta^{-s-1} q_{i+s+1}[p_s(n-1) + \sum_{k=0}^{N-s-1} q_k^{(s+1)} p_{k+s+1}(n-1)]\} .$$

Notice that with the identification (5.4), (5.12) is exactly the cut out of the expression (4.14) with all q_j's and $\frac{\delta H}{\delta q_j}$'s absent for $j>N$: the easiest way to see it is to observe that derivations of (4.14) and (5.12) can be identified step by step.

Thus (5.12) yields an explicit form of the second Hamiltonian structure for the case of the finite number of q's. It is now clear on what lines we must proceed. We again have to substitute p(n-2) instead of p(n-1) in (5.12) and try to make our choice between the competing candidates (5.5a) and (5.5b) in such a way that all unwanted p's will cancel each other out.

Let us begin with the linear in q terms in (5.12):

$$\sum_{s=0}^{N-i-1} (q_{i+s+1}\Delta^{i+i} - \Delta^{-s-1}q_{i+s+1})p_s(n-1) ,$$

which yields, for s>0, the following expression

$$\sum_{s=0}^{N-i-2} (q_{i+s+2}\Delta^{i+1}-\Delta^{-s-2}q_{i+s+2})p_{s+1}(n-1) .$$

We substitute the dangerous terms of (5.8) into this expression and get

$$\sum_{s=0}^{N-i-2} q_{i+s+2}\Delta^{i+1}\left\{\begin{matrix}\sum_{m=0}^{s} q^{(s+1)}_{N-s+m}\bar{p}_m \\ \underline{\text{or}} \\ \sum_{m=0}^{s} \Delta^{s-N-m}q_{N-s+m}\bar{p}_m\end{matrix}\right. - \sum_{s=0}^{N-i-2} \Delta^{-s-2}q_{i+s+2}\left\{\begin{matrix}\sum_{m=0}^{s} q^{(s+1)}_{N-s+m}\bar{p}_m \\ \underline{\text{or}} \\ \sum_{m=0}^{s} \Delta^{s-N-m}q_{N-s+m}\bar{p}_m\end{matrix}\right. \qquad (5.13)$$

where $\bar{p}_m$ now stands for $p_{N+m+1}(n-2)$. It is now obvious that no balancing could save (5.13) from nonvanishing: the first term has $q_j^{(0)}$'s and the second one does not. The moral is that these sums should not be present from the very beginning, that is, we must have N=1. Thus let us look at the operator for the Toda lattice,

$$L = \zeta + q_o + \zeta^{-1}q_1 . \qquad (5.14)$$

Then we can simplify (5.8) and (5.12) into

$$p_1(n) = p_o(n-1) + q_o^{(1)}p_1(n-1) + q_1^{(1)}p_2(n-1) , \qquad (5.15a)$$

$$p_1(n) = \Delta p_o(n-1) + q_op_1(n-1) + \Delta^{-1}q_1p_2(n-1) , \qquad (5.15b)$$

$$\partial_P(q_o) = (q_1\Delta - \Delta^{-1}q_1)p_o(n-1) + q_o(1-\Delta^{-1})q_1p_1(n-1) , \qquad (5.16a)$$

$$\partial_p(q_1) = q_1(\Delta-1)q_o p_o(n-1) + q_1(\Delta-1)(1 + \Delta^{-1})q_1 p_1(n-1) \ , \tag{5.16b}$$

while (5.6) is now

$$p_o(n) = q_o p_o(n-1) + (1 + \Delta^{-1})q_1 p_1(n-1) \ . \tag{5.17}$$

We can plug (5.17) in (5.16) to get rid of $p_o(n-1)$. With $p_1(n-1)$ we proceed as follows:

$$(1-\Delta^{-1})q_1 p_1(n-1) = q_1 p_1(n-1) - q_1^{(-1)}\Delta^{-1}p_1(n-1) =$$

$$= [q_1 \times (5.15b) - q_1^{(-1)}\Delta^{-1}(5.15a)] \Big|_{n=n-1} =$$

$$= q_1[\Delta p_o(n-2) + q_o p_1(n-2) + \underline{\Delta^{-1}q_1 p_2(n-2)}] -$$

$-q_1^{(-1)}\Delta^{-1}[p_o(n-2)+q_o^{(1)}p_1(n-2)+\underline{q_1^{(1)}p_2(n-2)}]$ = (underlined terms cancel each other out) = $(q_1\Delta-\Delta^{-1}q_1)p_o(n-2)+q_o(1-\Delta^{-1})q_1p_1(n-2)$.

Thus (5.16) becomes

$$\left\{\begin{aligned} \partial_p(q_o) &= \{(q_1\Delta-\Delta^{-1}q_1)q_o+q_o(q_1\Delta-\Delta^{-1}q_1)\}p_o(n-2) + \\ &\quad +\{(q_1\Delta -\Delta^{-1}q_1)(1+\Delta^{-1})q_1+q_oq_o(1-\Delta^{-1})q_1\}p_1(n-2) \ , \\ \partial_p(q_1) &= \{q_1(\Delta-1)q_oq_o+q_1(\Delta+1)(q_1\Delta-\Delta^{-1}q_1)\}p_o(n-2) + \end{aligned}\right. \tag{5.18a}$$

$$+ \{q_1(\Delta-1)q_o(1+\Delta^{-1})q_1+q_1(\Delta+1)q_o(1-\Delta^{-1})q_1\}p_1(n-2) \ ,$$

which provides the third Hamiltonian structure B^3 for (5.14) if we rewrite (5.18a) as

$$\partial_P(q_i) = \sum_{j=0}^{1} B^3_{ij} \frac{\delta H_{n-1}}{\delta q_j}, \quad i=0,1 . \tag{5.18b}$$

We shall prove that the matrix B^3 is Hamiltonian in Chap. X.

Thus we have 3 Hamiltonian structures for (5.14). Let us write down the first two, (5.7) and (5.16), for future reference:

$$B^1 = \begin{vmatrix} 0 & (1-\Delta^{-1})q_1 \\ q_1(\Delta-1) & 0 \end{vmatrix} , \tag{5.19}$$

$$B^2 = \begin{vmatrix} q_1\Delta-\Delta^{-1}q_1 & q_o(1-\Delta^{-1})q_1 \\ q_1(\Delta-1)q_o & q_1(\Delta-1)(1+\Delta^{-1})q_1 \end{vmatrix} . \tag{5.20}$$

Let us indicate the explicit dependence upon q_o of matrices (5.18)-(5.20), by writing $B^k(q_o)$, $k=1,2,3$. Comparing their respective matrix coefficients, we arrive at

Proposition 5.21.

$$B^3(q_o+\lambda) = B^3(q_o) + 2\lambda B^2(q_o) + \lambda^2 B^1(q_o), \forall \lambda \in k .$$

Remark 5.21′. The 3rd Hamiltonian structure (5.18) is valid, as it stands, only for $n \geq 2$ since it was derived by using $p_j(n-2) = \frac{\delta H_{n-1}}{\delta q_j}$.

For $n=1$, $H_o = \text{Res } L^o = 1$, and $\frac{\delta(1)}{\delta q_j} = 0$. However, the Lax equations (5.2) still exist for $n=1$ (being just the usual Toda equations), and the question immediately arises whether these equations can be cast into the third Hamiltonian form as well. The answer is yes.

To see this, let us take $H = \frac{1}{2} \ell n q_1$, so $\frac{\delta H}{\delta q_o} = 0$, $\frac{\delta H}{\delta q_1} = \frac{1}{2q_1}$. Substituting this into (5.18b), we get

$$\begin{cases} \partial_P(q_o) = (q_1\Delta-\Delta^{-1}q_1)(1) = (1-\Delta^{-1})q_1, \\ \partial_P(q_1) = q_1(\Delta-1)q_o, \end{cases} \tag{5.22}$$

which are indeed the Toda equations.

The reader may notice that the Hamiltonian $H = \ell n q_1$ produces a zero vector when operated upon by either one of the Hamiltonian forms (5.19) or (5.20). For $H = H_1 = \mathrm{Res}\, L^1 = q_o$, B^1 of (5.19) still produces zero while B^2 of (5.20) yields the Toda equations (5.22).

We may ask ourselves whence this nonpolynomial Hamiltonian $\ell n q_1$ come. The answer is not clear. On the other hand, the reason <u>why</u> it it a c.l. for (5.22) - and it is, since it is the Hamiltonian function of (5.22) - is clear from the second equation of (5.22), which is of the form $\partial_P(q_1) = q_1 *$ (something ~ 0). It follows at once that we can find analogous polynomial c.l. for other Lax operators (5.3). Indeed, the Lax equations with P=L are

$$\partial_P(L) = [L_+,L] = [\zeta+q_o,\ \zeta+q_o+ \ldots + \zeta^{-N}q_N] =$$

$$= [\zeta+q_o,\ \zeta^{-1}q_1 + \ldots + \zeta^{-N}q_N]\ .$$

Therefore

$$\partial_P(q_i) = q_i(\Delta^i-1)q_o + (1-\Delta^{-1})q_{i+1},\ i<N\ , \tag{5.23}$$

$$\partial_P(q_N) = q_N(\Delta^N-1)q_o\ .$$

We see that $\ell n q_N$ is in fact a c.l. It is quite natural to expect then, that the 2nd Hamiltonian structure (5.12) has $H = \ell n q_N$ in its Kernel, that is, produces trivial equations from this H. Let us show that this is true.

<u>Proposition 5.24.</u> $H=\ell n q_N$ belongs to the Kernel of the second Hamiltonian structure (5.12).

<u>Proof.</u> We have to show that the right-hand side of (5.12) vanishes

when $p_o = \dots = p_{N-1} = 0$, $p_N = \frac{\delta H}{\delta q_N} = \frac{1}{q_N}$. We begin with i = N. Then the only terms present are all in the first row, which gives

$$\partial_P(q_N) = q_N(\Delta^N-1)\,\frac{\Delta^{N+1}-1}{\Delta-1}\,\Delta^{-1}q_N\,\frac{1}{q_N} = 0\ .$$

By the same line of reasoning the first row yields zero also for i < N. We thus look for the remainder, which gives for i<N,

$$\sum_{s=0}^{N-i-1}\{q_{i+s+1}\Delta^i\Delta^{-N+s+1}q_{N-s-1}\,\frac{1}{q_N} - \\ -\Delta^{-s-1}q_{i+s+1}q_{N-s-1}^{(s+1)}\,\frac{1}{q_N}\}\ . \tag{5.25}$$

Consider first the case i=N-1. Then s=0, and (5.25) becomes

$$q_N\Delta^o q_{N-1}\,\frac{1}{q_N} - \Delta^{-1}q_Nq_{N-1}^{(1)}\,\frac{1}{q_N} = 0\ .$$

Now let $i\leq N-2$. We rewrite (5.25) as

$$\sum_{s=0}^{N-i-1} q_{i+s+1}\Delta^{i+s+1-N}q_{N-s-1}\,\frac{1}{q_N} - \sum_{s=0}^{N-i-1} q_{N-s-1}\Delta^{-s-1}q_{i+s+1}\,\frac{1}{q_N}\ .$$

After substituting $s=N-i-\bar{s}-2$, the second sum of (5.26) becomes

$$-\sum_{\bar{s}=-1}^{N-i-2} q_{i+\bar{s}+1}\Delta^{-N+i+\bar{s}+1}q_{N-\bar{s}-1}\,\frac{1}{q_N}\ ,$$

and therefore (5.26) is left only with its boundary terms for s=N-i-1 and s=-1:

$$(q_{i+s+1}\Delta^{i+s+1-N}q_{N-s-1}\,\frac{1}{q_N})\Big|_{s=-1}^{s=N-i-1} =$$

$$= q_N\Delta^o q_i\,\frac{1}{q_N} - q_i\Delta^{i-N}q_N\,\frac{1}{q_N} = q_i - q_i = 0\ . \qquad \square$$

<u>Remark 5.27.</u> Proposition 5.24 remains true also for the operator

$$L = \zeta(1 + \sum_{j=0}^{N}\zeta^{-\gamma(j+1)}q_j)\ ,$$

for which the Hamiltonian structure is given by (4.14). The same proof as the one just given goes through when one changes Δ to Δ^{γ} in (5.25).

The presence of the Kernel of the second Hamiltonian structure and the fact that this Kernel depends upon N, makes the possible existence of the third Hamiltonian structure even more mysterious.

Chapter IV. The Modified Equations

In this chapter we construct modified equations together with their maps into the (nonmodified) systems of the preceeding chapters and discuss some of their specializations and Hamiltonian forms.

1. Modifications in General

A reasonably general idea of modification of Lax equations is to factorize the Lax operator L. Specifically, let us fix some natural number $n \geq 2$, let the index i run over $\mathbb{Z}_n$ and let

$$\ell_i = y_{i,o} + y_{i,1} + \dots + y_{i,N_i} \ , \ 1 \leq N_i \leq \infty \ , \tag{1.1}$$

where the $y_{i,j}$ are associative generators of the graded ring $k[\bar{y}] = k[y_{i,j}]$ with weights $w(y_{i,j}) = \beta_i - \alpha j, \beta_i \in \mathbb{Z}_+, \alpha \in \mathbb{N}$, and not all β_i are zeros. Denote

$$\bar{L} = \begin{vmatrix} 0 & \ell_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \ell_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & & 0 & \ell_{n-1} \\ \ell_n & 0 & \dots & & 0 & 0 \end{vmatrix} \tag{1.2}$$

$$\Pi_i = \ell_i \ell_{i+1} \dots \ell_{i-1} \ , \tag{1.3}$$

so that

$$\bar{L}^n = \mathrm{diag}\,(\Pi_1, \Pi_2, \dots, \Pi_n) \ . \tag{1.4}$$

Let

$$\beta = \sum_{i=1}^{n} \beta_i \ , \ \gamma = \alpha/(\alpha,\beta) \ . \tag{1.5}$$

For each $k \in \mathbb{N}$, we define a derivation $\partial_{\bar{P}}$ of $k[\bar{y}]$, where $\bar{P} = \bar{L}^{n\gamma k}$, by

$$\partial_{\bar{P}}(\bar{L}) = [\bar{P}_+,\bar{L}] = [-\bar{P}_-,\bar{L}] \ , \ w(\partial_{\bar{P}}) = 0 \ , \tag{1.6}$$

which can be rewritten as

$$\partial_{\bar{P}}(\ell_i) = (\Pi_i^{k\gamma})_+\ell_i - \ell_i(\Pi_{i+1}^{k\gamma})_+ = \ell_i(\Pi_{i+1}^{k\gamma})_- - (\Pi_i^{k\gamma})_-\ell_i \ , \tag{1.7}$$

$$w(\partial_{\bar{P}}) = 0 \ ,$$

where the notations follow those of Chapter I.

Equations (1.7) make sense: the first expression on the right shows that weights increase from $w(y_{i,N_i})$ with the step α, and the second expression shows that the same weights decrease from $\beta_i-\alpha$ with the step α. Hence

$$\partial_{\bar{P}}(y_{i,0}) = 0 \ . \tag{1.8}$$

Equations (1.7) are our (abstract) modified Lax equations. The name is justified by the observation that (1.6) implies

$$\partial_{\bar{P}}(\bar{L}^n) = [\bar{P}_+,\bar{L}^n] = [-\bar{P}_-,\bar{L}^n \ ] \ , \ w(\partial_{\bar{P}}) = 0 \ , \tag{1.9}$$

which is equivalent to

$$\partial_{\bar{P}}(\Pi_i) = [(\Pi_i^{k\gamma})_+,\Pi_i] = [-(\Pi_i^{k\gamma})_-,\Pi_i], \ w(\partial_{\bar{P}}) = 0 \ , \tag{1.10}$$

which are the usual (nonmodified) Lax equations of Chapter I.

Thus for each $i\in\mathbb{Z}_n$, we get a "Miura map": $\mathcal{R}[\bar{x}]\to\mathcal{R}[\bar{y}]$ which has weight zero and sends $L = x_o+\ldots+x_N$ into $\Pi_i\in\hat{\mathcal{R}}[\bar{y}]$. The correspondences between the images of $\mathcal{R}[\bar{x}]$ for different i's are sometimes incorrectly called "Bäcklund transformations" in the physical literature.

The only restriction on the possibility of having a Miura map comes when L has only finite number of generators x_j's. In this case, the lowest weight in Π_j is

$$\Sigma(\beta_i-\alpha N_i) = \beta - \alpha\Sigma N_i = w(x_N) = \beta - \alpha N ,$$

and so our condition is

$$N = \Sigma N_i . \tag{1.11}$$

2. 2 x 2 Case

The simplest case of the modification scheme occurs when n=2. This case we will study below.

Let

$$\ell_1 = \zeta+u,\ \ell_2 = 1 + \sum_j \zeta^{-j-1} v_j , \tag{2.1}$$

so that $\bar{L}$ in (1.2) becomes

$$\bar{L} = \begin{vmatrix} 0 & \zeta+u \\ 1+\Sigma\zeta^{-j-1}v_j & 0 \end{vmatrix} . \tag{2.2}$$

We take $\bar{P} = \bar{L}^{2n} = \mathrm{diag}[(\ell_1\ell_2)^n,\ (\ell_2\ell_1)^n]$, $n \in \mathbb{N}$. Denote

$$(\ell_1\ell_2)^n = \sum_j p_j(n)\zeta^j, (\ell_2\ell_1)^n = \sum_j q_j(n)\zeta^j . \tag{2.3}$$

Then the Lax equations (1.7) become

$$\partial_{\bar{P}}(\ell_1) = \partial_{\bar{P}}(u) = (\ell_1\ell_2)^n{}_+\ell_1-\ell_1(\ell_2\ell_1)^n{}_+ = (\text{taking } \zeta^o\text{-term}) =$$

$$= p_o(n)u-uq_o(n) , \text{ so}$$

$$\partial_{\bar{P}}(u) = u[p_o(n)-q_o(n)] . \tag{2.4a}$$

Also,

$$\partial_{\bar{P}}(\ell_2) = (\ell_2\ell_1)^n{}_+\ell_2 - \ell_2(\ell_1\ell_2)^n{}_+ =$$

$$= \sum_{j=0}^{n} \{q_j(n)\zeta^j(1+\sum_r \zeta^{-r-1}v_r) - (1+\sum_r \zeta^{-r-1}v_r)p_j(n)\zeta^j\} \;, \quad \text{thus}$$

$$\partial_{\bar{P}}(v_m) = \sum_{j=0}^{n} [v_{m+j}\Delta^{m+1}q_j(n) - \Delta^{-j}v_{m+j}p_j(n)]. \tag{2.4b}$$

To cast the equations (2.4) into a Hamiltonian form, we have to re-express $p_j(n)$ and $q_j(n)$ through variational derivatives of a c.l. We will use the same technique as in Chapter III.

Let

$$H_n = \frac{1}{n}\,\mathrm{Res}[(\ell_1\ell_2)^n] \sim \frac{1}{n}\,\mathrm{Res}[(\ell_2\ell_1)^n] \;. \tag{2.5}$$

Since

$$\ell_1\ell_2 = \zeta+u+v_o+\sum_{m\geq 0}\zeta^{-m-1}[v_{m+1}+v_m\Delta^{m+1}(u)] \;, \tag{2.6}$$

$$\ell_2\ell_1 = \zeta+u+\Delta^{-1}(v_o)+\sum_{m\geq 0}\zeta^{-m-1}[\Delta^{-1}(v_{m+1})+v_m u] \;, \tag{2.7}$$

we can rewrite the identities

$$dH_n \sim \mathrm{Res}[(\ell_1\ell_2)^{n-1}d(\ell_1\ell_2)] \sim \mathrm{Res}[(\ell_2\ell_1)^{n-1}d(\ell_2\ell_1)]$$

in the following way:

$$dH_n \sim p_o(n-1)(du+dv_o) + \sum_{m\geq 0} p_{m+1}(n-1)[dv_{m+1}+\Delta^{m+1}(u)dv_m+v_m\Delta^{m+1}(du)] \sim$$

$$\sim q_o(n-1)[du+\Delta^{-1}(dv_o)] + \sum_{m\geq 0} q_{m+1}(n-1)[\Delta^{-1}(dv_{m+1})+v_m du+u dv_m] \;.$$

This implies that

$$\frac{\delta H_n}{\delta u} = p_o(n-1) + \sum_{m\geq 0} \Delta^{-m-1} v_m p_{m+1}(n-1) \ , \qquad (2.8a)$$

$$\frac{\delta H_n}{\delta u} = q_o(n-1) + \sum_{m\geq 0} v_m q_{m+1}(n-1) \ , \qquad (2.8b)$$

$$\frac{\delta H_n}{\delta v_m} = p_m(n-1) + p_{m+1}(n-1)\Delta^{m+1}(u) \ , \qquad (2.9a)$$

$$\frac{\delta H_n}{\delta v_m} = \Delta q_m(n-1) + u q_{m+1}(n-1) \ . \qquad (2.9b)$$

We need a few identities between the p's and q's. We use the following relations:

$(\ell_1\ell_2)^n = (\ell_1\ell_2)^{n-1}(\ell_1\ell_2) = (\ell_1\ell_2)(\ell_1\ell_2)^{n-1}$, $(\ell_2\ell_1)^n =$

$= (\ell_2\ell_1)^{n-1}(\ell_2\ell_1) = (\ell_2\ell_1)(\ell_2\ell_1)^{n-1}$, $(\ell_1\ell_2)^{n-1}\ell_1 = \ell_1(\ell_2\ell_1)^{n-1}$,

$\ell_2(\ell_1\ell_2)^{n-1} = (\ell_2\ell_1)^{n-1}\ell_2$. In terms of the components, we have

$$p_j(n) = p_{j-1}(n-1) + p_j(n-1)\Delta^j(u+v_o) + \sum_m p_{j+m+1}(n-1)\Delta^j(v_{m+1}+v_m\Delta^{m+1}u) \ , \qquad (2.10a)$$

$$p_j(n) = \Delta p_{j-1}(n-1) + p_j(n-1)(u+v_o) + \sum_m \Delta^{-m-1}[p_{m+j+1}(n-1)(v_{m+1}+v_m u^{(m+1)})] \ , \qquad (2.10b)$$

$$q_j(n) = q_{j-1}(n-1) + q_j(n-1)\Delta^j[u+\Delta^{-1}(v_o)] + \sum_m q_{j+m+1}(n-1)\Delta^j[\Delta^{-1}(v_{m+1})+v_m u] \ , \qquad (2.11a)$$

$$q_j(n) = \Delta q_{j-1}(n-1) + q_j(n-1)[u+\Delta^{-1}(v_o)] + \sum_m \Delta^{-m-1}[q_{j+m+1}(n-1)(v_{m+1}^{(-1)}+v_m u)] , \tag{2.11b}$$

$$p_{j-1}(n-1) + p_j(n-1)\Delta^j(u) = \Delta q_{j-1}(n-1) + uq_j(n-1) , \text{ [no sum on j] }, \tag{2.12}$$

$$p_j(n-1) + \sum_m \Delta^{-m-1} v_m p_{j+m+1}(n-1) = q_j(n-1) + \sum_m q_{j+m+1}(n-1)\Delta^j(v_m) . \tag{2.13}$$

<u>Lemma 2.14</u>. Let us write H instead of H_n in what follows. Then

$$q_{j+1}(n) = \frac{\delta H}{\delta v_j} + \sum_{r\geq 0} \Delta^{-r-1} v_r \frac{\delta H}{\delta v_{j+r+1}} , \tag{2.14a}$$

$$p_{j+1}(n) = \frac{\delta H}{\delta v_j} + \sum_{r\geq 0} v_r^{(j+1)} \frac{\delta H}{\delta v_{j+r+1}} . \tag{2.14b}$$

<u>Proof.</u> From (2.11b) we have

$$q_{j+1}(n) = [\Delta q_j(n-1) + q_{j+1}(n-1)u] + [v_o^{(-1)} q_{j+1}(n-1)+\Delta^{-1}v_o uq_{j+2}(n-1)] +$$

$$+ [v_1^{(-2)} q_{j+2}^{(-1)}(n-1)+\Delta^{-2}v_1 uq_{j+3}(n-1] + \ldots = [\text{by (2.9b)}] =$$

$$= \frac{\delta H}{\delta v_j} + \Delta^{-1} v_o \frac{\delta H}{\delta v_{j+1}} + \Delta^{-2} v_1 \frac{\delta H}{\delta v_{j+2}} + \ldots ,$$

which proves (2.14a). Analogously, from (2.10a) we get

$$p_{j+1}(n) = [p_j(n-1)+p_{j+1}(n-1)u^{(j+1)}] + [v_o^{(j+1)} p_{j+1}(n-1)+p_{j+2}(n-1)v_o^{(j+1)}u^{(j+2)}] +$$

$$+ [p_{j+2}(n-1)v_1^{(j+1)}+p_{j+3}(n-1)v_1^{(j+1)}u^{(j+3)}] + \ldots = [\text{by (2.9a)}] =$$

$$= \frac{\delta H}{\delta v_j} + v_o^{(j+1)} \frac{\delta H}{\delta v_{j+1}} + v_1^{(j+1)} \frac{\delta H}{\delta v_{j+2}} + \ldots . \qquad \square$$

Lemma 2.15.

$$q_o(n) = \Delta q_{-1}(n-1) + uq_o(n-1) + \sum_{r\geq 0} \Delta^{-r-1} v_r \frac{\delta H}{\delta v_r}, \tag{2.15a}$$

$$p_o(n) = p_{-1}(n-1) + up_o(n-1) + \sum_{r\geq 0} v_r \frac{\delta H}{\delta v_r}. \tag{2.15b}$$

Proof. The same as the proof of lemma 2.14, with j substituted instead of j+1. □

Lemma 2.16.

$$q_o(n) = u\frac{\delta H}{\delta u} + \sum_{r\geq 0} \frac{1-\Delta^{-r-1}}{\Delta-1} v_r \frac{\delta H}{\delta v_r}, \tag{2.16a}$$

$$p_o(n) = u\frac{\delta H}{\delta u} + \sum_{r\geq 0} \frac{1-\Delta^{-r-1}}{\Delta-1} \Delta v_r \frac{\delta H}{\delta v_r}. \tag{2.16b}$$

Proof. From (2.10b) we have

$$p_o(n) = u\{p_o(n-1)+\sum_m \Delta^{-m-1} v_m p_{m+1}(n-1)\} +$$

$$+ \{\Delta p_{-1}(n-1)+\sum_m \Delta^{-m} v_m p_m(n-1)\} = [\text{by } (2.8a)] =$$

$$= u\frac{\delta H}{\delta u} + \theta,\ \theta := \Delta p_{-1}(n-1) + \sum_m \Delta^{-m} v_m p_m(n-1). \tag{2.17}$$

On the other hand, from (2.11a) we get

$$q_o(n) = u\{q_o(n-1) + \sum_m q_{m+1}(n-1)v_m\} +$$

$$+\{q_{-1}(n-1) + \sum_m q_m(n-1)\Delta^{-1} v_m\} = [\text{by } (2.8b),\ (2.13)\big|_{j=-1}] =$$

$$= u\frac{\delta H}{\delta u} + \Delta^{-1}(\theta), \tag{2.18}$$

where θ is defined in (2.17). Applying Δ to (2.18) and subtracting (2.17), we find that

$$\Delta q_o(n) - p_o(n) = (\Delta-1)u \frac{\delta H}{\delta u} . \tag{2.19}$$

Now we subtract (2.15b) from (2.15a) and use (2.12) with j=0, which results in

$$q_o(n) - p_o(n) = \sum_{r\geq 0} (\Delta^{-r-1}-1)v_r \frac{\delta H}{\delta v_r} . \tag{2.20}$$

Solving the system of two equations (2.19) and (2.20), we get (2.16). □

Now we are ready to find a Hamiltonian form for the equations (2.4). Substituting (2.20) into (2.4a) we obtain

$$\partial_{\bar{P}}(u) = u \sum_{r\geq 0} (1-\Delta^{-r-1})v_r \frac{\delta H}{\delta v_r} . \tag{2.21a}$$

To transform (2.4b), we use (2.14) and (2.16):

$$\partial_{\bar{P}}(v_m) = v_m[\Delta^{m+1}q_o(n)-p_o(n)] + \sum_{j\geq 0} \{v_{m+j+1}\Delta^{m+1}q_{j+1}(n) -$$

$$- \Delta^{-j-1}v_{m+j+1}p_{j+1}(n)\} =$$

$$= v_m\{(\Delta^{m+1}-1)u \frac{\delta H}{\delta u} + \sum_{r\geq 0} \frac{1-\Delta^{-r-1}}{\Delta-1}(\Delta^{m+1}-\Delta)v_r \frac{\delta H}{\delta v_r}\} + \tag{2.21b}$$

$$+ \sum_{j\geq 0} \{v_{m+j+1}\Delta^{m+1}[\frac{\delta H}{\delta v_j} + \sum_{r\geq 0} \Delta^{-r-1}v_r \frac{\delta H}{\delta v_{r+j+1}}] -$$

$$- \Delta^{-j-1}v_{m+j+1} [\frac{\delta H}{\delta v_j} + \sum_{r\geq 0} v_r^{(j+1)} \frac{\delta H}{\delta v_{j+r+1}}]\} . \tag{2.21c}$$

The equations (2.21) represent the Hamiltonian form of the modified Lax equations (2.4). Notice the curious coincidence of the $(\bar{v},\bar{v})$ part of the matrix in (2.21) with the matrix of the second Hamiltonian structure of Lax equations III (4.14) with $\gamma = 1$.

Recall now that we have two Miura maps $L = \ell_1\ell_2$ and $L = \ell_2\ell_1$, from modified into unmodified Lax equations. Both systems of equations are Hamiltonian. The natural question is then to ask if the Miura maps are canonical transformations.

Theorem 2.22. Denote $C_1 = K[q_j^{(n_j)}]$, $C_2 = K[u^{(n)}, v_j^{(n_j)}]$ two rings with an automorphism Δ. Let M_1 and M_2 be two homomorphisms of C_1 into C_2 over K commuting with Δ, and given by

$$M_1(\zeta+\Sigma\zeta^{-j}q_j) = (\zeta+u)(1+\Sigma\zeta^{-j-1}v_j) = \ell_1\ell_2 \ , \ M_1(\zeta^s) = \zeta^s \ ,$$

$$M_2(\zeta+\Sigma\zeta^{-j}q_j) = (1+\Sigma\zeta^{-j-1}v_j)(\zeta+u) = \ell_2\ell_1 \ , \ M_2(\zeta^s) = \zeta^s \ .$$

Let $H\in C_1$, and let $\partial_H : C_1 \to C_1$ be an evolutionary derivation defined by the equations III (4.14) (with $\gamma = 1$). Let $H_i = M_i(H)\in C_2$, and let $\partial_{H_i} : C_2 \to C_2$ be an evolutionary derivation defined by the equations (2.21). Then ∂_{H_i} and ∂_H are compatible with respect to M_i (which is what it means to be a "canonical transformation" or "canonical map").

Proof will be given in Chapter X. Let us check here the simplest case when we have only one variable, $v = v_o$, in ℓ_2: $\ell_2 = 1 + \zeta^{-1}v$. Then equations (2.21) reduce to

$$\begin{pmatrix} \partial_{\bar{P}}(u) \\ \partial_{\bar{P}}(v) \end{pmatrix} = \begin{vmatrix} 0 & u(1-\Delta^{-1})v \\ v(\Delta-1)u & 0 \end{vmatrix} \begin{pmatrix} \delta H/\delta u \\ \delta H/\delta v \end{pmatrix} . \tag{2.23}$$

Let us denote by B the matrix which appears in (2.23). We have to check that JBJ^* is equal to the image under M_i of the matrix B^2 in III (5.20) (the second Hamiltonian structure of the Toda hierarchy), where J is the Fréchet derivative of the vector $M_i(\bar{q})$ (in (u,v)-space), see II 43. Let us begin with M_1. From (2.6) we have

$$M_1(q_o) = u+v \ , \ M_1(q_1) = vu^{(1)} \ , \tag{2.24}$$

thus

$$J = \begin{vmatrix} 1 & 1 \\ v\Delta & u^{(1)} \end{vmatrix} \ , \ J^* = \begin{vmatrix} 1 & \Delta^{-1}v \\ 1 & u^{(1)} \end{vmatrix} ,$$

and we get

$$JB = \begin{vmatrix} v(\Delta-1)u & u(1-\Delta^{-1})v \\ u^{(1)}v(\Delta-1)u & v\Delta u(1-\Delta^{-1})v \end{vmatrix} ,$$

$$JBJ^* = \begin{vmatrix} v(\Delta-1)u+u(1-\Delta^{-1})v & v(\Delta-1)u\Delta^{-1}v+u(1-\Delta^{-1})vu^{(1)} \\ \dots & u^{(1)}v[(\Delta-1)u\Delta^{-1}v+\Delta(1-\Delta^{-1})vu^{(1)}] \end{vmatrix} =$$

$$= \begin{vmatrix} M_1(q_1)\Delta-\Delta^{-1}M_1(q_1) & M_1(q_o)(1-\Delta^{-1})M_1(q_1) \\ \dots & M_1(q_1)[1-\Delta^{-1}+\Delta-1]M_1(q_1) \end{vmatrix} =$$

$$= M_1 \begin{vmatrix} q_1\Delta-\Delta^{-1}q_1 & q_o(1-\Delta^{-1})q_1 \\ \dots & q_1(\Delta-\Delta^{-1})q_1 \end{vmatrix} = M_1(B^2) \ ,$$

where "..." in the lower left corner means: "minus adjoint of the opposite entry, with respect to the diagonal."

Analogously, we have from (2.7)

$$M_2(q_o) = u+v^{(-1)} \ , \ M_2(q_1) = uv \ , \tag{2.25}$$

thus

$$J = \begin{vmatrix} 1 & \Delta^{-1} \\ v & u \end{vmatrix} \ , \ J^* = \begin{vmatrix} 1 & v \\ \Delta & u \end{vmatrix} ,$$

$$JB = \begin{vmatrix} \Delta^{-1}v(\Delta-1)u & u(1-\Delta^{-1})v \\ uv(\Delta-1)u & vu(1-\Delta^{-1})v \end{vmatrix} ,$$

$$JBJ^* = \begin{vmatrix} \Delta^{-1}v(\Delta-1)u+u(1-\Delta^{-1})v\Delta & \Delta^{-1}v(\Delta-1)uv+u(1-\Delta^{-1})vu \\ \dots & uv[(\Delta-1)uv+(1-\Delta^{-1})vu] \end{vmatrix} =$$

$$= \begin{vmatrix} v^{(-1)}u-\Delta^{-1}uv+uv\Delta-uv^{(-1)} & [v^{(-1)}(1-\Delta^{-1})+u(1-\Delta^{-1})]uv \\ \dots & uv[\Delta-1+1-\Delta^{-1}]uv \end{vmatrix}$$

$$= M_2 \begin{vmatrix} q_1\Delta-\Delta^{-1}q_1 & q_o(1-\Delta^{-1})q_1 \\ \dots & q_1(\Delta-\Delta^{-1})q_1 \end{vmatrix} = M_2(B^2) \ .$$

If we call the equations (2.23) the modified Toda hierarchy, what we have just checked is the property that both Miura maps M_1 and M_2 are canonical between the second Hamiltonian structure B^2 of the Toda hierarchy and the Hamiltonian structure (2.23) of the modified Toda hierarchy. This strongly resembles the property of the Miura maps between the modified and unmodified Korteweg - de Vries equations (see, e.g., [9] p. 405): the Hamiltonian structure $v_t = -\frac{1}{2}\partial\frac{\delta H}{\delta v}$ is canonically related to the second Hamiltonian structure $u_t = (\frac{1}{2}\partial^3+u\partial+\partial u)\frac{\delta H}{\delta u}$ with respect to the homomorphisms $u \to \pm v_x-v^2$. However, our situation is richer: the Toda hierarchy possesses one more Hamiltonian structure III (5.18). Since it is an experimental observation that modified equations in general have one Hamiltonian structure less than unmodified equations, and the Hamiltonian structures of modified and original equations are canonically related with respect to the same Miura map(s), it is natural to assume that our modified equations (2.23) have one more Hamiltonian structure which is canonically related through both M_1 and M_2 with the third Hamiltonian structure of the Toda hierarchy. This is indeed the case and we will study it in the next section.

3. The Modified Toda Hierarchy

We have now

$$\ell_1 = \zeta + u \ , \ \ell_2 = 1 + \zeta^{-1}v \ ,$$

$$\ell_1\ell_2 = \zeta+u+v+\zeta^{-1}vu^{(1)} \ , \ \ell_2\ell_1 = \zeta+u+v^{(-1)} + \zeta^{-1}uv \ . \tag{3.1}$$

Equations (2.3), (2.4), (2.8a), (2.9a) and (2.10) become

$$(\ell_1\ell_2)^n = \sum_j p_j(n)\zeta^j \ , \ (\ell_2\ell_1)^n = \sum_j q_j(n)\zeta^j \ , \tag{3.2}$$

$$\partial_{\bar{P}}(u) = u[p_o(n)-q_o(n)] \ , \tag{3.3a}$$

$$\partial_{\bar{P}}(v) = v[\Delta q_o(n)-p_o(n)] \ , \tag{3.3b}$$

$$\frac{\delta H_n}{\delta u} = p_o(n-1) + \Delta^{-1} \, vp_1(n-1) \ , \tag{3.4}$$

$$\frac{\delta H_n}{\delta v} = p_o(n-1) + u^{(1)}p_1(n-1) \ , \tag{3.5}$$

$$p_j(n) = p_{j-1}(n-1) + p_j(n-1)\Delta^j(u+v) + p_{j+1}(n-1)\Delta^j[vu^{(1)}] \ , \tag{3.6a}$$

$$p_j(n) = \Delta p_{j-1}(n-1) + p_j(n-1)(u+v) + \Delta^{-1}[p_{j+1}(n-1)vu^{(1)}] \ . \tag{3.6b}$$

Our next step is to express q(n) in terms of p(n-1), thus eliminating q's completely. For this, we use the identity $(\ell_2\ell_1)^n = \ell_2(\ell_1\ell_2)^{n-1}\,\ell_1$:

$$\sum_j q_j(n)\zeta^j = (1+\zeta^{-1}v) \sum_s p_s(n-1)\zeta^s(\zeta+u) =$$

$$= \sum_s \{(p_s(n-1)\zeta^s + [vp_s(n-1)]^{(-1)}\zeta^{s-1})(\zeta+u)\} =$$

$$= \sum_s \{p_s(n-1)\zeta^{s+1} + p_s(n-1)u^{(s)}\zeta^s + [vp_s(n-1)]^{(-1)}\zeta^s +$$

$$+ [vp_s(n-1)]^{(-1)}u^{(s-1)}\zeta^{s-1}\} \ .$$

Thus,

$$q_j(n) = p_{j-1}(n-1) + [u^{(j)}+\Delta^{-1}v]p_j(n-1) + u^{(j)}\Delta^{-1}vp_{j+1}(n-1) \ . \tag{3.7}$$

As we see from (3.3), we need only q_o, for which (3.7) provides us with

$$q_o(n) = p_{-1}(n-1) + (u+\Delta^{-1}v)p_o(n-1) + \Delta^{-1}u^{(1)}vp_1(n-1) \ . \tag{3.8}$$

Now we work out (3.3a) using (3.6a) for $p_o(n)$ and (3.8) for $q_o(n)$:

$$\partial_{\bar{P}}(u) = u\{[p_{-1}(n-1)+p_o(n-1)(u+v) + p_1(n-1)vu^{(1)}] -$$

$$- [p_{-1}(n-1)+(u+\Delta^{-1}v)p_o(n-1) + \Delta^{-1}u^{(1)}vp_1(n-1)]\} =$$

$$= u\{(1-\Delta^{-1})vp_o(n-1) + (1-\Delta^{-1})vu^{(1)}p_1(n-1)\} =$$

$$= u(1-\Delta^{-1})v[p_o(n-1) + u^{(1)}p_1(n-1)] \ . \tag{3.9a}$$

For (3.3b), we use (3.6b) for $p_o(n)$ and (3.8) for $q_o(n)$:

$$\partial_{\bar{p}}(v) = v\{[\Delta p_{-1}(n-1)+(\Delta u+v)p_o(n-1) + u^{(1)}vp_1(n-1)] -$$

$$- [\Delta p_{-1}(n-1) + p_o(n-1)(u+v) + \Delta^{-1}(p_1(n-1)vu^{(1)})]\} =$$

$$= v\{(\Delta-1)up_o(n-1) + (1-\Delta^{-1})[p_1(n-1)vu^{(1)}]\} =$$

$$= v(1-\Delta^{-1})\Delta u[p_o(n-1) + \Delta^{-1}vp_1(n-1)] \ . \tag{3.9b}$$

Equations (3.9) are the ones with which we are going to work. Notice that they at once provide the Hamiltonian form (2.23) if one uses (3.5) in (3.9a) and (3.4) in (3.9b):

$$\begin{cases} \partial_{\bar{p}}(u) = u(1-\Delta^{-1})v \dfrac{\delta H_n}{\delta v} \ , \\ \partial_{\bar{p}}(v) = v(\Delta-1)u \dfrac{\delta H_n}{\delta u} \ . \end{cases} \tag{3.10}$$

Now we have to use (3.6) and re-express $p_o(n-1)$ and $p_1(n-1)$ through $p_{...}(n-2)$. However, $p_o(n-1)$ involves $p_{-1}(n-2)$ and $p_1(n-1)$ involves $p_2(n-2)$, which are both absent in (3.4), (3.5). We manage as follows. For j = 0, apply Δ to (3.6a) and subtract (3.6b), getting

$$p_o(n-1) = p_o(n-2)(u+v) + (1+\Delta^{-1})vu^{(1)}p_1(n-2) \ . \tag{3.11}$$

Then, for j = 0, subtract (3.6b) from (3.6a):

$$(\Delta-1)p_{-1}(n-1) = (1-\Delta^{-1})p_1(n-1)vu^{(1)} \ .$$

Therefore,

$$p_{-1}(n-1) = \Delta^{-1}p_1(n-1)vu^{(1)} \ . \tag{3.12}$$

Now, for $j = -1$, apply Δ to (3.6a) and subtract (3.6b):

$$(\Delta-1)p_{-1}(n-1) = (u+v)(\Delta-1)p_{-1}(n-2) + [vu^{(1)}\Delta-\Delta^{-1}vu^{(1)}]p_o(n-2) ,$$

substitute (3.12), and get

$$\begin{aligned}(1-\Delta^{-1})vu^{(1)}p_1(n-1) = (u+v)(1-\Delta^{-1})vu^{(1)}p_1(n-2) + \\ + (v\Delta u-u\Delta^{-1}v)p_o(n-2) .\end{aligned} \tag{3.13}$$

Using (3.11) and (3.13) in (3.9), we find that

$$\begin{aligned}\partial_{\bar{p}}(u) = u\{(1-\Delta^{-1})v[p_o(u+v)+(1+\Delta^{-1})vu^{(1)}p_1] + \\ + [(u+v)(1-\Delta^{-1})vu^{(1)}p_1 + (v\Delta u-u\Delta^{-1}v)p_o]\} ,\end{aligned} \quad 3.14a)$$

$$\begin{aligned}\partial_{\bar{p}}(v) = v\{(\Delta-1)u[p_o(u+v)+(1+\Delta^{-1})vu^{(1)}p_1] + \\ + [(u+v)(1-\Delta^{-1})vu^{(1)}p_1 + (v\Delta u-u\Delta^{-1}v)p_o]\} ,\end{aligned} \tag{3.14b}$$

where the index n-2 has been dropped out of $p_i(n-2)$, $i = 0,1$.

