Astérisque

J. Ross
F. Brulois

The stability of axisymmetric rotating drops

Astérisque, tome 118 (1984), p. 219-226
<http://www.numdam.org/item?id=AST_1984__118__219_0>

Société Mathématique de France
Astérisque 118 (1984) p.219 à 226.

THE STABILITY OF AXISYMMETRIC ROTATING DROPS

by J. ROSS and F. BRULOIS (San Diego State University)

1. INTRODUCTION.

Let Ω be an open set in $\mathbb{R}^3$.

The variational problem

Minimize $E(\Omega) \equiv A(\partial\Omega) - cI(\Omega)$

Subject to i) Volume $(\Omega) = V_o$ = given constant ,

ii) The center of gravity of Ω at the origin

is an accepted mathematical model for studying the equilibrium shapes and the stability of an incompressible liquid drop driven at a constant angular velocity. An important physical assumption in this model, is that there is no internal motion. Here $A(\partial\Omega)$ is the surface area of $\partial\Omega$, $I(\Omega)$ is the moment of inertia of Ω with respect to the axis of rotation, and c is a positive constant which can be expressed in terms of physical quantities by the formula

$$c = \frac{\Delta p}{2T}\omega^2$$

where Δp is the pressure difference on the axis of rotation, T is the surface tension, and ω is the angular velocity.

There exists, mathematically, a well known family of axisymmetric equilibrium shapes whose surfaces are topologically the sphere [2,4,5,6]. These surfaces must satisfy the Euler-Lagrange equation for this problem. This means that at any point, p , on the surface we must have

$$2H = 2h + cr^2$$

where H is the mean curvature at p, h is a constant, and r is the distance from p to the axis at rotation. If we fix the volume, then c may be taken as the parameter in a smooth one parameter subfamily. In this subfamily, c ranges from 0 to c^*. When $c = 0$, we have the sphere. When $c = c^*$, we have $h = 0$. (h is never negative in this subfamily and is zero only at $c = c^*$). We denote this subfamily by $\tilde{f} = \{\Omega_c; 0 \leq c \leq c^*\}$. All members of this family are convex. Here c^* depends on V_o.

Besides these equilibrium shapes and another family of axisymmetric equilibrium shapes whose surfaces are tori, there are two other mathematical facts known about this problem. The first is that any equilibrium shape must have as a plane of symmetry the plane which passes through its center of gravity and is perpendicular to the axis of rotation. This follows from a variant of Aleksandrov's proof that the sphere is the only compact, imbedded, two dimensional surface of constant mean curvature. The second is the result of Albano and Gonzalez [1] which establishes the existence and regularity of a true local minimum provided that c is sufficiently small. As $c \to 0$, these local minima are known to converge in L^1 to the sphere.

Some experiments conducted at JPL by T. Wang and E. Trinh [7] suggest that there are families of equilibrium shapes which bifurcate from $\tilde{f}$. Brown and Scriven [3] did extensive numerical calculations which support the contention that these bifurcating families exist. Furthermore, they investigated numerically the stability of all these equilibrium shapes and found that only those shapes in $\tilde{f}$ with $0 \leq c < c_o < c^*$ are stable. Again, c_o depends on V_o. At the point $c = c_o$ a family of two-lobed shapes appears to bifurcate from $\tilde{f}$.

In this paper, we study, analytically the stability of the members of the family $\tilde{f}$. Although we cannot claim to have proved the stability of $\tilde{f}$ in the range $0 \leq c < c_o$ and instability in the range $c_o \leq c \leq c^*$, our results go quite far in this direction. Only a few rather standard problems remain.

2. DEFINITIONS AND NOTATION.

A point in $\mathbb{R}^3$ will be denoted by (x,y,z) where the z axis is assumed to be the axis of rotation. Thus $r = \sqrt{x^2+y^2}$.

<u>Definition 1</u>. A shape, Ω, will be said to belong to the admissible class, θ, if $\partial\Omega$ can be represented by a function $r = r(z)$ which is nonnegative, smooth, and such that the volume of Ω is finite and $E(\Omega)$ is finite. The center of gravity is required to be at the origin. We assume that there are numbers $\ell_1(\Omega) < 0 < \ell_2(\Omega)$ such that $r(z) = 0$ if $z < \ell_1$ or $z > \ell_2$.

