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ON THE STABILITY OF THE OBSTACLE PLATEAU PROBLEM ON LOCALLY PSEUDOCONVEX DOMAINS

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1. INTRODUCTION AND RESULTS.

§1.1 Minimal surfaces with obstacles have been studied intensively in the seventies from several points of views: in the parametric and non-parametric form; with smooth, discontinuous or thin obstacles; for classic and generalized surfaces (see [GP], [M1,2], [Ma] and [KS] for others references). Existence and regularity results have been well established and extended to surfaces with prescribed mean curvature (see [G]). The coincidence surface with the obstacle, and the free boundary, have been also deeply studied (see [KS], for references). However very little seems to have been considered from the point of view of the continuous dependence on the data. From the variational inequalities approach, there are general perturbations results (see [Mo], [KS]) with a few consequences for the obstacle Plateau problem. Recently, the author in [R] has considered the stability of the coincidence set for smooth minimal surfaces with respect to concave obstacles.

§1.2 Here we shall consider a family of bounded domains Ω_ν , in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundaries $\partial\Omega_\nu$ with generalized non-negative mean curvature, that is, the Ω_ν are locally pseudoconvex domains in the sense of Miranda [M2]. We shall consider the behaviour of minimal surfaces, which are graphs of functions u_ν , constrained by a fixed Lipschitz obstacle ψ. We assume $\max_{\Omega_\nu} \psi > 0$ and $\psi < 0$ on $\mathbb{R}^n \setminus \Omega_\nu$, for every ν , and we consider the following convex sets of Lipschitz functions

(1) $\mathbb{K}_\nu = \{v \in W_0^{1,\infty}(\Omega_\nu) : v \geq \psi \text{ in } \Omega_\nu\}$.

Then, u_ν will be the unique solution of the variational inequality

(2) $$u_\nu \in \mathbb{K}_\nu : \int_{\Omega_\nu} \frac{Du_\nu \cdot D(v - u_\nu)}{\sqrt{1 + |Du_\nu|^2}} dx \geq 0 \text{ , for all } v \in \mathbb{K}_\nu \text{ ,}$$

or, equivalently, u_ν will be the element of $\mathbb{K}_\nu$ of least area.

We shall consider the classical convergence of Kuratowski for sequences of sets, by assuming the existence of a domain Ω , such that, also $\max_\Omega \psi > 0$ and $\psi < 0$ on Ω^c , and

(3) $\Omega_\nu \to \Omega$ iff $\Omega^c = K - \lim \Omega_\nu^c$.

We recall that the convergence of the complements Ω_ν^c in the Kuratowski sense means that (i) for any $x \in \Omega^c$ there exists $x_\nu \in \Omega_\nu^c$ such that $x_\nu \to x$, and (ii) the limit of any subsequence $x_\mu \in \Omega_\mu^c$ is in Ω^c. Considering a smooth compact subset B of $\mathbb{R}^n$, containing strictly Ω and all Ω_ν , the convergence (3) holds if the $B \setminus \Omega_\nu$ are compact connected subsets converging to $B \setminus \Omega$ in the Hausdorff distance, since, then, this is equivalent to the Kuratowski convergence (see, for instance, [SW], pag. 22).

Let $\mathcal{O} = \text{int } B$ and $\tilde{\cdot}$ denote the extension in $\mathcal{O}$ by zero. Our first result, which we shall prove in the next section, is a continuous dependence result on the domain.

<u>Theorem 1</u>. *Let* Ω, Ω_ν *a family of locally pseudoconvex bounded domains in* $\mathcal{O}$. *If* $\Omega_\nu \to \Omega$, *then*

(4) $\tilde{u}_\nu \to \tilde{u}$ *in* $H_0^1(\mathcal{O}) \cap C^{0,\alpha}(\overline{\mathcal{O}})$, *for all* $0 \leq \alpha < 1$,

where the restriction $u = \tilde{u}|_\Omega \in \mathbb{K}$ *is the solution of* (2) *in* Ω , *and* $\tilde{u} = 0$ *in* $\mathcal{O} \setminus \Omega$. □

This theorem is an extension of [PS], where the asymptotic behaviour, with respect to the convergence of the domains in the Hausdorff distance, of solutions to linear partial differential equations is considered with a few applications to physical problems, optimum design and optimal control problems. We remember that classical results of continuous dependence on the boundary contour for parametric minimal surfaces without obstacles have been considered in [C], Ch. III §8.

§1.3 The solution u_ν divides Ω_ν into two subsets:

(5) $\quad I_\nu = I(u_\nu) = \{x \in \Omega_\nu \mid u_\nu(x) = \psi(x)\}$,

the coincidence set, which is a closed subset, and its complement $\Lambda_\nu =$ $= \{x \in \Omega_\nu \mid u_\nu(x) > \psi(x)\}$. The Theorem 1 enables us to apply the results of Theorems 1 and 2 of [R], which state that supplementary smoothness assumptions on the obstacle imply stability results for the coincidence set.