The only thing that now remains before we obtain the third Hamiltonian structure for the modified Toda hierarchy is to represent the expressions in the curly brackets of (3.14) through just those combinations of p_o and p_1 which appear in the right-hand sides of (3.4), (3.5). We begin with (3.14a). Suppose we manage to find two operators, A and B, say, such that

$$\begin{aligned}A \frac{\delta H}{\delta v} + B \frac{\delta H}{\delta u} = \{(1-\Delta^{-1})v[p_o(u+v)+(1+\Delta^{-1})vu^{(1)}p_1] + \\ + [(u+v)(1-\Delta^{-1})vu^{(1)}p_1 + (v\Delta u-u\Delta^{-1}v)p_o]\} , \ H: = H_{n-1} .\end{aligned}$$

Using (3.4), (3,5) we can rewrite this as a system,

$$\begin{cases} A+B = (1-\Delta^{-1})v(u+v) + v\Delta u - u\Delta^{-1}v , & (3.15a) \\ Au^{(1)} + B\Delta^{-1}v = (1-\Delta^{-1})v(1+\Delta^{-1})u^{(1)}v + (u+v)(1-\Delta^{-1})u^{(1)}v . & (3.15b) \end{cases}$$

From (3.15b), we see that $A = \alpha v$, $B = \beta u$ with some operators α,β. Then (3.15) simplifies to

$$\begin{cases} \alpha v + \beta u = (1-\Delta^{-1})v(u+v) + v\Delta u - u\Delta^{-1}v \ , & (3.16a) \\ \alpha + \beta\Delta^{-1} = (1-\Delta^{-1})v(1+\Delta^{-1}) + (u+v)(1-\Delta^{-1}) \ . & (3.16b) \end{cases}$$

Multiplying (3.16b) from the right by v and subtracting (3.16a), we get

$$\beta(u-\Delta^{-1}v) = (v\Delta-\Delta^{-1}v)(u-\Delta^{-1}v) \ ,$$

and so

$$\beta = v\Delta-\Delta^{-1}v \ , \ \alpha = u(1-\Delta^{-1}) + (1-\Delta^{-1})v \ ,$$

$$B = (v\Delta-\Delta^{-1}v)u \ , \ A = [u(1-\Delta^{-1}) + (1-\Delta^{-1})v]v \ ,$$

$$\partial_{\bar{P}}(u) = u(v\Delta-\Delta^{-1}v)u \frac{\delta H}{\delta u} +$$

$$+ u[u(1-\Delta^{-1}) + (1-\Delta^{-1})v]v \frac{\delta H}{\delta v} \ . \qquad (3.17a)$$

We transform (3.14b) along the same lines as (3.14a). If

$$A \frac{\delta H}{\delta v} + B \frac{\delta H}{\delta u} = (\Delta-1)u[p_o(u+v) + (1+\Delta^{-1})vu^{(1)}p_1] +$$

$$+ [(u+v)(1-\Delta^{-1})vu^{(1)}p_1 + (v\Delta u-u\Delta^{-1}v)p_o] \ ,$$

then

$$\begin{cases} A + B = (\Delta-1)u(u+v) + (v\Delta u-u\Delta^{-1}v) \ , & (3.18a) \\ Au^{(1)} + B\Delta^{-1}v = (\Delta-1)u(1+\Delta^{-1})vu^{(1)} + (u+v)(1-\Delta^{-1})vu^{(1)} \ . & (3.18b) \end{cases}$$

By putting $A = \alpha v$, $B = \beta u$, we rewrite (3.18) as

$$\begin{cases} \alpha v + \beta u = (\Delta-1)u(u+v) + v\Delta u - u\Delta^{-1}v \ , \\ \alpha + \beta\Delta^{-1} = (\Delta-1)u(1+\Delta^{-1}) + (u+v)(1-\Delta^{-1}) \ . \end{cases}$$

Solving this system produces

$$\alpha = \Delta u - u\Delta^{-1} \ , \ \beta = (\Delta-1)u + v(\Delta-1) \ ,$$

$$A = (\Delta u - u\Delta^{-1})v \ , \ B = [(\Delta-1)u + v(\Delta-1)]u \ ,$$

$$\partial_{\bar{P}}(v) = v[(\Delta-1)u+v(\Delta-1)]u \frac{\delta H}{\delta u} + v(\Delta u - u\Delta^{-1})v \frac{\delta H}{\delta v} \ . \tag{3.17b}$$

Equations (3.17) provide the third Hamiltonian structure for the Modified Toda hierarchy. The proof that they are Hamiltonian will be given in Chap. X. Recall that for the second Hamiltonian structure (2.23), both Miura maps M_1 and M_2 are canonical maps into the second Hamiltonian structure of the Toda hierarchy.

<u>Theorem 3.18</u>. For the third Hamiltonian structure (3.17) of the Modified Toda hierarchy, both Miura maps M_1 and M_2 are canonical with respect to the third Hamiltonian structure III (5.18) of the Toda hierarchy.

<u>Proof</u>. Denote by $\bar{B}^3$ the Hamiltonian matrix which corresponds to (3.17). We use the same computations as at the end of section 2. For M_1 we have from (2.24),

$$J\bar{B}^3 = \left\| \begin{array}{c|c} \left\{ \begin{array}{l} u(v\Delta - \Delta^{-1}v) + \\ +v[(\Delta-1)u+v(\Delta-1)] \end{array} \right\} u & \left\{ \begin{array}{l} u^2(1-\Delta^{-1})+u(1-\Delta^{-1})v + \\ +v(\Delta u - u\Delta^{-1}) \end{array} \right\} v \\ \hline v \left\{ \begin{array}{l} u^{(1)}\Delta(v\Delta-\Delta^{-1}v) + \\ +u^{(1)}[(\Delta-1)u+v(\Delta-1)] \end{array} \right\} u & v \left\{ \begin{array}{l} u^{(1)}\Delta[u(1-\Delta^{-1})+(1-\Delta^{-1})v] + \\ +u^{(1)}(\Delta u - u\Delta^{-1}) \end{array} \right\} v \end{array} \right\| ,$$

and $J\bar{B}^3J^*$ has the following components:

$$\begin{aligned} 1)\ B_{oo} &= [uv\Delta u + v\Delta u^2 + v^2\Delta u + v\Delta uv] + \\ &\quad + [-vu^2 - v^2u + (u^2+uv)v] + [-u\Delta^{-1}vu - u^2\Delta^{-1}v - u\Delta^{-1}v^2 - vu\Delta^{-1}v] = \\ &= vu^{(1)}(u\Delta + \Delta u + v\Delta + \Delta v) - \\ &\quad - (\Delta^{-1}u + u\Delta^{-1} + \Delta^{-1}v + v\Delta^{-1})vu^{(1)} = \\ &= M_1[q_1(q_o\Delta + \Delta q_o) - (q_o\Delta^{-1} + \Delta^{-1}q_o)q_1] \ ; \end{aligned}$$

$$\begin{aligned} 2)\ B_{10} &= vu^{(1)}\{(\Delta v\Delta u) + (\Delta u^2 + v\Delta u + \Delta uv + \Delta v^2 + \Delta uv) + \\ &\quad + (-vu - u^2 - vu - u^{(1)}v - v^2) - u\Delta^{-1}v\} = \\ &= M_1(q_1)\{\Delta vu^{(1)}\Delta + [\Delta(u+v)^2 + u^{(1)}v\Delta] - [(u+v)^2 + u^{(1)}v] - \Delta^{-1}u^{(1)}v\} = \end{aligned}$$

$$= M_1\{q_1[\Delta q_1\Delta+\Delta q_o^2+q_1\Delta-q_1-q_o^2-\Delta^{-1}q_1]\} =$$
$$= M_1\{q_1(\Delta-1)q_o^2+q_1(\Delta+1)(q_1\Delta-\Delta^{-1}q_1)\} ;$$

3) $$B_{11} = vu^{(1)}\{[\Delta(v\Delta-\Delta^{-1}v)+(\Delta-1)u+v(\Delta-1)]\Delta^{-1} + \Delta[u(1-\Delta^{-1})+(1-\Delta^{-1})v] + \Delta u-u\Delta^{-1}\}vu^{(1)} =$$
$$= vu^{(1)}\{(\Delta v+\Delta u+\Delta v+\Delta u) + (-v\Delta^{-1}-u\Delta^{-1}-v\Delta^{-1}-u\Delta^{-1}) + [u^{(1)}+v-u^{(1)}-v]\}vu^{(1)} =$$
$$= vu^{(1)}2\{\Delta(v+u)-(v+u)\Delta^{-1}\}vu^{(1)} =$$
$$= M_1\{2q_1(\Delta q_o-q_o\Delta^{-1})q_1\} =$$
$$= M_1\{q_1[(\Delta-1)q_o(1+\Delta^{-1}) + (\Delta+1)q_o(1-\Delta^{-1})]q_1\} ,$$

thus we get exactly the image of III (5.18).

The same computation goes through for M_2. Using (2.25), we get

$$J\bar{B}^3 = \left|\begin{array}{c|c} \begin{Bmatrix} u(v\Delta-\Delta^{-1}v) + \\ + \Delta^{-1}v[(\Delta-1)u+v(\Delta-1)] \end{Bmatrix} u & \begin{Bmatrix} u^2(1-\Delta^{-1})+u(1-\Delta^{-1})v + \\ + \Delta^{-1}v(\Delta u-u\Delta^{-1}) \end{Bmatrix} v \\ \hline vu\begin{Bmatrix} v\Delta-\Delta^{-1}v + \\ + (\Delta-1)u+v(\Delta-1) \end{Bmatrix} u & uv\begin{Bmatrix} u(1-\Delta^{-1})+(1-\Delta^{-1})v + \\ + \Delta u-u\Delta^{-1} \end{Bmatrix} v \end{array}\right| ,$$

and for $J\bar{B}^3J^*$ we have the following components:

1) $$B_{oo} = [uv\Delta u+u^2v\Delta+uv^2\Delta+v^{(-1)}uv\Delta] + [-u\Delta^{-1}vu-\Delta^{-1}vu^2-\Delta^{-1}v^2u-\Delta^{-1}vuv^{(-1)}] + [v^{(-1)}u^2+(v^{(-1)})^2u-u^2v^{(-1)}-u(v^{(-1)})^2] =$$
$$= uv[\Delta u+u\Delta+\Delta v^{(-1)}+v^{(-1)}\Delta] - [u\Delta^{-1}+\Delta^{-1}u+v^{(-1)}\Delta+\Delta^{-1}v^{(-1)}]uv =$$
$$= M_2\{q_1(q_o\Delta+\Delta q_o) - (q_o\Delta^{-1}+\Delta^{-1}q_o)q_1\} ;$$

2) $$B_{01} = \{u(v\Delta-\Delta^{-1}v) + \Delta^{-1}v[(\Delta-1)u+v(\Delta-1)] + u^2(1-\Delta^{-1}) + u(1-\Delta^{-1})v+\Delta^{-1}v(\Delta u-u\Delta^{-1})\}uv =$$
$$= \{uv\Delta-\Delta^{-1}vu\Delta^{-1} + [v^{(-1)}u+(v^{(-1)})^2+u^2+uv+v^{(-1)}u] + [-uv^{(-1)}\Delta^{-1}-\Delta^{-1}vu - (v^{(-1)})^2\Delta^{-1}-u^2\Delta^{-1}-uv^{(-1)}\Delta^{-1}]\}uv =$$
$$= \{uv\Delta-\Delta^{-1}uv\Delta^{-1} + [uv+(u+v^{(-1)})^2]-\Delta^{-1}uv-(u+v^{(-1)})^2\Delta^{-1}\}uv =$$

$$= M_2\{[(q_1\Delta-\Delta^{-1}q_1)(1+\Delta^{-1})+q_o^2(1-\Delta^{-1})]q_1\} ;$$

3) $$B_{11} = uv\{v\Delta-\Delta^{-1}v+(\Delta-1)u+v(\Delta-1)+u(1-\Delta^{-1})+(1-\Delta^{-1})v + \Delta u-u\Delta^{-1}\}uv = uv\{\Delta[2v^{(-1)}+2u] - [2u+2v^{(-1)}]\Delta^{-1}\}uv = M_2\{2q_1(\Delta q_o-q_o\Delta^{-1})q_1\} ,$$

as desired. □

4. Specialization to $\zeta + \zeta^{-1}q$

For the general operator $\bar{L}$ in (2.2), the modified Lax equations imply Lax equations for each of the operators $L_1 = \ell_1\ell_2$ and $L_2 = \ell_2\ell_1$. In both cases, the operators L_i have the $\gamma = 1$-form,

$$L = \zeta + \sum_{j\geq 0} \zeta^{-j}q_j . \tag{4.1}$$

As we know from Chapter I, we can put "gaps" of arbitrary size γ in L, requiring

$$\{q_j = 0 , j \not\equiv 0(\text{mod } \gamma)\} , \tag{4.2}$$

in which case our Lax equations have to be constructed from $P = L^n$ with $n \equiv 0$ (mod γ).

Unfortunately, if we look at the relations among $\{u,v_j\}$ in $\bar{L}$ (2.2) which result from the Miura maps being applied to (4.2), these relations cannot be resolved explicitly. Thus, for example, we would not know how to find modified equations with respect to the operator

$$\zeta + \zeta^{-1}q_o + \zeta^{-3}q_1 .$$

The origin of this difficulty seems clear enough: it is the size n = 2 of the matrix $\bar{L}$ of our modified equations. Apparently, one has to consider matrices with n > 2, but from section 3 we can appreciate what a nightmare a search for a Hamiltonian form would turn out to be.

There exists, however, one case for which the problem of specialization can be well understood: it is the case of the modified Toda hierarchy of the preceeding section:

$$\bar{L} = \begin{vmatrix} 0 & \zeta + u \\ 1+\zeta^{-1}v & 0 \end{vmatrix} . \tag{4.3}$$

Then $\ell_1\ell_2 = (\zeta+u)(1+\zeta^{-1}v) = \zeta + (u+v) + \zeta^{-1}u^{(1)}v$, and if we wish this operator to be of the form

$$L = \zeta + \zeta^{-1}q , \tag{4.4}$$

we have to specialize our $\bar{L}$ by requiring

$$v = -u . \tag{4.5}$$

Thus our $\bar{L}$ becomes

$$\bar{L} = \begin{vmatrix} 0 & \zeta + u \\ 1-\zeta^{-1}u & 0 \end{vmatrix} , \tag{4.6}$$

and we are faced with two typical problems of specializations (in the differential case these problems are discussed in considerable detail in [9], section 3.). The first problem is this: which Lax equations survive the specialization (4.5)? In other words, for which $\bar{P}$ will we have

$$[\partial_{\bar{P}}(u+v)] \in \left\{ \begin{matrix} \text{Ideal in } K[u^{(n)}, v^{(m)}] \\ \text{generated by } (u+v)^{(s)} \ , \ s \in \mathbb{Z} \end{matrix} \right\} ? \tag{4.7}$$

The second problem is: for which n do the conservation laws $H_n = \frac{1}{2n} \operatorname{Tr} \operatorname{Res} \bar{L}^{2n}$ remain nontrivial? After solving these two problems, we can consider the third one, which is to find a Hamiltonian form of the specialized equations.

We proceed as follows. Equations (3.3) are consistent iff

$$- \partial_{\bar{P}}(v) = \partial_{\bar{P}}(u) ,$$

or, equivalently,

$$u[p_o(n)-q_o(n)] = u[\Delta q_o(n)-p_o(n)] \ ,$$

or, equivalently again,

$$(\Delta+1)q_o(n) = 2p_o(n) \ , \tag{4.8}$$

where

$$\ell_1\ell_2 = \zeta-\zeta^{-1}uu^{(1)} \ , \ \ell_2\ell_1 = \zeta+(1-\Delta^{-1})u - \zeta^{-1}u^2 \ , \tag{4.9}$$

$$(\ell_1\ell_2)^n = \sum_j p_j(n)\zeta^j \ , \ (\ell_2\ell_1)^n = \sum_j q_j(n)\zeta^j \ . \tag{4.10}$$

To solve (4.8) we first use (3.8) to get

$$(\Delta+1)[p_{-1}(n-1)+(1-\Delta^{-1})up_o(n-1)-\Delta^{-1}up_1(n-1)] = 2p_o(n) \ . \tag{4.11}$$

Then we add (3.6a) and (3.6b) with j = 0, and substitute the result into the right-hand side of (4.11). After cancellations, the result is

$$(\Delta-\Delta^{-1})up_o(n-1) = 0$$

which holds iff $p_o(n-1) = 0$. This happens iff

$$n \equiv 0 \pmod 2) \ . \tag{4.12}$$

Indeed, if $n \not\equiv 0 \pmod 2$, then $(\ell_1\ell_2)^n$ has only odd powers of ζ present; on the other hand, if $n \equiv 0 \pmod 2$, $p_o(n) \neq 0$ by lemma III 1.13.

Thus, we get sensible specialized modified Lax equations only for $\bar{P} = \bar{L}^{4n}$, $n \in \mathbb{N}$. To sum up:

<u>Theorem 4.13</u>. i) The modified equations for the specialized $\bar{L}$ of (4.6) are consistent if and only if $\bar{P} = \bar{L}^{4n}$, $n \in \mathbb{N}$; ii) When they are consistent, the equations are nontrivial.

Proof. Part i) was proved above. To prove ii), notice that if $\partial_{\bar{P}}(u) = 0$, then $\partial_{\bar{P}}(v) = 0$, and from (3.3) it follows that this is possible only when $p_o(n) = q_o(n) \in k$, which is not true. □

Remark 4.14. It would make no difference if one tries to specialize $\ell_2\ell_1$, instead of $\ell_1\ell_2$: if we wish $\mathrm{Res}(\ell_2\ell_1) = 0$, it means $u+v^{(-1)} = 0$, i.e. $v^{(-1)} = -u$, which amounts to the same situation as before if we write $\ell_1 = \zeta+u$, $\ell_2 = 1+v^{(-1)}\zeta^{-1}$, consider ζ acting on the left and read our arguments in mirror-fashion.

Proposition 4.15. Let $H_n = \frac{1}{n}\mathrm{Res}(\ell_1\ell_2)^n$. Then $H_n \sim 0$ for $n \not\equiv 0 \pmod 2$, $H_n \not\sim 0$ for $n \equiv 0 \pmod 2$.

Proof. Again, lemma III 1.13 says that $H_{2n} \not\sim 0$ in $K[q^{(m)}]$ with $q = u^{(1)}u$. But the Miura map $M : q \to u^{(1)}u$ is injective (in every sense), therefore $H_{2n} \not\sim 0$ in $K[u^{(m)}]$ as well. On the other hand, $(\ell_1\ell_2)^{2n+1}$ has no terms of ζ-degree zero. □

Another proof of nontriviality of H_{2n} will follow from the Hamiltonian form (4.26) of our equations, which we shall begin to analyze at this point.

Proposition 4.16.

$$\frac{\delta H_{2n}}{\delta u} = -[u^{(1)}p_1(2n-1)+\Delta^{-1}up_1(2n-1)] .$$

Proof.

$$\frac{1}{2n}\, d\,\mathrm{Res}(\ell_1\ell_2)^{2n} \sim \mathrm{Res}[(\ell_1\ell_2)^{2n-1}\, d(\ell_1\ell_2)] = \mathrm{Res}[\sum_j p_j(2n-1)\zeta^j\, d(-\zeta^{-1}u^{(1)}u)] =$$

$$= -p_1(2n-1)[u^{(1)}du+udu^{(1)}] \sim$$

$$\sim -[p_1(2n-1)u^{(1)}+\Delta^{-1}(p_1(2n-1)u)]du . \qquad \square$$

Now let us look at the equation

$$\partial_{\bar{P}}(u) = u[p_o(2n)-q_o(2n)] . \tag{4.17}$$

From (3.8), taking into account that $p_{2s}(2n-1) = 0$, we get

$$q_o(2n) = p_{-1}(2n-1)-\Delta^{-1}u^{(1)}up_1(2n-1) ,$$

which becomes, with the help of (3.12),

$$q_o(2n) = - 2\Delta^{-1}u^{(1)}up_1(2n-1) . \tag{4.18}$$

On the other hand, (3.11) yields

$$p_o(2n) = -(1+\Delta^{-1})uu^{(1)} p_1(2n-1) ,$$

which together with (4.18) results in

$$p_o(2n) - q_o(2n) = (\Delta^{-1}-1)uu^{(1)}p_1(2n-1) , \tag{4.19}$$

and, thus,

$$\partial_{\bar{p}}(u) = u(\Delta^{-1}-1)uu^{(1)}p_1(2n-1) . \tag{4.20}$$

Since the expression $uu^{(1)}p$ in (4.20) cannot be expressed in terms of the combination $u^{(1)}p + \Delta^{-1}up$ in (4.16) (which is obvious and easy to prove), our equation (4.20) cannot be expressed through the Hamiltonian H_{2n}. Let us see if we can use $H_{2(n-1)}$ instead.

Denote $w = u^{(1)}u$, so that

$$\ell_1\ell_2 = \zeta-\zeta^{-1}w , (\ell_1\ell_2)^2 = \zeta^2 - (1+\Delta^{-1})w+\zeta^{-2} ww^{(1)} . \tag{4.21}$$

Consider the identities $(\ell_1\ell_2)^{2n-1} = (\ell_1\ell_2)^{2n-3}(\ell_1\ell_2)^2 = (\ell_1\ell_2)^2(\ell_1\ell_2)^{2n-3}$, and pick from all sides the ζ^1-coefficients. We get

$$p_1(2n-1) = p_{-1}(2n-3)-p_1(2n-3)(\Delta+1)w+p_3(2n-3)w^{(1)}w^{(2)} , \tag{4.22a}$$

$$p_1(2n-1) = \Delta^2p_{-1}(2n-3)-(w+w^{(-1)})p_1(2n-3)+\Delta^{-2}ww^{(1)}p_3(2n-3) . \tag{4.22b}$$

Let us apply Δ^2 to (4.22b), then multiply by $w^{(2)}$ and subtract from the result (4.22a) multiplied by w. We obtain

$$wp_1(2n-1)-w^{(2)}\Delta^2 p_1(2n-1) = wp_{-1}(2n-3)-w^{(2)}\Delta^4 p_{-1}(2n-3) +$$

$$+ w^{(2)}[w^{(2)}+w^{(1)}]\Delta^2 p_1(2n-3)-w[w^{(1)}+w]p_1(2n-3) \ . \tag{4.23}$$

Now use (3.12) to eliminate p_{-1} in (4.23):

$$(1-\Delta^2)wp_1(2n-1) = \{-w\Delta^{-1}w+w^{(2)}\Delta^4\Delta^{-1}w +$$

$$+ \Delta^2 w[w+w^{(-1)}] - w[w^{(1)}+w]\}p_1(2n-3) =$$

$$= (1+\Delta)[w^{(1)}\Delta^2-w](1+\Delta^{-1})wp_1(2n-3) \ .$$

Dividing from the left by $\Delta^{-1}(1+\Delta)$ and using (4.16) in the form

$$- u \frac{\delta H_{2n}}{\delta u} = (1+\Delta^{-1})wp_1(2n-1) \ , \tag{4.24}$$

we get

$$(\Delta^{-1}-1)wp_1(2n-1) = \Delta^{-1}(w-w^{(1)}\Delta^2)u \frac{\delta H_{2n-2}}{\delta u} =$$

$$= (u\Delta^{-1}u-u\Delta u)u \frac{\delta H_{2n-2}}{\delta u} \ .$$

Substituting this last expression into (4.20) we obtain the following theorem:

Theorem 4.25. The specialized equations (4.17) of the modified Toda hierarchy can be written in the Hamiltonian form

$$\partial_{\bar{P}}(u) = u^2(\Delta^{-1}-\Delta)u^2 \frac{\delta H}{\delta u} \ , \ H = H_{2n-2} \ . \tag{4.26}$$

Remark 4.26. At least now we don't have to make a forward reference to where the proof is given about our structure being a Hamiltonian structure indeed: if one introduces new "coordinate" $\tilde{u} = \frac{1}{u}$, then (4.26) can be written as

$$\partial_{\bar{P}}(\tilde{u}) = (\Delta^{-1}-\Delta) \frac{\delta H}{\delta \tilde{u}} \ ,$$

which is almost obviously Hamiltonian, having constant coefficients (the general result is theorem VIII 2.29).

Remark 4.27. The reader might have noticed an implicit assumption made in deriving (4.26): that $n > 1$; indeed, $H_{2(1-1)} = H_o \sim 0$ while equations (4.17) still make sense for $n = 1$. We thus have to check out whether we can cast (4.17) with $n = 1$ into the form (4.26). To do that, let us just compute $p_o(2)$ and $q_o(2)$. We have,

$$\begin{aligned} p_o(2) &= \text{Res } (\ell_1\ell_2)^2 = \text{Res}(\zeta-\zeta^{-1}uu^{(1)})^2 = \\ &= -(1+\Delta^{-1})uu^{(1)} , \\ q_o(2) &= \text{Res } (\ell_2\ell_1)^2 = \text{Res}[\zeta+(1-\Delta^{-1})u-\zeta^{-1}u^2]^2 = \\ &= [(1-\Delta^{-1})u]^2 - (1+\Delta^{-1})u^2 . \end{aligned}$$

Thus,

$$\begin{aligned} p_o(2) - q_o(2) &= (1+\Delta^{-1})(u^2-uu^{(1)})-[u-u^{(-1)}]^2 = \\ &= u^2-uu^{(1)} + (u^{(-1)})^2-u^{(-1)}u-[u^2-2uu^{(1)}+(u^{(-1)})^2] = u(u^{(-1)}-u^{(1)}) , \end{aligned}$$

and, therefore,

$$\partial_{\bar{P}}(u) = u^2(u^{(-1)}-u^{(1)}) , \tag{4.28}$$

which can be written as

$$\partial_{\bar{P}}(u) = u^2(\Delta^{-1}-\Delta)u^2 \frac{\delta}{\delta u} \ell nu . \tag{4.29}$$

At this point, having found the Hamiltonian form (4.26) of the modified equations, we could ask what happens with this structure under the Miura map

$$M : q \to -uu^{(1)} , \ \ell_1\ell_2 = M(\zeta+\zeta^{-1}q) . \tag{4.30}$$

The usual phenomenon is that a Hamiltonian structure of modified equations induces, under an appropriate Miura map, a Hamiltonian structure of unmodified equations. Let us see what happens in our case. Taking the Fréchet derivative of M, $J = -(u^{(1)}+u\Delta)$, we have to compute

$$J[u^2(\Delta^{-1}-\Delta)u^2]J^* = (u^{(1)}+u\Delta)u^2(\Delta^{-1}-\Delta)u^2(u^{(1)}+\Delta^{-1}u) =$$
$$= u^{(1)}u[(u+\Delta u)(\Delta^{-1}-\Delta)(u+u\Delta^{-1})]u^{(1)}u =$$
$$= u^{(1)}u[(1+\Delta)u(\Delta^{-1}-\Delta)u(1+\Delta^{-1})]u^{(1)}u =$$
$$= u^{(1)}u(1+\Delta)[\Delta^{-1}u^{(1)}u-uu^{(1)}\Delta](1+\Delta^{-1})u^{(1)}u =$$
$$= M\{q(1+\Delta)(q\Delta-\Delta^{-1}q)(1+\Delta^{-1})q\} .$$

We thus get

Theorem 4.31. Lax equations with $L = \zeta + \zeta^{-1}q$ have the third Hamiltonian structure

$$\partial_P(q) = q(1+\Delta)(q\Delta-\Delta^{-1}q)(1+\Delta^{-1})q\,\frac{\delta H}{\delta q} , \quad H = H_{2n-2} , \tag{4.32}$$

for $P = L^{2n}$. The Miura map (4.30) is canonical between (4.32) and (4.26).

Again, we have to check the lowest case of $P = L^2$. Then $P_- = \zeta^{-1}q\zeta^{-1}q$, so

$$\partial_P(L) = \zeta^{-1}\partial_P(q) = \zeta^{-1}q(q^{(1)}-q^{(-1)}) , \text{ so}$$

$$\partial_P(q) = q(q^{(1)}-q^{(-1)}) . \tag{4.33}$$

On the other hand,

$$H = \frac{1}{2}\,\ell n q \tag{4.33'}$$

in (4.32) yields

$$q(1+\Delta)(q\Delta-\Delta^{-1}q)(1+\Delta^{-1})q\,\frac{1}{2q} = q(1+\Delta)(q-q^{(-1)}) =$$
$$= q(q-q^{(-1)}+q^{(1)}-q) = q(q^{(1)}-q^{(-1)}) ,$$

as in (4.33).

Remark 4.34. The existence of the third Hamiltonian structure (4.32) for the specialized operator $L = \zeta + \zeta^{-1}q$ strongly suggests that analogous extra Hamiltonian forms exist for at least some other nongeneral operators of the form

$L = \zeta(1 + \sum_{j=0}^{N} \zeta^{-\gamma(j+1)} q_j)$, $\gamma > 1$. The simplest case must be when L is monomial $\zeta(1+\zeta^{-\gamma}q)$ or binomial $\zeta(1+\zeta^{-\gamma}q_o+\zeta^{-2\gamma}q_1)$. If we hope to induce the Hamiltonian structure from the Miura maps, the binomial case is actually simpler, and so at this point we shall analyze it.

5. Modification of $\zeta(1+\zeta^{-\gamma}q_o+\zeta^{-2\gamma}q_1)$

We have now

$$\ell_1 = \zeta(1+\zeta^{-\gamma}u) \ , \ \ell_2 = 1 + \zeta^{-\gamma}v \ , \ \gamma \geq 1 \ , \tag{5.1}$$

so that we recover the modified Toda situation for $\gamma = 1$. It would now seem appropriate to use the path of section 3 and eliminate q's at each step. This is indeed what we shall do. We have

$$\ell_1\ell_2 = \zeta[1+\zeta^{-\gamma}(u+v) + \zeta^{-2\gamma}u^{(\gamma)}v] \ , \tag{5.2a}$$

$$\ell_2\ell_1 = \zeta[1+\zeta^{-\gamma}(u+v^{(-1)}) + \zeta^{-2\gamma}uv^{(\gamma-1)}] \ , \tag{5.2b}$$

$$(\ell_1\ell_2)^n = \sum_j p_j(n)\zeta^j \ , \ (\ell_2\ell_1)^n = \sum_j q_j(n)\zeta^j \ . \tag{5.3}$$

Now we take $\bar{P} = \bar{L}^{2n\gamma} = \mathrm{diag}[(\ell_1\ell_2)^{n\gamma} \ , \ (\ell_2\ell_1)^{n\gamma}]$. Then the modified Lax equations (1.7) become

$$\partial_{\bar{P}}(u) = u[\Delta^{\gamma-1}p_o(n\gamma) - q_o(n\gamma)] \ , \tag{5.4a}$$

$$\partial_{\bar{P}}(v) = v[\Delta^{\gamma}q_o(n\gamma) - p_o(n\gamma)] \ . \tag{5.4b}$$

Next are functional derivatives. For

$$H_n = \frac{1}{n} \mathrm{Res} \ (\ell_1\ell_2)^n \ , \tag{5.5}$$

we have

$$dH_n \sim \mathrm{Res}[(\ell_1\ell_2)^{n-1}d(\ell_1\ell_2)] = \mathrm{Res}[\Sigma p_j(n-1)\zeta^j \ \zeta^{1-\gamma}\{du+dv+$$

$$+ \zeta^{-\gamma}[u^{(\gamma)}dv+vdu^{(\gamma)}]\}] = p_{\gamma-1}(n-1)(du+dv)+p_{2\gamma-1}(n-1)[u^{(\gamma)}dv+vdu^{(\gamma)}] .$$

Therefore,

$$\frac{\delta H_n}{\delta u} = p_{\gamma-1}(n-1) + \Delta^{-\gamma}vp_{2\gamma-1}(n-1) , \tag{5.6a}$$

$$\frac{\delta H_n}{\delta v} = p_{\gamma-1}(n-1) + u^{(\gamma)}p_{2\gamma-1}(n-1) . \tag{5.6b}$$

Now we use $(\ell_1\ell_2)^n = (\ell_1\ell_2)^{n-1}\ell_1\ell_2 = \ell_1\ell_2(\ell_1\ell_2)^{n-1}$ to get

$$p_j(n) = p_{j-1}(n-1) + p_{j-1+\gamma}(n-1)\Delta^j(u+v) + p_{j-1+2\gamma}(n-1)\Delta^j u^{(\gamma)}v , \tag{5.7a}$$

$$p_j(n) = \Delta p_{j-1}(n-1) + \Delta^{1-\gamma}p_{j-1+\gamma}(n-1)(u+v) + \Delta^{1-2\gamma}p_{j-1+2\gamma}(n-1)u^{(\gamma)}v . \tag{5.7b}$$

Taking ζ^o-term in $(\ell_2\ell_1)^n = \ell_2(\ell_1\ell_2)^{n-1}\ell_1$, we get

$$q_o(n) = p_{-1}(n-1) + (u+\Delta^{-\gamma}v)p_{\gamma-1}(n-1) + \Delta^{-\gamma}u^{(\gamma)}vp_{2\gamma-1}(n-1) . \tag{5.8}$$

For $j = 0$, apply Δ to (5.7a) and subtract (5.7b) to obtain

$$p_o(n) = \Delta \frac{1-\Delta^{-\gamma}}{\Delta-1} [p_{\gamma-1}(n-1)(u+v) + (1+\Delta^{-\gamma})p_{2\gamma-1}(n-1)u^{(\gamma)}v] . \tag{5.9}$$

For $j = 0$, subtract (5.7b) from (5.7a) and get

$$p_{-1}(n-1) = \frac{1-\Delta^{1-\gamma}}{\Delta-1} p_{\gamma-1}(n-1)(u+v) + \frac{1-\Delta^{1-2\gamma}}{\Delta-1} p_{2\gamma-1}(n-1)u^{(\gamma)}v . \tag{5.10}$$

Now substituting (5.9) and (5.10) in (5.8), we have

$$q_o(n) = \frac{1-\Delta^{-\gamma}}{\Delta-1} [(\Delta u+v)p_{\gamma-1}(n-1)+(1+\Delta^{1-\gamma})p_{2\gamma-1}(n-1)u^{(\gamma)}v] . \tag{5.11}$$

We can now handle (5.4). Substituting (5.9) and (5.11) into (5.4) we find that

$$\partial_{\bar{P}}(u) = u\,\frac{1-\Delta^{-\gamma}}{\Delta-1}\,\{[(\Delta^{\gamma}-\Delta)u+(\Delta^{\gamma}-1)v]p_{\gamma-1}(n\gamma-1)+(\Delta^{\gamma}-\Delta^{1-\gamma})u^{(\gamma)}vp_{2\gamma-1}(n\gamma-1)\}\ , \tag{5.12a}$$

$$\partial_{\bar{P}}(v) = v\,\frac{1-\Delta^{-\gamma}}{\Delta-1}\,\{[(\Delta^{\gamma+1}-\Delta)u+(\Delta^{\gamma}-\Delta)v]p_{\gamma-1}(n\gamma-1)+(\Delta^{\gamma}-\Delta^{1-\gamma})u^{(\gamma)}vp_{2\gamma-1}(n\gamma-1)\}\ , \tag{5.12b}$$

To represent expressions in the curly brackets through $\frac{\delta H}{\delta u}$ and $\frac{\delta H}{\delta v}$ of (5.6), $H = H_{n\gamma}$, we use the same device as in solving (3.14). Suppose, for (5.12a), we have found operators A and B such that

$$A\,\frac{\delta H}{\delta u} + B\,\frac{\delta H}{\delta v} = \{\ \cdots\ \}\ \text{in (5.12a)}\ .$$

Using (5.6), we get the system

$$\begin{cases} A + B = (\Delta^{\gamma}-\Delta)u + (\Delta^{\gamma}-1)v\ , \\ A\Delta^{-\gamma}v + Bu^{(\gamma)} = (\Delta^{\gamma}-\Delta^{1-\gamma})u^{(\gamma)}v\ . \end{cases}$$

Thus, $A = \alpha u$, $B = \beta v$, and

$$\begin{cases} \alpha u + \beta v = (\Delta^{\gamma}-\Delta)u + (\Delta^{\gamma}-1)v\ , \\ \alpha\Delta^{-\gamma} + \beta = \Delta^{\gamma} - \Delta^{1-\gamma}\ , \end{cases}$$

from which we readily find

$$\alpha = \Delta^{\gamma}-\Delta\ ,\ \beta = \Delta^{\gamma}-1\ ,$$

$$A = (\Delta^{\gamma}-\Delta)u\ ,\ B = (\Delta^{\gamma}-1)v\ .$$

Therefore

$$\partial_{\bar{P}}(u) = u\,\frac{1-\Delta^{-\gamma}}{\Delta-1}\,[(\Delta^{\gamma}-\Delta)u\,\frac{\delta H}{\delta u} + (\Delta^{\gamma}-1)v\,\frac{\delta H}{\delta v}]\ . \tag{5.13a}$$

The same computation works for (5.12b). If

$$A\,\frac{\delta H}{\delta u} + B\,\frac{\delta H}{\delta v} = \{\cdots\} \text{ in } (5.12b),$$

then

$$\begin{cases} A + B = (\Delta^{\gamma+1}-\Delta)u + (\Delta^{\gamma}-\Delta)v \ , \\ A\Delta^{-\gamma}v + Bu^{(\gamma)} = (\Delta^{\gamma}-\Delta^{1-\gamma})u^{(\gamma)}v \ , \end{cases}$$

and, therefore,

$$A = (\Delta^{\gamma+1}-\Delta)u \ , \ B = (\Delta^{\gamma}-\Delta)v \ ,$$

$$\partial_{\bar{P}}(v) = v\,\frac{1-\Delta^{-\gamma}}{\Delta-1}\,[(\Delta^{\gamma+1}-\Delta)u\,\frac{\delta H}{\delta u} + (\Delta^{\gamma}-\Delta)v\,\frac{\delta H}{\delta v}] \ . \tag{5.13b}$$

Formulae (5.13) provide the Hamiltonian form of the modified Lax equations with $\bar{L}$ defined by (5.1). The fact that equations (5.13) are indeed Hamiltonian, and the following theorem are particular cases of the results which will be discussed in section 7.

Theorem 5.14. The Miura maps $M_1(L) = \ell_1\ell_2$, $M_2(L) = \ell_2\ell_1$, where $L = \zeta(1+\zeta^{-\gamma}q_o+\zeta^{-2\gamma}q_1)$, are canonical between (5.13) and the $N = 1$-case of III (4.14).

6. Modification of $\zeta(1+\zeta^{-2\gamma}q)$.

In this section we specialize operators in (5.1) in such a way that

$$\ell_1\ell_2 = \zeta(1+\zeta^{-2\gamma}q) \ . \tag{6.1}$$

That is, we consider

$$\ell_1 = \zeta(1+\zeta^{-\gamma}u) \ , \ \ell_2 = 1-\zeta^{-\gamma}u \ . \tag{6.2}$$

Let us consider, along the lines of section 4, the problems of specialization of (5.1) into (6.2).

First we look at the Lax equations (5.4). For these to survive, we must have

$$\Delta^{\gamma-1}p_o(n\gamma) - q_o(n\gamma) = \Delta^{\gamma}q_o(n\gamma)-p_o(n\gamma) \ ,$$

or

$$(1+\Delta^{\gamma-1})p_o(n\gamma) = (1+\Delta^{\gamma})q_o(n\gamma) \ .$$

By using (5.9) and (5.11), this becomes

$$(1+\Delta^{\gamma-1})\Delta[-(1+\Delta^{-\gamma})p_{2\gamma-1}(n\gamma-1)u^{(\gamma)}u] = (1+\Delta^{\gamma})\{(\Delta-1)up_{\gamma-1}(n\gamma-1) - (1+\Delta^{1-\gamma})p_{2\gamma-1}(n\gamma-1)u^{(\gamma)}u\} \ . \tag{6.3}$$

Since

$$(1+\Delta^{\gamma-1})\Delta(1+\Delta^{-\gamma}) = (1+\Delta^{\gamma})(1+\Delta^{1-\gamma}) \ ,$$

(6.3) reduces to

$$p_{\gamma-1}(\gamma n-1) = 0 \ . \tag{6.4}$$

Since

$$p_{\gamma-1}(n\gamma-1) = \mathrm{Res}\{[\zeta(1+\zeta^{-2\gamma}q)]^{n\gamma-1}\ \zeta^{1-\gamma}\} = \zeta^{-(n-1)\gamma}\text{-coefficients in } (1+\zeta^{-2\gamma}q^{(1)})(1+\zeta^{-2\gamma}q^{(2)})\dots(1+\zeta^{-2\gamma}q^{(n\gamma-1)}) \ , \tag{6.5}$$

and since the product in (6.5) is polynomial in $\zeta^{-2\gamma}$ with all coefficients present being non-zero and belonging to the semiring $\mathbb{N}[q^{(\sigma)}]$, it follows that n-1 must be odd:

$$n \equiv 0 \pmod 2 \ , \tag{6.6}$$

which solves our first problem of specialization. As in section 4, it is clear that equations (5.4) do not degenerate because $p_o(2n\gamma)$ and $q_o(2n\gamma)$ do not vanish.

Next we consider c.l.'s. Obviously $0 = \mathrm{Res}(\ell_1\ell_2)^{(2n+1)\gamma}$, thus the c.l.'s $H_{(2n+1)\gamma}$ become trivial. For the same reason, $H_{2n\gamma}$ remain nontrivial.

Finally, let us look at the problem of converting the equations

$$\partial_{\bar{p}}(u) = u[\Delta^{\gamma-1}p_o(2n\gamma)-q_o(2n\gamma)] \tag{6.7}$$

into a Hamiltonian form. We begin with the conservation laws

$$H_{2n\gamma} = \frac{1}{2n\gamma} \operatorname{Res}(\ell_1\ell_2)^{2n\gamma} . \tag{6.8}$$

We have

$$dH_{2n\gamma} \sim \operatorname{Res}\{[\zeta(1+\zeta^{-2\gamma}w)]^{2n\gamma-1}d\zeta^{1-2\gamma}w\} = p_{2\gamma-1}(2n\gamma-1)dw =$$

$$= p_{2\gamma-1}(2n\gamma-1)[-u^{(\gamma)}du-udu^{(\gamma)}] , \tag{6.9}$$

where we denote $w = -uu^{(\gamma)}$ to conform with the notations of section 4. From (6.9), we get

$$-\frac{\delta H_{2n\gamma}}{\delta u} = p_{2\gamma-1}(2n\gamma-1)u^{(\gamma)} + \Delta^{-\gamma}up_{2\gamma-1}(2n\gamma-1) , \tag{6.10}$$

which becomes, after multiplying both sides by u:

$$u\frac{\delta H_{2n\gamma}}{\delta u} = (1+\Delta^{-\gamma})wp_{2\gamma-1}(2n\gamma-1) . \tag{6.11}$$

Using (6.4), our equation (5.12a) becomes

$$\partial_{\bar{p}}(u) = u\,\frac{1-\Delta^{-\gamma}}{\Delta-1}\,(\Delta^{\gamma}-\Delta^{1-\gamma})wp_{2\gamma-1}(2n\gamma-1) , \tag{6.12}$$

and since $(1-\Delta^{-1})(\Delta^{\gamma}-\Delta^{1-\gamma})$ is not divisible by $(1+\Delta^{-\gamma})$, we conclude that (6.11) cannot be used in (6.12), and, therefore, as in section 4, the second Hamiltonian structure does not exist and we have to look for the third one.