<u>Definition 2</u>. A perturbation, P, of an admissible shape, Ω, is a three parameter family of shapes, $\Omega(\varepsilon,\eta,v)$, whose boundaries can be represented in the following way:

$$X(\theta,z,\varepsilon,\eta,v) \equiv (\eta a(z) + R(\theta,z,\varepsilon,v)\cos\theta, \eta b(z) + R(\theta,z,\varepsilon,v)\sin\theta, z)$$

where $0 \le \theta \le 2\pi$, $\ell_1(v) \le z \le \ell_2(v)$, $-\infty < \eta < +\infty$, $-1 < v < +1$, and $|\varepsilon| \le \delta$ for some $\delta > 0$.

Here X is a position vector defined by its coordinates (x,y,z). The point $(\eta a(z_o), \eta b(z_o), z_o)$ is the center of gravity of the two dimensional region $\Omega(\varepsilon,\eta,v) \cap \{\text{plane } z = z_o\}$ for each z_o, $\ell_1(v) \le z_o \le \ell_2(z_o)$ and for all ε,η, and v.

$$R(\theta,z,\varepsilon,v) = r(z,v)(q(z,\varepsilon) + \varepsilon f(\theta,z))$$

where $\displaystyle\int_0^{2\pi} f(\theta,z)d\theta = 0$ for all z, $\min \ell_1(v) \le z \le \max \ell_2(v)$

$$h^2(z) \equiv \frac{1}{2\pi}\int_0^{2\pi} f^2(\theta,z)d\theta$$

$$q(z,\varepsilon) = \sqrt{1-\varepsilon^2 h^2(z)}$$

and $r(z,v)$ represents an admissible shape whose volume is V_0 for all v, $-1 \le v \le +1$. While $r(z,0)$ represents the member of $\tilde{f}$ corresponding to the value of c considered. We assume that $h^2(z)$ is a bounded function on $\min l_1(v) \le z \le \max l_2(v)$. This permits the existence of δ.

Definition 3.

$$E(\varepsilon,\eta,v) \equiv E(X(\varepsilon,\eta,v)) \quad .$$

We assume that the functions $r(z,v)$, $a(z)$, $b(z)$, and $f(\theta,z)$ are smooth enough so that the function $E(\varepsilon,\eta,v)$ can be written as

$$E(\varepsilon,\eta,v) \equiv \int_{\ell_1(v)}^{\ell_2(v)} \int_0^{2\pi} \left\{ \left(c_1 + (c_2 + \eta c_3)^2\right)^{1/2} - c\left(\frac{1}{2}\eta^2 (a^2 + b^2) R^2 + \frac{1}{4} R^4\right) \right\} d\theta dz$$

where $c_1 = R_\theta^2 + R^2$,

$$c_2 = R\,R_z$$

$$c_3 = R_\theta (a_z \sin\theta - b_z \cos\theta) + R(a_z \cos\theta + b_z \sin\theta)$$

and is a twice continuously differentiable function of ε, η , and v.

Remark. The assumptions made imply that

$$\int_{\ell_1(v)}^{\ell_2(v)} a(z) r^2(z,v)\,dz = \int_{\ell_1(v)}^{\ell_2(v)} b(z) r^2(z,v)\,dz = 0 \quad ,$$

and

$$\int_0^{2\pi} f(\theta,z) \cos\theta\, d\theta = \int_0^{2\pi} f(\theta,z) \sin\theta\, d\theta = 0 \text{ , for all } z \quad .$$

3. RESULTS OF A COMPUTATION.

With the above assumptions, we have

Theorem 1.

1) $\quad E_\varepsilon(0,0,v) = E_\eta(0,0,v) = 0$

2) $\quad E_v(0,0,0) = 0$

3) $E_{\varepsilon\eta}(0,0,v) = E_{\varepsilon v}(0,0,v) = E_{\eta v}(0,0,v) = 0$

4) $$E_{\varepsilon\varepsilon}(0,0,v) = \int_0^{2\pi}\int_{\ell_1}^{\ell_2}\left\{\frac{r}{\sqrt{1+r_z^2}}\,(f_\theta^2 - f^2) + \frac{r}{(1+r_z^2)^{3/2}}\,[(rf)_z]^2 - 2cr^4f^2\right\}d\theta dz$$

5) $$E_{\eta\eta}(0,0,v) = \int_{\ell_1}^{\ell_2}\left\{\frac{r}{(1+r_z^2)^{3/2}}\,(a_z^2 + b_z^2) - 2cr^2(a^2+b^2)\right\}dz$$

Proof. 1) follows by direct calculation

2) is an assumption

3) $E_{\varepsilon\eta}(0,0,v) = 0$ by direct calculation while $E_{\eta v}(0,0,v) = E_{\varepsilon v}(0,0,v) = 0$ by differentiating the formula in 1)

4) and 5) follows by direct calculation.