<u>Theorem 2</u>. *Under the conditions of the Theorem 1, and if*

(6) $\quad \psi \in W^{2,\infty}(\mathcal{O})$ *and* $D \cdot \left(\dfrac{D\psi}{\sqrt{1+|D\psi|^2}}\right) \neq 0$ *a.e. in* $\mathcal{O}$,

then $\operatorname{meas}(I_\nu \Delta I) \to 0$ *(here* Δ *denotes the symmetric difference).*

If, in addition, the limit coincidence set $I = I(u)$ *verifies*

(7) $\quad I = \overline{\operatorname{int} I}$,

then $I_\nu \overset{H}{\to} I$ *in the Hausdorff distance. In particular, if* I, I_ν *are convex sets, then also the free boundaries* $\partial I_\nu \overset{H}{\to} \partial I$. □

We observe that the convergence in the (Lebesgue) measure of the coincidence sets, in general, is not equivalent to the convergence in the Hausdorff distance. However, if I and I_ν are convex sets, the Hausdorff convergence implies the convergence in measure (see [B]). In that case, their boundaries also converge in the Hausdorff distance, which is the last assertion in the Theorem 2. We couldn't find a reference to this result, but it follows easily if we remind that the

Hausdorff distance between two closed sets is defined by $h(A,B) = $
$= \max\left[\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,B)\right]$. Indeed, since I_ν and I are non empty closed convex sets, we immediately conclude that $h(I,I_\nu) = h(\partial I,\partial I_\nu)$.

§1.4 In general, we don't know nontrivial conditions in order to assume the convexity of the coincidence sets. But there is a special case studied by Kinderlehrer (see [KS]) where the regularity of the free boundaries ∂I_ν is wellknown. Let $\Omega,\Omega_\nu \subset \mathbb{R}^2$ be strictly convex domains with $C^{2,\lambda}$-boundaries, and assume the obstacle $\psi \in C^4(\mathcal{O})$ and strictly concave in the following sense

$$D_{ij}\psi(x)\xi_i\xi_j < 0 \quad \text{for} \quad (0,0) \neq (\xi_1,\xi_2) \in \mathbb{R}^2 \quad \text{and} \quad x \in \mathcal{O} \quad .$$

In this case, by the results of Caffarelli, Kinderlehrer and Nirenberg (see [KS], for references) the free boundaries ∂I_ν and ∂I are $C^{2,\beta}$ curves $(0<\beta<1)$, which, locally, are graphs verifying (at least) a Lipschitz estimate independent of ν. Then, as in Theorem 4 of [R], from the convergence of the coincidence sets stated in Theorem 2, one can deduce:

Theorem 3. *Under the preceding assumptions, for strictly convex* $C^{2,\lambda}$*-domains of* $\mathbb{R}^2$ *and for a strictly concave* C^4 *obstacle, when* $\Omega_\nu \to \Omega$ *one has*

$$\partial I_\nu \to \partial I \quad in \quad C^{0,\alpha} \quad (for\ all\quad 0 \le \alpha < 1) \quad ,$$

where the convergence of the free boundaries is taken in the sense of their graphs. □

This theorem on the stability of the free boundary on the variation of the domain gives a positive answer to a conjecture formulated by Lewy, in page 562 of [L], for the Dirichlet integral with obstacle.

2. CONTINUOUS DEPENDENCE ON THE DOMAIN.

§2.1 The Theorems 2 and 3, which proofs we shall omit since they are essentially the same of [R], are based on the Theorem 1 which is new. Its proof follows some arguments of [PS] and [Mo] together with an apriori estimate of [Ma].

We begin with two lemmas.

Lemma 1. *For any* $v \in \mathbb{K}$ *, there exists* $v_\nu \in \mathbb{K}_\nu$ *, such that*

$$\tilde{v}_\nu \to \tilde{v} \quad in \quad W_0^{1,p}(\mathcal{O}) \ , \ for\ any \quad 1 \leq p < \infty \ . \tag{8}$$

Proof. First we remark that for any function w with support in Ω , there is some order ν_0 such that $\operatorname{supp} w \subset \Omega_\nu$ for all $\nu \geq \nu_0$. Indeed, if there exists an infinity of subsets Ω_ν^c verifying $\operatorname{supp} w \cap \Omega_\nu^c \neq \emptyset$, one can find a subsequence of points $x_\nu \in \operatorname{supp} w \cap \Omega_\nu^c$, such that $x_\nu \to x \in \operatorname{supp} w \subset \Omega$, which contradicts $\Omega^c = K - \lim \Omega_\nu^c$.