How are we to proceed? We need to express the right-hand side of (6.12) through

$$u\frac{\delta H_{2(n-1)\gamma}}{\delta u} = (1+\Delta^{-\gamma})wp_{2\gamma-1}(2(n-1)\gamma-1) . \tag{6.13}$$

We write $(\ell_1\ell_2)^m = (\ell_1\ell_2)^{m-1}\ell_1\ell_2 = \ell_1\ell_2(\ell_1\ell_2)^{m-1}$ in the form

$$p_j(m) = p_{j-1}(m-1) + p_{j-1+2\gamma}(m-1)w^{(j)} , \tag{6.14a}$$

$$p_j(m) = \Delta p_{j-1}(m-1) + \Delta^{1-2\gamma} p_{j-1+2\gamma}(m-1)w . \tag{6.14b}$$

Subtracting (6.14b) from (6.14a), we get

$$(1-\Delta)p_j(m) = [\Delta^{1-2\gamma}w - w^{(j+1)}]p_{j+2\gamma}(m) \tag{6.15}$$

which shows that $p_j(m)$ can be "almost expressed" through $p_{j+2\gamma}(m)$. In particular, for j = -1, we obtain the familiar equations

$$p_{-1}(m) = \frac{\Delta^{1-2\gamma}-1}{1-\Delta} w p_{2\gamma-1}(m) . \tag{6.16}$$

Now we have to write $(\ell_1\ell_2)^{2n\gamma-1} = (\ell_1\ell_2)^{2(n-1)\gamma-1}(\ell_1\ell_2)^{2\gamma} =$

$= (\ell_1\ell_2)^{2\gamma}(\ell_1\ell_2)^{2(n-1)\gamma-1}$, pick up the $\zeta^{2\gamma-1}$-terms to result in an analog of (4.22), and devise some elimination scheme which, using (6.15), will leave us with a desired expression of (6.12) through (6.13). We achieved this for $\gamma = 1$, in going from (4.22) to (4.24). For $\gamma > 1$, I want to argue that the task is impossible, at least along the proposed route. To simplify the arguments, notice that we are actually talking about the scalar operator $\zeta(1+\zeta^{-2\gamma}w)$ in our discussion, and that the modified origin of our problem is not important. Thus, we could begin with the scalar operator $\zeta(1+\zeta^{-\gamma}q)$ and restrict ourselves to this case. The first new case would be $\gamma = 3$, so let us take

$$L = \zeta(1+\zeta^{-3}q) . \tag{6.17}$$

Walking along the familiar route with $L^n = \sum_j p_j(n)\zeta^j$, we get

$$p_j(n) = p_{j-1}(n-1) + p_{j+2}(n-1)q^{(j)} , \tag{6.18a}$$

$$p_j(n) = \Delta p_{j-1}(n-1) + \Delta^{-2} p_{j+2}(n-1)q , \tag{6.18b}$$

$$p_{-1}(m) = (1+\Delta^{-1})\Delta^{-1} p_2(m)q , \tag{6.19}$$

$$p_o(n) = (1+\Delta^{-1}+\Delta^{-2})p_2(n-1)q \ , \tag{6.20}$$

$$H_n = \frac{1}{n}\,\mathrm{Res}\ L^n \ , \ \frac{\delta H_n}{\delta q} = p_2(n-1) \ , \tag{6.21}$$

$$\partial_p(q) = q(\Delta^2-1)p_o(m) = q(\Delta^2-1)(1+\Delta^{-1}+\Delta^{-2})qp_2(m-1),\ m = 3n \ . \tag{6.22}$$

Now we need a formula for $p_2(m-1)$ in terms of $p_2(m-4)$. First,

$$L^3 = \zeta^3 + [q+q^{(-1)}+q^{(-2)}] + \{[q^{(-1)}+q^{(-2)}]q^{(-3)}+q^{(-4)}q^{(-2)}\}\zeta^{-3} + \\ + q^{(-6)}q^{(-4)}q^{(-2)}\zeta^{-6} \ . \tag{6.23}$$

From $L^n = L^{n-3}L^3 = L^3L^{n-3}$, we obtain

$$p_s(n) = p_{s-3}(n-3) + p_s(n-3)[q^{(s)}+q^{(s-1)}+q^{(s-2)}] + \\ + p_{s+3}(n-3)\Delta^{s+3}\{[q^{(-1)}+q^{(-2)}]q^{(-3)} + q^{(-4)}q^{(-2)}\} + p_{s+6}(n-3)q^{(s+4)}q^{(s+2)}q^{(s)} \ , \tag{6.24a}$$

$$p_s(n) = \Delta^3p_{s-3}(n-3) + [q+q^{(-1)}+q^{(-2)}]p_s(n-3)+\{[q^{(-1)}+q^{(-2)}]q^{(-3)} + \\ + q^{(-4)}q^{(-2)}\}\Delta^{-3}p_{s+3}(n-3)+q^{(-6)}q^{(-4)}q^{(-2)}\Delta^{-6}p_{s+6}(n-3) \ , \tag{6.24b}$$

which becomes, after multiplying by q and putting s = 2:

$$qp_2(n) = qp_{-1} + qp_2[q+q^{(1)}+q^{(2)}] + qp_5\{[q^{(4)}+q^{(3)}]q^{(2)} + \\ + q^{(1)}q^{(3)}\} + p_8qq^{(2)}q^{(4)}q^{(6)} \ , \tag{6.25a}$$

$$qp_2(n) = q\Delta^3p_{-1} + q[q+q^{(-1)}+q^{(-2)}]p_2 + \{[q^{(-1)}+q^{(-2)}]q^{(-3)} + \\ + q^{(-4)}q^{(-2)}\}q\Delta^{-3}p_5 + \Delta^{-6}p_8qq^{(2)}q^{(4)}q^{(6)} \ , \tag{6.25b}$$

where p_i stands for $p_i(n-3)$.

Relations (6.15) for p_5 and p_8 become

$$(\Delta-1)p_5 = q^{(6)}p_8 - \Delta^{-2}qp_8 \ , \tag{6.26}$$

$$(\Delta-1)p_2 = q^{(3)}p_5 - \Delta^{-2}qp_5 \ . \tag{6.27}$$

Denote $R = p_8 qq^{(2)}q^{(4)}q^{(6)}$, then (6.26) becomes

$$qq^{(2)}q^{(4)}(\Delta-1)p_5 = (1-\Delta^{-2})R \ . \tag{6.28}$$

We need to get rid of p_5 and R, to be left with p_2 and $p_{-1} = \frac{\Delta^{-5}-1}{1-\Delta}\, qp_2$ only. The result must be an operator, with constant coefficients, acting on $qp_2(n)$, as is seen from (6.22). Thus, we need to apply some operators $A(\Delta)$ to (6.25a), $C(\Delta)$ to (6.25b), add them and, to be rid of p_5 and p_8, use (6.27), (6.28). Since R comes into (6.25) only as R in (6.25a) and $\Delta^{-6}R$ in (6.25b), then, in view of (6.28), we must have $C(\Delta) = [B(1-\Delta^{-2})-A]\Delta^6$, with some operator $B = B(\Delta)$.

Let us write $x \cong y$ if $(x-y)$ can be expressed in terms of p_2. Denote $p = p_5$. Then (6.27) can be rewritten as

$$qp \cong q^{(5)}p^{(2)} = \Delta^2(pq^{(3)}) \ . \tag{6.29}$$

We have

$$\begin{aligned} qq^{(2)}q^{(4)}p &\cong q^{(5)}p^{(2)}q^{(2)}q^{(4)} \cong \Delta^2[q^{(3)}q^{(2)}q^{(2)}p^{(5)}] \cong \\ &\cong \Delta^4[q^{(1)}q^{(3)}p^{(2)}q^{(5)}] = \Delta^6[pq^{(-1)}q^{(1)}q^{(3)}] = \Delta^6\sigma \ , \end{aligned} \tag{6.30}$$

where

$$\sigma := pq^{(-1)}q^{(1)}q^{(3)} \ . \tag{6.31}$$

Analogously, we have

$$qp_5\{[q^{(4)}+q^{(3)}]q^{(2)} + q^{(1)}q^{(3)}\} \cong (\Delta^2+\Delta^4+\Delta^6)\sigma \ , \tag{6.32}$$

$$p_5 q^{(3)}\Delta^3\{[q^{(-1)}+q^{(-2)}]q^{(-3)}+ q^{(-4)}q^{(-2)}\} \cong (1+\Delta^2+\Delta^4)\sigma \ . \tag{6.33}$$

Therefore,

$$\{A + [B(1-\Delta^{-2})-A]\Delta^6\}qp_2(n) \cong B\Delta\sigma - B\Delta^6\sigma + A[\Delta^2+\Delta^4+\Delta^6]\sigma +$$

$$+ [B(1-\Delta^{-2})-A]\Delta^3(1+\Delta^2+\Delta^4)\sigma \ , \tag{6.34}$$

and if we don't want σ, we must have

$$0 = B\Delta - B\Delta^6 + A(\Delta^2+\Delta^4+\Delta^6) + [B(1-\Delta^{-2})-A]\Delta^3(1+\Delta^2+\Delta^4) \ ,$$

or

$$B[\Delta-\Delta^6+(\Delta^2-1)\Delta(1+\Delta^2+\Delta^4)] = A(\Delta^3-\Delta^2)(1+\Delta^2+\Delta^4) \ .$$

Consequently,

$$B = A\Delta^{-4}(1+\Delta^2+\Delta^4) \ . \tag{6.35}$$

However, with this B we have

$$C = [B(1-\Delta^{-2})-A]\Delta^6 = A\{\Delta^{-6}(\Delta^2-1)(1+\Delta^2+\Delta^4)-1\}\Delta^6 =$$

$$= A\{\Delta^{-6}(\Delta^6-1)-1\}\Delta^6 = -A \ ,$$

which means that $Aqp_2(n) + Cqp_2(n) = 0$ and no equation for $qp_2(n)$ results. Notice that since we worked with (6.26) and (6.27) only, the arguments above show that there is no way one can express $D(\Delta)qp_2(n)$ through $qp_2(n-3)$ only, unless the operator $D(\Delta)$ vanishes. Thus the third Hamiltonian structure does not exist for $L = \zeta(1+\zeta^{-3}q)$. It probably does not exist for any $\gamma > 2$, $L = \zeta(1+\zeta^{-\gamma}q)$, but it would be hard to imitate the arguments above which work with a concrete form of L^γ.

7. Modified Form of $L = \zeta(1+\Sigma\zeta^{-\gamma(j+1)}q_j)$.

The results of section 5 suggest that it might be possible to analyze the modified equations with

$$\ell_1 = \zeta(1+\zeta^{-\gamma}u) \ , \ \ell_2 = 1 + \sum_j \zeta^{-\gamma(j+1)} v_j \ . \tag{7.1}$$

This is what we are going to do now, but instead of using the methods of section 5 which require us to solve systems of operator equations, we shall use the route given in section 3, holding on to the $q_j(n)$'s.

We have

$$\ell_1\ell_2 = \zeta\{1+\zeta^{-\gamma}(u+v_o) + \sum_{m\geq 0} \zeta^{-\gamma(m+2)}[v_{m+1}+v_m u^{(\gamma m+\gamma)}]\} \ , \tag{7.2a}$$

$$\ell_2\ell_1 = \zeta\{1+\zeta^{-\gamma}[u+v_o^{(-1)}] + \sum_{m\geq 0} \zeta^{-\gamma(m+2)}[v_{m+1}^{(-1)}+uv_m^{(\gamma-1)}]\} \ , \tag{7.2b}$$

$$(\ell_1\ell_2)^n = \sum_j p_j(n)\zeta^j \ , \ (\ell_2\ell_1)^n = \sum_j q_j(n)\zeta^j \ . \tag{7.3}$$

Equations $\partial_{\bar{P}}(\bar{L}) = [(\bar{L}^2)^{n\gamma}_+ \ , \ \bar{L}]$ become

$$\partial_{\bar{P}}(u) = u[\Delta^{\gamma-1}p_o(n\gamma)-q_o(n\gamma)] \ , \tag{7.4a}$$

$$\partial_{\bar{P}}(v_m) = \sum_{j\geq 0} [v_{m+j}\Delta^{\gamma(m+1)}q_{\gamma j}(n\gamma)-\Delta^{-\gamma j}v_{m+j}p_{\gamma j}(n\gamma)] \ . \tag{7.4b}$$

For $H_n = \frac{1}{n}\,\mathrm{Res}(\ell_1\ell_2)^n$, we have

$$dH_n \sim \mathrm{Res}[(\ell_1\ell_2)^{n-1}d(\ell_1\ell_2)] \sim \mathrm{Res}[(\ell_2\ell_1)^{n-1}d(\ell_2\ell_1)] \ ,$$

or, using (7.2),

$$dH_n \sim p_{\gamma-1}(n-1)(du+dv_o) + \sum_{m\geq 0} p_{\gamma(m+2)-1}(n-1)[dv_{m+1}+u^{(\gamma m+\gamma)}dv_m+v_m du^{(\gamma m+\gamma)}] \sim$$

$$\sim q_{\gamma-1}(n-1)[du+dv_o^{(-1)}] + \sum_{m\geq 0} q_{\gamma(m+2)-1}(n-1)[dv_{m+1}^{(-1)}+v_m^{(\gamma-1)}du+udv_m^{(\gamma-1)}] \ .$$

This implies that

$$\frac{\delta H_n}{\delta u} = p_{\gamma-1}(n-1) + \sum_{m\geq 0} \Delta^{-\gamma m-\gamma} v_m p_{\gamma(m+2)-1}(n-1) \ , \tag{7.5a}$$

$$\frac{\delta H_n}{\delta u} = q_{\gamma-1}(n-1) + \sum_{m\geq 0} q_{\gamma(m+2)-1}(n-1) v_m^{(\gamma-1)} \ , \tag{7.5b}$$

$$\frac{\delta H_n}{\delta v_m} = p_{\gamma(m+1)-1}(n-1) + u^{(\gamma m+\gamma)} p_{\gamma(m+2)-1}(n-1) \ , \tag{7.5c}$$

$$\frac{\delta H_n}{\delta v_m} = \Delta q_{\gamma(m+1)-1}(n-1) + \Delta^{1-\gamma} u q_{\gamma(m+2)-1}(n-1) \ . \tag{7.5d}$$

On expanding the identities $(\ell_1\ell_2)^n = (\ell_1\ell_2)^{n-1}\,\ell_1\ell_2 = \ell_1\ell_2(\ell_1\ell_2)^{n-1}$ and $(\ell_2\ell_1)^n = (\ell_2\ell_1)^{n-1}\ell_2\ell_1 = \ell_2\ell_1(\ell_2\ell_1)^{n-1}$, we get

$$p_j(n) = p_{j-1}(n-1) + p_{j-1+\gamma}(n-1)\Delta^j(u+v_o) + \sum_{m\geq 0} p_{j-1+\gamma(m+2)}(n-1)\Delta^j[v_{m+1}+u^{(\gamma m+\gamma)}v_m] \ , \tag{7.6a}$$

$$p_j(n) = \Delta p_{j-1}(n-1) + \Delta^{1-\gamma}(u+v_o)p_{j-1+\gamma}(n-1) + $$

$$+ \sum_{m\geq 0} \Delta^{1-\gamma(m+2)} p_{j-1+\gamma(m+2)}(n-1)[v_{m+1}+u^{(\gamma m+\gamma)}v_m] \ , \tag{7.6b}$$

$$q_j(n) = q_{j-1}(n-1) + q_{j-1+\gamma}(n-1)\Delta^j[u+v_o^{(-1)}] + $$

$$+ \sum_{m\geq 0} q_{j-1+\gamma(m+2)}(n-1)\Delta^j[v_{m+1}^{(-1)}+uv_m^{(\gamma-1)}] \ , \tag{7.7a}$$

$$q_j(n) = \Delta q_{j-1}(n-1) + \Delta^{1-\gamma}[u+v_o^{(-1)}]q_{j-1+\gamma}(n-1) + $$

$$+ \sum_{m\geq 0} \Delta^{1-\gamma(m+2)} q_{j-1+\gamma(m+2)}(n-1)[v_{m+1}^{(-1)}+uv_m^{(\gamma-1)}] \ . \tag{7.7b}$$

Now let us write down $(\ell_1\ell_2)^{n-1}\ell_1 = \ell_1(\ell_2\ell_1)^{n-1}$ and $(\ell_2\ell_1)^{n-1}\ell_2 = \ell_2(\ell_1\ell_2)^{n-1}$, thereby getting

$$p_{j-1}(n-1) + p_{j-1+\gamma}(n-1)u^{(j)} = \Delta q_{j-1}(n-1) + \Delta^{1-\gamma}uq_{j-1+\gamma}(n-1) , \tag{7.8a}$$

$$p_j(n-1) + \sum_{m\geq 0} \Delta^{-\gamma(m+1)}v_m p_{j+\gamma(m+1)}(n-1) = q_j(n-1) + \sum_{m\geq 0} q_{j+\gamma(m+1)}(n-1)\Delta^j v_m . \tag{7.8b}$$

Lemma 7.9.

$$q_{\gamma r}(n) = \Delta q_{\gamma r-1}(n-1) + \Delta^{1-\gamma}uq_{\gamma(r+1)-1}(n-1) + \sum_{m\geq 0} \Delta^{-\gamma(m+1)}v_m \frac{\delta H_n}{\delta v_{m+r}} , \tag{7.9a}$$

$$p_{\gamma r}(n) = p_{\gamma r-1}(n-1) + p_{\gamma(r+1)-1}(n-1)\Delta^{\gamma r}u + \sum_{m\geq 0} v_m^{(\gamma r)} \frac{\delta H_n}{\delta v_{m+r}} . \tag{7.9b}$$

Proof. From (7.7b) with $j = \gamma r$, we obtain

$$q_{\gamma r}(n) = \Delta q_{\gamma r-1}(n-1) + \Delta^{1-\gamma}uq_{\gamma(r+1)-1}(n-1) + \mu ,$$

where

$$\mu = \sum_{m\geq 0} \Delta^{1-\gamma(m+1)}v_m^{(-1)}q_{\gamma r-1+\gamma(m+1)}(n-1) + \sum_{m\geq 0} \Delta^{1-\gamma(m+2)}uv_m^{(\gamma-1)}q_{\gamma(m+r+2)-1}(n-1) =$$

$$= \sum_{m\geq 0} \Delta^{1-\gamma(m+1)}v_m^{(-1)}[q_{\gamma(m+r+1)-1}(n-1)+\Delta^{-\gamma}uq_{\gamma(m+r+2)-1}(n-1)] = [\text{by } (7.5d)] =$$

$$= \sum_{m\geq 0} \Delta^{-\gamma(m+1)}\Delta v_m^{(-1)}\Delta^{-1} \frac{\delta H_n}{\delta v_{m+r}} = \sum_{m\geq 0} \Delta^{-\gamma(m+1)}v_m \frac{\delta H_n}{\delta v_{m+r}} ,$$

which proves (7.9a). Similarly, from (7.6a) with $j = \gamma r$, we find that

$$p_{\gamma r}(n) = p_{\gamma r-1}(n-1) + p_{\gamma(r+1)-1}(n-1)\Delta^{\gamma r}u + \mu' ,$$

where

$$\mu' = \sum_{m\geq 0} p_{\gamma(m+r+1)-1}(n-1)\Delta^{\gamma r}v_m + \sum_{m\geq 0} p_{\gamma(m+r+2)-1}(n-1)\Delta^{\gamma r}u^{(\gamma m+\gamma)}v_m =$$

$$= \sum_{m\geq 0} v_m^{(\gamma r)}[p_{\gamma(m+r+1)-1}(n-1)+p_{\gamma(m+r+2)-1}(n-1)u^{(\gamma(m+r)+\gamma)}] = \text{[by (7.5c)]} =$$

$$= \sum_{m\geq 0} v_m^{(\gamma r)} \frac{\delta H_n}{\delta v_{m+r}} ,$$

which proves (7.9b). □

Comparing (7.9a) with (7.5d), and (7.9b) with (7.5c), we obtain the formulae

$$q_{\gamma(r+1)}(n) = \frac{\delta H_n}{\delta v_r} + \sum_{m\geq 0} \Delta^{-\gamma(m+1)} v_m \frac{\delta H_n}{\delta v_{r+m+1}} , \qquad (7.10a)$$

$$p_{\gamma(r+1)}(n) = \frac{\delta H_n}{\delta v_r} + \sum_{m\geq 0} v_m^{(\gamma r+\gamma)} \frac{\delta H_n}{\delta v_{r+m+1}} . \qquad (7.10b)$$

<u>Lemma 7.11</u>.

$$q_o(n) = \Delta \frac{1-\Delta^{-\gamma}}{\Delta-1} u \frac{\delta H_n}{\delta u} + \sum_{m\geq 0} \frac{1-\Delta^{-\gamma(m+1)}}{\Delta-1} v_m \frac{\delta H_n}{\delta v_m} , \qquad (7.11a)$$

$$p_o(n) = \Delta \frac{1-\Delta^{-\gamma}}{\Delta-1} u \frac{\delta H_n}{\delta u} + \sum_{m\geq 0} \Delta \frac{1-\Delta^{-\gamma(m+1)}}{\Delta-1} v_m \frac{\delta H_n}{\delta v_m} . \qquad (7.11b)$$

<u>Proof</u>. By subtracting (7.9b) from (7.9a) with $r = 0$ and using (7.8a) with $j = 0$, we get

$$q_o(n) - p_o(n) = \sum_{m\geq 0} (\Delta^{-\gamma(m+1)}-1)v_m \frac{\delta H_n}{\delta v_m} . \qquad (7.12)$$

For $j = 0$, (7.7a) yields

$$q_o(n) = u[q_{\gamma-1}(n-1)+ \sum_{m\geq 0} q_{\gamma(m+2)-1}(n-1)v_m^{(\gamma-1)}] +$$

$$+ q_{-1}(n-1) + \sum_{m\geq 0} v_m^{(-1)} q_{\gamma(m+1)-1}(n-1) = [\text{by } (7.5b)] =$$

$$= u \frac{\delta H_n}{\delta u} + \theta \ , \tag{7.13}$$

where

$$\theta = q_{-1}(n-1) + \sum_{m\geq 0} v_m^{(-1)} q_{\gamma(m+1)-1}(n-1) \ . \tag{7.14}$$

For $j = 0$, (7.6b) gives us

$$p_o(n) = \Delta^{1-\gamma} u[p_{\gamma-1}(n-1) + \sum_{m\geq 0} \Delta^{-\gamma(m+1)} v_m p_{\gamma(m+2)-1}(n-1)] +$$

$$+ [\Delta p_{-1}(n-1) + \Delta \sum_{m\geq 0} \Delta^{-\gamma(m+1)} v_m p_{\gamma(m+1)-1}(n-1)] = [\text{by } (7.5a), \text{ and}$$

$$(7.14) \text{ together with } (7.8b) \text{ for } j = -1] = \Delta^{1-\gamma} u \frac{\delta H_n}{\delta u} + \Delta\theta \ . \tag{7.15}$$

Applying Δ to (7.13) and subtracting (7.15) to eliminate θ, we get

$$\Delta q_o(n) - p_o(n) = (\Delta - \Delta^{1-\gamma}) u \frac{\delta H_n}{\delta u} \ . \tag{7.16}$$

Upon solving (7.12) and (7.16), we recover (7.11). □

Substituting (7.10) and (7.11) into (7.4), we obtain the second Hamiltonian form of our modified equations:

$$\partial_{\bar{p}}(u) = u[(\Delta^{\gamma-1}-1)\Delta \frac{1-\Delta^{-\gamma}}{\Delta-1} u \frac{\delta H}{\delta u} + \sum_{m\geq 0} (\Delta^{\gamma}-1) \frac{1-\Delta^{-\gamma(m+1)}}{\Delta-1} v_m \frac{\delta H}{\delta v_m}] \ , \tag{7.17a}$$

$$\partial_{\bar{P}}(v_m) = v_m[(\Delta^{\gamma(m+1)}-1)\Delta \frac{1-\Delta^{-\gamma}}{\Delta-1} u \frac{\delta H}{\delta u} + \sum_{r\geq 0} (\Delta^{\gamma(m+1)}-\Delta) \frac{1-\Delta^{-\gamma(r+1)}}{\Delta-1} v_r \frac{\delta H}{\delta v_r}] +$$

$$+ \sum_{r\geq 0} \{v_{m+r+1}\Delta^{\gamma(m+1)}[\frac{\delta H}{\delta v_r} + \sum_{s\geq 0} \Delta^{-\gamma(s+1)} v_s \frac{\delta H}{\delta v_{r+s+1}}] -$$

$$- \Delta^{-\gamma(r+1)} v_{m+r+1}[\frac{\delta H}{\delta v_r} + \sum_{s\geq 0} v_s^{(\gamma r+\gamma)} \frac{\delta H}{\delta v_{r+s+1}}]\} , \qquad (7.17b)$$

where $H = H_{n\gamma}$.

Notice that for $\gamma = 1$ equations (7.17) degenerate strongly into (2.21). Also, the $(\bar{v},\bar{v})$ part of the Hamiltonian form (7.17) is exactly the Hamiltonian form III (4.14).

Theorem 7.18. i) Equations (7.17) are Hamiltonian. ii) Both Miura maps, $M_1(L) = \ell_1\ell_2$, $M_2(L) = \ell_2\ell_1$, are canonical between (7.17) and III (4.14).

The proof will be given in Chap. X.

Chapter V. Deformations

We discuss deformations in general, find some curves for the Lax equations with $L = \zeta + \sum_j \zeta^{-j} q_j$, and a surface for the Toda lattice.

1. Basic Concepts

The Korteweg-de Vries equation,

$$u_t = 6uu_x - u_{xxx} , \tag{1.1}$$

can serve as a convenient example to discuss the general phenomenon of deformations.

Consider the following equations:

$$v_t = 6v^2 v_x - v_{xxx} , \tag{1.2}$$

$$w_t = 6ww_x - w_{xxx} + 6\varepsilon^2 w^2 w_x , \tag{1.3}$$

$$q_t = 6\left(\frac{\sinh 2\varepsilon q}{2\varepsilon}\right)^2 q_x - q_{xxx} + 2\varepsilon^2 q_x^3 , \tag{1.4}$$

$$\begin{aligned} p_t &= 6(1+\varepsilon^2 C)p_x - p_{xxx} + 2\varepsilon^2 \nu^2 p_x^3 , \\ C: &= \frac{\sinh(2\varepsilon\nu p)}{2\varepsilon\nu} + \frac{\cosh(2\varepsilon\nu p)-1}{2\varepsilon^2} . \end{aligned} \tag{1.5}$$

If v satisfies the modified Korteweg-de Vries equation (or mKdV for short) (1.2), then

$$u = M(v) = v^2 + v_x \tag{1.6}$$

satisfies the KdV equation (1.1). The map M in (1.6) is called the Miura map.

Now one can easily check out, that if w satisfies (1.3), then

$$u = (G(\varepsilon))(w) = w + \varepsilon^2 w^2 + \varepsilon w_x \tag{1.7}$$

satisfies (1.1). Since all c.l.'s of the KdV equation (1.1) come back via (1.7) to become c.l.'s of (1.3), we can consider (1.3) as a deformation of (1.1), that is, a one-parameter curve of equations, which goes through our original equation (1.1) when the parameter $\varepsilon = 0$. In addition, we have a contraction (1.7) of our curve into its base point (1.1).

This example indicates that integrable systems occur in families, which are sometimes contractible. There is enough evidence already accumulated in differential Lax equations [6,7], to believe that Lax equations themselves and the basic morphisms in the theory of Lax equations, can be viewed as base points in the curves which deform them. Let us look again at the KdV equation. One can check that if q satisfies (1.4) then

$$v = (g(\varepsilon))(q) = \frac{\sinh(2\varepsilon q)}{2\varepsilon} + \varepsilon q_x \,, \tag{1.8}$$

satisfies (1.2), and

$$w = (M(\varepsilon))(q) = \frac{\sinh^2(\varepsilon q)}{\varepsilon^2} + q_x \tag{1.9}$$

satisfies (1.3). Thus (1.4) is a mKdV-curve, (1.8) is its contraction, and (1.9) is a deformation of the Miura map (1.6), since $\lim_{\varepsilon \to 0} (M(\varepsilon))(q) = q^2 + q_x = M(q)$. In addition, we have the commutative diagram

$$G(\varepsilon) \cdot M(\varepsilon) = M \cdot g(\varepsilon) \,. \tag{1.10}$$

I remark in passing, that the origin of the map $M(\varepsilon)$ in (1.9), is not known even in the simplest case of the KdV equation (1.1).

Finally, let me mention that there exists a deformation of the diagram (1.10), from which the simplest part is as follows: if p satisfies (1.5) then

$$w = (G(\varepsilon,\nu))(p) = C + \nu p_x \tag{1.11}$$

where C is given in (1.5), satisfies (1.3). Thus we have at least a surface over the KdV memorabilia.

Incidentally, deformed equations usually acquire discrete symmetries which depend singularly upon the deformation parameter and are thus absent from the original equations. Probably, the simplest case provides (1.3): if w satisfies (1.3), then $\bar{w} = s(w) := -w - \varepsilon^{-2}$ satisfies (1.3) also. Naturally, this symmetry can be lifted up through (1.9) into (1.4), and also can be deformed through (1.11) into (1.5).

What is a general origin of these deformation phenomenon? The answer is not known, and apparently there is no common origin. From the computational point of view, let us notice that the usual idea of considering first the infinitesimal deformations doesn't work. Indeed, if one has <u>any</u> regular map near the identity, say

$$a = (f(\varepsilon))(b) = b + O(\varepsilon) \ , \tag{1.12}$$

then one can <u>formally</u> invert the map (1.12) in the appropriate ring of formal power series in ε, say,

$$b = (f^{-1}(\varepsilon))(a) = a + O(\varepsilon) \ . \tag{1.13}$$

Then, whatever the original equation for a is, say,

$$a_t = F(a) \ , \tag{1.14}$$

we find from (1.13) that

$$b_t = \frac{\partial}{\partial t} [(f^{-1}(\varepsilon))(a)]\Big|_{a=(f(\varepsilon))(b)} = F(b) + O(\varepsilon) \ . \tag{1.15}$$

Consequently, $\varepsilon^2 = 0$ or $\varepsilon^N = 0$ won't help, since the essential condition that (1.15) is a "finite" equation (e.g., in the sense that it involves only a finite number of derivatives), is automatically satisfied when one cuts off higher terms in ε.

Let us now review the known methods of finding deformations so we can see which ones are applicable for discrete equations which are the ones with which we are working in these lectures.

The first method heavily depends on the fact that the equation under consideration is of the Lax type $L_t = [P_+, L]$, where we now write L_t instead of $\partial_P(L)$ in order to make the reasoning more informal. The importance of this representation comes from its interpretation as a compatibility condition for the following system

$$\begin{cases} L\psi = \lambda\psi , & (1.16a) \\ \psi_t = P_+\psi , & (1.16b) \end{cases}$$

where λ is a formal parameter which commutes with everything. If we could find a representation for (1.16a) which gives a resolution of the coefficients of L in terms of ψ, then (1.16b) becomes an autonomous equation and we could hope to interprete it as a deformation and use the resolution just mentioned as a contruction of this deformation. Let us see how this works. We take

$$L = \zeta + \zeta^{-1}q, \; P_+ = (L^2)_+ = \zeta^2 + [q+q^{(-1)}] ,$$

$$q_t = q^{(1)}q - qq^{(-1)} , \tag{1.17}$$

as in IV (4.33). An auxilliary problem for (1.17) would be

$$\begin{cases} ((\zeta+\zeta^{-1}q)\psi = \lambda\psi , & (1.18a) \\ \psi_t = [\zeta^2+q+q^{(-1)}]\psi . & (1.18b) \end{cases}$$

From (1.18a) we have

$$\psi^{(2)} = \lambda\psi^{(1)} - q\psi , \tag{1.19}$$

$$q = \lambda v - v^{(1)}v , \tag{1.20}$$

which is the desired "resolution" of the coefficient of L in terms of ψ. Here

$$v: = \psi^{(1)} \frac{1}{\psi} , \tag{1.21}$$

and we treat all letters (except λ) as noncommuting so as to cover the matrix case at no extra cost. Using (1.19), we have from (1.18b)

$$\psi_t = \lambda\psi^{(1)} + q^{(-1)}\psi ,$$

$$\psi_t^{(1)} = \lambda\psi^{(2)} + q\psi^{(1)} ,$$

and thus

$$\begin{aligned} v_t &= [\psi^{(1)}\psi^{-1}]_t = \psi_t^{(1)}\psi^{-1} - \psi^{(1)}\psi^{-1}\psi_t\psi^{-1} = \\ &= [\lambda\psi^{(2)}+q\psi^{(1)}]\psi^{-1} - v[\lambda\psi^{(1)}+q^{(-1)}\psi]\psi^{-1} = \\ &= \lambda v^{(1)}v + qv - v[\lambda v+q^{(-1)}] = \quad [\text{by } (1.20)] = \\ &= \lambda v^{(1)}v + [\lambda v-v^{(1)}v]v - \lambda v^2 - v[\lambda v^{(-1)}-vv^{(-1)}] = \\ &= \lambda[v^{(1)}v-vv^{(-1)}] + v^2v^{(-1)} - v^{(1)}v^2 . \end{aligned} \tag{1.22}$$

Now put

$$\varepsilon = \lambda^{-2} , \ \rho = v\lambda .$$

Then (1.20) and (1.22) become

$$q = \rho - \varepsilon\rho^{(1)}\rho , \tag{1.23}$$

$$\rho_t = \rho^{(1)}\rho - \rho\rho^{(-1)} + \varepsilon[\rho^2\rho^{(-1)}-\rho^{(1)}\rho^2] . \tag{1.24}$$

Thus (1.24) deforms (1.17) while (1.23) contracts (1.24) into (1.17).

Although we used crude computational force to derive a deformation of (1.17), a little bit of reasoning will show that the same device produces deformations for all Lax equations associated with the operator $L = \zeta + \zeta^{-1}q$. We leave this to the reader as an exercise.

Next we describe some Hamiltonian machinery which is useful in deformational analysis.

<u>Theorem 1.25</u>. Suppose we have a bi-Hamiltonian system of evolution equations, which we can symbolically write as

$$\bar{q}_t = B^2 \frac{\delta H_n}{\delta \bar{q}} = B^1 \frac{\delta H_{n+1}}{\delta \bar{q}} , \tag{1.26}$$

where B^1 and B^2 are matrices of operators "in $\bar{q}$-space." Suppose that B^1 and B^2 are compatible; that is, $\alpha B^1 + \beta B^2$ is a Hamiltonian matrix for any constants α and β. Assume also that $B^1 \frac{\delta H_o}{\delta \bar{q}} = 0$.

Now consider another space with variables $\bar{v}$. Let $\phi\colon \bar{v} \to \bar{q}$ be a map of the form $\phi(\bar{v}) = \bar{v} + O(\varepsilon)$. Let B be a Hamiltonian structure in $\bar{v}$-space such that ϕ is canonical between $B \frac{\delta}{\delta \bar{v}}$ and $(B^1+\varepsilon B^2) \frac{\delta}{\delta \bar{q}}$.

Then any equation (1.26) has the following deformation

$$\bar{v}_t = B \frac{\delta}{\delta \bar{v}} \phi^*[\sum_{k=0}^{n} (-1)^k \varepsilon^{-k-1} H_{n-k}] , \tag{1.27}$$

and ϕ is its contraction.

<u>Proof</u>. First we check that our original equation (1.26) can also be written as

$$\bar{q}_t = (B^1+\varepsilon B^2) \frac{\delta \tilde{H}_n}{\delta \bar{q}} , \tilde{H}_n : = \sum_{k=0}^{n} (-1)^k \varepsilon^{-k-1} H_{n-k} . \tag{1.28}$$

Indeed,

$$\varepsilon B^2 \frac{\delta \tilde{H}_n}{\delta \bar{q}} = \sum_{k=0}^{n} (-1)^k \varepsilon^{-k} B^2 \frac{\delta H_{n-k}}{\delta \bar{q}} = \sum_{k=0}^{n} (-1)^k \varepsilon^{-k} B^1 \frac{\delta H_{n-k+1}}{\delta \bar{q}} =$$

$$= B^1 \frac{\delta H_{n+1}}{\delta \bar{q}} + \sum_{k=0}^{n-1} (-1)^{k+1} \varepsilon^{-k-1} B^1 \frac{\delta H_{n-k}}{\delta \bar{q}} = [\text{since } B^1 \frac{\delta H_o}{\delta \bar{q}} = 0] =$$

$$= B^1 \frac{\delta H_{n+1}}{\delta \bar{q}} + \sum_{k=0}^{n} (-1)^{k+1} \varepsilon^{-k-1} B^1 \frac{\delta H_{n-k}}{\delta \bar{q}} = B^2 \frac{\delta H_n}{\delta \bar{q}} - B^1 \frac{\delta \tilde{H}_n}{\delta \bar{q}} .$$

Now we show that equations (1.27) are indeed regular in ε; that is, no negative powers of ε are involved in the equation itself, regardless of the fact that $\phi^*(\tilde{H}_n)$ is heavily singular. But this is obvious: since ϕ is near identity, we can invert it and get

$$\bar{v} = \bar{q} + O(\varepsilon) ,$$

thus we can have $\bar{v}_t$ expressed regularity through $\bar{q}$ and $\bar{q}_t$, i.e. $\bar{v}_t = \bar{q}_t + O(\varepsilon)$, and again, since ϕ is regular, and $\bar{q}_t$ is given by (1.26), we see that $\bar{v}_t$ is regular in ε as well. □

The simplest case of the theorem provides the map (1.7), in which $B = \partial$, $B^1 = \partial$, $B^2 = -\partial^3 + 2u\partial + 2\partial u$, ϕ: $w \to u = w^2 + \varepsilon^2 w^2 + \varepsilon w_x$ and ϕ is canonical between $\partial \frac{\delta}{\delta w}$ and $[\partial + \varepsilon^2(-\partial^3+2u\partial+2\partial u)] \frac{\delta}{\delta u}$. Then

$$u_t = 6uu_x - u_{xxx} = [\partial+\varepsilon^2(-\partial^3+2u\partial+2\partial u)] \frac{\delta}{\delta u} (\varepsilon^{-2} \frac{u^2}{2} - \varepsilon^{-4} \frac{u}{2}) .$$

The third method is very similar to the Hamiltonian one of theorem 1.25 and involves the renormalization of modified variables. For the case of the KdV equation (1.1) for instance, one can proceed as follows. If u is a solution of any <u>linear combination</u> of KdV fields in the KdV hierarchy $\{u_t = X_r(u) | r = 0, 1, \ldots\}$, say $u_t = \sum_i \alpha_i X_i(u)$, then $\bar{u} = u+c$, (c = const) is also a solution, of another linear combination $\bar{u}_t = \sum_i \bar{\alpha}_i X_i(\bar{u})$, where $\bar{\alpha}_i = \alpha_i + \sum_{j<i} f_{ij}(c)\alpha_j$ for some polynomials $f_{ij}(c)$. Now let v be a solution of a linear combination of mKdV fields, say, $v_t = \sum_i \alpha_i Y_i(v)$. Then

$$u = v^2 + v_x = M(v)$$

is a solution of $u_t = \sum_i \alpha_i X_i(u)$. Now put

$$v = \varepsilon w + \frac{1}{2\varepsilon} .$$

Then

$$u = v^2 + v_x = \frac{1}{2\varepsilon^2} + (w+\varepsilon^2 w^2+\varepsilon w_x) ,$$

take $c = -\frac{1}{2\varepsilon^2}$, and we are done.

We apply this method in the next section.

2. The Operator $\zeta + \sum_j \zeta^{-j} q_j$ and its Specializations

Let $\alpha = (\alpha_1,\ldots,\alpha_n)$ be a vector, $\alpha_i \in k$. Consider a Lax equation

$$\partial_t(L) = [\sum_i \alpha_i L^i{}_+, L] , \tag{2.1α}$$

with $L = \zeta + \sum_j \zeta^{-j} q_j$. Let us concentrate on the dependence of our constructions upon the variable q_o only, and for this reason we will write $L = L(q_o)$. Since $L(q_o+c) = L(q_o) + c$, then $[L(q_o+c)]^i = \sum_{k=0}^{i} \binom{i}{k} L(q_o)^{i-k} c^k$, and thus if $(q_o,q_1,\ldots)$ satisfy (2.1α), then $(\bar{q}_o = q_o+c,q_1,\ldots)$ satisfy (2.1$\bar{\alpha}$) where $\bar{\alpha} = \Omega^c\alpha$, Ω^c being n×n lower triangular matrix with ones on the diagonal and polynomially dependent upon c.

Now consider the modified Lax equations of section 2, Chapter IV:

$$\partial_t(\bar{L}) = [\sum_i \beta_i \bar{L}^{2i}{}_+, \bar{L}] , \tag{2.2β}$$

where

$$\bar{L} = \begin{vmatrix} 0 & \ell_1 \\ \ell_2 & 0 \end{vmatrix}, \ \ell_1 = \zeta+u, \ \ell_2 = 1 + \sum_j \zeta^{-j-1} v_j . \tag{2.3}$$

As we know, (2.2β) implies (2.1β) for $L = \ell_1\ell_2$ or $L = \ell_2\ell_1$; these two maps are denoted M_1,M_2:

$$M_1 : (u,\bar{v}) \to \begin{cases} q_o = u+v_o , \\ q_{m+1} = v_{m+1} + v_m u^{(m+1)} , \end{cases} \tag{2.4}$$

$$M_2 : (u,\bar{v}) \to \begin{cases} q_o = u + v_o^{(-1)} , \\ q_{m+1} = v_{m+1}^{(-1)} + v_m u . \end{cases} \tag{2.5}$$

Now let us change variables in the $(u,\bar{v})$-space by:

$$u = U + \varepsilon^{-1} , \quad v_m = \varepsilon V_m , \tag{2.6}$$

so that M_1 and M_2 become

$$\tilde{M}_1 : (U,\bar{V}) \to \begin{cases} \tilde{q}_o = U + \varepsilon V_o , \\ q_{m+1} = V_m(1+\varepsilon U^{(m+1)}) + \varepsilon V_{m+1} , \end{cases} \tag{2.7}$$

$$\tilde{M}_2 : (U,\bar{V}) \to \begin{cases} \tilde{q}_o = U + \varepsilon V_o^{(-1)} , \\ q_{m+1} = V_m(1+\varepsilon U) + \varepsilon V_{m+1}^{(-1)} , \end{cases} \tag{2.8}$$

where $\tilde{q}_o = q_o - \varepsilon^{-1}$.

Thus, a β-combination in $(U,\bar{V})$-space produces a $\bar{\beta} = \Omega^{-\varepsilon^{-1}}\beta$ -combination in $\bar{q}$-space. Since the matrix $\Omega^{-\varepsilon^{-1}}$ is invertible, we can find β such that $\bar{\beta} = (0,0,\ldots,1)$. Using the same arguments as in the proof of theorem 1.25, we deduce that the resulting deformed equations in $(U,\bar{V})$-space depend regularly upon ε. Thus we have proved

Theorem 2.9. For any Lax equation $L_t = [L^n{}_+,L]$ with $L = \zeta + \sum_i \zeta^{-j} q_j$, there exists a curve of equations in $(U,\bar{V})$-space, polynomially dependent upon ε, such that both maps (2.7) and (2.8) are contractions of this curve.