Conjecture. It is surely true that $Evv(0,0,0) > 0$. We believe that if only axially symmetric perturbations are allowed the classical methods in the calculus variations would establish that $\Omega(0,0,0)$ is a local minimum in a weak neighborhood, the volume constraint notwithstanding.

4. DISCUSSION.

The ε perturbation changes the shape of cross sections $z = \text{constant}$ while preserving their area and center of gravity. The η perturbation preserves the shape of the cross sections $z = \text{const}$ but shifts their centers of gravity. The v perturbation changes the areas of the cross sections of the admissible shape but preserves the axisymmetry. Theorem 1 shows that in a natural sense these three perturbations are always orthogonal thus stability is reduced to studying $E_{\varepsilon\varepsilon}$, $E_{\eta\eta}$, and E_{vv}.

5. RESULTS ABOUT $E_{\varepsilon\varepsilon}$.

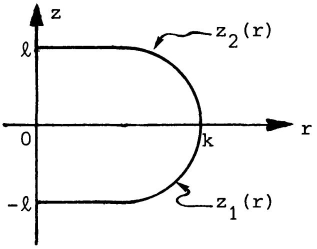
Theorem 2. $E_{\varepsilon\varepsilon}(0,0,0) > 0$ *provided* $ck^3 < 3/2$ *where* k *is the equatorial radius at* Ω_c.

Proof.

$$E_{\varepsilon\varepsilon}(0,0,0) > 2\pi\int_{-\ell}^{\ell} r^2\left[\frac{3}{r(1+r_z^2)^{1/2}} - 2cr^2\right]h^2 dz =$$

$$= 4\pi\int_0^k r^2\left(\frac{3}{r(1+r_z^2)^{1/2}} - 2cr^2\right)\left[h^2(z_1(r)) + h^2(z_2(r))\right]\frac{dz_1}{dr}dr$$

where $z_1(r)$ and $z_2(r)$ are the inverses of $r(z,0)$ as indicated below.



Since h is an arbitrary function, we must have

$$\frac{3}{r(1+r_z^2)^{1/2}} - 2cr^2 > 0 \quad \text{for all} \quad r\ ,\ 0<r<k \quad .$$

The Euler-Lagrange equation is

$$\frac{1}{r}\,(rf(z_r))_r = 2h + cr^2\ , \text{ where } \ f(x) = \frac{x}{\sqrt{1+x^2}}\ ;\ f^{-1}(x) = \frac{x}{\sqrt{1-x^2}}$$

So $z_r = f^{-1}(hr + \frac{1}{4}cr^3)$. And

$$\frac{3}{r\left(1+\left(\frac{1}{z_r}\right)^2\right)^{1/2}} - 2cr^2 = 3h - \frac{5}{4}\,cr^2 \geq 3h - 5/4\ ck^2 \quad .$$

Now $3h - r/4\ ck^2 > 0$ iff $ck^3 < 3/2$ where we have used the relationship $hk + \frac{1}{4}\,ck^3 = f(z_r(k)) = 1$. This completes the proof.

The above theorem cannot locate the precise value of c_o since a definitely positive term is dropped in the first step of the proof. Indeed the above sufficient condition for ε-stability, namely $ck^3 < 1.5$ is stronger than the necessary and sufficient condition obtained by Chandrasekhar [4] and Brown and Scriven [3], namely $ck^3 < 1.8$. (N.B.: for comparison purposes, Chandrasekhar's dimensionless parameter Σ is, in our notation, $\frac{1}{4}ck^3$; and Brown and Scriven's dimensionless parameter Σ or Ω^2 is $\frac{1}{4}c\tilde{r}^3$ where $\tilde{r}$ is given by $V_0 = \frac{4}{3}\pi\tilde{r}^3$; recall that V_0 is the prescribed volume of the drop).