Due to the extreme simplicity of the contraction maps (2.7) and (2.8), we can easily handle the problem of specialization. Consider the operator $L(\gamma) = \zeta(1+\sum_j \zeta^{-\gamma(j+1)}\hat{q}_j)$. As we know, every flow $L_t = [L^n{}_+,L]$ of our original operator L leaves the submanifold $I^\gamma := \{q_j = 0 | j \not\equiv 0 (\text{mod } \gamma)\}$ invariant for all

$n \in \mathbb{Z}_+\gamma$; the corresponding deformed flows in $(U,\bar{V})$-space leave invariant the pre-image under either $\tilde{M}_1$ or $\tilde{M}_2$ of I^γ. Let us take $\tilde{M}_1$, for definiteness. From (2.7) we easily find $\tilde{M}_1^{-1}(I^\gamma)$:

$$U = -\varepsilon V_o \ , \ V_m = -\varepsilon V_{m+1}(1-\varepsilon^2 V_o^{(m+1)})^{-1} \ , \ m + 1 \not\equiv 0 \ (\mathrm{mod}\ \gamma) \ , \tag{2.10}$$

which provides a deformation of the flows for the operator $\zeta(1+\sum_j \zeta^{-\gamma(j+1)}\hat{q}_j)$. For example, for $\gamma = 2$ and $L = \zeta(1+\zeta^{-1}\hat{q})$, from (2.7), (2.10) we get

$$\hat{q} = V(1-\varepsilon^2 V^{(1)}) \ , \ V := V_1 \ , \tag{2.11}$$

which is, of course, (1.23) in its commutative version.

Remark 2.12. Once the contraction maps (2.7) and (2.8) have been found, one can apply theorem 1.25 to construct deformed equations. Indeed, the original Miura maps (2.4) and (2.5) are canonical between the Hamiltonian structures IV (2.21) and III (4.14) with $\gamma = 1$, in $(u,\bar{v})$- and $\bar{q}$-spaces respectively. After the change of variables (2.6), we get the structure $\varepsilon^{-1} B(U,\bar{V})$ in the $(U,\bar{V})$-space, where the matrix elements of the matrix $B = B(U,\bar{V})$ are given by

$$B_{oo} = 0 \ , \ B_{o,r+1} = (1+\varepsilon U)(1-\Delta^{-r-1})V_r \ ,$$

$$B_{r+1,k+1} = V_{r+k+1}\Delta^{r+1} - \Delta^{-k-1}V_{r+k+1} + \varepsilon\{V_r \frac{(\Delta^r-1)(1-\Delta^{k+1})}{(1-\Delta)\Delta^k} V_k +$$

$$+ \sum_{m+s=k-1} (V_{r+s+1}\Delta^{r-m}V_m - V_m\Delta^{-s-1}V_{r+s+1}) \ . \tag{2.13}$$

Simultaneously, the change of variables

$$\tilde{q}_o = q_o - \varepsilon^{-1} \ , \ q_{i+1} = q_{i+1} \ ,$$

makes in new $\bar{q} = (\tilde{q}_o, q_1, \ldots)$-space the matrix $B^2 + \varepsilon^{-1}B^1$ out of B^2 (lemma III 4.22). Multiplying both new matrices ($\varepsilon^{-1}B(U,\bar{V})$ and $B^2 + \varepsilon^{-1}B^1$) by ε, we see that both contractions (2.7) and (2.8) are canonical between (2.13) and $B^1 + \varepsilon B^2$.

Hence we can apply theorem 1.15 for the explicit construction of deformed equations.

Remark 2.14. Looking at the matrix (2.13), we observe from its first row, that equations for U are

$$U_t = (1+\varepsilon U) \sum_{r\geq 0} (1-\Delta^{-r-1})V_r \frac{\delta H}{\delta V_r} ,$$

therefore, whatever H is, we have

$$\frac{\partial}{\partial t} \ell n(1+\varepsilon U) \sim 0$$

that is, $\ell n(1+\varepsilon U)$ is an universal c.l. Thus we can, following the historical development of the deformations-related observations, invert either (2.7) or (2.8) and get

$$\ell n(1+\varepsilon U) = \sum_{n=0}^{\infty} \varepsilon^{n+1} G_n ,$$

where $G_n \in \mathbb{Q}[q_j^{(\sigma_j)}]$ will be c.l.'s for Lax equations in the $\bar{q}$-variables.

In conclusion, I'd like to point out that we don't have any analog of (1.8) for the deformation of the Miura maps (2.4), (2.5). The reason, I think, reflects the absence of a convenient form of modified-modified equations. There is one exception, though, where a deformation of the Miura map can be found: it is the Toda lattice case. Since it is a three-Hamiltonian system, we can take advantage of the Hamiltonian formalism. Without going into details, I simply write down what the deformations look like.

$$\begin{cases} \dot{q}_o = (1-\Delta^{-1})q_1 , \\ \dot{q}_1 = q_1(\Delta-1)q_o . \end{cases} \qquad \text{Toda equations} \qquad (2.15)$$

$$\begin{cases} \dot{u} = u(1-\Delta^{-1})v , \\ \dot{v} = v(\Delta-1)u . \end{cases} \qquad \text{Modified Toda equations.} \qquad (2.16)$$

$$M_1 : \begin{cases} q_o = u + v , \\ q_1 = u^{(1)} v . \end{cases} ; \quad M_2 : \begin{cases} q_o = u + v^{(-1)} , \\ q_1 = uv . \end{cases} \qquad \text{Miura maps.} \tag{2.17}$$

$$\begin{cases} \dot{U} = (1+\varepsilon U)(1-\Delta^{-1})V , \\ \dot{V} = V(\Delta-1)U . \end{cases} \qquad \text{Deformed Toda equations .} \tag{2.18}$$

$$\begin{cases} \dot{p} = p(1+\varepsilon p)(1-\Delta^{-1})q , \\ \dot{q} = q(1+\varepsilon q)(\Delta-1)p . \end{cases} \qquad \text{Deformed modified Toda equations .} \tag{2.19}$$

$$\tilde{M}_1 : \begin{cases} q_o = U + \varepsilon V , \\ q_1 = V + \varepsilon U^{(1)} V \end{cases} ; \quad \tilde{M}_2 : \begin{cases} q_o = U + \varepsilon V^{(-1)} \\ q_1 = U + \varepsilon VU . \end{cases} \tag{2.20}$$

Contractions on Toda equations

$$D_1 : \begin{cases} u = p(1+\varepsilon q) , \\ v = q(1+\varepsilon p^{(1)}) . \end{cases} ; \quad D_2 : \begin{cases} u = p(1+\varepsilon q^{(-1)}) , \\ v = q(1+\varepsilon p) . \end{cases} \tag{2.21}$$

Contractions on modified Toda equations .

$$\tilde{M}_1(\varepsilon) : \begin{cases} U = p + q + \varepsilon pq , \\ V = p^{(1)} q . \end{cases} ; \tag{2.22}$$

Deformations of Miura maps .

$$\tilde{M}_2(\varepsilon) : \begin{cases} U = p + q^{(-1)} + \varepsilon pq^{(-1)} , \\ V = pq . \end{cases}$$

Commutative diagrams:

$$\tilde{M}_i \circ \tilde{M}_i(\varepsilon) = M_i \circ D_i , \quad i = 1,2 \tag{2.23}$$

$$\tilde{M}_{i+1} \circ \tilde{M}_i(\varepsilon) = M_i \circ D_{i+1} , \quad i \quad {}_2 . \tag{2.24}$$

Second parameter in the Toda equations:

$$\begin{cases} \dot{P} = (1+\nu P)(1+\varepsilon P)(1-\Delta^{-1})Q , \\ \dot{Q} = Q(1+\varepsilon\nu Q)(\Delta-1)P . \end{cases} \tag{2.25}$$

Contractions of (2.25):

$$B_1 : \begin{cases} U = P + \nu Q(1+\varepsilon P) , \\ V = Q + \nu P^{(1)} Q . \end{cases} ; \quad B_2 : \begin{cases} U = P + \nu Q^{(-1)}(1+\varepsilon P) , \\ V = Q + \nu PQ . \end{cases} \tag{2.26}$$

Singular symmetries:

$$U \to -U - 2\varepsilon^{-1} \; ; \; V \to -V \; ; \; t \to -t \; , \tag{2.27}$$

where t is time-coordinate, and $t \to -t$ means that the "flow changes direction," or the derivation or the corresponding vector field changes direction.

$$P \to P \; ; \; Q \to -Q - \varepsilon^{-1}\nu^{-1} \; ; \; t \to -t \; . \tag{2.28}$$

Chapter VI. Continuous Limit

In this chapter we discuss some features of a passage from discrete to continuous points of view.

1. Examples

Let us begin with the first nontrivial equation associated with the simplest possible operator

$$L = \zeta + \zeta^{-1}q_o \ , \tag{1.1}$$

for $P = L^2$. It is $\partial_P(L) = [P_+,L]$, i.e.

$$\partial_P(q_o) = q_o(\Delta-\Delta^{-1})q_o \ . \tag{1.2}$$

Let us imagine that $q_o = q_o(x)$ is a function on $\mathbb{R}^1$ and Δ is the automorphism of $C^\infty(\mathbb{R}^1)$ generated by the diffeomorphism $S_\lambda : \mathbb{R}^1 \to \mathbb{R}^1$, $x \to x+\lambda$. Now extend everything in the formal power series in λ, which commutes with everything, so we can take $\Delta = \exp(\lambda\partial)$, with $\partial = d/dx$.

If we now put

$$q_o = 1 + \lambda^2 v \ , \tag{1.3}$$

then (1.2) becomes

$$\lambda^2\partial_P(v) = (1+\lambda^2 v)2[\lambda\partial + \frac{\lambda^3\partial^3}{3!} + O(\lambda^5)](1+\lambda^2 v) =$$

$$= 2\lambda(1+\lambda^2 v)\lambda^2[\partial + \frac{\lambda^2\partial^3}{3!} + O(\lambda^4)](v) =$$

$$= 2\lambda^3(1+\lambda^2 v)[v_x + \frac{\lambda^2}{6} v_{xxx} + O(\lambda^4)] =$$

$$= 2\lambda^3 \ \{v_x + \lambda^2(vv_x + \frac{1}{6} v_{xxx}) + O(\lambda^4)\} \ .$$

Thus if we put

$$\partial_P = 2\lambda(\partial_\tau+\partial) \ , \ t = \tau\lambda^2 \ , \tag{1.4}$$

we obtain

$$v_t = vv_x + \frac{1}{6} v_{xxx} + O(\lambda^2) \ , \tag{1.5}$$

which reduces to the Korteweg-de Vries equation at the zero-order in λ.

The heuristic derivation above has two main ingredients: we treat Δ as $\exp(\lambda \frac{d}{dx})$; and we renormalize q (1.3) and ∂_P (1.4). Since this renormalization appears as a somewhat less natural operation, we postpone our discussion of it until the next section.

Consider the following Lax operator

$$L = \zeta + \zeta^{-1}q_o + \zeta^{-3}q_1 \ . \tag{1.6}$$

For $P = L^2$, the Lax equations $\partial_P(L) = [P_+,L]$ become

$$\begin{cases} \partial_P(q_o) = q_o(\Delta-\Delta^{-1})q_o + (1-\Delta^{-2})q_1 \ , \\ \partial_P(q_1) = q_1(\Delta^3-1)(1+\Delta^{-1})q_o \ , \end{cases} \tag{1.7}$$

which are Hamiltonian equations with the matrix B given by III(4.15) for $\gamma = 2$:

$$B_{oo} = q_o(\Delta-\Delta^{-1})q_o + q_1\Delta^2 - \Delta^{-2}q_1 \ ; \ B_{01} = q_o(\Delta+1)(1-\Delta^{-3})q_1 \ ;$$
$$B_{10} = q_1(\Delta^3-1)(1+\Delta^{-1})q_o \ ; \ B_{11} = q_1(\Delta^3-1)\ \frac{1-\Delta^{-4}}{1-\Delta^{-1}}\ q_1 \ , \tag{1.8}$$

and the Hamiltonian

$$H = q_o(\sim \frac{1}{2} \operatorname{Res} L^2) \ . \tag{1.9}$$

Now let $\Delta = \exp(\lambda\partial)$, $\partial = \frac{d}{dx}$. Then (1.7) becomes, in the first order of λ,

$$\begin{cases} \partial_P(q_o) = 2\lambda(q_oq_o'+q_1') \ , \\ \partial_P(q_1) = 2\lambda(3q_1q_o') \ , \end{cases} \quad ' \equiv \frac{\partial}{\partial x} \ , \tag{1.10}$$

and to avoid the definition of precise relations between ∂_P and $\frac{\partial}{\partial x}$, we change ∂_P into $2\lambda \frac{\partial}{\partial t}$, so (1.10) turns into

$$\begin{cases} \dot{q}_o = \partial(\frac{q_o^2}{2} + q_1) , \\ \dot{q}_1 = 3q_1 q_o' , \end{cases} \qquad (\dot{\ }) \equiv \frac{\partial}{\partial t} () . \tag{1.11}$$

The system (1.11) certainly seems unfamiliar, and it obviously has nothing to do with the differential scalar Lax equations (although its prelimited parent (1.7) is derived from the discrete scalar Lax equations). It surely has an infinity of c.l.'s, as any continuum limit system should, namely the limits of the original c.l.'s.

Let us analyze (1.11) a bit closer. First let us introduce the conservation coordinates

$$u = q_o \ , \ h = q_1 + \frac{3}{2} q_o^2 , \tag{1.12}$$

so (1.11) becomes

$$\begin{cases} \dot{u} = \partial(h-u^2) , \\ \dot{h} = \partial(3uh-\frac{7}{2}u^3) . \end{cases} \tag{1.13}$$

Let us cast (1.13) into a Hamiltonian form, with the Hamiltonian $H = q_o = u$. If we look for evolution systems of the form

$$\begin{pmatrix} \dot{u} \\ \dot{h} \end{pmatrix} = \begin{vmatrix} a\partial+\partial a & f\partial+\partial g \\ \partial f+g\partial & b\partial+\partial b \end{vmatrix} \begin{pmatrix} \delta/\delta u \\ \delta/\delta h \end{pmatrix} (H) , \tag{1.14}$$

with some $a,b,f,g \in C^\infty(u,h)$, then a necessary and sufficient condition for (1.14) to be Hamiltonian is the following system of equations

$$2b\frac{\partial g}{\partial h} + (f+g)\frac{\partial g}{\partial u} = 2a\frac{\partial b}{\partial u} + (f+g)\frac{\partial b}{\partial h},$$

$$2a\frac{\partial f}{\partial u} + (f+g)\frac{\partial f}{\partial h} = 2b\frac{\partial a}{\partial h} + (f+g)\frac{\partial a}{\partial u},$$

$$\frac{\partial g}{\partial h}\left(\frac{\partial a}{\partial u} - \frac{\partial f}{\partial h}\right) = \frac{\partial a}{\partial h}\left(\frac{\partial g}{\partial u} - \frac{\partial b}{\partial h}\right), \tag{1.15}$$

$$\frac{\partial b}{\partial u}\left(\frac{\partial a}{\partial u} - \frac{\partial f}{\partial h}\right) = \frac{\partial f}{\partial u}\left(\frac{\partial g}{\partial u} - \frac{\partial b}{\partial h}\right),$$

$$\frac{\partial a}{\partial h}\frac{\partial b}{\partial u} = \frac{\partial g}{\partial h}\frac{\partial f}{\partial u}.$$

This statement follows from the methods of Hamiltonian formalism (see, e.g. chapter VIII, sect. 2) applied to (1.14). We do not need the proof right now.

Since we would like (1.13) to be generated by (1.14) with $H = u$, we immediately find that $a = h-u^2$, $f = 3uh - \frac{7}{2}u^3$. Solving (1.15) we finally obtain

$$\begin{aligned} a &= h-u^2, & f &= 3uh - \tfrac{7}{2}u^3, \\ b &= 3h^2 + 9(u^2h - \tfrac{7}{4}u^4), & g &= 6uh - 7u^3. \end{aligned} \tag{1.16}$$

Now let us return to our original variables q_o, q_1. To do this, we must multiply the matrix in the right-hand side of (1.14): from the left by J, and from the right by J^t, where

$$J = \begin{vmatrix} 1 & 0 \\ -3u & 1 \end{vmatrix}$$

is the Fréchet derivative of the vector $\begin{pmatrix} q_o = u \\ q_1 = h - \frac{3}{2}u^2 \end{pmatrix}$.

The result is

$$\begin{pmatrix} \dot q_o \\ \dot q_1 \end{pmatrix} = \begin{vmatrix} (\frac{q_o^2}{2}+q_1)\partial + \partial(\frac{q_o^2}{2}+q_1) & 3q_o\partial q_1 \\ 3q_1\partial q_o & 3(q_1^2\partial+\partial q_1^2) \end{vmatrix} \begin{pmatrix} \delta/\delta q_o \\ \delta/\delta q_1 \end{pmatrix}(H), \tag{1.17}$$

and it is obvious that (1.17) produces (1.11) for $H = q_o$.

To interprete (1.17), let us take the continuous limit of the matrix (1.8). Keeping lowest order terms only and using tilde for the resulting matrix elements, we get

$$\tilde{B}_{oo} = 2\lambda(q_o\partial q_o + q_1\partial + \partial q_1) \ ; \ \tilde{B}_{01} = 2\lambda(3q_o\partial q_1) \ ;$$
$$\tilde{B}_{10} = 2\lambda(3q_1\partial q_o) \ ; \ \tilde{B}_{11} = 2\lambda(6q_1\partial q_1) \ . \tag{1.18}$$

Thus we get 2λ times the matrix of (1.17)! This fact may be explained as follows. As we prove in the next chapter, under continuous limit <u>functional derivatives go into functional derivatives</u> (Theorem VII 3.4). Therefore, equations $\dot{\bar{q}} = B \dfrac{\delta H}{\delta \bar{q}}$ go into equations $\dot{\bar{q}} = B^c \dfrac{\delta}{\delta \bar{q}} (H^c)$, where "c" stands for continuous limit. It is clear that the matrix B^c is also Hamiltonian (assuming, that B is), as follows, for example, from the characteristic equation (lemma VIII 2.20) for B in order for it to be Hamiltonian. Now, B^c is a formal power series in λ, therefore its lowest term in λ is also Hamiltonian, since to be Hamiltonian is a quadratic property. Therefore, the lowest order equations of the continuous limit, like (1.10), have the Hamiltonian form

$$\dot{\bar{q}} = (\text{lowest part of } B^c) \frac{\delta}{\delta \bar{q}} (\text{lowest part of } H^c) \ . \tag{1.19}$$

In particular, all Hamiltonian structures of Chapters III-V provide new Hamiltonian structures for the limiting equations (1.19). Notice, that the matrix elements of these new structures do not have the operator ∂ in powers more than first, since $\Delta^k = 1 + \kappa\lambda\partial + O(\lambda^2)$. Thus these new matrices are of order ≤ 1 in ∂.

The importance of these new systems follows from the fact that until now there were no known 1st order integrable systems with more than 2 components, so they provide a set of convenient first examples.

Let us write down these new matrices, the lowest order of the various B^c's, for the following Hamiltonian matrices of Chapter III:

1) B_{rs} from III (3.4) becomes

$$\tilde{B}_{rs} = \lambda(rq_{s+r}\partial+\partial sq_{s+r}) \ . \tag{1.20}$$

This matrix is important in two-dimensional hydrodynamics.

2) B_{rs} from III (3.12) becomes

$$\tilde{B}_{js} = \lambda[(1+j)R_{j+s+1}\partial+\partial(1+s)R_{j+s+1}+\beta\delta_\beta^{s+j+2}\partial] \ , \ 0 \leq j,s \leq \beta-2 \ . \tag{1.21}$$

3) B_{rs} from III (4.15) becomes

$$\tilde{B}_{rs} = \gamma\lambda\{(r+1)q_{s+r+1}\partial+\partial(s+1)q_{r+s+1}+q_r[\gamma(r+1)-1]\partial(s+1)q_s + \\ + \sum_{j+k+1=s} [q_{k+r+1}(r-j)\partial q_j+q_j(k+1)\partial q_{k+r+1}]\} \ . \tag{1.22}$$

4) The third Hamiltonian structure for the Toda hierarchy, III (5.18), becomes

$$\tilde{B}_{oo} = \lambda[2(q_1\partial+\partial q_1)q_o+2q_o(q_1\partial+\partial q_1)] \ , \ \tilde{B}_{o1} = \lambda[(q_1\partial+\partial q_1)2q_1+q_o^2\partial q_1] \ , \\ \tilde{B}_{1o} = \lambda[q_1\partial q_o^2+2q_1(\partial q_1+q_1\partial)] \ , \ \tilde{B}_{11} = \lambda[2q_1(q_o\partial+\partial q_o)q_1] \ , \tag{1.23}$$

while the first III (5.19) and the second III (5.20) become respectively

$$\tilde{B}^1 = \lambda \begin{vmatrix} 0 & \partial q_1 \\ q_1\partial & 0 \end{vmatrix} , \tag{1.24}$$

$$\tilde{B}^2 = \lambda \begin{vmatrix} q_1\partial + \partial q_1 & q_o\partial q_1 \\ q_1\partial q_o & 2q_1\partial q_1 \end{vmatrix} \ . \tag{1.25}$$

From what has been said above, it follows, for example, that the lowest limit of the Toda equations III (5.22):

$$\begin{cases} \partial_P(q_o) = \lambda q_1' , \\ \partial_P(q_1) = \lambda q_1 q_o' , \end{cases} \tag{1.26}$$

is a three-Hamiltonian system with respect to three structures (1.23)-(1.25). The same is also true, of course, for the higher equations of the Toda hierarchy, which don't look so silly as (1.26); the next equation is

$$\begin{cases} \partial_P(q_o) = 4\lambda(q_o q_1)' \\ \partial_P(q_1) = \lambda[(q_1^2)'+q_1(q_o^2)'] \end{cases} = \tilde{B}^3 \frac{\delta}{\delta \bar{q}} (q_o) = \tilde{B}^2 \frac{\delta}{\delta \bar{q}} [2(\frac{q_o^2}{2}+q_1)] . \tag{1.27}$$

2. Approximating Differential Lax Operators

In this section we prove a generalization of the renormalization formula (1.3) which provides correctly defined frameworks appropriate in considering continuous limits. In contrast to the preceeding section, we no longer look at equations such as (1.2), but only at their Lax operators, such as (1.1).

Let F be a differential algebra over $\mathscr{k}$ with a derivation $\partial : F \to F$. Let $C_p = F[p_o^{(j_o)},\dots,p_N^{(j_N)}]$, $C_u = F[u_o^{(j_o)},\dots,u_N^{(j_N)}]$ be two differential rings with the derivation ∂ acting on them by $\partial : p_j^{(n)} \to p_j^{(n+1)}$, $\partial : u_j^{(n)} \to u_j^{(n+1)}$. Denote $K_p = C_p((\lambda))$, $K_u = C_u((\lambda))$, where λ is a formal parameter commuting with everything. We make K_p and K_u rings with automorphisms by defining $\Delta = \exp(-\lambda\partial)$. Let $K_u[\zeta,\zeta^{-1}]$ be the set of finite polynomials in ζ,ζ^{-1} over K_u with the usual commutation relations $\zeta^s b = \Delta^s(b)\zeta^s$. We denote by

$$\phi_u : K_u[\zeta,\zeta^{-1}] \to C_u[\partial]((\lambda))$$

the monomorphism of associative rings which sends ζ^k into $\exp(-k\lambda\partial)$ and is identical on C_u,λ.

Denote by $\psi : K_u \to K_p$ an isomorphism of differential rings over F which commutes with ∂, is identical on λ and is given on generators by

$$\psi : u_j \to \frac{(-1)^j}{(2j+1)} \binom{N+1}{j+1} + \sum_{s=0}^{j} (-1)^{j-s} \binom{N-s}{j-s} p_s \lambda^{s+2} . \tag{2.1}$$

Denote by the same letter ψ the natural extension of ψ from $K_u = C_u((\lambda))$ to $C_u[\partial]((\lambda))$ by allowing ψ to act identically on ∂.

Theorem 2.2. Let $L \in K_u[\zeta,\zeta^{-1}]$ be given by

$$L = \zeta + \sum_{i=0}^{N} u_i \zeta^{-2i-1} . \tag{2.3}$$

Then

$$\psi\phi_u(L) = \theta_N + \lambda^{N+2} \bar{L} + O(\lambda^{N+3}) , \tag{2.4}$$

where

$$\theta_N = 1 + \sum_{i=0}^{N} \frac{(-1)^i}{2i+1} \binom{N+1}{i+1} , \tag{2.5}$$

$$\bar{L} = \frac{(-2\partial)^{N+2}}{2N+4} + \sum_{s=0}^{N} p_{N-s}(-2\partial)^s . \tag{2.6}$$

Remark 2.7. Apart from an unessential constant θ_N, the lowest order image $\bar{L}$ of the discrete Lax operator (2.3) is a typical differential scalar Lax operator (2.6). However, it does not immediately follow that the discrete Lax equations collapse into differential Lax equations because we do not yet know the precise structure of the λ-series for the operators $\{P = L^n\}$. Another problem we must consider with care is that we can no longer use weights in which $w(\Delta) = 1$ since $\Delta = \exp(-\lambda\partial)$, and we clearly have to use weights of differential Lax equations where ∂ has weight 1. The way round this obstacle is to notice that one can use another grading, let us call it rk, in constructing abstract Lax equations of Chapter I. Namely, by putting $rk(x_j) = \alpha j$, $\deg(x_{i_1}\cdots x_{i_k}) = k$, we find that $rk = \beta\cdot\deg - w$ and so the condition $w(\partial_P) = 0$ in I(1.26) is equivalent to $rk(\partial_P) = \beta \deg(\partial_P)$, which is βn for $P = L^n$. However, we seem to meet a new problem in (2.1) which insists on u_j having weight zero. Let us turn to the proof.

Proof of theorem 2.2. We write

$$\psi(u_i) = \alpha_i + \sum_{s=0}^{i} \beta_{i,s}p_s\lambda^{s+2} \ , \ 0 \le i \le N \ , \ \beta_{i,i} = 1 \ ,$$

where $\alpha_i, \beta_{i,s}$ can be read off (2.1). We have

$$\psi\phi_u(L) \equiv \sum_{r=0}^{N+2} \{(-1)^r + \sum_{i=0}^{N} (\alpha_i + \sum_{s=0}^{i} \beta_{i,s}p_s\lambda^{s+2})(2i+1)^r\} \frac{\lambda^r\partial^r}{r!} \pmod{\lambda^{N+3}} \ , \tag{2.8}$$

where we have used $\Delta^{2j+1} = \sum_{r=0}^{\infty} (-1)^r(2j+1)^r \frac{\lambda^r\partial^r}{r!}$.

Let us firstly consider the terms with p_s present:

$$\sum_{r=0}^{N+2}\sum_{i=0}^{N}\sum_{s=0}^{i} \beta_{i,s}p_s\lambda^{s+2}(2i+1)^r \frac{\lambda^r\partial^r}{r!} \equiv \sum_{s=0}^{N} \lambda^{s+2}p_s \sum_{r=0}^{N-s} \frac{\lambda^r\partial^r}{r!} \sum_{i=s}^{N} \beta_{i,s}(2i+1)^r \pmod{\lambda^{N+3}} \tag{2.9}$$

Lemma 2.10.

$$\sum_{i=s}^{N} \beta_{i,s}(2i+1)^r = \begin{cases} 0 \ , & r < N-s \ , \\ (-2)^r r! \ , & r = N-s \ . \end{cases}$$

Proof.

$$\sum_{i=s}^{N} \beta_{i,s}(2i+1)^r = \sum_{i=s}^{N} (-1)^{i-s}\binom{N-s}{i-s}(2i+1)^r =$$

$$= \sum_{j=0}^{M=N-s} (-1)^j \binom{M}{j}(2j+2s+1)^r \ . \tag{2.11}$$

Now, $(x-1)^M = \sum_{j=0}^{M} x^j \binom{M}{j}(-1)^{M-j}$, hence $\sum_{j=0}^{M} x^j \binom{M}{j}(-1)^j = (1-x)^M$,

so $\sum_{j=0}^{M} x^{2j}\binom{M}{j}(-1)^j = (1-x^2)^M$, thus $\sum_{j=0}^{M} x^{2j+2s+1}\binom{M}{j}(-1)^j = (1-x^2)^M x^{2s+1}$.

Applying $(x \frac{d}{dx})^r \Big|_{x=1}$ to both sides of the last equation we get the statement of the lemma. □

Substituting (2.10) into (2.9) we get

$$\sum_{s=0}^{N} \lambda^{s+2} p_s \sum_{r=0}^{N-s} \frac{\lambda^r \partial^r}{r!} (-2)^r r!\ \delta^r_{N-s} = \sum_{s=0}^{N} \lambda^{s+2} p_s \lambda^{N-s} \partial^{N-s} (-2)^{N-s} =$$

$$= \sum_{s=0}^{N} \lambda^{N+2} [p_s (-2\partial)^{N-s}] , \qquad (2.12)$$

which gives us λ^{N+2} times ($\bar{L}$ without its highest term).

Consider now the rest in (2.8):

$$\sum_{r=0}^{N+2} \{(-1)^r + \sum_{i=0}^{N} \alpha_i (2i+1)^r\} \frac{\lambda^r \partial^r}{r!} . \qquad (2.13)$$

For r=0 we get

$$1 + \sum_{i=0}^{N} \alpha_i = 1 + \sum_{i=0}^{N} \frac{(-1)^i}{2i+1} \binom{N+1}{i+1} = \theta_N ,$$

which takes care of (2.5). For the rest of r's, we get from (2.13):

$$\sum_{r=0}^{N+1} \{(-1)^{r+1} + \sum_{i=0}^{N} \alpha_i (2i+1)^{r+1}\} \frac{\lambda^{r+1} \partial^{r+1}}{(r+1)!} =$$

$$= \sum_{r=0}^{N+1} [(-1)^{r+1} + \sum_{i=0}^{N} (-1)^i (2i+1)^r \binom{N+1}{i+1}] \frac{\lambda^{r+1} \partial^{r+1}}{(r+1)!} \qquad (2.14)$$

Lemma 2.15.

$$\sum_{i=0}^{N} (-1)^i \binom{N+1}{i+1}(2i+1)^r = \begin{cases} (-1)^r , & r < N+1 , \\ (-1)^{N+1} - (-2)^{N+1}(N+1)! , & r = N+1 . \end{cases}$$

Given the lemma, (2.14) reduces to

$$-(-2)^{N+1}(N+1)! \, \frac{\lambda^{N+2}\partial^{N+2}}{(N+2)!} = \lambda^{N+2} \, \frac{(-2\partial)^{N+2}}{2(N+2)} \text{ , which is the last piece of (2.6).}$$

Proof of the lemma. We have

$$\frac{1-(1-y^2)^{N+1}}{y} = y^{-1} \, [1 - \sum_{k=0}^{N+1} (-y^2)^k \binom{N+1}{k}] =$$

$$= y^{-1}[1-(-y^2)^o - \sum_{k=0}^{N} (-y^2)^{k+1}\binom{N+1}{k+1}] = \sum_{k=0}^{N} (-1)^k y^{2k+1}\binom{N+1}{k+1} .$$

Applying $(y \frac{d}{dy})^p \Big|_{y=1}$ to this equality, we get

$$\sum_{k=0}^{N} (-1)^k(2k+1)^p\binom{N+1}{k+1} = (y \frac{d}{dy})^p \, [\frac{1-(1-y^2)^{N+1}}{y}] \Big|_{y=1} . \tag{2.16}$$

Define $p_r \in \mathbb{Z}[y]$ by

$$(y \frac{d}{dy})^r \, \frac{1-(1-y^2)^{N+1}}{y} = \frac{p_r}{y} . \tag{2.17}$$

Since $y \frac{d}{dy} (\frac{p_r}{y}) = \dfrac{y^2 \frac{dp_r}{dy} - yp_r}{y^2} = \frac{p_{r+1}}{y}$, we obtain

$$p_{r+1} = y \frac{dp_r}{dy} - p_r . \tag{2.18}$$

Set

$$p_r = (-1)^r + (1-y^2)^{N+1-r}\pi_r \ , \ \pi_o = -1 \ . \tag{2.19}$$

Substituting (2.18) into (2.19) we find

$$(-1)^{r+1} + (1-y^2)^{N-r}\pi_{r+1} = (-1)^{r+1} - \pi_r(1-y^2)^{N+1-r} +$$

$$+ y\{(1-y\ )^{N+1-r}\frac{d\pi_r}{dy} + (N+1-r)\pi_r(-2y)(1-y^2)^{N-r}\} \ ,$$

thus

$$\pi_{r+1} = (y^2-1)\pi_r + y(1-y^2)\frac{d\pi_r}{dy} - 2y^2(N+1-r)\pi_r \ ,$$

and therefore, since $\pi_r(y)$ is obviously regular at y = 1, we get

$$\pi_{r+1}(1) = -2(N+1-r)\pi_r(1) \ , \tag{2.20}$$

and since $\pi_o = -1$, it follows that

$$\pi_{N+1}(1) = -(-2)^{N+1}(N+1)! \ . \tag{2.21}$$

By (2.16), (2.17), the sum we are interested in equals $p_r(1)$. By (2.19), $P_r(1) = (-1)^r$ for $r < N+1$, and by (2.21), $p_{N+1}(1) = (-1)^{N+1} + (-1)(-2)^{N+1}(N+1)!$, as stated. □

Remark 2.22. For N = 0, (2.1) becomes $\psi : u_o \to 1+\lambda^2 p_o$, which is (1.3).

Remark 2.23. The map ψ in (2.1) is not the only map which produces a differential Lax operator in the image of its lowest order. Consider, for example, another map $\bar\psi$, given as

$$\bar\psi : u_j \to \alpha_j + \sum_{s=0}^{N-j} \lambda^{2+s} p_s \gamma_{i,s} \ , \tag{2.24}$$

where α_j are the same as before, and

$$\gamma_{j,s} = \binom{N-s}{j}(-1)^j \ . \tag{2.25}$$

Then

$$\bar{\psi}\,\phi_u(L) = \theta_N + \lambda^{N+2}\bar{L} + O(\lambda^{N+3}) \,, \tag{2.26}$$

where θ_N is as in (2.5), and $\bar{L}$ is as in (2.6).

Indeed, since the α_j are the same as in (2.1), we get the same θ_N and the same constant term in $\bar{L}$. Consider then, only those terms where p_s appear, such as in (2.9):

$$\sum_{r=0}^{N+2}\sum_{i=0}^{N}\sum_{s=0}^{N-i} \gamma_{i,s}p_s\lambda^{s+2}(2i+1)^r \frac{\lambda^r\partial^r}{r!} \equiv \sum_{s=0}^{N}\lambda^{s+2}p_s\sum_{r=0}^{N-s}\frac{\lambda^r\partial^r}{r!}\sum_{i=0}^{N-s}\gamma_{i,s}(2i+1)^r \,. \tag{2.27}$$

We have,

$$\sum_{i=0}^{N-s}\gamma_{i,s}(2i+1)^r = \sum_{i=0}^{N-s}\binom{N-s}{i}(-1)^i(2i+1)^r = \left(x\frac{d}{dx}\right)^r\Bigg|_{x=1}\sum_{i=0}^{M=N-s} x^{2i+1}\binom{M}{i}(-1)^i =$$

$$= \left(x\frac{d}{dx}\right)^r\Bigg|_{x=1} x(1-x^2)^M = \begin{cases} 0 \,, & r < M \\ (-2)^M M! \,, & r = M \,. \end{cases}$$

Substituting this into (2.27) we get

$$\sum_{s=0}^{N}\lambda^{s+2}p_s\frac{\lambda^{N-s}\partial^{N-s}}{(N-s)!}(-2)^{N-s}(N-s)! = \lambda^{N+2}\sum_{s=0}^{N}p_s(-2\partial)^{N-s} \,,$$

as stated in (2.26).

Chapter VII. Differential-Difference Calculus

We study a model of calculus which incorporates derivations into discrete calculus of Chapter II. This universal calculus behaves naturally with respect to the "continuous limit" - maps.

1. Calculus

We will use a route in this section which is parallel to that of Chapter II in order to make the presentation more clear.

Let k, as usual, be a field of characteristic zero. Let K be a commutative algebra over k, and let $\Delta_1,\dots,\Delta_r : K \to K$ be mutually commuting automorphisms of K over k. Let $\partial_1,\dots,\partial_m : K \to K$ be mutually commuting derivations of K over k, and assume the Δ's commute with the ∂'s too.

Let C denote the ring of polynomials

$$C = K[q_j^{(\sigma_j|\nu_j)}] \ , \ j\in J \ , \ \sigma_j\in\mathbb{Z}^r \ , \ \nu_j\in\mathbb{Z}_+^m \ , \tag{1.1}$$

with independent commuting variables $q_j^{(\sigma|\nu)}$. Denote $\Delta^\sigma = \Delta_1^{\sigma_1} \cdots \Delta_r^{\sigma_r}$, $(\pm\partial)^\nu = (\pm\partial_1)^{\nu_1} \cdots (\pm\partial_m)^{\nu_m}$ for $\sigma\in\mathbb{Z}^r$, $\nu\in\mathbb{Z}_+^m$. We extend the action of the Δ's and the ∂'s from K to C by the formulae

$$\Delta^{\sigma'}(q_j^{(\sigma|\nu)}) = q_j^{(\sigma+\sigma'|\nu)} \ , \ \partial^{\nu'}(q_j^{(\sigma|\nu)}) = q_j^{(\sigma|\nu+\nu')} \ .$$

Thus all the actions continue to commute. We denote

$$q_j = q_j^{(0|0)}$$

and let Der(C) be the C-module of derivations of C over K.

<u>Definition 1.2</u>. A derivation $\hat{X}\in$Der(C) is called <u>evolutionary</u> if it commutes with Δ's and Δ's. Thus

$$\hat{X} = \Sigma \ [\Delta^\sigma\partial^\nu(\hat{X}(q_j))] \cdot \frac{\partial}{\partial q_j^{(\sigma|\nu)}} \ ,$$

and any evolutionary derivation is uniquely defined by an arbitrary vector

$$\bar{X} = \{X_j\} \ , \ X_j \ := \hat{X}(q_j) \ . \tag{1.3}$$

Let $\Omega^1(C)$ be the universal C-module of 1-forms

$$\Omega^1(C) = \{\Sigma f_j^{(\sigma|\nu)} dq_j^{(\sigma|\nu)} | f_j^{(\sigma|\nu)} \in C, \text{ finite sums}\} \ ,$$

together with the universal derivation

$$d : C \to \Omega^1(C) \ , \ d : q_j^{(\sigma|\nu)} \to dq_j^{(\sigma|\nu)}$$

over K.

For $\omega = \Sigma \ f_j^{(\sigma|\nu)} \ dq_j^{(\sigma|\nu)} \in \Omega^1(C)$, and $Z \in \mathrm{Der}(C)$, we denote

$$\omega(Z) = \Sigma \ f_j^{(\sigma|\nu)} \ Z(q_j^{(\sigma|\nu)}) \ .$$

The action of Der(C) is uniquely lifted to $\Omega^1(C)$ such that it commutes with d:

$$Z(f dq_j^{(\sigma|\nu)}) = Z(f) dq_j^{(\sigma|\nu)} + f d(Z(q_j^{(\sigma|\nu)})) \ .$$

Denote by $D^{ev} = D^{ev}(C)$ the Lie algebra of evolution derivations of C. The properties of d, Δ's, ∂'s and $D^{ev}(C)$ can be summarized as follows:

Proposition 1.4. The actions of d, Δ's, ∂'s and $D^{ev}(C)$ all commute.

Denote $\mathrm{Im}\mathfrak{D} = \sum_i \mathrm{Im}(\Delta_i - 1) + \sum_i \mathrm{Im}\partial_i$. Elements of $\mathrm{Im}\mathfrak{D}$ are called trivial. We write $a \sim b$ if $(a-b) \in \mathrm{Im}\mathfrak{D}$. Finally, denote $\Omega_o^1(C) = \{\Sigma \ f_j dq_j | f_j \in C, \text{ finite sums}\}$ and let us introduce the operators

$$\frac{\delta}{\delta q_j} = \sum_{\sigma,\nu} \Delta^{-\sigma}(-\partial)^\nu \frac{\partial}{\partial q_j^{(\sigma|\nu)}} : C \to C \ . \tag{1.5}$$

We define the map $\hat{\delta} : \Omega^1(C) \to \Omega_o^1(C)$ by

$$\hat{\delta}(dq_j^{(\sigma|\nu)} f) = dq_j \Delta^{-\sigma}(-\partial)^\nu(f) \ , \tag{1.6}$$

and let

$$\delta = \hat{\delta}d : C \to \Omega_o^1(C) \ . \tag{1.7}$$

For $H \in C$, we can compute δH:

$$\delta H = \hat{\delta}dH = \hat{\delta}(dq_j^{(\sigma|\nu)} \frac{\partial H}{\partial q_j^{(\sigma|\nu)}}) = dq_j \Delta^{-\sigma}(-\partial)^{\nu} (\frac{\partial H}{\partial q_j^{(\sigma|\nu)}}) \ ,$$

thus

$$\delta H = \sum_j \frac{\delta H}{\delta q_j} dq_j \ . \tag{1.8}$$

From (1.6) we have

$$(\hat{\delta}-1)\Omega^1(C) \subset \mathrm{Im}\mathfrak{D} . \tag{1.9}$$

<u>Proposition 1.10</u>.

$$\hat{\delta}(\mathrm{Im}\mathfrak{D}) = 0 \ .$$

<u>Proof</u>. a) $\hat{\delta}\Delta_i(fdq_j^{(\sigma|\nu)}) = \hat{\delta}[\Delta_i(f)dq_j^{(\sigma+1_i|\nu)}] =$

$= dq_j(-\partial)^{\nu} \Delta^{-\sigma-1_i}\Delta_i(f) = \hat{\delta}(fdq_j^{(\sigma|\nu)})$, thus $\hat{\delta}(\Delta_i-1) = 0$;

b) $\hat{\delta}\partial_i(fdq_j^{(\sigma|\nu)}) = \hat{\delta}[\partial_i(f)dq_j^{(\sigma|\nu)}+fdq_j^{(\sigma|\nu+1_i)}] =$

$= dq_j\{\Delta^{-\sigma}(-\partial)^{\nu}\partial_i(f)+\Delta^{-\sigma}(-\partial)^{\nu+1_i}(f)\} = dq_j\Delta^{-\sigma}(-\partial)^{\nu}[\partial_i+(-1)^1\partial_i](f) = 0$

thus $\hat{\delta}\partial_i = 0$. □

Corollary 1.11.

$$\frac{\delta}{\delta q_j}(\mathrm{Im}\mathcal{D}) = 0 \; .$$

Proof. $\Sigma \frac{\delta H}{\delta q_j} dq_j = \hat{\delta}\, dH$, and if $H \in \mathrm{Im}\mathcal{D}$, then $dH \in \mathrm{Im}\mathcal{D}$, and hence $\hat{\delta}(dH) = 0$ by Proposition 1.10. □

To make sure that we indeed factor out $\mathrm{Im}\mathcal{D}$ using the map $(\hat{\delta} - 1)$, we need some analog of theorem II 15.

Lemma 1.12. If $g \in C$ and $gC \sim 0$, then $g = 0$.

Proof. For $m = 0$, the statement reduces to lemma II 17. So, suppose $m > 0$. Assume $g \neq 0$. Let $M \in \mathbb{N}$ be such that $\frac{\partial g}{\partial q_j^{(\sigma|\nu)}} = 0$ for $\nu_m \geq M$. We have

$$0 = \frac{\partial}{\partial q_j^{(0|1_{2m})}} \frac{\delta}{\delta q_j} [(q_j^{(0|1_m)})^2 g] = (-1)^m 2g \quad [\text{no sum on } j] \; ,$$

thus $g = 0$, which is a contradiction. □

Corollary 1.13. If $\omega \in \Omega^1_o(C)$ and $\omega(\mathrm{Der}(C)) \sim 0$, then $\omega = 0$.

Proof. If $\omega \neq 0$, then there exists $Z \in \mathrm{Der}(C)$ such that $g = \omega(Z) \neq 0$. Then for any $f \in C$, $\omega(fZ) = fg \sim 0$, which implies that $g = 0$ by the lemma 1.12, which is a contradiction. □

Theorem 1.14. a) If $\omega \in \Omega^1_o(C)$ and $\omega \sim 0$, then $\omega = 0$; b) If $\omega \in \Omega^1(C)$, then $\omega \in \mathrm{Im}\mathcal{D}$ if and only if $\omega(D^{ev}) \in \mathrm{Im}\mathcal{D}$; c) The map $\hat{\delta} : \Omega^1(C) \to \Omega^1_o(C)$ can be uniquely defined by

$$(\hat{\delta}\omega)(\hat{X}) \sim \omega(\hat{X}) \; , \; \forall \hat{X} \in D^{ev} \; .$$

Proof. c): Uniqueness follows from the corollary 1.13, and existence follows from (1.9) and the "if" part of b); a): follows from the "if" part of b) and corollary 1.13; b): We have,

$$[(\Delta_i-1)(fdq_j^{(\sigma|\nu)})](\hat{X}) = [\Delta_i(f)dq_j^{(\sigma+1_i|\nu)}-fdq_j^{(\sigma|\nu)}](\hat{X}) =$$

$$= [\Delta_i(f)\Delta^{\sigma+1_i}\partial^\nu-f\Delta^\sigma\partial^\nu](\hat{X}(q_j)) = (\Delta_i-1)\Delta^\sigma\partial^\nu(\hat{X}(q_j)) \ ,$$

and also

$$[\partial_i(fdq_j^{(\sigma|\nu)})](\hat{X}) = [\partial_i(f)dq_j^{(\sigma|\nu)}+fdq_j^{(\sigma|\nu+1_i)}](\hat{X}) =$$

$$= [\partial_i(f)\Delta^\sigma\partial^\nu+f\Delta^\sigma\partial^{\nu+1_i}](\hat{X}(q_j)) = \partial_i[f\Delta^\sigma\partial^\nu(\hat{X}(q_j))] \ ,$$

which proves the "if" part of b. To prove the "only if" part, notice that $\Omega^1(C) = \Omega^1_o(C) \oplus \operatorname{Ker}\hat{\delta} = \operatorname{Im}\hat{\delta} \oplus \operatorname{Ker}\hat{\delta}$, and $\Omega^1_o(C) \cap \operatorname{Im}\mathfrak{D} = \{0\}$ by a). Therefore

$$\operatorname{Ker}\hat{\delta} = \operatorname{Im}\mathfrak{D} . \tag{1.15}$$

Now let $\omega \in \Omega^1(C)$ be such that $\omega(D^{ev}) \sim 0$. Since $(\hat{\delta}-1)(\omega) \sim 0$ by (1.9), then $[(\hat{\delta}-1)(\omega)](D^{ev}) \sim 0$ by the "if" part of b). Therefore, $[\hat{\delta}(\omega)](D^{ev}) \sim 0$ and so $\hat{\delta}(\omega) = 0$ by a). Hence $\omega \sim 0$ by (1.15). □

Denote by $\frac{\delta H}{\delta \bar{q}}$ the vector with the components $\frac{\delta H}{\delta q_j}$. Let us agree to write all vectors as columns and to use the letter "t" for transpose. For instance, we write $\frac{\delta H}{\delta \bar{q}^t}$ instead of $(\frac{\delta H}{\delta \bar{q}})^t$. We shall often use theorem 1.14 in the form

$$\hat{X}(H) = dH(\hat{X}) \sim \delta H(\bar{X}) = \bar{X}^t \frac{\delta H}{\delta \bar{q}} \ , \ \bar{X} = \{X_j\} \ , \tag{1.16}$$

for any $H \in C$ and any $\hat{X} \in D^{ev}$.

Remark 1.17. The same proof as that used for theorem II 31 provides the result

$$\operatorname{Ker}\delta = \operatorname{Im}\mathfrak{D} + K \ .$$

We leave this as an exercise to the reader.

2. The First Complex for the Operator δ.

We are going to construct an operator $\delta^1 : \Omega^1_o(C) \to ?$ which makes

$$C \xrightarrow{\delta} \Omega^1_o(C) \xrightarrow{\delta^1} ?$$

into a complex. The name "first" comes from an analogous geometric situation [5], Ch. II, §6, where there exists also a second complex; the reader can ignore the word "first" in what follows.

Perhaps a few words would be helpful about the idea behind the construction below. As we have seen in Chapter II, the operator δ^1 (in theorem II 54) was essentially the same operator δ but in a situation extended by the presence of a new derivation even though there were no derivations present initially. Thus the derivations seem to be forced into the play by the logic of the calculus. This is one of the primary reasons for studying them jointly with automorphisms in this chapter. Our plan, then, will be to make exactly the same extensions of the basic variables.

$$\text{Let } \bar{C} = K[q_j^{(\sigma_j|\bar{\nu}_j)}] \ , \ j \in J \ , \ \sigma_j \in \mathbb{Z}^r \ , \ \bar{\nu}_j \in \mathbb{Z}_+^{m+1} \ ,$$

and let $\partial_{m+1} : \bar{C} \to \bar{C}$ be a new derivation which acts trivially on K and takes $q_j^{(\sigma|\bar{\nu})}$ into $q_j^{(\sigma|\bar{\nu}+1_{m+1})}$. We shall write $q_j^{(\sigma|\nu|p)}$ instead of $q_j^{(\sigma|\bar{\nu})}$ if $\bar{\nu} = \nu \oplus p$, $\nu \in \mathbb{Z}_+^m$, $p \in \mathbb{Z}_+$, and preserve the notation $q_j^{(\sigma|\nu)}$ for $q_j^{(\sigma|\nu|0)}$. All other ∂'s and Δ's are extended on $\bar{C}$ as before. This allows us to consider C as sitting inside $\bar{C}$, the actions of Δ's and $\partial_1, \ldots, \partial_m$ being compatible with this imbedding.

Let $\bar{\tau}$ be the homomorphism of C-modules

$$\bar{\tau} : \Omega^1(C) \to \bar{C} \ , \ \bar{\tau} : f dq_j^{(\sigma|\nu)} \mapsto f q_j^{(\sigma|\nu|1)} \ , \tag{2.1}$$

which covers the above imbedding $C \to \bar{C}$. Since $\bar{\tau}$ commutes with Δ's and $\partial_1, \dots, \partial_m$, we have

$$\bar{\tau}(\operatorname{Im}\mathfrak{B}) \subset \operatorname{Im}\mathfrak{B}. \tag{2.2}$$

Proposition 2.3.

$$\partial_{m+1}(H) = \bar{\tau}d(H), \quad \forall H \in C.$$

Proof. $\bar{\tau}d(H) = \bar{\tau}[dq_j^{(\sigma|\nu)} \frac{\partial H}{\partial q_j^{(\sigma|\nu)}}] = q_j^{(\sigma|\nu|1)} \frac{\partial H}{\partial q_j^{(\sigma|\nu)}}$. □

Denote by δ^1 the operator $\delta^1 : \bar{C} \to \Omega_o^1(\bar{C})$, which was denoted by δ for C. The same meaning let be given to $\hat{\delta}^1$, so $\delta^1 = \hat{\delta}^1 d$.

Theorem 2.4.

$$\delta^1 \bar{\tau} \delta = 0 .$$

Proof. $\delta(H) = \hat{\delta}d(H) \sim d(H)$, thus $\bar{\tau}\delta(H) \sim \bar{\tau}d(H) = \partial_{m+1}(H)$ by (2.2), (2.3). Hence $\delta^1\bar{\tau}\delta(H) = \delta^1\partial_{m+1}(H) = \hat{\delta}^1 d\partial_{m+1}(H) = \hat{\delta}^1\partial_{m+1}(dH) = 0$ by (1.10). □

Theorem 2.4 provides us with the first complex for the operator δ. As we may expect from Chapter II, the coordinate version of this complex must assert the symmetry property of the operator $D(\frac{\delta H}{\delta \bar{q}})$. This is indeed the case.

First recall that $D(\bar{R})$ is the matrix with the matrix elements $D(\bar{R})_{ij} = D_j(R_i)$, where

$$D_j(f) = \Sigma \frac{\partial f}{\partial q_j^{(\sigma|\nu)}} \Delta^\sigma \partial^\nu .$$

The adjoint operator $A^* : C^m \to C^n$ with respect to an operator $A : C^n \to C^m$, is defined in the usual manner by

$$u^t A v \sim (A^* u)^t v , \quad u \in C^m , \quad v \in C^n .$$

If the matrix elements of A are given by

$$A_{ij} = \Sigma\, f_{ij}^{\sigma|\nu} \Delta^{\sigma} \partial^{\nu} \ , \quad f_{ij}^{\sigma|\nu} \in C \ ,$$

then

$$(A^*)_{ji} = (A_{ij})^* = \Sigma \Delta^{-\sigma} (-\partial)^{\nu} f_{ij}^{\sigma|\nu} \ .$$

Let $D(\frac{\delta H}{\delta \bar{q}})$ be the matrix with elements $D(\frac{\delta H}{\delta \bar{q}})_{ij} = D_j(\frac{\delta H}{\delta q_i})$.

<u>Theorem 2.5</u>. The operator $D(\frac{\delta H}{\delta \bar{q}})$ is symmetric, $\forall H \in C$.

<u>Proof</u>. We simply rewrite theorem 2.4 in components. We have

$$0 = \delta^1 \bar{\tau} \delta(H) = \delta^1 \bar{\tau}(dq_j \frac{\delta H}{\delta q_j}) = \delta^1 (\frac{\delta H}{\delta q_j} q_j^{(0|0|1)}) =$$

$$= dq_i \frac{\delta}{\delta q_i} (\frac{\delta H}{\delta q_j} q_j^{(0|0|1)}) = dq_i \Delta^{-\sigma} (-\partial)^{\nu} (-\partial_{m+1})^p \frac{\partial}{\partial q_i^{(\sigma|\nu|p)}} (\frac{\delta H}{\delta q_j} q_j^{(0|0|1)}) =$$

$$= dq_i \{\Delta^{-\sigma} (-\partial)^{\nu} [q_j^{(0|0|1)} \frac{\partial}{\partial q_i^{(\sigma|\nu)}} (\frac{\delta H}{\delta q_j})] - \partial_{m+1} (\frac{\delta H}{\delta q_i})\} \ ,$$

thus

$$\Delta^{-\sigma} (-\partial)^{\nu} [\frac{\partial}{\partial q_i^{(\sigma|\nu)}} (\frac{\delta H}{\delta q_j}) \cdot q_j^{(0|0|1)}] = \partial_{m+1} (\frac{\delta H}{\delta q_i}) = \frac{\partial}{\partial q_j^{(\sigma|\nu)}} (\frac{\delta H}{\delta q_i}) \cdot q_j^{(\sigma|\nu|1)} =$$

$$= \frac{\partial}{\partial q_j^{(\sigma|\nu)}} (\frac{\delta H}{\delta q_i}) \cdot \Delta^{\sigma} \partial^{\nu} (q_j^{(0|0|1)}) = [D_j (\frac{\delta H}{\delta q_i})] (q_j^{(0|0|1)}) \ .$$

Since $q_j^{(0|0|1)}$ are free independent variables, we drop them to arrive at the operator identity

$$[D(\frac{\delta H}{\delta \bar{q}})]_{ij} = D_j(\frac{\delta H}{\delta q_i}) = \Delta^{-\sigma}(-\partial)^{\nu} \frac{\partial}{\partial q_i^{(\sigma|\nu)}} (\frac{\delta H}{\delta q_j}) =$$

$$= [\frac{\partial}{\partial q_i^{(\sigma|\nu)}} (\frac{\delta H}{\delta q_j}) \cdot \Delta^{\sigma}\partial^{\nu}]^* = [D_i(\frac{\delta H}{\delta q_j})]^* = [D(\frac{\delta H}{\delta \bar{q}})_{ji}]^* = [D(\frac{\delta H}{\delta \bar{q}})^*]_{ij} . \qquad \square$$

3. Continuous Limit

Relations between continuous and discrete events can be viewed from different perspectives: equations and Lax operators (as in Chapter VI), solutions etc. Here we look at the calculus. Our aim is to show that the calculus we are dealing with in this chapter, behaves <u>naturally</u> with respect to continuous limit.

Let $C_1 = K[q_j^{(\bar{\sigma}_j|\nu_j)}]$, $C_2 = K[q_j^{(\sigma_j|\bar{\nu}_j)}]$, $j \in J$, $\bar{\sigma}_j \in \mathbb{Z}^{r+1}$, $\nu_j \in \mathbb{Z}_+^m$, $\sigma_j \in \mathbb{Z}^r$,

$\bar{\nu}_j \in \mathbb{Z}_+^{m+1}$, with $\Delta_1,\dots,\Delta_r$, $\partial_1,\dots,\partial_m$ acting on K, $\Delta_1,\dots,\Delta_{r+1}$, $\partial_1,\dots,\partial_m$ acting on C_1, and $\Delta_1,\dots,\Delta_r$, $\partial_1,\dots,\partial_{m+1}$ acting on C_2 in the usual way. Let λ be a formal parameter, commuting with everything. We denote

$$\tilde{C}_i = C_i((\lambda)) \ , \ \tilde{\Omega}_i = \Omega^1(C_i)((\lambda)) \ , \ i = 1,2,$$

and extend the differential d in the obvious way, $d_i : \tilde{C}_i \to \tilde{\Omega}_i$. We also denote $\delta_i : \tilde{C}_i \to \Omega_o^1(C_i)((\lambda))$.

Consider the homomorphism $\ell : \tilde{C}_1 \to \tilde{C}_2$ over $K((\lambda))$, given on generators as

$$\ell : q_j^{(\sigma|p|\nu)} \mapsto [\exp(p\lambda\partial_{m+1})](q_j^{(\sigma|\nu|0)}) \ , \ p \in \mathbb{Z}, \ \sigma \in \mathbb{Z}^r, \ \nu \in \mathbb{Z}_+^m . \qquad (3.1)$$

We denote by ℓ the unique extension of (3.1) into the map $\ell : \tilde{\Omega}_1 \to \tilde{\Omega}_2$ such that $d_2\ell = \ell d_1$.

Proposition 3.2.

$$\ell\Delta_{r+1} = \exp(\lambda\partial_{m+1})\ell \ .$$

Proof. It is enough to check this equality on generators. We have,

$$\ell\Delta_{r+1}(q_j^{(\sigma|p|\nu)}) = \ell(q_j^{(\sigma|p+1|\nu)}) = [\exp((p+1)\lambda\partial_{m+1})](q_j^{(\sigma|\nu|0)}) =$$

$$= [\exp(\lambda\partial_{m+1})\exp(p\lambda\partial_{m+1})](q_j^{(\sigma|\nu|0)}) = \exp(\lambda\partial_{m+1})\ell(q_j^{(\sigma|p|\nu)}) \ . \qquad \square$$

Let $\mathrm{Im}\,\mathcal{D}_i$ refer to the situation with index i, where i = 1,2.

Lemma 3.3.

$$\ell(\mathrm{Im}\,\mathcal{D}_1) \subset \mathrm{Im}\,\mathcal{D}_2 \ .$$

Proof. Since ℓ commutes with $\Delta_1,\ldots,\Delta_r,\partial_1,\ldots,\partial_m$, we have to take care only of Δ_{r+1}. We have,

$$\ell(1-\Delta_{r+1}) = \ell - \ell\Delta_{r+1} = [\text{by } (3.2)] = \ell - \exp(\lambda\partial_{m+1})\ell =$$

$$= [1-\exp(\lambda\partial_{m+1})]\ell \subset \mathrm{Im}\,\partial_{m+1} \ . \qquad \square$$

Theorem 3.4.

$$\ell\delta_1 = \delta_2\ell \ .$$

Remark. If $H \in C_1$, then the theorem says that

$$\ell\left(\frac{\delta H}{\delta q_j}\right) = \frac{\delta}{\delta q_j}(\ell H) \ ,$$

not only to first order of λ, but to all orders. Thus, functional derivatives go under ℓ into functional derivatives.

Proof. For any $H \in \tilde{C}_1$, we have $\delta_1(H) \sim d(H)$. Therefore, by lemma 3.3, $\ell\delta_1(H) \sim \ell d(H) = d\ell(H) \sim \delta_2\ell(H)$. Since both $\ell\delta_1(H)$ and $\delta_2\ell(H)$ belong to $\Omega_o^1(C_2)((\lambda))$, they are equal by theorem 1.14a, which remains obviously true in the presence of λ. □

Chapter VIII. Dual Spaces of Lie Algebras Over Rings with Calculus

We begin to develop the machinery of the Hamiltonian formalism and prove a one-to-one correspondence between Lie algebras and linear Hamiltonian operators.

1. Classical Case: Finite-Dimensional Lie Algebras over Fields

In this section we briefly review the construction of Poisson brackets for functions on dual spaces of finite-dimensional Lie algebras (for more details, see, e.g. [4]).

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic zero. Denote by $\mathfrak{g}^*$ the dual space to the vector space of $\mathfrak{g}$, and let $S(\mathfrak{g})$ be the algebra of symmetric tensors on $\mathfrak{g}$ understood as polynomials on $\mathfrak{g}^*$.

If $f \in S(\mathfrak{g})$, then $df|_y \in T^*_y(\mathfrak{g}^*) \cong \mathfrak{g}$, for any point $y \in \mathfrak{g}^*$. Therefore, if $f,g \in S(\mathfrak{g})$, then we can form the commutator $[df|_y, dg|_y]$ of the covectors $df|_y$ and $dg|_y$ understood as vectors in $\mathfrak{g}$. Thus we can form the following (Kirillov's) bracket

$$\{f,g\}(y) = \langle y,[df|_y \, , \, dg|_y]\rangle \, , \tag{1.1}$$

which makes $S(\mathfrak{g})$ into a Lie algebra (indeed: the bracket is skew-symmetric and a derivation with respect to each argument; on $\mathfrak{g} \subset S(\mathfrak{g})$ it coincides with the Lie bracket on $\mathfrak{g}$ and $S(\mathfrak{g})$ is generated by $\mathfrak{g}$).

The Poisson bracket on $\mathfrak{g}^*$ is natural: If $\mathfrak{g}_1$ is another Lie algebra and ϕ: $\mathfrak{g} \to \mathfrak{g}_1$ is a homomorphism of Lie algebras, then the dual to ϕ map ϕ^*: $\mathfrak{g}_1^* \to \mathfrak{g}^*$ induces dual to it map on the functions $(\phi^*)^*$: $S(\mathfrak{g}) \to S(\mathfrak{g}_1)$. Then

$$(\phi^*)^*(\{f,g\}_{\mathfrak{g}^*}) = \{(\phi^*)^*(f),(\phi^*)^*(g)\}_{\mathfrak{g}_1^*} \, , \forall f,g \in S(\mathfrak{g}) \, . \tag{1.2}$$

Let us write down the bracket (1.1) in coordinates. Let $(e_1,\dots,e_n)$ be a basis in $\mathfrak{g}$ and $(e_1^*,\dots,e_n^*)$ be the dual basis in $\mathfrak{g}^*$. Let c^k_{ij} be the structure constants of $\mathfrak{g}$ in the basis $(e_1,\dots,e_n)$: if $X = X_i e_i$, $Y = Y_j e_j$ (we sum throughout over repeated indices) then

$$[X,Y]_k = c^k_{ij} X_i Y_j \, . \tag{1.3}$$

Let $u_1, \ldots, u_n$ be coordinate functions on $\mathfrak{g}^*$: $u_i(y) = \langle y, e_i \rangle$. Expanding (1.1) we get

$$\{f,g\}(y) = \langle y, [\frac{\partial f}{\partial u_i} du_i, \frac{\partial g}{\partial u_j} du_j]\Big|_y \rangle =$$

$$= \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial u_j}\Big|_y \langle y, c^k_{ij} du_k|_y \rangle = c^k_{ij} u_k \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial u_j}\Big|_y ,$$

thus

$$\{f,g\} = \frac{\partial f}{\partial u_i} c^k_{ij} u_k \frac{\partial g}{\partial u_j} . \tag{1.4}$$

If we denote by $B = (B^{ij})$ the matrix which defines the bracket in (1.4):

$$B^{ij} = c^k_{ij} u_k , \tag{1.5}$$

then we can rewrite (1.1) into the following definition of B: for any two (column-) vectors X and Y,

$$X^t B Y = \langle \bar{u}, [X,Y] \rangle , \tag{1.6}$$

where $\bar{u}$ is the row-vector $\bar{u} = (u_1, \ldots, u_n)$. The right-hand side of (1.6) means $u_i[X,Y]_i$.

In the forthcoming sections we consider an infinite-dimensional analog of the Kirillov bracket. This generalization is based on two observations. First, given <u>any</u> algebra, not necessarily a Lie algebra, the matrix B defined by (1.6) still makes sense. Secondly, the thus defined matrix is Hamiltonian (that is, the bracket (1.4) satisfies a condition very near to the Jacobi identity), if and only if the original algebra is actually a <u>Lie</u> algebra. (The "only if" part follows from the fact that the algebra $\mathfrak{g}$ itself is isomorphic to the algebra of linear functions on $\mathfrak{g}^*$ under the Poisson bracket.)

Perhaps I should stress that we are working in fixed bases and local coordinates for brevity only; the reader with geometrical inclinations will have no trouble in translating our calculations into notions.

2. Hamiltonian Formalism

In this section we discuss the Hamiltonian formalism and derive a few formulae for future use.

The idea of the Hamiltonian formalism is very simple in its purest form. Let S be an abelian group and End S = Hom(S,S). If Γ: S $\to$ End S is an additive map, it makes S into a ring through the multiplication

$$\{s_1,s_2\} = \Gamma(s_1)(s_2) , \tag{2.1}$$

where $\{s_1,s_2\}$ can be called the Poisson bracket. We call Γ Hamiltonian if

$$\Gamma(\{s_1,s_2\}) = [\Gamma(s_1) , \Gamma(s_2)] , \forall s_1,s_2 \in S , \tag{2.2}$$

where the bracket on the right-hand side is the commutator. In other words, we want Γ to be a homomorphism into the Lie ring.

Denote

$$\text{Ker}\,\Gamma = \{s \in S|\Gamma(s) = 0\} . \tag{2.3}$$

Then from (2.2) we have

$$\{S, \text{Ker}\Gamma\} \subset \text{Ker}\,\Gamma, \{\text{Ker}\Gamma,S\} \subset \text{Ker}\,\Gamma , \tag{2.4}$$

$$(\{s_1,s_2\} + \{s_2,s_1\}) \in \text{Ker}\,\Gamma , \forall s_1,s_2 \in S , \tag{2.5}$$

which means that Ker Γ is stable under multiplication and multiplication in S is skew-symmetric modulo Ker Γ. Finally, to get the Jacobi identity we remark that (2.2) yields

$$\Gamma(\{\{s_1,s_2\},s_3\}) = [\Gamma(\{s_1,s_2\}), \Gamma(s_3)] =$$

$$= [[\Gamma(s_1), \Gamma(s_2)] , \Gamma(s_3)] ,$$

and therefore

$$(\{\{s_1,s_2\},s_3\} + \text{c.p.}) \in \operatorname{Ker} \Gamma \ , \ \forall s_1,s_2,s_3 \in S \ , \tag{2.6}$$

where "c.p." stands for "cyclic permutation".

If desired, one can pass to the center-free Lie algebra S/Ker Γ, but we will not do this.

<u>Remark 2.7</u>. Let S_o be a subgroup in S generated by $(\{s_1,s_2\} + \{s_2,s_1\})$, $s_1,s_2 \in S$. Suppose that S_o is stable under multiplication:

$$\{S_o,S\} \subset S_o \ , \ \{S,S_o\} \subset S_o \ . \tag{2.8}$$

Then

$$(\{\{s_1,s_2\},s_3\} + \text{c.p.}) \in S_o \ , \forall s_1,s_2,s_3 \in S \ . \tag{2.9}$$

<u>Proof</u>. Let us write $a \simeq b$ if $(a-b) \in S_o$. Then by (2.8),

$$\begin{aligned} &\{\{s_2,s_3\},s_1\} \simeq -\{s_1,\{s_2,s_3\}\} \ , \\ &\{\{s_3,s_1\},s_2\} \simeq -\{s_2,\{s_3,s_1\}\} \simeq \{s_2,\{s_1,s_3\}\} \ , \end{aligned} \tag{2.10}$$

and thus

$$\{\{s_1,s_2\},s_3\} + \{\{s_2,s_3\},s_1\} + \{\{s_3,s_1\},s_2\} \simeq [\text{by (2.2) and (2.10)}] =$$

$$= \Gamma(\{s_1,s_2\})(s_3) - \Gamma(s_1)\,\Gamma(s_2)(s_3) + \Gamma(s_2)\Gamma(s_1)(s_3) = [\text{by (2.2)}] =$$

$$= ([\Gamma(s_1),\Gamma(s_2)] - [\Gamma(s_1),\Gamma(s_2)])(s_3) = 0 \ . \qquad \square$$

In practice, S is a vector space and Γ is a linear map. In calculus, S is of the type VII (1.1): $C = K[q_j^{(\sigma_j|\nu_j)}]$, and we require that $\operatorname{Im}\Gamma \subset D^{ev} = D^{ev}(C)$. In addition, we want

$$\Gamma(\operatorname{Im}\mathfrak{D}) = 0 \ , \tag{2.11}$$

thus the map Γ must be of the form

$$\Gamma = B\delta \ , \tag{2.12}$$

where

$$B : \Omega^1_o(C) \to D^{ev}(C) \ . \tag{2.13}$$

For this map two further requirements are made: 1) If we identify $\Omega^1_o(C)$ and $D^{ev}(C)(\bar{q})$ with C^N, where $N = |J|$, then B is given as a matrix with matrix elements $B_{ij} \in D_c(C)$, where

$$D_c(C) = \{\Sigma\, f^{\sigma|\nu}\Delta^{\sigma}\partial^{\nu} \,|\, f^{\sigma|\nu} \in C\} \ ; \tag{2.14}$$

2) We want the subspace S_o from remark 2.7 to lie inside $\mathrm{Im}\,\mathfrak{D}$ which amounts to

$$B^* = -B \tag{2.15}$$

by lemma VII 1.12.

We speak of B being Hamiltonian too (when Γ is Hamiltonian). From now on, we work with the Hamiltonian formalism in calculus; that is, over the ring C of VII (1.1). If $H \in C$, we denote

$$\hat{X}_H = \Gamma(H) \ , \tag{2.16}$$

$$\bar{X}_H = \hat{X}_H(\bar{q}) = B\,\frac{\delta H}{\delta \bar{q}} \ , \tag{2.17}$$

where $\bar{q}$ is a (column-) vector with components q_j, and $\frac{\delta H}{\delta \bar{q}}$ is a vector with components $\frac{\delta H}{\delta q_j}$. We also denote

$$\frac{\delta H}{\delta \bar{q}^t} = \left(\frac{\delta H}{\delta \bar{q}}\right)^t \ ,$$

where "t" denotes "transpose". The basic definition (2.2) now becomes

$$\hat{X}_{\{H,F\}} = [\hat{X}_H, \hat{X}_F] \ , \forall H, F \in C \ , \tag{2.18}$$

where

$$\{H,F\} = \hat{X}_H(F) \sim \frac{\delta F}{\delta \bar{q}^t} B \frac{\delta H}{\delta \bar{q}} \ , \tag{2.19}$$

by VII (1.16) and (2.17).

Lemma 2.20. Equation (2.18) is equivalent to

$$B \frac{\delta}{\delta \bar{q}} \left(\frac{\delta F}{\delta \bar{q}^t} B \frac{\delta H}{\delta \bar{q}}\right) = D\left(B \frac{\delta F}{\delta \bar{q}}\right) B \frac{\delta H}{\delta \bar{q}} - D\left(B \frac{\delta H}{\delta \bar{q}}\right) B \frac{\delta F}{\delta \bar{q}} \ ,$$

where $D(\bar{R})$ is the Fréchet derivative of Sect. 2, Chap. VII:

$$D(\bar{R})(\hat{X}(\bar{q})) = \hat{X}(\bar{R}) \ , \forall \hat{X} \in D^{ev} \ . \tag{2.21}$$

Proof. Two evolution fields coincide if they yield the same result acting on vector $\bar{q}$. Therefore let us apply both sides of (2.18) to $\bar{q}$. For the left-hand side we obtain

$$\hat{X}_{\{H,F\}}(\bar{q}) = B \frac{\delta\{H,F\}}{\delta \bar{q}} = [\text{by (2.19) and VII 1.11}] =$$

$$= B \frac{\delta}{\delta \bar{q}} \left(\frac{\delta F}{\delta \bar{q}^t} B \frac{\delta H}{\delta \bar{q}}\right) .$$

For the right-hand side we get

$$[\hat{X}_H, \hat{X}_F](\bar{q}) = \hat{X}_H(\bar{X}_F) - \hat{X}_F(\bar{X}_H) =$$

$$= \hat{X}_H\left(B \frac{\delta F}{\delta \bar{q}}\right) - \hat{X}_F\left(B \frac{\delta H}{\delta \bar{q}}\right) = [\text{by (2.21) and (2.17)}] =$$

$$= D\left(B \frac{\delta F}{\delta \bar{q}}\right) B \frac{\delta H}{\delta \bar{q}} - D\left(B \frac{\delta H}{\delta \bar{q}}\right) B \frac{\delta F}{\delta \bar{q}} \ .$$

□

Denote $C_o = K[q_j]_{j\in J}$.

Lemma 2.22. Let $B = b - b^*$ where $b \in Mat_N(C_o)[\Delta^{\pm1},\partial]$. Let us define an object $\frac{\partial b}{\partial \bar{q}}$ as a matrix whose (ij)-entry is a vector

$$\sum_{\sigma,\nu} \frac{\partial b_{ij}^{\sigma|\nu}}{\partial \bar{q}} \Delta^{\sigma}\partial^{\nu} , \qquad (2.23)$$

where $(b)_{ij} = \sum_{\sigma,\nu} b_{ij}^{\sigma|\nu} \Delta^{\sigma}\partial^{\nu}$.

Then

$$\frac{\delta}{\delta\bar{q}} \left(\frac{\delta F}{\delta\bar{q}^t} B \frac{\delta H}{\delta\bar{q}}\right) = D\left(\frac{\delta F}{\delta\bar{q}}\right) B \frac{\delta H}{\delta\bar{q}} - D\left(\frac{\delta H}{\delta\bar{q}}\right) B \frac{\delta F}{\delta\bar{q}} +$$

$$+ \frac{\delta F}{\delta\bar{q}^t} \frac{\partial b}{\partial\bar{q}} \frac{\delta H}{\delta\bar{q}} - \frac{\delta H}{\delta\bar{q}^t} \frac{\partial b}{\partial\bar{q}} \frac{\delta F}{\delta\bar{q}} . \qquad (2.24)$$

Proof. We use theorem VII.1.14c). For any $\hat{X} \in D^{ev}$,

$$\bar{X}^t \frac{\delta}{\delta\bar{q}} \left(\frac{\delta F}{\delta\bar{q}^t} B \frac{\delta H}{\delta\bar{q}}\right) \sim \hat{X}\left(\frac{\delta F}{\delta\bar{q}^t} B \frac{\delta H}{\delta\bar{q}}\right) =$$

$$= \hat{X}\left(\frac{\delta F}{\delta\bar{q}^t}\right)\cdot B \frac{\delta H}{\delta\bar{q}} + \frac{\delta F}{\delta\bar{q}^t} \hat{X}(B) \frac{\delta H}{\delta\bar{q}} + \frac{\delta F}{\delta\bar{q}^t} B \hat{X}\left(\frac{\delta H}{\delta\bar{q}}\right) , \qquad (2.25)$$

where, for $B = b\text{-}b^* = \Sigma[b^{\sigma|\nu}\Delta^{\sigma}\partial^{\nu} - \Delta^{-\sigma}(-\partial)^{\nu} (b^{\sigma|\nu})^t]$,

$$\hat{X}(B) = \Sigma[\hat{X}(b^{\sigma|\nu})\Delta^{\sigma}\partial^{\nu} - \Delta^{-\sigma}(-\partial)^{\nu} \hat{X}(b^{\sigma|\nu})^t] =$$

$$= \Sigma\left[X_k \frac{\partial b^{\sigma|\nu}}{\partial q_k} \Delta^{\sigma}\partial^{\nu} - \Delta^{-\sigma}(-\partial)^{\nu} X_k \frac{\partial (b^{\sigma|\nu})^t}{\partial q_k}\right] ,$$

therefore the second term in (2.25) can be rewritten as

$$\frac{\delta F}{\delta q_i}\left[X_k \frac{\partial b^{\sigma|\nu}_{ij}}{\partial q_k} \Delta^{\sigma}\partial^{\nu} - \Delta^{-\sigma}(-\partial)^{\nu} X_k \frac{\partial b^{\sigma|\nu}_{ji}}{\partial q_k}\right] \frac{\delta H}{\delta q_j} \sim$$

$$\sim X_k\left\{\frac{\delta F}{\delta q_i} \frac{\partial b^{\sigma|\nu}_{ij}}{\partial q_k} \Delta^{\sigma}\partial^{\nu} \frac{\delta H}{\delta q_j} - \frac{\delta H}{\delta q_j} \frac{\partial b^{\sigma|\nu}_{ji}}{\partial q_k} \Delta^{\sigma}\partial^{\nu} \frac{\delta F}{\delta q_i}\right\} =$$

$$= \bar{X}^t \left\{\frac{\delta F}{\delta \bar{q}^t} \frac{\partial b}{\partial \bar{q}} \frac{\delta H}{\delta \bar{q}} - \frac{\delta H}{\delta \bar{q}^t} \frac{\partial b}{\partial \bar{q}} \frac{\delta F}{\delta \bar{q}}\right\},$$

which yields the last two terms in the right-hand side of (2.24).

The remaining first and third terms in (2.25) we transform as

$$\hat{X}\left(\frac{\delta F}{\delta \bar{q}^t}\right)\cdot B \frac{\delta H}{\delta \bar{q}} + \frac{\delta F}{\delta \bar{q}^t} B\hat{X}\left(\frac{\delta H}{\delta \bar{q}}\right) \sim \text{ [by (2.21) and (2.15)] } \sim$$

$$\sim\left(B \frac{\delta H}{\delta \bar{q}}\right)^t\cdot D\left(\frac{\delta F}{\delta \bar{q}}\right)\bar{X} - \left(B \frac{\delta F}{\delta \bar{q}}\right)^t\cdot D\left(\frac{\delta H}{\delta \bar{q}}\right)\bar{X} \sim \text{ [by theorem VII 2.5] } \sim$$

$$\sim \left[D\left(\frac{\delta F}{\delta \bar{q}}\right)\left(B \frac{\delta H}{\delta \bar{q}}\right) - D\left(\frac{\delta H}{\delta \bar{q}}\right)\left(B \frac{\delta F}{\delta \bar{q}}\right)\right]^t \bar{X},$$

which provides the first two terms in the right-hand side of (2.24). □

Applying the operator B to (2.24), we find

<u>Corollary 2.26</u>. Let B be as in lemma 2.22. Then

$$B \frac{\delta}{\delta \bar{q}} \left(\frac{\delta F}{\delta \bar{q}^t} B \frac{\delta H}{\delta \bar{q}}\right) = BD\left(\frac{\delta F}{\delta \bar{q}}\right) B \frac{\delta H}{\delta \bar{q}} - BD\left(\frac{\delta H}{\delta \bar{q}}\right) B \frac{\delta F}{\delta \bar{q}} +$$

$$+ B\left(\frac{\delta F}{\delta \bar{q}^t} \frac{\partial b}{\partial \bar{q}} \frac{\delta H}{\delta \bar{q}} - \frac{\delta H}{\delta \bar{q}^t} \frac{\partial b}{\partial \bar{q}} \frac{\delta F}{\delta \bar{q}}\right).$$

Comparing lemma 2.20 with corollary 2.26, we get

<u>Lemma 2.27</u>. Let B be as in lemma 2.22. Then B is Hamiltonian if for any $F, H \in C$,

$$[D,B](\frac{\delta F}{\delta \bar{q}})B\frac{\delta H}{\delta \bar{q}} - [D,B](\frac{\delta H}{\delta \bar{q}})B\frac{\delta F}{\delta \bar{q}} =$$

$$= B(\frac{\delta F}{\delta \bar{q}^t}\frac{\partial b}{\partial \bar{q}}\frac{\delta H}{\delta \bar{q}} - \frac{\delta H}{\delta \bar{q}^t}\frac{\partial b}{\partial \bar{q}}\frac{\delta F}{\delta \bar{q}}) . \tag{2.28}$$

We can now describe the first large class of Hamiltonian operators.

Theorem 2.29. Let $B \in \mathrm{Mat}(K)[\Delta^{\pm 1},\partial]$ and $B^* = -B$. Then B is Hamiltonian.

Proof. Let us show that $[D,B] = 0$; then (2.28) will become $0 = 0$. Since $B \in \mathrm{Mat}(K)[\Delta^{\pm 1},\partial]$, $\hat{X}B = B\hat{X}$ for any $\hat{X} \in D^{ev}(C)$. Therefore $\hat{X}B(\bar{R}) = B\hat{X}(\bar{R})$ for any vector $\bar{R} \in C^N$ which we rewrite, using (2.21), as $(DB(\bar{R}))(\bar{X}) = BD(\bar{R})(\bar{X})$. Since $\bar{X}$ and $\bar{R}$ are arbitrary, we find that $DB = BD$, Q.E.D.

3. Linear Hamiltonian Operators and Lie Algebras

In this section we study relations between Lie algebras and linear Hamiltonian operators.

Let K be as in Sect. 1, Chap. VII, and let $L = K^N$ have a structure of an algebra in the following sense: if $X = (X_1,\dots,X_N)$, $Y = (Y_1,\dots,Y_N) \in L$, then multiplication $\triangle$ in L is given by

$$(X\triangle Y)_k = \Sigma c^k_{ij,\sigma^1|\nu^1,\sigma^2|\nu^2}\,\Delta^{\sigma^1}\partial^{\nu^1}(X_i)\cdot\Delta^{\sigma^2}\partial^{\nu^2}(Y_j) , \tag{3.1}$$

where $c^k_{ij,\dots} \in K$. We require the sum in (3.1) to be finite even if $N = \infty$.

We construct "functions on L*" as follows. Let $q_1,\dots,q_N$ be free independent variables and let $C = K[q_j^{(\sigma_j|\nu_j)}]$ be as in Sect. 1, Chap. VII. We can think of $q_1,\dots,q_N$ as providing "coordinates on L*".

Let us denote

$$\langle q,X\rangle = \Sigma\, q_jX_j \tag{3.2}$$

for $X = (X_1,\dots,X_N) \in L$.

An analog of (1.5), (1.6) is provided by

Definition 3.3. Matrix $B \in Mat_N(C)[\Delta^{\pm 1}, \partial]$ is defined by the equation

$$X^t BY \sim \langle q, X\,Y\rangle \ , \ \forall X,\ Y \in L \ , \tag{3.4}$$

where ~ means "equal modulo Im$\mathfrak{D}$ in C."

Let us compute B. We have

$$\langle q, X \circ Y\rangle = q_k (X \circ Y)_k = [\text{by } (3.1)] =$$

$$= q_k \, c^k_{ij,\sigma^1|\nu^1,\sigma^2|\nu^2} \, \Delta^{\sigma^1} \partial^{\nu^1}(X_i) \cdot \Delta^{\sigma^2} \partial^{\nu^2}(Y_j) \sim$$

$$\sim X_i [\Delta^{-\sigma^1} (-\partial)^{\nu^1} \, c^k_{ij,\sigma^1|\nu^1,\sigma^2|\nu^2} \, q_k \, \Delta^{\sigma^2} \partial^{\nu^2}] Y_j \ ,$$

thus

$$B^{ij} = \Sigma \Delta^{-\sigma^1} (-\partial)^{\nu^1} \, c^k_{ij,\sigma^1|\nu^1,\sigma^2|\nu^2} \, q_k \Delta^{\sigma^2} \partial^{\nu^2} \ . \tag{3.5}$$

This is an analog of the innocent-looking (1.5).

Our goal is now to find out when the matrix B is Hamiltonian. First,

Proposition 3.6. Matrix B is skew-symmetric iff the multiplication in L is skew-commutative.

Proof. By definition of the adjoint operator (Sect. 2, Chap. VII), we have

$$Y^t BX \sim (B^* Y)^t X = X^t B^* Y \ ,$$

thus

$$\langle q, X \circ Y + Y \circ X\rangle \sim X^t BY + Y^t BX \sim X^t (B + B^*) Y \ .$$

If $B + B^* = 0$, then $X \circ Y + Y \circ X = 0$ by theorem VII 1.14a applied to

$d\langle q, X\blacktriangle Y + Y\vartriangle X\rangle$. If $X\vartriangle Y + Y\vartriangle X = 0$, then $(B+B^*)Y = 0, \forall Y\in L$ by lemma VII 1.12. To conclude that $B + B^* = 0$ we require the following "relations-free" property of K: if an operator in $K[\Delta^{\pm 1,\partial}]$ annihilates K, this operator is zero.

Thereafter we assume B and L to be skew, and K to have the above mentioned relations-free property.

Lemma 3.7. For any $F, H \in C$,

$$\frac{\delta}{\delta\bar{q}}\left(\frac{\delta F}{\delta\bar{q}^t} B \frac{\delta H}{\delta\bar{q}}\right) = D\left(\frac{\delta F}{\delta\bar{q}}\right) B \frac{\delta H}{\delta\bar{q}} - D\left(\frac{\delta H}{\delta\bar{q}}\right) B \frac{\delta F}{\delta\bar{q}} +$$

$$+ \frac{\delta F}{\delta\bar{q}} \vartriangle \frac{\delta H}{\delta\bar{q}} . \tag{3.8}$$

Remark. For any free $\tilde{K}$-module E on which operators Δ's and ∂'s are acting in accord with the $\tilde{K}$-module structure, where $\tilde{K}$ is a K-module and a ring where Δ's and ∂'s act K-compatibly, the structure constants $c^k_{ij},\ldots$ make E into a differential-difference algebra by the same formula (3.1). In this sense the expression $\frac{\delta F}{\delta\bar{q}} \vartriangle \frac{\delta H}{\delta\bar{q}}$ is understood in (3.8).

Proof. From the proof of lemma 2.22 we see that we have to check out that

$$\frac{\delta F}{\delta\bar{q}^t} \hat{X}(B) \frac{\delta H}{\delta\bar{q}} \sim \bar{X}^t\left(\frac{\delta F}{\delta\bar{q}} \vartriangle \frac{\delta H}{\delta\bar{q}}\right) , \forall \hat{X} \in D^{ev}(C) . \tag{3.9}$$

Let us write $B_{\bar{q}}$ instead of B in (3.9) to indicate explicite dependence of B.

Lemma 3.10. $\hat{X}(B_{\bar{q}}) = B_{\bar{X}}, \forall \hat{X} \in D^{ev}(C)$.

Granted the lemma, (3.9) follows at once from definition 3.3. □

Proof of the lemma 3.10. Since $\hat{X}$ commutes with Δ's, ∂'s and K, we get from (3.5):

$$\hat{X}(B^{ij}_{\bar{q}}) = \Sigma\Delta^{-\sigma^1}(-\partial)^{\nu^1} c^k_{ij,\sigma^1|\nu^1,\sigma^2|\nu^2} X_k\Delta^{\sigma^2}\partial^{\nu^2} = B^{ij}_{\bar{X}} .$$ □

Now we can derive a property which discriminates in favor of Lie algebras.