The next theorem is the first step in determining the exact point at which ε instability occurs.

<u>Theorem 3.</u> *Normalize* f *by the condition that* $\int_0^{2\pi}\int_{\ell_1}^{\ell_2} f^2(\theta,z)\,d\theta dz = \pi$.

Let

$$Q_n(g) \equiv \int_{\ell_1}^{\ell_2} \frac{r}{(1+r_z^2)^{3/2}}([rg]_z)^2 + \left[(n^2-1)\frac{r}{(1+r_z^2)^{1/2}} - 2cr^4\right]g^2 dz$$

Suppose that

$$\inf_k Q_2(g) > -\infty \quad \textit{where} \quad k = \left\{ g \,\middle|\, \int_{\ell_1}^{\ell_2} g^2 dt = 1 \right\} .$$

Then the $\inf E_{\varepsilon\varepsilon}(0,0,v)$ *over normalized admissible* f *may be attained as the limit of a sequence of functions* $\{f_i\}$ $i = 1,2,3...$ *where* $f_i(\theta,z) = g_i(z)\cos 2\theta$ *and* $\int_{\ell_1}^{\ell_2} g_i^2(z)\,dz = 1$.

<u>Proof.</u> We know that $f(\theta,z) = \sum_{n=2}^{\infty} a_n(z)\cos n\theta + b_n(z)\sin n\theta$.

Now

$$E\varepsilon\varepsilon(0,0,v) = \pi \sum_{n=1}^{\infty} (Q_n(a_n) + Q_n(b_n)) \quad .$$

Let $a_n = d_n\,\bar{a}_n(z)$; $h_n = e_n\,\bar{b}_n(z)$ where d_n, e_n are constants such that

$$\int_{\ell_1}^{\ell_2} \bar{a}_n^2 dz = \int_{\ell_1}^{\ell_2} \bar{b}_n^2 dz = 1 \quad .$$

So

$$\sum_{n=2}^{\infty} (Q_n(a_n) + Q_n(b_n)) = \sum_{n=1}^{\infty} (d_n^2 Q_n(\bar{a}_n) + e_n^2 Q_n(\bar{b}_n)) \quad \text{where} \quad \sum_{n=2}^{\infty} (d_n^2 + e_n^2) = 1 \quad .$$

Let $\bar{Q}_n \equiv \inf_{g\in k} Q_n(g)$. We have

$$E\varepsilon\varepsilon(0,0,v) \geq \pi \sum_{n=2}^{\infty} (d_n^2 + e_n^2)\overline{Q}_n \geq \pi\, \overline{Q}_2 \quad ,$$

since $\overline{Q}_n \leq \overline{Q}_{n+1}$ for all n. This completes the proof.

6. CONCLUSION.

We have not obtained any concrete estimates about η stability. We wish to point out that it is expected that the shape of the perturbation which first induces η instability will *not* be symmetric with respect to the xy plane. If $r(z,v) \equiv \sqrt{1-z^2}$, then we have a singular Sturm-Liouville problem which leads to Jacobi's differential equation. The first eigenfunction (after the constant function) is odd. The numerical work of Brown and Scriven [3] does not consider such perturbations while the experiments of T. Wang [7] produce shapes which appear to have resulted from such an instability.

REFERENCES

[1] S. ALBANO and E.H.A. GONZALEZ, Rotating drops, Indiana Univ. Math. J. (1983).

[2] P. APPEL, Traité de Mécanique Rationelle, vol. 4, chap. 9, Paris, Gauthier-Villars, (1932).

[3] R.A. BROWN and L.E. SCRIVEN, The shape and stability of rotating liquid drops, Proc. R. Soc. Lond. A371 (1980), 331-357.

[4] S. CHANDRASEKHAR, The stability of rotating liquid drops, Proc. R. Soc. Lond. A286 (1965), 1-26.

[5] H. POINCARE', Capillarité, Editeur Georges Garre, Paris, (1895).

[6] LORD RAYLEIGH, The equilibrium of revolving liquid under capillary force, Phil. Mag. 28 (1914), 161-170.

[7] T.G. WANG and E. TRINH, Study of drop oscillation and rotation in immiscible liquid systems, these proceedings.

James ROSS - Frederic BRULOIS
Department of Mathematical Sciences
San Diego State University
San Diego, California 92182
U.S.A.