<u>Theorem 3.11</u>. For any F_1, F_2, $F_3 \in C$, $(\{F_1,\{F_2,F_3\}\} + \text{c.p.}) \sim 0$ iff L is a Lie algebra.

<u>Remark</u>. As usual, the Poisson bracket $\{F,G\}$ equals $\hat{X}_F(G)$, where $\bar{X}_F = B \frac{\delta F}{\delta \bar{q}}$.

<u>Proof</u>. If $F \sim G$, then $\{F,H\} \sim \{G,H\} \sim -\{H,G\}$, for any F, G, H $\in$ C. Hence $\{F,G\} = \hat{X}_F(G) \sim \frac{\delta G}{\delta \bar{q}^t} \bar{X}_F = \frac{\delta G}{\delta \bar{q}^t} B \frac{\delta F}{\delta \bar{q}}$. Denote $X = \frac{\delta F_1}{\delta \bar{q}}$, $Y = \frac{\delta F_2}{\delta \bar{q}}$, $Z = \frac{\delta F_3}{\delta \bar{q}}$. Using (3.8), we obtain

$$\{F_1,\{F_2,F_3\}\} \sim \frac{\delta F_1}{\delta \bar{q}^t} B \frac{\delta\{F_2,F_3\}}{\delta \bar{q}} =$$

$$= X^t B[D(Y)BZ-D(Z)BY+Y\ Z] \ . \tag{3.12a}$$

Analogously,

$$\{F_3,\{F_1,F_2\}\} \sim Z^t B[D(X)BY-D(Y)BX+X\ Y] \ , \tag{3.12b}$$

$$\{F_2,\{F_3,F_1\}\} \sim Y^t B[D(Z)BX-D(X)BZ+Z\ X] \ . \tag{3.12c}$$

Let us take the first term in (3.12a) and transform it into minus the second term of (3.12b). We have $X^t BD(Y)BZ \sim$ (B is skew)$\sim -(BX)^t D(Y)BZ \sim$ [D(Y) is symmetric by theorem VII 2.5] $\sim -(BZ)^t D(Y)BX = -Z^t B^t D(Y)BX =$ (B is skew) $= Z^t BD(Y)B(X)$.

Thus, on adding (3.12a) through (3.12c) we are left, modulo Im$\mathfrak{D}$, with

$$X^t B(Y \triangle Z) + Z^t B(X \triangle Y) + Y^t B(Z \triangle X) \sim \text{[by definition of B]} \sim$$

$$\sim \langle q, X\triangle(Y\triangle Z) + \text{c.p.}\rangle \ .$$

Thus if L is a Lie algebra, then $X\triangle(Y\triangle Z)$ + c.p. vanishes, and $\{F_1,\{F_2,F_3\}\}$ + c.p. ~ 0.

Conversely, if $\{F_1,\{F_2,F_3\}\}$ + c.p. $\sim 0, \forall F_1,F_2,F_3 \in C$, and if we are given $X^1,X^2,X^3 \in L$, we take $F_i = \langle q,X^i\rangle$. Then $\frac{\delta F_i}{\delta \bar{q}} = X^i$ and $\{F_1,\{F_2,F_3\}\}$ + c.p. $\sim$ $\langle q,X^1 \blacktriangle (X^2 \blacktriangle X^3)$ + c.p.$\rangle \sim 0$. Therefore $X^1 \blacktriangle (X^2 \blacktriangle X^3)$ + c.p. = 0 by theorem VII.1.14a) applied to $d\langle q,X^1 \blacktriangle (X^2 \blacktriangle X^3)$ + c.p.$\rangle$. □

Corollary 3.13. If B is Hamiltonian then L is a Lie algebra.

Proof. If B is Hamiltonian then $\{F_1,\{F_2,F_3\}\}$ + c.p. ~ 0 and we can apply theorem 3.11. □

Suppose now that L is a Lie algebra. Can we be sure that B is Hamiltonian? From theorem 3.11 we find that $\{F_1,\{F_2,F_3\}\}$ + c.p. $\sim 0, \forall F_1,F_2,F_3 \in C$, but we want the much stronger equation (2.18) instead. Let us see where the problem lies. We have

$$\{F,\{H,G\}\} = \hat{X}_F\hat{X}_H(G) ,$$

$$\{G,\{F,H\}\} \sim -\{\{F,H\},G\} = -\hat{X}_{\{F,H\}}(G) ,$$

$$\{H,\{G,F\}\} \sim -\{H,\{F,G\}\} = -\hat{X}_H\hat{X}_F(G) ,$$

and theorem 3.11 yields

$$(\hat{X}_{\{F,H\}}-[\hat{X}_F,\hat{X}_H])(G) \sim 0 , \forall F,H,G \in C . \tag{3.14}$$

We can't, however, deduce (2.18) from (3.14) without additional analysis, because there could conceivably exist evolution derivations sending C into Im$\mathfrak{D}$. The simplest example provides an evolution field $\bar{X} = q^{(1)}$ in the differential ring $k[q^{(n)}]$ with the derivation ∂, $\partial: q^{(n)} \to q^{(n+1)}$, $\partial: k \to 0$. It is clear why the trouble occurs: because our base field k consists only of constants. The remedy, then, is obvious: we have to throw in some "independent variable(s)."

Lemma 3.15. Let $\hat{X} \in D^{ev}(C)$ be such that $\hat{X}(C) \sim 0$ for any differential-difference extension $\tilde{K}$ of K over k. Then $\hat{X} = 0$.

Granted the lemma, which we shall prove below, we can deduce the main result of this section.

Theorem 3.16. If L is a Lie algebra, then B is Hamiltonian.

Proof. Since K is assumed to be relations-free, the property of L to be a Lie algebra is the property of the structure constants $c^k_{ij,\dots}$ in (3.1). Therefore, upon extending K to $\tilde{K}$, $\tilde{L} = \tilde{K}^N$ still remains a Lie algebra. This implies (3.14), which implies (2.18) by lemma (3.15) applied to $\hat{X} = \hat{X}_{\{F,H\}} - [\hat{X}_F, \hat{X}_H]$. □

Remark 3.17. If the structure constants $c^k_{ij,\dots}$ are such that they produce a Lie algebra, we don't need to bother whether K is relations-free or not (for example K could be $\mathcal{k}$). The proof of theorem 3.16 will still be valid. These are the circumstances in which one applies theorem 3.16 in practice.

Proof of lemma 3.15. We let $\tilde{K} = K[x,\tilde{x}]$ where new variables $x_1,\dots,x_r$, $\tilde{x}_1,\dots,\tilde{x}_m$ are introduced subject to following relations:

$$\partial_i x_j = 0 \ , \ \Delta_i(x_j) = x_j + \delta^i_j c_j \quad \text{(no sum on j)}, \ c_j \in \mathcal{k} \ ,$$

$$\Delta_i \tilde{x}_j = \tilde{x}_j \ , \ \partial_i \tilde{x}_j = \delta^i_j \ , \tag{3.18}$$

where δ^i_j is the usual Kronecker symbol. Obviously, Δ's and ∂'s still commute.

Let $\hat{X} \in D^{ev}(C)$ and $\hat{X}(C) \sim 0$. We want to show that $\hat{X} = 0$. Suppose $\hat{X} \neq 0$, then $\hat{X}(q_j) \neq 0$ for some j. Let j = 1, say, and denote $Z = \hat{X}(q_1)$.

Let us fix some $\sigma^o \in \mathbb{Z}^r_+$, $\nu^o \in \mathbb{Z}^m_+$ and denote

$$x^{\sigma^o} = x_1^{\sigma_1^o} \cdots x_r^{\sigma_r^o} \ , \ \tilde{x}^{\nu^o} = \tilde{x}_1^{\nu_1^o} \cdots \tilde{x}_m^{\nu_m^o} \ , \ \text{for } \sigma^o = (\sigma_1^o,\dots,\sigma_r^o) \ , \ \nu^o = (\nu_1^o,\dots,\nu_m^o) \ .$$

We have

$$\hat{X}(q_1 x^{\sigma^o} \tilde{x}^{\nu^o}) = x^{\sigma^o} \tilde{x}^{\nu^o} Z \sim 0$$

by the condition of the lemma. Therefore

$$0 = \frac{\delta}{\delta q_j} (x^{\sigma^o} \tilde{x}^{\nu^o} Z) = \sum_{\sigma,\nu} \Delta^{-\sigma}(-\partial)^\nu x^{\sigma^o} \tilde{x}^{\nu^o} \frac{\partial Z}{\partial q_j^{(\sigma|\nu)}} =$$

$$= \sum_\sigma (x_1 - c_1\sigma_1)^{\sigma_1^o} \cdots (x_r - c_r\sigma_r)^{\sigma_r^o} f_\sigma , \tag{3.19}$$

where $\sigma = (\sigma_1,\ldots,\sigma_r)$ and

$$f_\sigma : = \Delta^{-\sigma} \sum_\nu (-\partial)^\nu \tilde{x}^{\nu^o} \frac{\partial Z}{\partial q_j^{(\sigma|\nu)}} . \tag{3.20}$$

Since Z and f_σ do not depend upon x, and (3.19) is an identity, we put x = 0, $c_1 = \ldots = c_r = -1$ and get

$$\sum_\sigma \sigma_1^{\sigma_1^o} \cdots \sigma_r^{\sigma_r^o} f_\sigma = 0 . \tag{3.21}$$

Since σ^o is arbitrary element of $\mathbb{Z}_+^r$, easy arguments of analysis show that $f_\sigma = 0, \forall \sigma \in \mathbb{Z}_+^r$. Therefore, $\Delta^\sigma f_\sigma = 0$ (no sum on σ) and we have

$$\sum_\nu (-\partial)^\nu \tilde{x}^{\nu^o} \frac{\partial Z}{\partial q_j^{(\sigma|\nu)}} = 0 . \tag{3.22}$$

<u>Remark</u>. If we had only derivations ∂'s present in K, then we would have begun with (3.22). On the other hand, if only automorphisms Δ's are present, our job would have been almost finished and reduced to (3.23) below.

<u>Claim</u>: $\dfrac{\partial Z}{\partial q_j^{(\sigma|\nu)}} = 0$, for all ν (j and σ are fixed). Indeed, suppose $\exists \nu = \nu^o$ such that $\dfrac{\partial Z}{\partial q_j^{(\sigma|\nu^o)}} \neq 0$ but $\dfrac{\partial Z}{\partial q_j^{(\sigma|\nu)}} = 0$ for all $|\nu| > |\nu^o|$ where $|(\nu_1,\ldots,\nu_m)| = \nu_1 + \ldots + \nu_m$. Since Z does not depend upon $\tilde{x}$, we put $\tilde{x} = 0$ in (3.22) and obtain

$$0 = \frac{\partial Z}{\partial q_j^{(\sigma|\nu^o)}} (-\partial)^{\nu^o} \tilde{x}^{\nu^o} = \frac{\partial Z}{\partial q_j^{(\sigma|\nu^o)}} (-1)^{|\nu^o|} \nu_1^o! \ldots \nu_m^o! \ ,$$

which proves our claim that

$$\frac{\partial Z}{\partial q_j^{(\sigma|\nu)}} = 0 \ , \forall j,\sigma,\nu \ . \tag{3.23}$$

Thus $Z \in K$. Therefore

$$\hat{X}\left(\frac{q_1^2}{2}\right) = q_1 Z \sim 0$$

which implies that $Z = 0$ by theorem VII 1.14a) applied to Zdq_1. □

As in the finite-dimensional case, L is imbedded in $D^{ev}(C)$:

Proposition 3.24. Let L be a Lie algebra and let θ: $L \to C$ be the map defined by

$$\theta(X) = -\langle q,X\rangle \ , \ X \in L \ . \tag{3.25}$$

Then θ induces a Lie algebra homomorphism $\bar{\theta}$: $L \to D^{ev}(C)$ given by

$$\bar{\theta}(Y) = \hat{X}_{\theta(Y)} \ , \forall Y \in L \ . \tag{3.26}$$

Proof. Let $Y, Z \in L$. Then

$$\{\theta(Y),\theta(Z)\} = \{\langle q,Y\rangle \ , \ \langle q,Z\rangle\} \sim \frac{\delta\langle q,Z\rangle}{\delta\bar{q}^t} B \frac{\delta\langle q,Y\rangle}{\delta\bar{q}} =$$

$$= Z^t BY \sim \langle q,Z\triangle Y\rangle \sim -\langle q,Y\triangle Z\rangle = \theta(Y\triangle Z) \ . \tag{3.27}$$

Hence

$$[\bar{\theta}(Y),\bar{\theta}(Z)] = [\hat{X}_{\theta(Y)},\hat{X}_{\theta(Z)}] = (\text{B is Hamiltonian}) =$$

$$= \hat{X}_{\{\theta(Y),\theta(Z)\}} = [\text{by } (3.27)] = \hat{X}_{\theta(Y\triangle Z)} = \bar{\theta}(Y\triangle Z) \ .$$ □

As the first example of the application of theorem 3.16, consider the Lie algebra L generated by the associative algebra K[Δ] of polynomials in Δ over the ring K, with r = 1, m = 0. If $X = \sum_{i\geq 0} X_i\Delta^i$, $Y = \sum_{j\geq 0} Y_j\Delta^j$, then

$$X\circ Y = \sum_{i,j} [X_iY_j^{(i)}-Y_iX_j^{(i)}]\Delta^{i+j} .$$

Therefore, writing X and Y as vectors, we have

$$X^tBY \sim \langle q,X\circ Y\rangle = q_{i+j}(X_iY_j^{(i)}-Y_iX_j^{(i)}) \sim$$

$$\sim X_i[q_{i+j}\Delta^i-\Delta^{-j}q_{i+j}]Y_j ,$$

and thus

$$B^{ij} = q_{i+j}\Delta^i - \Delta^{-j}q_{i+j} , \tag{3.28}$$

which is exactly the matrix III (3.4) of the first Hamiltonian structure of Lax equations.

For our second example, let K be a differential ring with a derivation $\partial: K \to K$; so r = 0, m = 1. Let L be one-dimensional Lie algebra with the multiplication

$$X\circ Y = X\partial Y - Y\partial X$$

(If $K = C^\infty(\mathbb{R}^1)$, then $L \cong \mathfrak{D}(\mathbb{R}^1)$ = {vector fields on $\mathbb{R}^1$}.) Let us compute B:

$$X^tBY \sim \langle q,X\circ Y\rangle = \langle q,X\partial Y-Y\partial X\rangle \sim X(q\partial+\partial q)Y ,$$

thus

$$B = q\partial + \partial q . \tag{3.29}$$

Evolution equations with this B are

$$q_t = (q\partial + \partial q) \frac{\delta H}{\delta q} , \tag{3.30}$$

which becomes

$$u_t = \partial \frac{\delta H}{\delta u} \tag{3.31}$$

after the change of variables

$$u = \sqrt{2q} . \tag{3.32}$$

Thus we attach the Hamiltonian structure (3.31) of the Korteweg-de Vries equation (0.11) to the Lie algebra of vector fields on the line. It would be interesting to find an interpretation of c.l.'s of the Korteweg-de Vries equation from the point of view of this Lie algebra.

We end this section with a discussion of the natural properties of the matrix B associated with the Lie algebra L. First some preliminaries.

Suppose we have two differential-difference rings over K: $C_1 = K[q_j^{(\sigma_j|\nu_j)}]$ and $C_2 = K[p_i^{(\sigma_i|\nu_i)}]$, and suppose we have Hamiltonian structures Γ_1 and Γ_2 in C_1 and C_2 respectively, $\Gamma_i\colon C_i \to D^{ev}(C_i)$. Let $\phi\colon C_1 \to C_2$ be a homomorphism of rings over K, which commutes with the actions of Δ's and ∂'s. We call the map ϕ <u>canonical</u>, or Γ_1 and Γ_2 <u>ϕ-compatible</u>, if the evolution fields $\Gamma_1(H)$ and $\Gamma_2(H)$ are ϕ-compatible, $\forall\, H \in C_1$. That is,

$$\phi \cdot \Gamma_1(H) = \Gamma_2(\phi H) \cdot \phi \ , \forall H \in C_1 , \tag{3.32}$$

or

$$\phi[\Gamma_1(H)(G)] = (\Gamma_2(\phi H))(\phi G) \ , \forall H,G \in C_1 , \tag{3.33}$$

or in other words

$$\phi(\{H,G\}_{C_1}) = \{\phi H, \phi G\}_{C_2} \ , \forall H,G \in C_1 . \tag{3.34}$$

Let us transform (3.32) into more transparent form.

Lemma 3.35. For any $H \in C_1$

$$\frac{\delta(\phi H)}{\delta \bar{p}} = [D(\bar{\phi})^*] \, \phi(\frac{\delta H}{\delta \bar{q}}) , \tag{3.36}$$

where $\bar{\phi} = \phi(\bar{q}) = (\phi_1, \ldots, \phi_N)^t$, $\phi_j = \phi(q_j)$.

Proof. We have

$$d(\phi H) = \phi(dH) \sim \phi(\delta H) = \phi(dq_j \frac{\delta H}{\delta q_j}) = \phi(\frac{\delta H}{\delta q_j}) \, d\phi(q_j) =$$

$$= \phi(\frac{\delta H}{\delta q_j}) \frac{\partial \phi_j}{\partial p_i^{(\sigma|\nu)}} dp_i^{(\sigma|\nu)} \sim dp_i \, \Delta^{-\sigma}(-\partial)^\nu \frac{\partial \phi_j}{\partial p_i^{(\sigma|\nu)}} \phi(\frac{\delta H}{\delta q_j}) ,$$

and so

$$\frac{\delta(\phi H)}{\delta p_i} = \Delta^{-\sigma}(-\partial)^\nu \frac{\partial \phi_j}{\partial p_i^{(\sigma|\nu)}} \phi(\frac{\delta H}{\delta q_j}) =$$

$$= (\frac{\partial \phi_j}{\partial p_i^{(\sigma|\nu)}} \Delta^\sigma \partial^\nu)^* \, \phi(\frac{\delta H}{\delta q_j}) = [D(\bar{\phi})^*]_{ij} \, \phi(\frac{\delta H}{\delta q_j}) . \tag{3.37}$$

□

Let $B_i \in \mathrm{Mat}(C_i)[\Delta^{\pm 1}, \partial]$ be the Hamiltonian matrix corresponding to Γ_i, $i = 1,2$. Denote by $\phi(B_1)$ the matrix-elements-wise image of B_1 in $\mathrm{Mat}(C_2)[\Delta^{\pm 1}, \partial]$. Since $\Gamma_1(H)$ and $\Gamma_2(H)$ are evolutionary derivations, (3.32) is satisfied if (3.33) is satisfied when G runs over $q_1, q_2, \ldots, q_N$. Thus it is enough to apply (3.32) to the vector $\bar{q}$. We get

$$\phi(B_1 \frac{\delta H}{\delta \bar{q}}) = (\Gamma_2(\phi H))\bar{\phi} = D(\bar{\phi}) B_2 \frac{\delta(\phi H)}{\delta \bar{p}} ,$$

which can be rewritten with the help of (3.36) as

$$\phi(B_1)\ \phi(\frac{\delta H}{\delta \bar{q}}) = D(\bar{\phi})B_2\ D(\bar{\phi})^*\ \phi(\frac{\delta H}{\delta \bar{q}})\ . \tag{3.38}$$

This implies, since H is arbitrary and K is relations-free, that

$$\phi(B_1) = D(\bar{\phi})B_2\ D(\bar{\phi})^*\ . \tag{3.39}$$

Equation (3.39) gives us a convenient tool to analyze maps suspected of being canonical.

We consider now an analog of (1.2). Let $\mathcal{G} = K^M$ be another differential-difference Lie algebra and let ϕ: $L \to \mathcal{G}$ be a linear map over k. If $(e_1,\dots,e_N)$ and $(\bar{e}_1,\dots,\bar{e}_M)$ are natural bases in L and $\mathcal{G}$ respectively, we assume that ϕ has the form

$$Y = \Phi X, \qquad Y = (Y_1,\dots,Y_M)^t\ ,\ L \ni X = (X_1,\dots,X_N)^t\ ,\ \Phi \in \mathrm{Mat}(K)[\Delta^{\pm 1},\partial]\ . \tag{3.39}$$

We shall write

$$Y_i = \phi_{ij}(X_j)\ ,\ \phi_{ij} \in K[\Delta^{\pm 1},\partial]\ , \tag{3.40}$$

for $Y = Y_i\bar{e}_i$, $X = X_je_j$.

Denote by $C_2 = K[p_i^{(\sigma_i|\nu_i)}]$ the ring which plays for $\mathcal{G}$ the same role which $C_1 = K[q_j^{(\sigma_j|\nu_j)}]$ plays for L. Since we are avoiding such objects as "L*" and are working with C_1 = "functions on L*", we proceed to define the homomorphism ϕ: $C_1 \to C_2$,

$$\phi(q_j) = \phi^*_{ij}(p_i)\ , \tag{3.42}$$

which we denote by the same letter ϕ as the map ϕ: $L \to \mathcal{G}$ (and which was denoted $(\phi^*)^*$ in section 1); we also require ϕ to be identical on K and to commute with Δ's, ∂'s.

The origin of the formula (3.42) can be explained by the following

Proposition 3.43. For any $X\in L$, denote $H_X = -\langle q,X\rangle$, so that Lie algebra L is isomorphically imbedded into the Lie algebra of "functions on L^*". Then

$$\phi(H_X) \sim H_{\phi(X)} \ , \ \forall X\in L \ . \tag{3.44}$$

Proof. We have,

$$\phi(H_X) = \phi(-\langle q,X\rangle) = -\phi(q_j X_j) = -X_j \phi^*_{ij}(p_i) \sim$$

$$\sim -p_i \phi_{ij}(X_j) = -\langle p,\phi(X)\rangle = H_{\phi(X)} \ . \qquad \square$$

Remark. Formula (3.44) can be rewritten as

$$\langle p,\phi(X)\rangle \sim \langle\bar{\phi},X\rangle \ , \ \bar{\phi} = \phi(\bar{q}) \ . \tag{3.45}$$

Denote by $B_{\bar{q}}$ and $B_{\bar{p}}$ the Hamiltonian matrices generated by L and $\mathcal{G}$ respectively; (we used the notations B_1 and B_2 before). To check (3.39), we need $D(\bar{\phi})$ and $\phi(B_{\bar{q}})$. Notice that evidently we have

Proposition 3.46. $\phi(B_{\bar{q}}) = B_{\bar{\phi}}$.

Lemma 3.47. $D(\bar{\phi}) = \Phi^*$.

Proof. We have,

$$[D(\bar{\phi})]_{ji} = D_i(\phi_j) = D_i(\phi(q_j)) = [\text{by } (3.42)] =$$

$$= D_i(\phi^*_{kj}(p_k)) = \phi^*_{ij} = (\Phi^*)_{ji} \ . \qquad \square$$

Now we can formulate the main relation between properties of the maps $\phi: L \to \mathcal{G}$ and $\phi: C_1 \to C_2$.

Theorem 3.48. The map $\phi: C_1 \to C_2$ is canonical iff $\phi: L \to \mathcal{G}$ is a Lie algebra homomorphism.

Proof. For any $X,\tilde{X}\in L$, we have

$$\langle p,\phi(X\triangle\tilde{X})\rangle \sim [\text{by } (3.45)] \sim \langle\bar{\phi},X\triangle\tilde{X}\rangle \sim X^t B_{\bar{\phi}}\tilde{X}\ , \tag{3.49a}$$

$$\langle p,\phi(X)\triangle\phi(\tilde{X})\rangle \sim \phi(X)^t\ B_{\bar{p}}'\ \phi(\tilde{X}) = \ [\text{by } (3.40)] =$$

$$= (\Phi X)^t\ B_{\bar{p}}'(\Phi\tilde{X}) \sim X^t\Phi^* B_{\bar{p}}'\Phi\tilde{X}\ . \tag{3.49b}$$

Now, if ϕ is a Lie algebra homomorphism, i.e. $\phi(X\triangle\tilde{X}) = \phi(X)\triangle\phi(\tilde{X})$, then (3.49a) ~ (3.49b), therefore $B_{\bar{\phi}} = \Phi^*\ B_{\bar{p}}'\Phi$, since K is relations-free; hence $B_{\bar{\phi}} = D(\bar{\phi})\ B_{\bar{p}}'D(\bar{\phi})^*$ by lemma 3.47. Conversely, if ϕ is canonical, then $\langle p,\phi(X\triangle\tilde{X}) - \phi(X)\triangle\phi(\tilde{X})\rangle \sim 0$, which implies $\phi(X\triangle\tilde{X}) = \phi(X)\triangle\phi(\tilde{X})$ by theorem VII 1.14a; that is, ϕ is a Lie algebra homomorphism.

4. Canonical Quadratic Maps Associated with Representations of Lie Algebras (Generalized Clebsch Representations)

Let G be a finite-dimensional Lie group with the Lie algebra $\mathfrak{g}$. Then the cotangent bundle $T^*(G)$ of G is a symplectic manifold and taking the left invariant part of the Hamiltonian formalism on $T^*(G)$ results in the Hamiltonian structure in the ring $C^\infty(\mathfrak{g}^*)$ which we discussed in section 1. In general, to trace a symplectic origin of a given Hamiltonian structure, is important aesthetically, conceptually, and technically. In this section we discuss this problem for the Hamiltonian structures associated with Lie algebras.

To begin with, it is clear that the classical mechanical route $C^\infty(T^*(G)) \to C^\infty(\mathfrak{g}^*)$ mentioned above is of no use since we have no infinite-dimensional groups (and we don't want to have them). We thus have to look for other ways.

Let us begin with the elementary finite-dimensional situation first. Let $\mathfrak{g}$ be a Lie algebra over k, let V be a vector space over k, and let

$$\rho: \mathfrak{g} \to \text{End } V \tag{4.1}$$

be a representation of $\mathfrak{g}$. Denote by $\mathfrak{g}\ominus V$ the semidirect product of $\mathfrak{g}$ and V; it is a Lie algebra with the multiplication

$$[(\ell_1 \otimes v_1),(\ell_2 \otimes v_2)] = [\ell_1,\ell_2] \otimes (\rho(\ell_1)v_2-\rho(\ell_2)v_1), \ell_i \in \mathcal{G}, v_i \in V . \tag{4.2}$$

Consider the map

$$R: V \oplus V^* \to (\mathcal{G} \otimes V)^* \tag{4.3}$$

given by the formula

$$[R(\alpha \oplus \alpha^*)](\ell \otimes v) = \langle \alpha^*, v-\rho(\ell)\alpha \rangle, \alpha, v \in V, \alpha^* \in V^*, \ell \in \mathcal{G}. \tag{4.4}$$

<u>Theorem 4.5</u>. The map R^*: $S(\mathcal{G} \otimes V) \to S(V \oplus V^*)$ is <u>canonical</u>.

<u>Proof</u>. Recall that both the rings of functions: $S(V \oplus V^*)$ on $V \oplus V^*$ and $S(\mathcal{G} \otimes V)$ on $(\mathcal{G} \otimes V)^*$ possess natural Poisson brackets: $V \oplus V^* \cong T^*(V)$ which is a symplectic space, and $(\mathcal{G} \otimes V)^*$ has the bracket (1.1). We have to check that

$$R(\{f,g\}_{(\mathcal{G} \otimes V)^*}) = \{R^*f, R^*g\}_{V \oplus V^*} , \quad f,g \in S(\mathcal{G} \otimes V) . \tag{4.6}$$

Since we are dealing with the finite-dimensional case, the Poisson brackets are derivations with respect to each entry. Thus it is enough to check (4.6) for elements $\ell \otimes v$ only. We have, for $f = \ell_1 \otimes v_1$, $g = \ell_2 \otimes v_2$:

$$\{(\ell_1 \otimes v_1),(\ell_2 \otimes v_2)\} = [(\ell_1 \otimes v_1),(\ell_2 \otimes v_2)] = [\ell_1,\ell_2] \otimes (\ell_1(v_2)-\ell_2(v_1)) , \tag{4.7}$$

where we suppress ρ from the notations.

<u>Remark 4.8</u>. The reader may have noticed that the Poisson bracket (4.7) has the opposite sign than the one we used in the infinite-dimensional case (cf. (3.27)). The difference is unimportant and is due to historical reasons.

Since, by (4.4),

$$[R^*(\ell \otimes v)](\alpha \oplus \alpha^*) = \langle \alpha^*, v-\ell(\alpha) \rangle , \tag{4.8}$$

we can compute the value of the left-hand side of (4.6) at the point $(\alpha \oplus \alpha^*) \in V \oplus V^*$:

$$\langle \alpha^*, \ell_1(v_2)-\ell_2(v_1)-[\ell_1,\ell_2](\alpha) \rangle . \tag{4.9}$$

Let us compute the right-hand side of (4.6).

Let A: V → V be any polynomial (or "smooth") map. We associate to it a vector field $\hat{A}\in\mathcal{D}(V)$ by the formula

$$(\hat{A}\phi)(w) = \frac{d}{dt}\Big|_{t=0} \phi[w+tA(w)], \forall \phi\in S(V^*), \forall w\in V . \tag{4.10}$$

Denote be $f_A \in S(V\oplus V^*)$ the following function:

$$f_A(\alpha\oplus\alpha^*) = \langle\alpha^*, A(\alpha)\rangle.$$

Lemma 4.12. For any maps A,B: V → V,

$$\{f_A, f_B\}_{V\oplus V^*} = f_{[A,B]} , \tag{4.13}$$

where

$$\widehat{[A,B]} = [\hat{A},\hat{B}] . \tag{4.14}$$

Proof. This is the standard fact from classical mechanics: if $X,Y\in\mathcal{D}(M)$ and $\rho\in\Lambda^1(T^*M)$ is the universal form, then $\{\rho(X),\rho(Y)\} = \rho([X,Y])$. In our situation, we have $M = V$, $X = \hat{A}$, $Y = \hat{B}$, $\rho(X)(\alpha\oplus\alpha^*) = \langle\alpha^*,A(\alpha)\rangle$, etc. □

From (4.8) we have $R^*f = f_A$, $R^*g = f_B$, where

$$A = v_1 - \ell_1(\alpha) , B = v_2 - \ell_2(\alpha) \tag{4.15}$$

To compute [A,B], we need $[\hat{A},\hat{B}]$. For this we have

$$[\hat{A}(\hat{B}\phi)](w) = \frac{d}{dt}\Big|_{t=0} (\hat{B}\phi) [w+tA(w)] =$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi[w+tA(w)+\varepsilon B(w+tA(w))] =$$

$$= \frac{d}{dt}\frac{d}{d\varepsilon}\Big|_{\varepsilon=t=0} \phi[w+tA(w)+\varepsilon B(w)+\varepsilon t(\hat{A}(B))(w)+O(\varepsilon t^2)] =$$

$$= (\widehat{\hat{A}(B)\phi})(w) \;, \tag{4.16}$$

where

$$(\hat{A}(B))(w) = \frac{d}{dt}\Big|_{t=0} B(w+tA(w)) \;. \tag{4.17}$$

Thus

$$[\hat{A},\hat{B}] = \widehat{\hat{A}(B) - \hat{B}(A)} \;,$$

and therefore

$$[A,B] = \hat{A}(B) - \hat{B}(A) \;. \tag{4.18}$$

For A and B given by (4.15) we obtain

$$(\hat{A}(B))(\alpha) = \frac{d}{dt}\Big|_{t=0} B(\alpha+tA(\alpha)) =$$

$$= \frac{d}{dt}\Big|_{t=0} [v_2-\ell_2(\alpha+t(v_1-\ell_1(\alpha)))] = -\ell_2(v_1-\ell_1(\alpha)) =$$

$$= -\ell_2(v_1) + \ell_2\ell_1(\alpha) \;,$$

therefore

$$[A,B](\alpha) = -\ell_2(v_1) + \ell_2\ell_1(\alpha) - [-\ell_1(v_2)+\ell_1\ell_2(\alpha)] =$$

$$= \ell_1(v_2) - \ell_2(v_1) - [\ell_1,\ell_2](\alpha) \;. \tag{4.19}$$

Substituting (4.19) into (4.13), we obtain for the right-hand side of (4.6),

$$\{R^*f,R^*g\}_{V\oplus V^*} = \{f_A,f_B\}_{V\oplus V^*} = f_{[A,B]} =$$

$$= \langle\alpha^*,[A,B](\alpha)\rangle = \langle\alpha^*,\ell_1(v_2)-\ell_2(v_1) - [\ell_1,\ell_2](\alpha)\rangle \;,$$

which is exactly the left-hand side of (4.6) given by (4.9). □

<u>Corollary 4.20</u>. The map

$$r\colon V\otimes V^* \to \mathfrak{g}^*, \quad [r(\alpha\otimes\alpha^*)](\ell) = \langle\alpha^*, -\ell(\alpha)\rangle , \tag{4.21}$$

is canonical.

<u>Proof</u>. Let $\psi\colon \mathfrak{g} \to \mathfrak{g} \oplus V$ be the Lie algebra homomorphism defined by $\psi(\ell) = \ell \oplus 0$. Then the dual map $\psi^*\colon (\mathfrak{g}\oplus V)^* \to \mathfrak{g}^*$ is canonical by the finite-dimensional degeneration of theorem 3.48. Since the map $R\colon V\otimes V^* \to (\mathfrak{g}\oplus V)^*$ is canonical by theorem 4.5, the composition ψ^*R is canonical too. Let us show that $\psi^*R = r$. We have

$$[(\psi^*R)(\alpha\otimes\alpha^*)](\ell) = [R(\alpha\otimes\alpha^*)](\psi(\ell)) = [R(\alpha\otimes\alpha^*)](\ell\oplus 0) =$$

$$= \langle a^*, 0-\ell(\alpha)\rangle = [r(\alpha\otimes\alpha^*)](\ell) . \qquad \square$$

<u>Remark 4.22</u>. Taking $V = \mathfrak{g}$ and $\rho = \mathrm{ad}$ in the corollary 4.20, we obtain a symplectic representation for the usual Poisson structure on the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$.

<u>Remark 4.23</u>. Let $\psi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism, and let $\rho_i\colon \mathfrak{g}_i \to \mathrm{End}\, V$ be two representations compatible with ψ; that is, $\rho_1 = \rho_2\psi$. Denote by $R(\mathfrak{g}_i)$ the map $R\colon V\otimes V^* \to (\mathfrak{g}_i\oplus V)^*$ in (4.3), and let $\psi^*\colon (\mathfrak{g}_2\oplus V)^* \to (\mathfrak{g}_1\oplus V)^*$ be the dual map to the Lie algebra homomorphism $\psi\oplus 1\colon \mathfrak{g}_1\oplus V \to \mathfrak{g}_2\oplus V$. Then

$$R(\mathfrak{g}_1) = \psi^* R(\mathfrak{g}_2) . \tag{4.24}$$

In other words, <u>the map R is natural</u>.

<u>Proof of (4.24)</u>. For any $(\alpha\otimes\alpha^*)\in V\otimes V^*$, $(\ell\oplus v)\in \mathfrak{g}_1\oplus V$, we have

$$([\psi^*R(\mathfrak{g}_2)](\alpha\otimes\alpha^*))(\ell\oplus v) = ([R(\mathfrak{g}_2)](\alpha\otimes\alpha^*))(\psi(\ell\oplus v)) =$$

$$=([R(\mathfrak{g}_2)](\alpha\otimes\alpha^*))(\psi(\ell)\oplus v) = \langle\alpha^*, v-\rho_2(\psi(\ell))(\alpha)\rangle =$$

$$= \langle\alpha^*, v-\rho_1(\ell)(\alpha)\rangle = ([R(\mathfrak{g}_1)](\alpha\otimes\alpha^*))(\ell\oplus v) . \qquad \square$$

We now turn to the general case. Let $L = K^N$ be a Lie algebra of the type considered in section 3, and let

$$\rho: L \to Mat_M(K)[\Delta^{\pm 1},\partial]$$

be a representation of L such that for any $X \in L$, the matrix elements of $\rho(X)$ are given by the formula

$$\rho(X)_{ij} = \rho_{ij}^{k,\sigma,\nu}(X_K)\Delta^{\sigma}\partial^{\nu} , \tag{4.25}$$

where

$$\rho_{ij}^{k,\sigma,\nu} \in K[\Delta^{\pm 1},\partial] . \tag{4.25´}$$

We make $\widetilde{L}: = K^{N+M} \cong K^N \oplus K^M$ into a semidirect product Lie algebra letting

$$(X;u)\ast(Y;v) = (X\ast Y;\rho(X)v-\rho(Y)u), \forall X,Y \in K^N, \forall u,v \in K^M , \tag{4.26}$$

which is an analog of (4.2). Let $q = (q_1,\dots,q_N)$, $c = (c_1,\dots,c_M)$ be free variables which generate the ring $C_1 = K[q_j^{(\sigma_j|\nu_j)}, c_i^{(\sigma_i|\nu_i)}]$ which is an analog of "functions on $\widetilde{L}^*$" which we had in section 3. We denote

$$<(q;c) , (X;u)> = q_jX_j + c_iu_i , \tag{4.27}$$

as in (3.2). Now we need an analog of "functions on $V\oplus V^*$". Let $C_2 = K[a_i^{(\sigma_i|\nu_i)}, b_i^{(\sigma_i'|\nu_i')}]$ be a differential-difference ring generated by letters a_i, b_i, $i = 1,\dots,M$. We make C_2 into a Hamiltonian ring by imposing on it the Hamiltonian matrix

$$B_2 = \begin{vmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{vmatrix} . \tag{4.28}$$

In other words, for any $H \in C_2$, the evolutionary derivation $\hat{X}_H$ acts as follows:

$$\hat{X}_H(a_s) = -\frac{\delta H}{\delta b_s} , \quad \hat{X}_H(b_s) = \frac{\delta H}{\delta a_s} . \tag{4.29}$$

Now we can construct an analog of the map R* from (4.3).

Let us introduce multiplication ∇ on K^M with values in K^N by the formula

$$(u \nabla v)_k = (\rho_{ij}^{k,\sigma,\nu})^* \, (v_i \Delta^\sigma \partial^\nu u_j) , \tag{4.30}$$

where operators $\rho_{ij}^{k,\sigma,\nu}$ are taken from (4.26). This multiplication comes from the following property:

Proposition 4.31.

$$v^t \rho(X) u \sim X^t (u \nabla v) , \ \forall X \in L , \ \forall u, v \in K^M . \tag{4.32}$$

Proof. We have from (4.25),

$$v^t \rho(X) u = v_i \rho(X)_{ij} u_j = v_i \rho_{ij}^{k,\sigma,\nu} (X_K) \Delta^\sigma \partial^\nu u_j \sim$$

$$\sim X_K (\rho_{ij}^{k,\sigma,\nu})^* (v_i \Delta^\sigma \partial^\nu u_j) = X_k (u \nabla v)_k . \qquad \square$$

Theorem 4.33. Let ϕ: $C_1 \to C_2$ be the homomorphism of differential-difference rings given on generators by the formulae

$$\phi(q) = -(a \nabla b) , \ \phi(c) = b , \tag{4.34a}$$

that is,

$$\phi(q_k) = -(a \nabla b)_k , \ \phi(c_k) = b_k . \tag{4.34b}$$

Then ϕ is a canonical map.

Remark 4.35. This is the desired generalization of theorem 4.5, since

$$\langle\phi(q;c)\ ,\ (X;u)\rangle = [\text{by } (4.27) \text{ and } (4.34a)] =$$

$$= \langle -a\nabla b, X\rangle + \langle b,u\rangle \sim [\text{by } (4.32)] \sim \langle b,-\rho(X)a\rangle + \langle b,u\rangle =$$

$$= \langle b,u-\rho(X)a\rangle\ ,$$

which is an analog of (4.8′).

Proof of the theorem. We have to check out the equality (3.39) with $B_1 = B_{q;c}$ being the matrix associated with the Lie algebra $\tilde{L}$, B_2 being given by (4.28) and ϕ provided by (4.34). To do this, we take arbitrary elements (Y;v) and (X;u) from $\tilde{L} \cong K^{N+M}$, apply each side in (3.39) to (Y;v), then multiply the result from the left by $(X;u)^t$ and show that the resulting expressions are equal modulo Im$\mathfrak{D}$. Since K is relations-free and (X;u) and (Y;v) are arbitrary, this equality modulo Im$\mathfrak{D}$ implies equality (3.39).

Now for the details. We begin with the left-hand side of (3.39). We have,

$$(X;u)^t\ \phi(B_{q;c})(Y;v) = (X;u)^t B_{\phi(q;c)}(Y;v) \sim [\text{by the definition (3.4) of B}] \sim$$

$$\sim \langle\phi(q;c)\ ,\ (X;u)\vartriangle(Y;v)\rangle = [\text{by } (4.26)] =$$

$$= \langle\phi(q;c)\ ,\ (X\vartriangle Y;\rho(X)v-\rho(Y)u)\rangle = [\text{by } (4.27)] =$$

$$= \langle\phi(q),X\vartriangle Y\rangle + \langle\phi(c),\rho(X)v-\rho(Y)u\rangle = [\text{by } (4.34)] =$$

$$= \langle -a\nabla b,X\vartriangle Y\rangle + \langle b,\rho(X)v-\rho(Y)u\rangle \sim [\text{by } (4.32)] \sim$$

$$\sim \langle b,-\rho(X\vartriangle Y)a\rangle + \langle b,\rho(X)v-\rho(Y)u\rangle =$$

$$= \langle b,\rho(Y)\rho(X)a-\rho(X)\rho(Y)a+\rho(X)v-\rho(Y)u\rangle =$$

$$= \langle b,\rho(X)(v-\rho(Y)a)\rangle + \langle b,\rho(Y)(\rho(X)a-u)\rangle \sim [\text{by } (4.32)] \sim$$

$$\sim X^t[(v-\rho(Y)a)\nabla b] + \langle\rho(Y)^*b,\rho(X)a-u\rangle \sim [\text{by } (4.32)] \sim$$

$$\sim X^t[(v-\rho(Y)a)\nabla b] + X^t(a\nabla\rho(Y)^*b)-u^t\rho(Y)^*b =$$

$$= (X;u)^t \begin{pmatrix} (v-\rho(Y)a)\nabla b + a\nabla\rho(Y)^*b \\ -\rho(Y)^*b \end{pmatrix} .$$

Therefore,

$$[\phi(B_{q;c})] \begin{pmatrix} Y \\ v \end{pmatrix} = \begin{pmatrix} (v-\rho(Y)a)\nabla b+a\nabla\rho(Y)^*b \\ \\ -\rho(Y)^*b \end{pmatrix}. \quad \begin{matrix} (4.36a) \\ \\ (4.36b) \end{matrix}$$

Now let us turn to the right-hand side of (3.39). Denote $\phi_k = \phi(q_k)$. Then the Fréchet derivative $D(\bar{\phi})$ is the following matrix

$$\begin{array}{c|cc|} & a_s & b_s \\ \phi(q_k) & \frac{D\phi_k}{Da_s} & \frac{D\phi_k}{Db_s} \\ \phi(c_k) & 0 & \delta_k^s \end{array} ,$$

therefore the matrix $D(\bar{\phi})B_2D(\bar{\phi})^*$ is equal to

$$\begin{array}{c|cc|} & \phi(q_s) & \phi(c_s) \\ \phi(q_k) & \frac{D\phi_k}{Db_n}\left(\frac{D\phi_s}{Da_n}\right)^* - \frac{D\phi_k}{Da_n}\left(\frac{D\phi_s}{Db_n}\right)^* & -\frac{D\phi_k}{Da_s} \\ \phi(c_k) & \left(\frac{D\phi_s}{Da_k}\right)^* & 0 \end{array} \quad (4.37)$$

From this we obtain

$$D(\bar{\phi})B_2D(\bar{\phi})^* \begin{pmatrix} Y \\ v \end{pmatrix} =$$

$$= \begin{pmatrix} \text{component \# } k\text{: } \left[\frac{D\phi_k}{Db_n}\left(\frac{D\phi_s}{Da_n}\right)^* - \frac{D\phi_k}{Da_n}\left(\frac{D\phi_s}{Db_n}\right)^*\right]Y_s - \frac{D\phi_k}{Da_s} v_s \\ \\ \text{component \# } k\text{: } \left(\frac{D\phi_s}{Da_k}\right)^* Y_s \end{pmatrix} \quad \begin{matrix} (4.38a) \\ \\ (4.38b) \end{matrix}$$

We need formulae for $\frac{D\phi_k}{Db_n}$, $\frac{D\phi_k}{Da_n}$. From (4.30) we have, using (4.25′):

$$\frac{D\phi_k}{Db_n} = \frac{D[-(a\nabla b)_k)}{Db_n} = -(\rho_{nj}^{k,\sigma,\nu})^* (\Delta^\sigma \partial^\nu (a_j)) = -(\rho_{nj}^{k,\sigma,\nu})^* a_j^{(\sigma|\nu)} , \tag{4.39b}$$

$$\frac{D\phi_k}{Da_n} = -(\rho_{in}^{k,\sigma,\nu})^* b_i \Delta^\sigma \partial^\nu . \tag{4.39b}$$

Now we can compare (4.36b) and (4.38b). We have, for the component #k in (4.36b):

$$(-\rho(Y)^*b)_k = -[\rho(Y)^*]_{ki} b_i = -[\rho(Y)_{ik}]^* b_i = [\text{by } (4.25)] =$$

$$= -[\rho_{ik}^{s,\sigma,\nu}(Y_s)\Delta^\sigma \partial^\nu]^* b_i = -\Delta^{-\sigma}(-\partial)^\nu \rho_{ik}^{s,\sigma,\nu}(Y_s) b_i . \tag{4.40}$$

On the other hand, substituting (4.39b) into (4.38b) we find that

$$\left(\frac{D\phi_s}{Da_k}\right)^* Y_s = [-(\rho_{ik}^{s,\sigma,\nu})^* b_i \Delta^\sigma \partial^\nu]^* Y_s =$$

$$= -\Delta^{-\sigma}(-\partial)^\nu b_i \rho_{ik}^{ss,\sigma,\nu}(Y_s) ,$$

as in (4.40).

Thus the lower halves in the matrix $\phi(B_{q;c})$ and in the matrix $D(\bar{\phi})B_2 D(\bar{\phi})^*$ are the same. Since both of the matrices are skew-symmetric, it remains only to check that they have the same upper-left corner. Using (4.36a) and (4.38a), this amounts to the identity

$$[-\rho(Y)a\nabla b + a\nabla\rho(Y)^* b]_k =$$

$$= \left[\frac{D\phi_k}{Db_n}\left(\frac{D\phi_s}{Da_n}\right)^* - \frac{D\phi_k}{Da_n}\left(\frac{D\phi_s}{Db_n}\right)^*\right] Y_s . \tag{4.41}$$

This identity, in turn, follows from the following two formulae:

$$\frac{D\phi_k}{Db_n}\left(\frac{D\phi_s}{Da_n}\right)^* Y_s = [a\nabla\rho(Y)^*b]_k , \tag{4.42}$$

$$\frac{D\phi_k}{Da_n}\left(\frac{D\phi_s}{Db_n}\right)^* Y_s = [\rho(Y)a\nabla b]_k . \tag{4.43}$$

We begin with (4.42). Analyzing (4.36b) and (4.38b), we have proved above that

$$\left(\frac{D\phi_s}{Da_n}\right)^* Y_s = -[\rho(Y)^*b]_n . \tag{4.44}$$

On the other hand, for any $f \in K^M$, we find from (4.39a) that

$$\frac{D\phi_k}{Db_n} f_n = -(\rho_{nj}^{k,\sigma,\nu})^* a_j^{(\sigma|\nu)} f_n = -(\rho_{nj}^{k,\sigma,\nu})^* f_n \Delta^\sigma \partial^\nu a_j =$$

$$= [\text{by } (4.30)] = -(a\nabla f)_k . \tag{4.45}$$

Combining (4.44) and (4.45) for $f = -\rho(Y)^*b$, we obtain (4.42). It remains to prove (4.43), which can be deduced from (4.42). Let G be the matrix operator with the matrix elements

$$G_{ks} = \frac{D\phi_k}{Da_n}\left(\frac{D\phi_s}{Db_n}\right)^* .$$

We can transform (4.42) as follows

$$X_k G_{ks} Y_s = X^t[a\nabla\rho(Y)^*b] \sim [\text{by } (4.32)] \sim$$

$$\sim [\rho(Y)^*b]^t \rho(X)a \sim b^t\rho(Y)\rho(X)a . \tag{4.42'}$$

Therefore, for (4.43) we find that

$$X_k(G_{sk})^* Y_s \sim Y_s G_{sk} X_k \sim [\text{by } (4.42')] \sim b^t\rho(X)\rho(Y)a \sim$$

$$\sim [\text{by } (4.32)] \sim X^t(\rho(Y)a\nabla b) ,$$

which is equivalent to (4.43). Q.E.D.

Remark 4.46. The theorem provides us with the symplectic representation (4.28), (4.34) for the Hamiltonian structure associated with arbitrary semi-direct product (4.26). It also provides us with a symplectic representation for the Hamiltonian structure of Lie algebras themselves as in corollary 4.20. Such representations are important in physical theories connected with compressible hydrodynamics, where they are called Clebsch representations in honor of Euler who was the first to use them.

5. Affine Hamiltonian Operators and Generalized 2-cocycles

In this section we develop a simple machinery which reduces the Hamiltonian analysis of affine operators - such as III (3.12) - to the problem of whether a given skew-symmetric bilinear form on a Lie algebra represents a generalized 2-cocycle.

We begin with a simple case which often occurs in practice (see, e.g.,[10]). Suppose $B = B_{\bar{q}}$ is a Hamiltonian matrix which depends linearly upon variables q_j, $j \in J$, and suppose the variables q_j are divided into three different groups $j \in J_1$, $j \in J_2$, $j \in J_3$ such that:

$$B^{J_2,J} \text{ depends only upon } q_j,\ j \in J_3 \ , \tag{5.1a}$$

$$B^{J_3,J} \text{ depends only upon } q_j,\ j \in J_3 \ . \tag{5.1b}$$

This implies that each "submanifold" $S(\beta)$:

$$S(\beta) := \{q_j = 0,\ j \in J_3;\ q_j = \beta_j \in K,\ j \in J_2\} \ , \tag{5.2}$$

is invariant with respect to the Hamiltonian "flow"

$\bar{q}_t = B \dfrac{\delta H}{\delta \bar{q}}$, for every $H \in C_1 = K[q_j^{(\sigma_j | \nu_j)}]_{j \in J}$. It is tempting then, to consider only the remaining variables $u_j := q_j$, $j \in J_1$, with the new matrix

$$B^1 = B^1_{\bar{u}} = B^1_{\bar{u}}(\beta) = B^{J_1,J_1}\Big|_{q_j = 0,\ j\in J_3;\ q_j = \beta_j,\ j\in J_2} . \tag{5.3}$$

It is by no means obvious that the new matrix B^1 is Hamiltonian, since the operation of specialization of a part of variables, like (5.2), destroys (= does not commute with the reasoning of) the calculus.

Theorem 5.4. The matrix $B^1_{\bar{u}}$ defines a Hamiltonian structure in the ring $C_2 = K[u_j^{(\sigma_j|\nu_j)}]_{j\in J_1}$.

Proof. Let $L = K^{N_1} \oplus K^{N_2} \oplus K^{N_3}$ be the Lie algebra which corresponds to the matrix $B_{\bar{q}}$ by corollary 3.13, where $N_i = |J_i|$, $i = 1,2,3$. Then conditions (5.1) mean, by (3.5), that

$$K^{N_2} \vartriangle L \subset K^{N_3} , \tag{5.5a}$$

$$K^{N_3} \vartriangle L \subset K^{N_3} . \tag{5.5b}$$

Thus K^{N_3} is an ideal in L. Let $L_1 = K^{N_1} \oplus K^{N_2}$ be the factoralgebra L/K^{N_3} and let B^2 be the Hamiltonian matrix which corresponds to L_1. Then obviously

$$B^2 = B^{\tilde{J},\tilde{J}}\Big|_{q_j = 0,\ j\in J_3} ,\ \tilde{J} := J_1 \cup J_2 . \tag{5.6}$$

Therefore $B^1 = B^2\Big|_{q_j = \beta_j,\ j\in J_2}$, and we can consider the case when K^{N_3} is absent. Then (5.5a) becomes

$$K^{N_2} \vartriangle L = \{0\} , \tag{5.7}$$

which means that K^{N_2} belongs to the center of a central extension of the Lie algebra $(K^{N_1}\oplus\{0\})\wedge(K^{N_1}\oplus\{0\}) \xrightarrow{\text{Proj}} K^{N_1}\oplus\{0\}$. This is equivalent to having a set of N_2 2-cocycles on $K^{N_1}\oplus\{0\}$ but we will not pursue this analogy here since the notion of a 2-cocycle must be generalized, as we shall see below. Instead let us write down the formulae for $\bar{X}_H$ for any $H\in C_3 = K[q_j^{(\sigma_j|\nu_j)}]_{j\in\tilde{J}}$:

$$\hat{X}_H(\bar{u}) = (b_{\bar{u}}+\tilde{b}_{\bar{v}})\frac{\delta H}{\delta \bar{u}}\ , \tag{5.8a}$$

$$\bar{u} = \{q_j\}_{j\in J_1}\ ,\ \bar{v} = \{q_j\}_{j\in J_2}\ ,$$

$$\hat{X}_H(\bar{v}) = 0\ , \tag{5.8b}$$

where we explicitly separated $\bar{u}$- and $\bar{v}$-dependence in the matrix B^2 using (5.7) and (3.5). We want to show that if we substitute $v_j = \beta_j$, $j\in J_2$ into $\tilde{b}_{\bar{v}}$, then the resulting map Γ: $H \to \hat{X}_H$, $H\in C_2$, given by the equations

$$\hat{X}_H(\bar{u}) = (b_{\bar{u}}+\tilde{b}_{\bar{\beta}})\frac{\delta H}{\delta \bar{u}}\ , \tag{5.9}$$

is Hamiltonian.

Let us take $H,F\in C_2$ and consider them as belonging to C_3. We know that equations (5.9) do define a Hamiltonian system; that is

$$[\hat{X}_H,\hat{X}_F] = \hat{X}_{\{H,F\}}\ . \tag{5.10}$$

Let us apply both sides of (5.10) to the vector $\bar{u}$. For the left-hand side we obtain

$$[\hat{X}_H,\hat{X}_F](\bar{u}) = \hat{X}_H(\hat{X}_F(\bar{u})) - \hat{X}_F(\hat{X}_H(\bar{u})) = [\text{by (5.8a)}] =$$

$$= \hat{X}_H[(b_{\bar{u}}+\tilde{b}_{\bar{v}})\frac{\delta F}{\delta \bar{u}}] - \hat{X}_F[(b_{\bar{u}}+\tilde{b}_{\bar{v}})\frac{\delta H}{\delta \bar{u}}] = [\text{by (5.8b)}] =$$

$$= [\hat{X}_H(b_{\bar{u}})]\frac{\delta F}{\delta \bar{u}} + (b_{\bar{u}}+\tilde{b}_{\bar{v}})\hat{X}_H(\frac{\delta F}{\delta \bar{u}}) - (F \leftrightarrow H) \ , \tag{5.11a}$$

where "$-(F \leftrightarrow H)$" means: "minus the same expression with F and H interchanged."

For the right-hand side of (5.10) we have

$$\{H,F\} = \hat{X}_H(F) \sim [\text{by } (5.8b)] \sim \frac{\delta F}{\delta \bar{u}^t}(b_{\bar{u}}+\tilde{b}_{\bar{v}})\frac{\delta H}{\delta \bar{u}} \ ,$$

and so

$$\hat{X}_{\{H,F\}}(\bar{u}) = (b_{\bar{u}}+\tilde{b}_{\bar{v}})\frac{\delta\{H,F\}}{\delta \bar{u}} = (b_{\bar{u}}+\tilde{b}_{\bar{v}})\frac{\delta}{\delta \bar{u}}[\frac{\delta F}{\delta \bar{u}^t}(b_{\bar{u}}+\tilde{b}_{\bar{v}})\frac{\delta H}{\delta \bar{u}}] \ . \tag{5.11b}$$

Notice now that there are <u>no functional derivatives with respect to $\bar{v}$</u> present in either (5.11a) or (5.11b). It is thus an identity with respect to variables $\bar{v}$. Substituting $v_j = \beta_j$ we still will have an identity, which this time means that the Hamiltonian property (5.10) is satisfied for the system (5.9). Q.E.D.

<u>Remark 5.12</u>. The same reasoning as above shows also that if we have a Hamiltonian structure of the form

$$\hat{X}_H(\bar{u}) = B_{\bar{u},\bar{v}}\frac{\delta H}{\delta \bar{u}} \ ,$$
$$\hat{X}_H(\bar{v}) = 0 \ , \tag{5.13}$$

where $B_{\bar{u},\bar{v}}$ depends arbitrarily upon $\bar{u},\bar{v}$ (not necessarily linearly), then its reduction

$$\hat{X}_H(\bar{u}) = B_{\bar{u},\bar{\beta}}\frac{\delta H}{\delta \bar{u}} \tag{5.14}$$

is again a Hamiltonian structure in $\bar{u}$-variables, for any choice of $\beta_j \in K$ for which the matrix $B_{\bar{u},\bar{\beta}}$ exists.

Suppose now that we are given two Hamiltonian matrices $B = B_{\bar{q}}$ and $b \in \text{Mat}_N(K)[\Delta^{\pm 1},\partial]$, where B is linear in $\bar{q}$. We want to know when $B^1 = B+b$ is

Hamiltonian too. If we could find a central extension of the Lie algebra L which correponds to the matrix B, such that $b = \tilde{b}_{\bar{\beta}}$, we could apply theorem 5.4, but there is no reason why we could succeed in doing it. Instead, let us analyze the problem directly.

Using (2.24) and (3.8) for a pair $H,F \in C = K[q_j^{(\sigma_j|\nu_j)}]$, we get

$$\frac{\delta}{\delta\bar{q}}\left(\frac{\delta F}{\delta\bar{q}}^t b\frac{\delta H}{\delta\bar{q}}\right) = D\left(\frac{\delta F}{\delta\bar{q}}\right) b\frac{\delta H}{\delta\bar{q}} - D\left(\frac{\delta H}{\delta\bar{q}}\right) b\frac{\delta F}{\delta\bar{q}} , \tag{5.15}$$

$$\frac{\delta}{\delta\bar{q}}\left(\frac{\delta F}{\delta\bar{q}}^t B\frac{\delta H}{\delta\bar{q}}\right) = D\left(\frac{\delta F}{\delta\bar{q}}\right) B\frac{\delta H}{\delta\bar{q}} - D\left(\frac{\delta H}{\delta\bar{q}}\right) B\frac{\delta F}{\delta\bar{q}} + \frac{\delta F}{\delta\bar{q}} \blacktriangle \frac{\delta H}{\delta\bar{q}} , \tag{5.16}$$

where $\blacktriangle$ denotes the multiplication in the Lie algebra L which corresponds to the matrix $B = B_{\bar{q}}$. Adding (5.15) and (5.16) we obtain

$$\frac{\delta}{\delta\bar{q}}\left(\frac{\delta F}{\delta\bar{q}}^t B^1 \frac{\delta H}{\delta\bar{q}}\right) = D\left(\frac{\delta F}{\delta\bar{q}}\right) B^1 \frac{\delta H}{\delta\bar{q}} - D\left(\frac{\delta H}{\delta\bar{q}}\right) B^1 \frac{\delta F}{\delta\bar{q}} + \frac{\delta F}{\delta\bar{q}} \blacktriangle \frac{\delta H}{\delta\bar{q}} . \tag{5.17}$$

Let us define a bilinear form ω on L by setting

$$\omega(X,Y) = X^t bY . \tag{5.18}$$

<u>Definition 5.19</u>. A bilinear (over k) form θ on L is called a generalized 2-cocycle if

$$\theta(X,Y) \sim -\theta(Y,X), \forall X,Y \in L , \tag{5.20}$$

$$[\theta(X_1,X_2 X_3) + \text{c.p.}] \sim 0, \forall X_1,X_2,X_3 \in L . \tag{5.21}$$

We shall always assume that all bilinear forms we deal with are differential-difference operators over K with respect to each variable. This allows us to identify skew-symmetric 2-forms satisfying (5.20) with skew- symmetric operators (or matrices) by the formula

$$\theta(X,Y) \sim X^t\tilde{\theta}Y \tag{5.22}$$

with some $\tilde{\theta} \in \mathrm{Mat}_N(K)[\Delta^{\pm 1},\partial]$.

Theorem 5.23. For any F_1, F_2, $F_3 \in C$, $(\{F_1,\{F_2,F_3\}\} + \text{c.p.}) \sim 0$ iff ω is a generalized 2-cocycle on L.

Proof. Denote $X = \frac{\delta F_1}{\delta \bar{q}}$, $Y = \frac{\delta F_2}{\delta \bar{q}}$, $Z = \frac{\delta F_3}{\delta \bar{q}}$. Comparing the proof of theorem 3.11 with the formulae (3.8) and (5.17), we immediately obtain

$$\{F_1,\{F_2,F_3\}\} + \text{c.p.} \sim X^tB^1(Y\triangle Z) + \text{c.p.} =$$

$$= X^tB(Y\triangle Z) + \text{c.p.} + X^tb(Y\triangle Z) + \text{c.p.} \sim$$

$$\sim \langle q, X\triangle(Y\triangle Z) + \text{c.p.}\rangle + \omega(X,Y\triangle Z) + \text{c.p.} \sim$$

$$\sim \omega(X,Y\triangle Z) + \text{c.p.}$$

Thus ω is a generalized 2-cocycle iff $(\{F_1,\{F_2,F_3\}\} + \text{c.p.}) \sim 0$. □

Analogous to the derivation of theorem 3.16 from theorem 3.11, we find

Theorem 5.24. The matrix $B^1 = B + b$ is Hamiltonian iff ω is a generalized 2-cocycle on L.

Our next goal is to find a definition of the generalized 2-cocycle such that it is an equation, as opposite to the equality modulo $\mathrm{Im}\,\mathcal{D}$ in (5.21). There are two possible routes, both instructive.

First method. We transform (5.21). Let $\tilde{\theta}$ be a skew-symmetric operator from (5.22). For any elements $X,Y,Z \in L$, we have

$$\theta(Y,Z\triangle X) = Y^t\tilde{\theta}(Z\triangle X) \sim -(\tilde{\theta}Y)^t(Z\triangle X) = (\tilde{\theta}Y)^t(X\triangle Z) \sim$$

$$\sim X^t(Z\nabla\tilde{\theta}Y) \;, \tag{5.25}$$

where the new multiplication ∇ in L is taken from (4.32) for the adjoint representation $\rho = \mathrm{ad}$. Analogously,

$$\theta(Z,X\triangle Y) = Z^t\tilde{\theta}(X\triangle Y) \sim -(\tilde{\theta}Z)^t(X\triangle Y) \sim -X^t(Y\nabla\tilde{\theta}Z) \tag{5.26}$$

Using (5.25) and (5.26), we transform (5.21):

$$\theta(X,Y\vartriangle Z) + \theta(Y,Z\vartriangle X) + \theta(Z,X\vartriangle Y) \sim$$

$$\sim X^t[\tilde{\theta}(Y\vartriangle Z)+Z\nabla\tilde{\theta}Y-Y\nabla\tilde{\theta}Z] .$$

Thus θ is a generalized 2-cocycle iff

$$\tilde{\theta}(Y\vartriangle Z) = Y\nabla\tilde{\theta}Z - Z\nabla\tilde{\theta}Y , \tag{5.27}$$

which is the desired definition. Notice that

$$Y\nabla Z = B_Z Y . \tag{5.28}$$

Indeed, from (4.32) we have

$$X^t(Y\nabla Z) \sim Z^t(X\vartriangle Y) \sim X^t B_Z Y ,$$

and X is arbitrary. Using (5.28) we can rewrite (5.27) in equivalent form

$$\tilde{\theta}(Y\vartriangle Z) = B_{\tilde{\theta}Z} Y - B_{\tilde{\theta}Y} Z . \tag{5.29}$$

Remark. In a finite dimensional situation ($K=k$), (4.32) becomes $\langle v,[X,u]\rangle = \langle -ad_u^* v,X\rangle = \langle u\nabla v,X\rangle$.

Therefore

$$u\nabla v = -ad_u^* v , \tag{5.30}$$

and (5.27) turns into

$$\tilde{\theta}([Y,Z]) = ad_Z^*\tilde{\theta}Y - ad_Y^*\tilde{\theta}Z . \tag{5.31}$$

Second method. We analyze directly the equation of the Hamiltonian property given by lemma 2.20. Applying $B^1 = B + b$ to (5.17) and subtracting from the result b applied to (5.15) and B applied to (5.16), we obtain

$$B\frac{\delta}{\delta\bar{q}}\left(\frac{\delta F}{\delta\bar{q}}^{t} b\frac{\delta H}{\delta\bar{q}}\right) + b\frac{\delta}{\delta\bar{q}}\left(\frac{\delta F}{\delta\bar{q}}^{t} B\frac{\delta H}{\delta\bar{q}}\right) =$$

$$= BD\left(\frac{\delta F}{\delta\bar{q}}\right)b\frac{\delta H}{\delta\bar{q}} + bD\left(\frac{\delta F}{\delta\bar{q}}\right)B\frac{\delta H}{\delta\bar{q}} - (F\leftrightarrow H) + b\left(\frac{\delta F}{\delta\bar{q}} \vartriangle \frac{\delta H}{\delta\bar{q}}\right) . \tag{5.32}$$

On the other hand, since B and b are both Hamiltonian matrices, the Hamiltonian condition of lemma 2.20 results in

$$B\frac{\delta}{\delta\bar{q}}\left(\frac{\delta F}{\delta\bar{q}}^{t} b\frac{\delta H}{\delta\bar{q}}\right) + b\frac{\delta}{\delta\bar{q}}\left(\frac{\delta F}{\delta\bar{q}}^{t} B\frac{\delta H}{\delta\bar{q}}\right) =$$

$$= D\left(B\frac{\delta F}{\delta\bar{q}}\right)b\frac{\delta H}{\delta\bar{q}} + D\left(b\frac{\delta F}{\delta\bar{q}}\right)B\frac{\delta H}{\delta\bar{q}} - (F\leftrightarrow H) . \tag{5.33}$$

Subtracting (5.32) and (5.33) and using the formula $[D,b] = 0$ established in the course of the proof of theorem 2.29, we get

$$b(X\vartriangle Y) = [D,B](X)bY - [D,B](Y)bX , \tag{5.34}$$

where $X = \frac{\delta F}{\delta\bar{q}}$, $Y = \frac{\delta H}{\delta\bar{q}}$. Thus our operator $B^1 = B + b$ is Hamiltonian iff (5.34) is satisfied for any two vectors $X = \frac{\delta F}{\delta\bar{q}}$, $Y = \frac{\delta H}{\delta\bar{q}}$.

<u>Lemma 5.35</u>. For any $X,Y \in L$,

$$[D,B](X)Y = X\nabla Y . \tag{5.36}$$

<u>Proof</u>. $[D,B](X)Y = D(BX)Y - BD(X)Y =$
$= \hat{Y}(BX) - B\hat{Y}(X) = [\hat{Y}(B)](X) =$ [by lemma 3.10] $=$

$= B_Y X =$ [by (5.28)] $= X\nabla Y$. □

Using (5.36) we can rewrite (5.34) as

$$b(X\vartriangle Y) = X\nabla bY - Y\nabla bX , \tag{5.37}$$

and this time (5.37) must be an identity <u>in L</u>; that is, it must be true for all X,Y not necessarily vectors of functional derivatives. (Indeed, there are only differential-difference operators involved in (5.37), and we can take $F = \langle q,X\rangle$, $H = \langle q,Y\rangle$, for any $X,Y \in L$).

Equation (5.37) is the same as (5.27) if we remember that our generalized 2-cocycle ω defined in (5.18), involves $\tilde{\theta} = b$. This, incidentally, provides another proof of theorem 5.24.

We now apply theorem 5.24 to the matrix III (3.12) involved in the first Hamiltonian structure for the Lax operator $L = \zeta^{\beta}(1 + \sum_{j\geq 0} \zeta^{-j-1} q_j)$.

<u>Theorem 5.38</u>. The matrix III (3.12) is Hamiltonian.

<u>Proof</u>. Let L be the Lie algebra generated by the associative algebra $\{X = \sum_{i\geq 0} X_i \zeta^{-i-1}\}$. We have

$$(X\triangle Y)_o = 0 \ , \ (X\triangle Y)_{k+1} = \sum_{j+s=k} (X_j Y_s^{(-1-j)} - Y_s X_j^{(-1-s)}) \ . \tag{5.39}$$

For the matrix elements of the corresponding Hamiltonian matrix B, we have

$$X^t BY = X_j B_{js} Y_s \sim q_{j+s+1}(X_j Y_s^{(-1-j)} - Y_s X_j^{(-1-s)}) \sim$$

$$\sim X_j (q_{j+s+1}\Delta^{-1-j} - \Delta^{1+s} q_{j+s+1}) Y_s \ .$$

Therefore,

$$B_{js} = q_{j+s+1}\Delta^{-1-j} - \Delta^{1+s} q_{j+s+1} \ . \tag{5.40}$$

Now let us fix a natural number $\beta \geq 2$ and consider the following bilinear form on L:

$$\omega(X,Y) = \mathrm{Res}[X(1-\Delta^{\beta})Y\zeta^{\beta}] \ . \tag{5.41}$$

We have

$$\omega(X,Y) = \mathrm{Res}[X(1-\Delta^{\beta})Y\zeta^{\beta}] \sim \mathrm{Res}[(1-\Delta^{-\beta})X\cdot Y\zeta^{\beta}] \sim$$

$$\sim \mathrm{Res}[Y\zeta^{\beta}(1-\Delta^{-\beta})X] = \mathrm{Res}[Y(\Delta^{\beta}-1)X\zeta^{\beta}] = -\omega(Y,X) \;,$$

thus ω is skew-symmetric. Let us show that ω is in fact a generalized 2-cocycle. Let $X,Y,Z\in L$. Then

$$\omega(X\blacktriangle Y,Z) = \mathrm{Res}[(XY-YX)(1-\Delta^{\beta})Z\zeta^{\beta}] \sim$$

$$\sim \mathrm{Res}\{X[Y(1-\Delta^{\beta})Z - (1-\Delta^{\beta})Z\cdot\Delta^{\beta}Y]\zeta^{\beta}\} \;, \tag{5.42a}$$

$$\omega(Y\blacktriangle Z,X) \sim -\omega(X,Y\ Z) = -\mathrm{Res}\{X[(1-\Delta^{\beta})(YZ-ZY)]\zeta^{\beta}\} \;, \tag{5.42b}$$

$$\omega(Z\blacktriangle X,Y) = \mathrm{Res}[(ZX-XZ)(1-\Delta^{\beta})Y\zeta^{\beta}] \sim$$

$$\sim \mathrm{Res}\{X[(1-\Delta^{\beta})Y\cdot\Delta^{\beta}Z-Z(1-\Delta^{\beta})Y]\zeta^{\beta}\} \;. \tag{5.42c}$$

Adding expressions in (5.42) we find that

$$\omega(X\blacktriangle Y,Z) + \text{c.p.} \sim \mathrm{Res}\{X[\dots]\zeta^{\beta}\} \;,$$

where

$$[\dots] = Y(1-\Delta^{\beta})Z-(1-\Delta^{\beta})Z\cdot\Delta^{\beta}Y-(1-\Delta^{\beta})(YZ-ZY) +$$

$$+ (1-\Delta^{\beta})Y\cdot\Delta^{\beta}Z-Z(1-\Delta^{\beta})Y =$$

$$= YZ - Y\Delta^{\beta}Z-Z\Delta^{\beta}Y+\Delta^{\beta}ZY-YZ+ZY+\Delta^{\beta}(YZ-ZY) +$$

$$+ Y\Delta^{\beta}Z-\Delta^{\beta}YZ-ZY+Z\Delta^{\beta}Y = 0,$$

Thus ω is indeed a generalized 2-cocycle. Its corresponding matrix b from (5.18) can be computed as follows:

$$X^{t}bY = \mathrm{Res}[X(1-\Delta^{\beta})Y\zeta^{\beta}] = \mathrm{Res}\{X_j\zeta^{-1-j}(1-\Delta^{\beta})Y_s\zeta^{-1-s}\zeta^{\beta}\} =$$

$$= \sum_{j+s=\beta-2} X_j\,\Delta^{-1-j}(1-\Delta^{\beta})Y_s = \sum_{j=0}^{\beta-2} X_j(\Delta^{-\beta}-1)\Delta^{\beta-1-j}Y_{\beta-2-j} \;,$$

thus

$$b_{jk} = 0,\ k \neq \beta-2-j;\ b_{j,\beta-2-j} = (\Delta^{-\beta}-1)\Delta^{\beta-1-j}\ ,\ 0 \leq j \leq \beta - 2\ . \tag{5.43}$$

Hence for the Hamiltonian matrix $B^1 = B + b$, the evolution equations corresponding to a Hamiltonian H are

$$\dot{q}_j = (\Delta^{-\beta}-1)\Delta^{\beta-1-j} \frac{\delta H}{\delta q_{\beta-2-j}} + \tag{5.44a}$$

$$+ \sum_{s\geq 0} (q_{j+s+1}\Delta^{-1-j}-\Delta^{1+s}q_{j+s+1})\frac{\delta H}{\delta q_s}\ , \tag{5.44b}$$

where we agree to drop the term (5.44a) for $j > \beta - 2$.

Equations (5.44) are almost the same as equations III (3.12), when we restrict j to run between 0 and β-2. To get the form III (3.12) exactly we make a few remarks.

Define

$$I = \{X\in L | X_j = 0\ ,\ 0 \leq j \leq \beta - 2\}\ . \tag{5.45}$$

Then (5.39) shows that I is an ideal in L. In addition, $\omega(I,L) = 0$ as follows from (5.43). Therefore ω can be correctly restricted on the Lie algebra $L_1 = L/I$ to yield a new generalized 2-cocycle given by the same formula (5.43). The matrix B corresponding to the Lie algebra L_1 will be given also by (5.40) with the understanding that $0 \leq j,\ s \leq \beta - 2$ and $q_k = 0$ for $k > \beta - 2$. This way we arrive exactly at equations III (3.12), with an unessential minus sign and R's renamed by q's. □

Corollary 5.46. [The first Hamiltonian structure for the Lax operator $L = \zeta^{\beta}(1+ \sum_{j\geq 0} \zeta^{-j-1}q_j)$.] The system {III (3.8) plus III (3.12)} is Hamiltonian.

Proof. We proved by (3.28) that III (3.8) is Hamiltonian, and theorem 5.38 asserts that III (3.12) is Hamiltonian too. Thus we have two Hamiltonian structures in two different subspaces, with variables Q and R respectively. They both belong to the type described in lemma 2.22. Therefore the criterion (2.28) of lemma 2.27 is satisfied since both sides of (2.28) have block-diagonal form, with variables Q and R separated in their respective blocks. □

We conclude this section with a discussion of the natural properties of generalized 2-cocycles. We use the notation of the end of section 3, after formula (3.39).

Let $\phi: L \to \mathcal{G}$ be a homomorphism of Lie algebras and $\phi: C_1 \to C_2$ be the corresponding canonical map from theorem 3.48. Suppose we have two generalized 2-cocycles ω_1 and ω_2 on L and $\mathcal{G}$ respectively. Let b_1 and b_2 be associated skew-symmetric operators:

$$\omega_i(X,Y) \sim X^t b_i Y \ , \ i = 1,2 \ . \tag{5.47}$$

We want to know when the map ϕ is canonical between the operators $B_1 + b_1$ and $B_2 + b_2$.

<u>Theorem 5.48</u>. The map ϕ is canonical iff generalized 2-cocycles ω_1 and ω_2 are ϕ-compatible, that is,

$$\omega_1 \sim \phi^* \omega_2 \ . \tag{5.49}$$

<u>Proof</u>. By theorem 3.48, ϕ is canonical between B_1 and B_2. Therefore, by (3.39), ϕ is canonical between $B_1 + b_1$ and $B_2 + b_2$ iff ϕ is canonical between b_1 and b_2, which happens, in view of (3.39), when

$$b_1 = D(\bar{\phi}) b_2 \ D(\bar{\phi})^* \ ,$$

which is equivalent, by lemma 3.47, to

$$b_1 = \Phi^* b_2 \Phi \ . \tag{5.50}$$

If $X,Y \in L$ are arbitrary, then (5.50) is equivalent to

$$X^t b_1 Y = X^t \Phi^* b_2 \Phi Y \ ,$$

which can be transformed to

$$\omega_1(X,Y) = X^t\Phi^*b_2\Phi Y \sim (\Phi X)^t b_2\Phi Y = \omega_2(\Phi X,\Phi Y) =$$

$$= (\Phi^*\omega_2)(X,Y) ,$$

which is (5.49). □

Chapter IX. Formal Eigenfunctions and Associated Constructions

In this chapter we treat the variable L in the Lax equations as an operator. We construct formal eigenfunctions of L which enable us to find new constructions of conservation laws for Lax equations.

1. Formal Eigenfunctions

The Lax equations

$$\partial_P(L) = [P_+, L] \tag{1.1}$$

can often be interpreted as the integrability conditions for the system

$$\begin{cases} L\hat{\psi} = \lambda\hat{\psi} , & (1.2a) \\ \partial_P(\hat{\psi}) = P_+\hat{\psi} , & (1.2b) \\ \partial_P(\lambda) = 0 . & (1.2c) \end{cases}$$

Thus we can think of $\hat{\psi}$ as being an "eigenfunction" of the operator L and one may use it for various purposes in the study of the Lax equations (1.1) and their solutions.

Some instances of the above use will be seen in the subsequent sections. In this section we construct $\hat{\psi}$ itself. The reason why such a construction is required is that $\hat{\psi}$ does not belong to the difference ring C_L generated by the coefficients of L, but to some nontrivial extension of C_L (analogously to the differential case [12], [13]).

First, let us see informally what the nature of $\hat{\psi}$ is. In this chapter we restrict ourselves to the operator L of the form

$$L = \zeta + \sum_{j\geq 0} \zeta^{-j} q_j . \tag{1.3}$$

Let

$$K = 1 + \chi_1\zeta^{-1} + \chi_2\zeta^{-2} + \cdots = \sum_{j\geq 0} \chi_j\zeta^{-j}, \ \chi_o = 1 , \tag{1.4}$$

be such that

$$L = K\zeta K^{-1} . \tag{1.5}$$

In other words, K is the "dressing operator" for L. Let ϕ be such that

$$\Delta(\phi) = \lambda\phi \tag{1.6}$$

which is an analog of $\frac{d}{dx}(e^{\lambda x}) = \lambda e^{\lambda x}$ in the differential case, and of $\Delta(e^{\lambda n}) = \lambda e^{\lambda n}$ in the discrete $\mathbb{Z}$-case. Then

$$\tilde{\psi}: = K(\phi) = (\Sigma\chi_i\lambda^{-i})\phi \tag{1.7}$$

satisfies

$$L\tilde{\psi} = LK\phi = K\zeta\phi = K\lambda\phi = \lambda\tilde{\psi} , \tag{1.8}$$

that is, $\tilde{\psi}$ is a (formal) eigenfunction of L. It differs from $\hat{\psi}$ because (1.2b) fails for $\tilde{\psi}$, as we shall see shortly.

Thus $\tilde{\psi}$ and K carry the same informational value. On the other hand, rewriting (1.5) in the form

$$LK = (\zeta+\Sigma\zeta^{-j}q_j)\Sigma\chi_i\zeta^{-i} = (\Sigma\chi_i\zeta^{-i})\zeta = K\zeta , \tag{1.9}$$

we obtain

$$q_o = -(\Delta-1)\chi_1 ,$$

$$q_m = -(\Delta-1)\chi_{m+1}^{(m)} + R_m(\chi_1,\ldots,\chi_m), \ m \geq 1 , \tag{1.10}$$

where R_m's are some difference polynomials.

Let C_K be the difference ring $k[\chi_i^{(n_i)}]$, $i \geq 1$, $n_i \in \mathbb{Z}$, over the field k of characteristic zero, with the automorphism Δ acting identically on k and in the usual way on the $\chi_i^{(n)}$'s. Let C_L be the analogous difference ring $k[q_j^{(n_j)}]$, and let $h: C_L \to C_K$ be the difference embedding over k, given by (1.10). We suppress h from notations and consider C_L as the difference subring in C_K. As we know from Chap. III, sec. 1, for each $P \in Z(L) = \{L^n | n \in \mathbb{Z}_+\}$, the equations (1.1) define an evolutionary derivation ∂_P of C_L. Our goal is to extend ∂_P to C_K in a manner compatible with the embedding h. The procedure required for such extension is a bit tedious even in the differential case [12]. To avoid it, we take a circuitous route considering, in the spirit of Chapt. I, an algebraic scheme which afterwards can be specialized to produce derivations ∂_P of C_K compatible with those of C_L. (This scheme is important for the theory of matrix equations as well). Here are the details.

Let $k[\bar{z}]$ be the associative graded algebra over k,

$$k[\bar{z}] = k[z_o, z_1, \ldots] \tag{1.11}$$

with generators $z_o, z_1, \ldots$ and weights

$$w(z_o) = \beta \ , \ w(z_i) = -\alpha i, \ \alpha, \beta, i \in \mathbb{N} \ ; w(k) = 0 \ . \tag{1.12}$$

Let $\hat{k}[\bar{z}]$ be the completion of $k[\bar{z}]$ with respect to the above grading, and let $K \in \hat{k}[\bar{z}]$ be given as

$$K = 1 + z_1 + z_2 + \ldots \tag{1.13}$$

Obviously, $K^{-1} \in \hat{k}[\bar{z}]$:

$$K^{-1} = 1 + (1-K) + (1-K)^2 + \ldots = 1 - z_1 + (z_1^2 - z_2) + \ldots \tag{1.14}$$

For $P \in \hat{k}[\bar{z}]$, $P = \Sigma p_s$, with $w(p_s) = s$, we define

$$P_+ = \sum_{s \geq 0} p_s, \ P_- = P - P_+, \ \text{Res } P = p_o \ . \tag{1.15}$$

Now let us define

$$L = K z_o K^{-1} = z_o + [z_1, z_o] + \ldots \in \hat{k}[\bar{z}] \ . \tag{1.16}$$

Thus, if we write

$$L = x_o + x_1 + \dots , \tag{1.17}$$

we will have

$$w(x_i) = \beta - \alpha i , \tag{1.18}$$

in accordance with I (1.17).

Let $\gamma = \frac{\alpha}{(\beta,\alpha)}$, as in I (1.21). For each $k \in \mathbb{N}$ we define the derivation ∂_P of $\hat{k}[\bar{z}]$ with $P = L^{k\gamma}$, by the properties

$$\partial_P(K) = -P_- K , \tag{1.19a}$$

$$w(\partial_P) = 0 \ , \ \partial_P(z_o) = 0 \ , \ \partial_P(k) = 0 \ . \tag{1.19b}$$

Obviously, ∂_P is well-defined (see I(1.22)).

Proposition 1.20. For L given by (1.16), we have

$$\partial_P(L) = [-P_-, L] = [P_+, L] \ . \tag{1.21}$$

Proof. Since $[P,L] = 0$, the second equality in (1.21) follows from the first. Now, the equality

$$\partial_P(K^{-1}) = -K^{-1}\partial_P(K)K^{-1}$$

together with (1.16) and (1.19) implies

$$\partial_P(L) = \partial_P(Kz_oK^{-1}) = -P_-Kz_oK^{-1} - Kz_oK^{-1}(-P_-K)K^{-1} =$$

$$= [-P_-, Kz_oK^{-1}] = [-P_-, L] \ . \qquad \square$$

Now let $k' \in \mathbb{N}$ and $Q = L^{k'\gamma} \in \hat{k}[\bar{x}] \subset \hat{k}[\bar{z}]$. Proposition 1.20 tells us how to restrict ∂_P to $\hat{k}[\bar{x}]$. In particular, by I(2.2) we have

$$\partial_P(Q) = [-P_-, Q] \ . \tag{1.22}$$

Theorem 1.23. ∂_P and ∂_Q commute in $\hat{k}[\bar{z}]$.

Proof. It is enough to show that

$$[\partial_P,\partial_Q](K) = 0 \ . \tag{1.24}$$

From (1.19a) and (1.22) we get

$$\partial_P(\partial_Q(K)) = \partial_P(-Q_-K) = -[-P_-,Q]_-K - Q_-(-P_-K) \ ,$$

and analogously for $\partial_Q(\partial_P(K))$. Thus

$$[\partial_P,\partial_Q](K) = \{[P_-,Q]_- + [P,Q_-]_- - [P_-,Q_-]\}K \ .$$

But the expression in the curly brackets is identically zero, as can be seen at once by expanding the relation

$$[P_++P_-,Q_++Q_-] = 0 \ ,$$

and taking the negative part of it. □

Now, as in Chapt. III, sec. 1, we can specialize the foregoing scheme for the case

$$z_o = \zeta \ , \ z_{i+1} = \chi_i\zeta^{-i} \ , \ w(\zeta) = 1 \ , \ w(C_K) = 0 \ , \ \alpha = \beta = 1 \ . \tag{1.25}$$

Then the derivations ∂_P of $C_K((\zeta^{-1}))$ given by (1.19), define the evolutionary derivations ∂_P of C_K which commute and extend the corresponding evolutionary derivations of C_L.

2. The Second Construction of Conservation Laws

Consider the variable ϕ of the preceeding section as a new formal variable, and let us define the difference rings

$$C_{K,\phi}: = C_K[\phi,\phi^{-1}]((\lambda^{-1})) \ , \ C_{L,\phi} = C_L[\phi,\phi^{-1}]((\lambda^{-1})) \ , \tag{2.1}$$

where λ is a formal parameter commuting with everything and Δ is acting on ϕ by

$$\Delta^s(\phi^k) = \lambda^{ks}\phi^k \ . \tag{2.2}$$

Let $K\epsilon C_K((\zeta^{-1}))$ be given by (1.4) and $\tilde{\psi}\epsilon C_{K,\phi}$ be given by (1.7). Then (1.8) shows that $\tilde{\psi}$ is a formal eigenfunction of L, sometimes also called a formal

Baker-Akhiezer function (in the differential case).

We are now going to use $\tilde{\psi}$ to derive conservation laws for the Lax equations (1.1). This construction is called the second to distinguish it from the conservation laws given by the formulae $\mathrm{Res}L^n$; the latter formulae are called the first construction. This terminology and the main steps in the proof of the equivalence of two constructions are adopted from Wilson's treatment of the differential case [13]. For the reader's convenience I keep the notations and the line of reasoning as close to his notations and arguments as possible.

Fix $n \in \mathbb{N}$, let $P = L^n$ (remember that $\alpha = \beta = \gamma = 1$). Let us represent P_- and ζ as elements of $C_L((L^{-1}))$:

$$P_- = \sum_{i \geq 1} d_i L^{-i} , \quad d_i \in C_L , \tag{2.3}$$

$$\zeta = L - \sum_{j \geq 0} b_j L^{-j} , \quad b_j \in C_L . \tag{2.4}$$

Obviously, both decompositions (2.3) and (2.4) exist and are unique; (2.4) can be arrived at by inverting the equation $L = \zeta + \sum_{j \geq 0} \zeta^{-j} q_j$ step by step.

<u>Lemma 2.5</u>. Let us extend the derivation ∂_P to $C_{K,\phi}$ by $\partial_P(\phi^k) = 0$, $\partial_P(\lambda) = 0$. Then

$$\partial_P(\tilde{\psi}) \cdot \tilde{\psi}^{-1} = - \sum_{i \geq 1} d_i \lambda^{-i} , \tag{2.6}$$

$$\Delta(\tilde{\psi}) \cdot \tilde{\psi}^{-1} = \lambda(1 - \sum_{j \geq 0} b_j \lambda^{-j-1}) . \tag{2.7}$$

<u>Proof</u>. Notice that by (1.7) $\tilde{\psi} = [1 + O(\lambda^{-1})]\phi$, hence $\tilde{\psi}^{-1}$ makes sense in $C_{K,\phi}$. Now $L^s = K\zeta^s K^{-1}$ by (1.5), therefore $L^s \tilde{\psi} = \lambda^s \tilde{\psi}$ by (1.7) and (2.2), hence by (2.3) we obtain

$$\partial_P(\tilde{\psi}) = \partial_P(K\phi) = \partial_P(K)\phi = -P_- K\phi = -P_- \tilde{\psi} =$$

$$= - \sum_{i \geq 1} d_i L^{-i} \tilde{\psi} = - \sum_{i \geq 1} d_i \lambda^{-i} \tilde{\psi} , \tag{2.6a}$$

which proves (2.6). Analogously, we get (2.7) after applying (2.4) to $\tilde{\psi}$. □

Theorem 2.8. With d_i, b_j as above, we have

$$\partial_P[\ell n(1-\sum_{j\geq 0} b_j\lambda^{-j-1})] = (1-\Delta)\sum_{i\geq 1} d_i\lambda^{-i} . \tag{2.9}$$

Hence, denoting

$$-\ell n(1-\sum_{j\geq 0} b_j\lambda^{-j-1}) = \sum_{i\geq 1} \rho_i\lambda^{-i} , \rho_i \in C_L , \tag{2.10}$$

we obtain

$$\partial_P(\rho_i) \sim 0 , \tag{2.11}$$

and thus the ρ_i's provide us with the new set of conservation laws.

Proof of the theorem. We have

$$\partial_P(\ell n(1-\Sigma b_j\lambda^{-j-1})] = [\text{by (2.7) and since } \partial_P(\lambda) = 0]$$

$$= \partial_P(\ell n \frac{\Delta\tilde{\psi}}{\tilde{\psi}}) = \partial_P[(\Delta-1)\ell n\tilde{\psi}] =$$

$$= (\Delta-1)[\partial_P(\ell n\tilde{\psi})] = (\Delta-1)[\partial_P(\tilde{\psi})\cdot\tilde{\psi}^{-1}] = [\text{by (2.6)}]$$

$$= (\Delta-1)(-\Sigma d_i\lambda^{-i-1}) .$$ □

Let us see the first few new c.l.'s explicitly. For L given by (1.3) we have

$$L^{-1} = \zeta^{-1}-q_o^{(-1)}\zeta^{-2}+[-q_1^{(-2)}+q_o^{(-1)}q_o^{(-2)}]\zeta^{-3} +$$

$$+ [-q_2^{(-3)}+q_o^{(-1)}q_1^{(-3)}+q_o^{(-3)}q_1^{(-2)}-q_o^{(-1)}q_o^{(-2)}q_o^{(-3)}]\zeta^{-4} + \dots \tag{2.12}$$

and equating corresponding ζ-powers in

$$L = \zeta + b_o + b_1L^{-1} + b_2L^{-2} + \dots ,$$

we get

$$b_o = q_o \; ; \; b_1 = q_1^{(-1)} \; ; \; b_2 = q_2^{(-2)} + q_o^{(-1)}q_1^{(-1)} \; ; \; \ldots \tag{2.13}$$

Now

$$-\ell n(1-b_o\lambda^{-1}-b_1\lambda^{-2}-b_2\lambda^{-3}-\ldots) =$$

$$= \lambda^{-1}\{b_o+(b_1+\frac{b_o^2}{2})\lambda^{-1}+(b_2+b_ob_1+\frac{b_o^3}{3})\lambda^{-2} + \ldots\} ,$$

hence

$$\rho_1 = b_o = q_o ,$$

$$\rho_2 = b_1 + \frac{b_o^2}{2} = q_1^{(-1)} + \frac{q_o^2}{2} , \tag{2.14a}$$

$$\rho_3 = b_2 + b_ob_1 + \frac{b_o^3}{3} = q_2^{(-2)} + q_o^{(-1)}q_1^{(-1)} + q_oq_1^{(-1)} + \frac{q_o^3}{3} , \ldots$$

On the other hand, we have

$$H_1 = \mathrm{Res}L = q_o ,$$

$$H_2 = \frac{1}{2}\,\mathrm{Res}L^2 = \frac{1}{2}(q_1+q_1^{(-1)}) + \frac{q_o^2}{2} , \tag{2.14b}$$

$$H_3 = \frac{1}{3}\,\mathrm{Res}L^3 = \frac{1}{3}[q_2+q_2^{(-1)}+q_2^{(-2)}] + \frac{1}{3}[2q_oq_1+q_o^{(-1)}q_1^{(-1)}] +$$

$$+ \frac{1}{3}[2q_oq_1^{(-1)}+q_o^{(1)}q_1] + \frac{q_o^3}{3} , \ldots$$

Comparing (2.14a) with (2.14b) we are led to conjecture that c.l.'s H_i and ρ_i are equivalent: $H_i \sim \rho_i$. This is indeed the case and the rest of this section is devoted to the proof of the conjecture.

It will be convenient to use the following notations for the polynomial difference ring generated over a difference ring $\mathcal{k}$ by variables $p_i^{(n)}$:

$$k^{\Delta}[\bar{p}] := k[p_1^{(n_1)}, p_2^{(n_2)}, \dots] , \tag{2.15a}$$

as opposed to the usual polynomial ring

$$k[\bar{p}] := k[p_1, p_2, \dots] . \tag{2.15b}$$

Also, for any $r \in \mathbb{N}$ we denote

$$k_r^{\Delta}[\bar{p}] := k[p_1^{(n_1)}, \dots, p_r^{(n_r)}] , \tag{2.15c}$$

$$k_r[\bar{p}] := k[p_1, \dots, p_r] . \tag{2.15d}$$

If the numbering of the p-variables starts with zero instead of one, i.e., if we have $p_o^{(n_o)}$, etc., then the same notational conventions hold.

From (2.4) we get $(b_i - \Delta^{-i} q_i) \in k^{\Delta}_{i-1}[\bar{q}]$ and from (2.10) we obtain $(\rho_{i+1} - b_i) \in \in k_{i-1}[\bar{b}]$. Thus

$$k^{\Delta}[\bar{q}] \cong k^{\Delta}[\bar{b}] \cong k^{\Delta}[\bar{\rho}] , \tag{2.16a}$$

$$k_i^{\Delta}[\bar{q}] \cong k_i^{\Delta}[\bar{b}] , k_{i+1}^{\Delta}[\bar{b}] \cong k_i^{\Delta}[\bar{\rho}] . \tag{2.16b}$$

Let us introduce a few objects to make the reasoning clearer. First we define

$$\psi := 1 + \chi_1 \lambda^{-1} + \chi_2 \lambda^{-2} + \cdots \in (k^{\Delta}[\bar{\chi}])[[\lambda^{-1}]] , \tag{2.17}$$

so that

$$\tilde{\psi} = \psi \phi$$

by (1.7). Introduce variables β_j by

$$\ell n \psi = \ell n(1 + \chi_1 \lambda^{-1} + \dots) = \sum_{j \geq 1} \beta_j \lambda^{-j} , \tag{2.18}$$

so that

$$(\chi_s-\beta_s)\in\mathbb{Q}[\beta_1,\ldots,\beta_{s-1}] \cong \mathbb{Q}[\chi_1,\ldots,\chi_{s-1}] \tag{2.19}$$

(for s = 1, (2.19) should naturally read as $\chi_1 = \beta_1$). Finally, introduce variables η_i by the formula

$$K^{-1} = 1 + \sum_{r\geq 1} \zeta^{-r}\eta_r = \sum_{r\geq 0} \zeta^{-r}\eta_r \ , \ \eta_o = 1 \ . \tag{2.20}$$

Obviously,

$$(\chi_r+\eta_r)\in \mathcal{k}^{\Delta}_{r-1}[\bar{\chi}] \ . \tag{2.21}$$

Lemma 2.22. With the foregoing notations, for any $q\in\mathbb{N}$,

$$\mathrm{Res}L^q = (1-\Delta)[\sum_{s=0}^{q-1} \Delta^s\chi_q + \sum_{\alpha=1}^{q-1}\sum_{s=0}^{\alpha-1} \Delta^s\chi_\alpha\eta_{q-\alpha}] \ . \tag{2.23}$$

Proof. By (1.5), $L^q = K\zeta^qK^{-1}$, therefore

$$\mathrm{Res}L^q = \mathrm{Res}K\zeta^qK^{-1} = \mathrm{Res}[K,\zeta^qK^{-1}] = [\text{by } (1.4), (2.20)]$$

$$= \mathrm{Res}[\chi_1\zeta^{-1}+\ldots+\chi_q\zeta^{-q}, \ \zeta^q+\zeta^{q-1}\eta_1+\ldots+\zeta^1\eta_{q-1}] =$$

$$= (1-\Delta^q)\chi_q + \sum_{\alpha=1}^{q-1} (1-\Delta^\alpha)\chi_\alpha\eta_{q-\alpha} \ ,$$

from which (2.23) follows. □

Theorem 2.24.

$$[\sum_{s=0}^{q-1} \Delta^s\chi_q + \sum_{\alpha=1}^{q-1}\sum_{s=0}^{\alpha-1} \Delta^s\chi_\alpha\eta_{q-\alpha}-q\beta_q] \in \mathcal{k}^{\Delta}[\bar{\rho}] \ . \tag{2.25}$$

Corollary 2.26.

$$(\mathrm{Res}L^q-q\rho_q) \in \mathrm{Im}(\Delta-1) \quad \text{in} \quad \mathcal{k}^{\Delta}[\bar{q}] \ .$$

Proof of the corollary. Applying the operator 1-Δ to (2.25) and using (2.23) and (2.16a) we obtain

$$[\mathrm{Res}L^q-(1-\Delta)q\beta_q] \in \mathrm{Im}(\Delta-1) \quad \text{in} \quad k^{\Delta}[\bar{q}] \ ,$$

from which the corollary follows if we notice that

$$(1-\Delta)\beta_j = \rho_j \ . \tag{2.27}$$

This last equality can be seen as follows:

$$(\Delta-1) \sum_{j\geq 1} \beta_j\lambda^{-j} = (\Delta-1)\ln\psi \quad [\text{by } (2.18)]$$

$$= \ln \frac{\psi^{(1)}}{\psi} = \ln \frac{(\tilde{\psi}/\phi)^{(1)}}{\tilde{\psi}/\phi} = \ln \frac{\tilde{\psi}^{(1)}}{\lambda\tilde{\psi}} = \quad [\text{by } (2.7)]$$

$$= \ln(1-\sum_{j\geq 0} b_j\lambda^{-j-1}) = \quad [\text{by } (2.10)] \tag{2.27a}$$

$$= -\sum_{j\geq 1} \rho_j\lambda^{-j} \ . \qquad \square$$

The corollary establishes the equivalence of the two constructions of conservation laws. It remains to prove theorem 2.24, and we break the proof into a few lemmas.

<u>Lemma 2.28</u>.

$$k^{\Delta}[\bar{\chi}] \cong (k[\bar{\chi}])^{\Delta}[\bar{q}] = (k^{\Delta}[\bar{q}])[\bar{\chi}] \ .$$

In other words, the difference ring $k^{\Delta}[\bar{\chi}]$ is the ring of polynomials in variables $\chi_1, \chi_2, \ldots$ over the difference ring $k^{\Delta}[\bar{q}]$.

<u>Proof</u>. Since $k^{\Delta}[\bar{\chi}] = \bigcup_r k^{\Delta}[\bar{q};\chi_1,\ldots,\chi_r] = \bigcup_r (k^{\Delta}[q])^{\Delta}_r[\bar{\chi}]$, and, obviously, $k^{\Delta}[\bar{\chi}] \supset (k[\bar{\chi}])^{\Delta}[\bar{q}]$, it is enough to show that $(k_r[\bar{\chi}])^{\Delta}[\bar{q}] \supset (k^{\Delta}[\bar{q}])^{\Delta}_r[\bar{\chi}]$, which is equivalent, by (2.16a), to

$$(k_r[\bar{\chi}])^{\Delta}[\bar{\rho}] \supset (k^{\Delta}[\bar{\rho}])^{\Delta}_r[\bar{\chi}] \ . \tag{2.29}$$

We prove (2.29) by induction on r.

For r = 1, (2.18) yields $\chi_1 = \beta_1$, thus by (2.27)

$$\rho_1 = (1-\Delta)\beta_1 = (1-\Delta)\chi_1 = \chi_1 - \Delta\chi_1 ,$$

hence

$$\Delta\chi_1 = \chi_1 - \rho_1 , \quad \Delta^{-1}\chi_1 = \chi_1 + \Delta^{-1}\rho_1 .$$

This implies

$$\Delta^k\chi_1 \in (\mathbb{Z}^\Delta[\rho_1])(\chi_1) , \forall k \in \mathbb{Z},$$

which proves (2.29) for r = 1. Assume now that (2.29) is true for all $r \le (s-1)$. To prove it for r = s, write (2.19) in the form

$$\chi_s = \beta_s + R_s(\chi_1, \ldots, \chi_{s-1}) , \tag{2.30}$$

where R_s is some polynomial. Applying the operator $(1-\Delta)$ to (2.30) and using (2.27) and the induction assumption, we get

$$[\Delta\chi_s - (\chi_s - \rho_s)] \in (k_{s-1}[\bar{\chi}])^\Delta[\bar{\rho}] ,$$

which can be rewritten as

$$(\Delta\chi_s - \chi_s) \in (k_{s-1}[\bar{\chi}])^\Delta[\bar{\rho}] .$$

This, as above, implies

$$(\Delta^\kappa \chi_s - \chi_s) \in (k_{s-1}[\bar{\chi}])^\Delta[\bar{\rho}] , \forall k \in \mathbb{Z},$$

which proves (2.29) for r = s. Thus, the induction step is completed. □

Lemma 2.31. (i) The variables q_j are Δ-independent, that is, the variables $q_j^{(n)}$ are algebraically independent over k. (ii) The variables χ_i are algebraically independent over $k^\Delta[\bar{q}]$.

Proof. (i) By (2.16b), the statement is equivalent to the fact that the variables ρ_i are Δ-independent. Suppose that this is not so, and that there exists some polynomial f in the variables $\rho_i^{(n)}$, $i \le N$, which vanishes:

$$f = \Sigma f_i(\rho_N)a_i(<N) = 0 ,$$

with some $f_i(\rho_N) \in k^{\Delta}[\rho_N]$ and $a_i(<N) \in k^{\Delta}_{N-1}[\bar{\rho}]$. We choose the maximal $s \in \mathbb{Z}$ such that $\rho_N^{(s)}$ can be still met in f, and then we pick the maximal power ℓ of $\rho_N^{(s)}$ in f:

$$f = (\rho_N^{(s)})^{\ell} \, g[\rho_N^{(s-1)}, \rho_N^{(s-2)}, \ldots]a(<N) + \ldots$$

Substituting $\rho_i = \beta_i - \beta_i^{(1)}$, we see that the maximal power of $\beta_N^{(s+1)}$ in f is ℓ and the term $(-\beta_N^{(s+1)})^{\ell}$ is multiplied by the coefficient $\{g[\rho_N^{(s-1)}, \ldots]a(<N)\}^*$, which thus must vanish, since β_i's are Δ-independent (by * we denote the result of the substitution $\beta_i^{(k)} - \beta_i^{(k+1)}$ instead of $\rho_i^{(k)}$). Continuing further, we get rid of all the variables $\rho_N, \rho_{N-1}, \ldots$ etc., concluding that f = 0. (ii) We prove the equivalent statement that the variables β_i are algebraically independent over $k^{\Delta}[\bar{\rho}]$. For each i, and for each $N \in \mathbb{N}$, we have a linear invertible transformation between variables $(\beta_i, \rho_i^{(-N)}, \rho_i^{(-N+1)}, \ldots, \rho_i^{(N)})$ and $(\beta_i^{(-N)}, \ldots, \beta_i^{(N+1)})$, generated by the relations $\rho_i^{(s)} = \beta_i^{(s)} - \beta_i^{(s+1)}$, $-N \leq s \leq N$. This transformation induces the isomorphism of the rings

$$\{k[\rho_i^{(a_i)}, \beta_i] \mid -N \leq a_i \leq N , i \leq N\} \cong$$

$$\cong \{k[\beta_i^{(a_i)}] \mid -N \leq a_i \leq N + 1 , i \leq N\} .$$

But this last ring is the subring of $k^{\Delta}[\bar{\beta}]$ where the variables β_i are Δ-independent. □

By lemma 2.28, $k^{\Delta}[\bar{\chi}] \cong (k^{\Delta}[\bar{q}])[\bar{\chi}] \cong (k^{\Delta}[\bar{\rho}])[\bar{\beta}]$, and by lemma 2.31 (ii) the variables β_i are algebraically independent over $k^{\Delta}[\bar{\rho}]$. Thus we can introduce derivations $\frac{\partial}{\partial \beta_i}$ of the ring $(k^{\Delta}[\bar{\rho}])[\bar{\beta}]$ by the relations

$$\frac{\partial}{\partial \beta_i}: k^{\Delta}[\bar{\rho}] \mapsto 0 , \quad \frac{\partial}{\partial \beta_i}: \beta_j \mapsto \delta_j^i . \tag{2.32}$$

Lemma 2.33.

$$\frac{\partial}{\partial\beta_i}(1-\Delta) = (1-\Delta)\frac{\partial}{\partial\beta_i} . \tag{2.34}$$

Proof. We check out that

$$\frac{\partial}{\partial\beta_i}\Delta = \Delta\frac{\partial}{\partial\beta_i} , \tag{2.35}$$

from which (2.34) follows.

Denote $(k^\Delta[\bar\rho])[\hat\beta_i] = \{f\in(k^\Delta[\bar\rho])[\bar\beta] \mid \frac{\partial f}{\partial\beta_i} = 0\}$. Then $(k^\Delta[\bar\rho])[\bar\beta] \cong ((k^\Delta[\bar\rho])[\hat\beta_i])[\beta_i]$. Let us take an arbitrary element $f\beta_i^r$, $f\in(k^\Delta[\bar\rho])[\hat\beta_i]$. Then

$$\frac{\partial}{\partial\beta_i}\Delta(f\beta_i^r) = \frac{\partial}{\partial\beta_i}[\Delta(f)(\beta_i-\rho_i)^r] = \Delta(f)r(\beta_i-\rho_i)^{r-1} =$$

$$= \Delta(fr\beta_i^{r-1}) = \Delta\frac{\partial}{\partial\beta_i}(f\beta_i^r) ,$$

since $\frac{\partial}{\partial\beta_i}(\Delta f) = 0$ and $\beta_i - \Delta\beta_i = \rho_i$. □

Lemma 2.36. Let $R \in (k^\Delta[\bar\rho])[\bar\beta]$, $(1-\Delta)R \in k^\Delta[\bar\rho]$ and R not contain terms of the form $c_i\beta_i$, $0 \neq c_i \in k$. Then $R\in k^\Delta[\bar\rho]$.

Proof. Since $(1-\Delta)R \in k^\Delta[\bar\rho]$, then $\frac{\partial}{\partial\beta_i}(1-\Delta)R = 0$. By (2.34) we have $0 = (1-\Delta)\frac{\partial R}{\partial\beta_i}$, and lemma 2.37 below implies that $\frac{\partial R}{\partial\beta_i} = c_i \in k$. □

Lemma 2.37. Let A be a difference ring with commuting automorphisms $\Delta_1,\ldots, \Delta_r: A \to A$. Let $A^\Delta[\bar q]$ be the polynomial difference ring $A[q_j^{(\sigma)}]$ with free generators $q_j^{(\sigma)}$, $j \in J$, $\sigma\in\mathbb{Z}^r$. Let $R \in A^\Delta[\bar q]$ be such that $(\Delta_i-1) R = 0$, $i = 1,\ldots,r$. Then $R\in \bigcap_i \operatorname{Ker} (\Delta_i-1)|_A$.

Proof. Since $A^{\Delta}[q_1,\dots,q_r] \cong (A^{\Delta}[q_1,\dots,q_{r-1}])^{\Delta}\ [q_r]$, it is enough to prove the lemma for $r = 1$. Let us denote $q = q_1$, and assume that $R \notin A$. Then there exists $s \in \mathbb{Z}$ such that $\frac{\partial R}{\partial q^{(s)}} \neq 0$, $\frac{\partial R}{\partial q^{(s')}} = 0, \forall s' > s$. But $0 = \frac{\partial}{\partial q^{(s+1)}}(\Delta-1)R = \Delta \frac{\partial R}{\partial q^{(s)}}$, in contradiction with the assumption that $\frac{\partial R}{\partial q^{(s)}} \neq 0$. □

Proof of theorem 2.24. We have to prove that the expression in the square brackets in (2.25), let us call it ω, belongs to $k^{\Delta}[\bar{\rho}]$. From (2.23) and (2.27) we know that $(1-\Delta)\omega \in k^{\Delta}[\bar{\rho}]$. To apply lemma 2.36 it is enough to show that ω, as an element of $(k^{\Delta}[\bar{\rho}])[\bar{\beta}]$ does not contain any nonzero terms of the form $c_i\beta_i$, $c_i \in k$.

For elements in $k^{\Delta}[\bar{p}]$ let us write $O(p^2)$ to denote elements of degree at least 2 in the p-variables with the usual degree defined by $\deg(p_i^{(j)}) = 1$. Then: $\eta_r = -\chi_r + O(\chi^2)$ in $k^{\Delta}[\bar{\chi}]$ by (2.20); $\chi_s = \beta_s + O(\beta^2)$ in $k^{\Delta}[\bar{\beta}]$ by (2.18). Therefore $\Delta^s\eta_\alpha\chi_{q-\alpha} = O(\beta^2)$ in $k^{\Delta}[\bar{\beta}]$. Since the isomorphism $(k^{\Delta}[\bar{\rho}])[\bar{\beta}] \cong k^{\Delta}[\bar{\beta}]$ is induced by the linear transformation in the variables involved (see proof of lemma 2.31 (ii)), the notion of $O(\beta^2)$ is the same in both rings. Thus it only remains to look at the element $\omega' :=$

$\sum_{s=0}^{q-1} \Delta^s\chi_q - q\beta_q$ in ω. Again, $\chi_q = \beta_q + O(\beta^2)$, so $\omega' = \sum_{s=0}^{q-1} \Delta^s\beta_q - q\beta_q + O(\beta^2) =$

$$= [\beta_q + \sum_{s=1}^{q-1}(\beta_q - \sum_{k=0}^{q-2}\Delta^k\rho_q) - q\beta_q + O(\beta^2)] \in k^{\Delta}[\bar{\rho}] + O(\beta^2). \qquad \square$$

3. The Third Construction of Conservation Laws

For the differential Lax equations, the equivalence of the two constructions of conservation laws can be established, at least in the scalar case, by a procedure which differs significantly from the original method of Wilson [13]. This procedure was devised by Flaschka [3] who used Cherednik's arguments [1].

In this section we apply the analogous procedure to the discrete Lax equations. As we shall see below, instead of collapsing into a relation formula between the first two constructions of conservation laws, our procedure yields a seemingly new construction of conservation laws together with a formula which relates this third construction to the second one. By a separate argument we then show that the third construction in fact provides the same formulae as the first one, thus enabling us to find a simpler proof of the equivalence of the first and second constructions than the one presented in the preceeding section. As in the differential case, the formulae met during the derivation of the third construction, also yield the so-called "τ-function" type of relation.

First some notations. For any $n \in \mathbb{Z}_+$, write

$$L^n = (L^n)_+ + \varepsilon_n \zeta^{-1} + O(\zeta^{-2}) \ , \tag{3.1}$$

where $O(\zeta^{-2})$ denotes all terms of the form $p\zeta^k$, $k \leq -2$, with $p \in k^{\Delta}[\bar{q}]$ or $p \in k^{\Delta}[\bar{\chi}]$. Obviously, the ε_n's are uniquely defined and $\varepsilon_n \in k^{\Delta}[\bar{q}]$. Set

$$E = 1 + \sum_{m \geq 0} \lambda^{-m-1} \varepsilon_m^{(1)} \ , \tag{3.2}$$

$$\Lambda = \sum_{m \geq 0} \lambda^{-m-1} (L^m)_+ \ , \tag{3.3}$$

$$b = b(\lambda) = \lambda - \sum_{i \geq 0} b_i \lambda^{-i} \ , \tag{3.4}$$

with b_i's defined by (2.4).

Lemma 3.5.

$$(\zeta - b)\Lambda = -E \ . \tag{3.6}$$

Proof. Using (2.4) we obtain

$$L^{n+1} = LL^n = (\zeta + \sum_{j \geq 0} b_j L^{-j}) L^n = \zeta L^n + \sum_{j \geq 0} b_j L^{n-j} \ ,$$

which yields, upon substituting (3.1),

$$(L^{n+1})_+ = \zeta (L^n)_+ + \varepsilon_n^{(1)} + \sum_{j=0}^{n} b_j (L^{n-j})_+ \ .$$

Therefore, by (3.2)-(3.4),

$$\Lambda = \sum_{n\geq 0} \lambda^{-n-1}(L^n)_+ = \lambda^{-1} + \lambda^{-1} \sum_{n\geq 0} \lambda^{-n-1}(L^{n+1})_+ =$$

$$= \lambda^{-1} + \lambda^{-1}[\zeta\Lambda + (E-1) + \sum_{j\geq 0} \lambda^{-j} b_j \sum_{m\geq 0} (L^m)_+ \lambda^{-1-m}] =$$

$$= \lambda^{-1}\{\zeta\Lambda + E + (\lambda-b)\Lambda\} =$$

$$= \Lambda + \lambda^{-1}\{(\zeta-b)\Lambda + E\} . \qquad \square$$

For $P = L^n$, $n \in \mathbb{Z}_+$, denote

$$D_n = D_n(\lambda) = \partial_P(\tilde{\psi})\cdot\tilde{\psi}^{-1} , \tag{3.7}$$

$$D = D(\lambda) = \sum_{n\geq 0} \lambda^{-n-1} D_n . \tag{3.8}$$

Theorem 3.9. With the foregoing notations,

$$\frac{\partial}{\partial\lambda} \ell n b + (\Delta-1)D = \frac{E}{b} . \tag{3.10}$$

Corollary 3.11. The series $(\frac{E}{b} - \lambda^{-1})$ yields the third construction of conservation laws. Since $\frac{\partial}{\partial\lambda} \ell n b = \lambda^{-1} + \sum_{i\geq 1} i\rho_i\lambda^{-1}$ by (2.10), the third construction is equivalent to the second one.

Proof of the theorem. Let k be a formal parameter which commutes with everything, $\phi(k)$ be analogous to ϕ in (2.2), with $\Delta\phi(k) = k\phi(k)$. Denote $\tilde{\psi}(k) = K\phi(k)$, so that $L^n\tilde{\psi}(k) = k^n\tilde{\psi}(k)$. Also, we have

$$\Delta\tilde{\psi}(k)\cdot\tilde{\psi}(k)^{-1} = b(k) , \quad \partial_P\tilde{\psi}(k)\cdot\tilde{\psi}(k)^{-1} = D_n(k), \quad (P = L^n) \tag{3.12}$$

as in (2.6), (2.7), (3.7).

Now apply (3.6) to $\tilde{\psi}(k)$ and multiply the result by $\tilde{\psi}(k)^{-1}$. On the right hand side we obtain -E. On the left hand side we have

$$\zeta\Lambda\tilde{\psi}(k)\cdot\tilde{\psi}(k)^{-1} = \Sigma\lambda^{-m-1}\zeta(L^m)_+\tilde{\psi}(k)\cdot\tilde{\psi}(k)^{-1} =$$

$$= \Sigma\lambda^{-m-1}\{\Delta[(L^m)_+(\tilde{\psi}(k))\cdot\tilde{\psi}(k)^{-1}]\cdot[\Delta\tilde{\psi}(k)\cdot\tilde{\psi}(k)^{-1}]\} = \quad \text{[by (2.7)]}$$

$$= \Sigma\lambda^{-m-1}b(k)\{\Delta([L^m-(L^m)_-](\tilde{\psi}(k))\cdot\tilde{\psi}(k)^{-1})\} =$$

$$= \Sigma\lambda^{-m-1}b(k)\{\Delta[k^m-\varepsilon_m k^{-1}+O(k^{-2})]\} =$$

$$= b(k)\ \Sigma\lambda^{-m-1}\Delta[k^m-\varepsilon_m k^{-1} + O(k^{-2})] \ , \tag{3.13}$$

$$-b\Lambda\tilde{\psi}(k)\cdot\tilde{\psi}(k)^{-1} = -b(\lambda)\Sigma\lambda^{-m-1}[(L^m)_+\tilde{\psi}(k)\cdot\tilde{\psi}(k)^{-1}] =$$

$$= -b(\lambda)\Sigma\lambda^{-m-1}\{[L^m-(L^m)_-](\tilde{\psi}(k))\cdot\tilde{\psi}(k)^{-1}\} =$$

$$= -b(\lambda)\Sigma\lambda^{-m-1}[k^m-\varepsilon_m k^{-1}+O(k^{-2})] \ . \tag{3.14}$$

Adding (3.13) and (3.14) we get

$$[b(k)-b(\lambda)] \sum_{m\geq 0} \lambda^{-m-1}k^m + \tag{3.15a}$$

$$+ \Sigma\lambda^{-m-1}\{b(k)\Delta[-\varepsilon_m k^{-1}+O(k^{-2})]-b(\lambda)[-\varepsilon_m k^{-1}+O(k^{-2})]\} \ . \tag{3.15b}$$

Since $\sum_{m\geq 0} \lambda^{-m-1}k^m = \lambda^{-1}\dfrac{1}{1-k\lambda^{-1}} = \dfrac{1}{\lambda-k}$, (3.15a) becomes

$$\frac{b(k)-b(k)}{\lambda-k} \tag{3.16a}$$

Now let $k \to \lambda$. Then (3.16) turns into $-\dfrac{\partial b}{\partial\lambda}$ and (3.15b) becomes

$$\Sigma\lambda^{-m-1}b(\lambda)(\Delta-1)[-(L^m)_-\tilde{\psi}\cdot\tilde{\psi}^{-1}] = \quad \text{[by (3.7)]}$$

$$= b(\lambda)(\Delta-1)\Sigma\lambda^{-m-1}(-D_m) = b(\lambda)(1-\Delta)D \ . \tag{3.16b}$$

Altogether, we get

$$-\frac{\partial b}{\partial\lambda} + b(1-\Delta)D = -E \ , \tag{3.17}$$

which yields (3.10) after dividing both parts by -b. □

Lemma 3.18.

$$\frac{E}{b} = \text{Res}\ \Lambda\ . \tag{3.19}$$

Corollary 3.20. Since Res $\Lambda = \Sigma\lambda^{-n-1}$ Res L^n by (3.3), we see that (3.10) provides us with the new proof of the formula $n\rho_n \sim nH_n$.

Proof of lemma 3.18. Take Res of both parts of (3.6). □

We now turn to the derivation of the τ-function formula. For this we change our point of view on the derivation ∂_P and write instead $\frac{\partial}{\partial x_n}$ for $P = L^n$, thus considering all our objects as functions of infinitely many "time"-variables $x_1, x_2, ..$

Lemma 3.21.

$$-\frac{\partial b}{\partial \lambda} + \sum_{m\geq 1} \lambda^{-m-1} \frac{\partial b}{\partial x_m} = -E\ . \tag{3.22}$$

Proof. Formulae (3.17) and (3.22) differ only in the second term which can be transformed with the help of (3.16b) as

$$b(\lambda) \sum_{m\geq 0} \lambda^{-m-1}(\Delta-1)[-(L^m)_-\tilde{\psi}\cdot\tilde{\psi}^{-1}] = \qquad [\text{by } (2.6a)]$$

$$= b(\lambda) \sum_{m\geq 1} \lambda^{-m-1}(\Delta-1)\ [\frac{\partial\tilde{\psi}}{\partial x_m} \cdot \tilde{\psi}^{-1}] =$$

$$= b(\lambda) \sum_{m\geq 1} \lambda^{-m-1}(\Delta-1)\ \frac{\partial}{\partial x_m}\ \ell n\tilde{\psi} =$$

$$= b(\lambda) \sum_{m\geq 1} \lambda^{-m-1}\ \frac{\partial}{\partial x_m}\ \ell n\ \frac{\Delta\tilde{\psi}}{\tilde{\psi}} = \qquad [\text{by } (2.7)]$$

$$= b(\lambda) \sum_{m\geq 1} \lambda^{-m-1}\ \frac{\partial}{\partial x_m}\ \ell n\ b(\lambda) = \sum_{m\geq 1} \lambda^{-m-1}\ \frac{\partial b}{\partial x_m}\ . \qquad \square$$

Corollary 3.23. For $s \in \mathbb{N}$,

$$\varepsilon_s^{(1)} = sb_s + \sum_{m=1}^{s} \frac{\partial b_{s-m}}{\partial x_m} \ . \tag{3.24}$$

Proof. Substituting (3.2) and (3.4) into (3.22), and taking into account that $\varepsilon_o = 0$, we obtain (3.24). □

Lemma 3.25. For $m,j \in \mathbb{N}$,

$$\frac{\partial \varepsilon_m}{\partial x_j} = \frac{\partial \varepsilon_j}{\partial x_m} \ . \tag{3.26}$$

Corollary 3.27. Thus there exists a function, call it $-\sigma(x_1,x_2,\ldots)$, such that

$$\varepsilon_m^{(1)} = \frac{\partial(-\sigma)}{\partial x_m} \ . \tag{3.28}$$

Then, (3.24) becomes

$$\frac{\partial \sigma}{\partial x_s} = -sb_s - \sum_{m=1}^{s} \frac{\partial b_{s-m}}{\partial x_m} \ . \tag{3.29}$$

Proof of the lemma. By (2.6a),

$$\frac{\partial \tilde{\psi}}{\partial x_m} = -(L^m)_- \tilde{\psi} = -[\varepsilon_m \zeta^{-1} + O(\zeta^{-2})]\tilde{\psi} = -[\varepsilon_m \lambda^{-1} + O(\lambda^{-2})]\tilde{\psi} \ ,$$

hence

$$\frac{\partial}{\partial x_j} \frac{\partial \tilde{\psi}}{\partial x_m} = -[\frac{\partial \varepsilon_m}{\partial x_j} \lambda^{-1} + O(\lambda^{-2})]\tilde{\psi} \ ,$$

and the left hand side is symmetric with respect to the order of indices j and m. □

Remark 3.30. Formula (3.26) is a particular instance of a general algebraic fact. Let K, L, P and Q be as in section 1. Then from (1.19a) we obtain, as in the proof of theorem 1.23,

$$\partial_Q\partial_P(K) = \partial_Q(-P_-K) = -[\partial_Q P_- + P_-Q_-]K ,$$

hence

$$0 = [\partial_P,\partial_Q](K) = \{\partial_P Q_- - \partial_Q P_- + [Q_-,P_-]\}K ,$$

and the expression in the curly brackets vanishes as we have seen in the proof of theorem 1.23:

$$\partial_P Q_- - \partial_Q P_- + [Q_-,P_-] = 0 . \tag{3.31}$$

For any $R = \sum_s R_s \in \hat{k}[\bar{z}]$, with weights $w(R_s) = s$, define

$$\varepsilon(R) := R_{-1} \tag{3.32}$$

Applying this operator ε to (3.31) we obtain

$$\partial_P\ \varepsilon(Q) = \partial_Q\ \varepsilon(P) , \tag{3.33}$$

which reduces to (3.26) when one specializes to the set-up of the discrete Lax equations.

The operator ε in (3.32) resembles the residue in the ring of pseudo-differential operators. For the operator Res which is relevant in the discrete theory, one has the following result. Let $L \in \hat{k}[\bar{x}]$ be as in section 1 Chap. I. Let $k[\bar{x}]_o$ be the subring of $\hat{k}[\bar{x}]$ consisting of those elements whose Res equals zero. Let Tr: $k[\bar{x}]_o \to \hat{k}[\bar{x}]$ be a "character", i.e. a linear map which vanishes on commutators and commutes with all derivations ∂_P, $P \in Z(L)$.

Lemma 3.34. For $P = L^n$, $Q = L^m \in Z(L)$, we have

$$\partial_P\ \mathrm{Tr}\ \mathrm{Res}L^m = \partial_Q\ \mathrm{Tr}\ \mathrm{Res}L^n . \tag{3.35}$$

Proof. We have

$$\partial_Q \mathrm{Res}L^n = \mathrm{Res}\ \partial_Q L^n = \mathrm{Res}[L^m_+,L^n] = \mathrm{Res}[L^n,L^m_-] = \mathrm{Res}[L^n_+,L^m_-] , \tag{3.36a}$$

$$\partial_p \mathrm{Res} L^m = \mathrm{Res}[L^n_+, L^m] = \mathrm{Res}[L^n_+, L^m_-] + [\mathrm{Res} L^n, \mathrm{Res} L^m] \ . \tag{3.36b}$$

Taking Tr of both parts in (3.36) yields (3.35). □

Our next step is to invert the infinite system (3.29).

Lemma 3.37. For any set of smooth functions $A(x)$ and $\{B_\ell(x), \ell = 1,2,\ldots\}$ of variables $x = (x_1, x_2, \ldots)$, the relations

$$\frac{\partial A}{\partial x_\ell} = -\ell B_\ell - \sum_{m=1}^{\ell-1} \frac{\partial B_{\ell-m}}{\partial x_m} \ , \ \ell=1,2,\ldots \tag{3.38}$$

can be inverted by a single formal identity

$$\sum_{\ell\geq 1} B_\ell \lambda^{-\ell} = A(x_1 - \frac{1}{\lambda}, x_2 - \frac{1}{2\lambda^2}, \ldots) - A(x) \ , \tag{3.39}$$

where the expression on the right hand side of (3.39) must be understood in the sense of the corresponding Taylor series.

Remark. The lemma is apparently well known. The following proof was found jointly with H. Flaschka.

Proof. For $\ell = 1$, (3.38) yields $\frac{\partial A}{\partial x_1} = -B_1$, and taking λ^{-1}-coefficients in (3.39) yields the same result. For $\ell = 2$, we obtain from (3.38)

$$B_2 = -\frac{1}{2}\left(\frac{\partial A}{\partial x_2} + \frac{\partial B_1}{\partial x_1}\right) = -\frac{1}{2}\frac{\partial A}{\partial x_2} + \frac{1}{2}\frac{\partial^2 A}{\partial x_1^2} \ ,$$

which can be gotten from (3.39) by taking λ^{-2}-coefficients of both parts. At this point it becomes clear that the nature of the functions A and $\{B_\ell\}$ is not important, since the inversion of (3.38) can be performed at each step in finite terms, and our lemma is in fact the statement about linear differential operators of finite order. Thus it is enough to check (3.39) for a sufficiently large class of functions A, and for this purpose $A(x) = \exp\langle c,x\rangle := \exp(\sum_{i\geq 1} c_i x_i)$,

$c_i \in \mathbb{Q}$, will do.

Set $B_\ell = f_\ell \exp\langle c,x\rangle$. Then (3.38) becomes

$$c_\ell = -\ell f_\ell - \sum_{m=1}^{\ell-1} f_{\ell-m} c_\ell , \tag{3.40}$$

which is equivalent to

$$-\ell f_\ell \lambda^{-\ell-1} = c_\ell \lambda^{-\ell-1} + \sum_{i+j=\ell} f_i c_j \lambda^{-i-j-1} ,$$

which is equivalent to

$$\frac{\partial}{\partial\lambda} (1 + \sum_{\ell\geq 1} f_\ell \lambda^{-\ell}) = (1 + \sum_{\ell\geq 1} f_\ell \lambda^{-\ell})(\sum_{i\geq 1} c_i \lambda^{-i-1}) ,$$

which can be rewritten as

$$\frac{\partial}{\partial\lambda} \ell n(1+\Sigma f_\ell \lambda^{-\ell}) = \Sigma\, c_i \lambda^{-i-1} ,$$

which implies

$$\ell n(1+ \sum_{\ell\geq 1} f_\ell \Lambda^{-\ell}) = - \sum_{i\geq 1} c_i \frac{\lambda^{-i}}{i} ,$$

which is equivalent to

$$1 + \sum_{\ell\geq 1} f_\ell \lambda^{-\ell} = \exp(- \sum_{i\geq 1} c_i \frac{\lambda^{-i}}{i}) ,$$

or

$$\sum_{\ell\geq 1} f_\ell \exp\langle c,x\rangle \lambda^{-\ell} = \exp[\sum_{i\geq 1} c_i (x_i - \frac{1}{i\lambda^i})] - \exp\langle c,x\rangle ,$$

and this is exactly (3.39). □

Now we can invert (3.29), if we notice that it can be rewritten as

$$\frac{\partial(\sigma+b_o)}{\partial x_s} = -sb_s - \sum_{m=1}^{s} \frac{\partial b_{s-m}}{\partial x_m} . \tag{3.41}$$

Applying lemma (3.37) to (3.41) we get

$$\sum_{\ell\geq 1} b_\ell \lambda^{-\ell} = (\sigma+b_o)(\ldots,x_i - \frac{1}{i\lambda^i},\ldots) - (\sigma+b_o)(x) ,$$

or

$$\sum_{i\geq 0} b_i \lambda^{-i} = \ell n \frac{\tau(\ldots,x_i - \frac{1}{i\lambda^i},\ldots)}{\tau(x)} + b_o(\ldots,x_i - \frac{1}{i\lambda^i},\ldots) , \tag{3.42}$$

where we introduced the function τ by the relation

$$\sigma = \ell n \tau . \tag{3.43}$$

With (2.17) and (2.27a), (3.42) becomes

$$(\Delta-1)\ell n\psi = \ell n\{1 - \lambda^{-1}[\ell n \frac{\tau(\ldots,x_i - \frac{1}{i\lambda^i},\ldots)}{\tau(x)} + b_o(\ldots,x_i - \frac{1}{i\lambda^i},\ldots)]\} , \tag{3.44}$$

which is the desired analog of the τ-function formula in the differential case (see (5) in [3]).

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RÉSUMÉ

Ce texte est la première introduction détaillée à la théorie des systèmes intégrables discrets infinis et aux idées mathématiques associées.

Il décrit la construction des principales classes d'équations, leurs lois de conservation et leurs structures Hamiltoniennes, les applications canoniques entre elles, les limites continues, les valeurs propres formelles des opérateurs de Lax et une représentation en τ-fonctions.

Le langage de base de la théorie est le calcul des variations discret, qui se comporte naturellement sous limite continue.

L'auteur donne un exposé complet du formalisme Hamiltonien abstrait et du formalisme des espaces duaux d'algèbres de Lie sur les anneaux de fonction.

Ce volume sera utile aux mathématiciens et aux physiciens intéressés dans les solitons et dans le formalisme Hamiltonien ; il est accessible aux étudiants de 3ème cycle.