For any $v \in \mathbb{K}$ and any $\varepsilon > 0$, consider $\phi_\varepsilon \in \mathcal{D}(\Omega)$ such that $\|\phi_\varepsilon - v\|_{W_0^{1,p}(\Omega)} < \varepsilon$. Then $v_\varepsilon = \max(\phi_\varepsilon, \psi)$ also satisfies $\|v_\varepsilon - v\|_{W_0^{1,p}(\Omega)} < \varepsilon$, since $v \geq \psi$ and $\psi < 0$ in Ω^c implies $\operatorname{supp} v_\varepsilon \subset \Omega$. Hence, by the remark above, it follows $v_\varepsilon \in \mathbb{K}_\nu$ for all ν sufficiently large. □

Lemma 2 [Ma]. *Assume each boundary* $\partial\Omega_\nu \in C^2$ *, with non-negative mean curvature. Then, we have the following a priori estimate for the solution* u_ν *of* $(2)_\nu$ *,*

$$\|Du_\nu\|_{L^\infty(\Omega_\nu)} \leq e^R\left(1 + \|D\psi\|_{L^\infty(\mathcal{O})}\right) = M \quad , \tag{9}$$

where $R > 0$ *is a fixed constant bounding the internal radius of all* Ω_ν. □

The proof of this Lemma can be found in pages 125-126 of [Ma] for the case $\partial\Omega_\nu \in C^3$ and $\psi \in C^3$. As matter of fact, only the C^2-regularity for the boundary is needed. The extension to Lipschitz obstacles can be done by a limit argument as in [KS] page 134. We observe that for convex domains Ω_ν the estimate (9) becames $\|Du_\nu\|_{L^\infty} \leq \|D\psi\|_{L^\infty}$ and is due to [GP].

§2.2 Proof of Theorem 1. We assume first $\partial\Omega_\nu \in C^2$, for each ν. From Lemma 2, the functions $\tilde{u}_\nu$ are uniformly bounded in $W_0^{1,\infty}(\mathcal{O})$, and we can select a subsequence, still denoted by $\tilde{u}_\nu$, such that

$$\tilde{u}_\nu \to \tilde{u} \ , \text{ in } W_0^{1,\infty}(\mathcal{O})\text{-weakly* and in } C^{0,\alpha}(\overline{\mathcal{O}})\text{-strong} \quad ,$$

(for all $0 \le \alpha < 1$), by Rellich-Kondratchov theorem. It is clear that $\tilde{u} \ge \psi$ and, if $\Omega_\nu^c \ni x_\nu \to x \in \Omega^c$, from

$$|\tilde{u}(x)| = |\tilde{u}(x) - \tilde{u}_\nu(x_\nu)| \le |\tilde{u}(x) - \tilde{u}(x_\nu)| + |\tilde{u}(x_\nu) - \tilde{u}_\nu(x_\nu)| \quad ,$$

it follows $\tilde{u}(x) = 0$ for all $x \in \Omega^c$. Hence $u = \tilde{u}|_\Omega \in \mathbb{K}$. To show that u solves (2) in Ω, we use Lemma 1 and Minty's Lemma for the monotone vector field $a(p) = p/(1+p^2)^{1/2}$, $p \in \mathbb{R}^n$. For any $v \in \mathbb{K}$ take $v_\nu \in \mathbb{K}_\nu$ verifying (8), in

$$\int_{\mathcal{O}} a(D\tilde{v}_\nu) \cdot D(\tilde{v}_\nu - \tilde{u}_\nu)dx = \int_{\Omega_\nu} a(Dv_\nu) \cdot D(v_\nu - u_\nu)dx \ge 0 \quad ,$$

which follows from $(2)_\nu$. In the limit, we obtain

$$\int_\Omega a(Dv) \cdot D(v-u)dx = \int_{\mathcal{O}} a(D\tilde{v}) \cdot D(\tilde{v} - \tilde{u})dx \ge 0 \ , \text{ for all } v \in \mathbb{K}$$

which, by Minty's Lemma, is equivalent to (2). By uniqueness, the whole sequence $\tilde{u}_\nu$ converges to $\tilde{u}$. Finally, this convergence is also strong in $H_o^1(\mathcal{O})$, by the local coerciveness of $a(\cdot)$: using Lemma 1, take $\mathbb{K}_\nu \ni \tilde{w}_\nu \to \tilde{u} \in \mathbb{K}$ verifying (8), in the inequality (2_ν), in order to obtain

$$\limsup \int_{\mathcal{O}} a(D\tilde{u}_\nu) \cdot D(\tilde{u}_\nu - \tilde{u})dx \le \lim \int_{\mathcal{O}} a(D\tilde{u}_\nu) \cdot D(\tilde{w}_\nu - \tilde{u})dx = 0 \quad ,$$

which, then, can be used in $(\gamma = \gamma(M) > 0)$

$$0 \le \gamma \|\tilde{u}_\nu - \tilde{u}\|^2_{H_o^1(\mathcal{O})} \le \int_{\mathcal{O}} \left[a(D\tilde{u}_\nu) - a(D\tilde{u})\right] \cdot \left[D\tilde{u}_\nu - D\tilde{u}\right] dx \quad .$$

For general pseudoconvex domains Ω_ν, we first extend Lemma 2 by approximating, for each fixed ν, the solutions u_ν by solutions $u_{\nu\mu}$ corresponding to smooth domains $\Omega_{\nu\mu} \to \Omega_\nu$. Then the proof is exactly the same as before. □

§2.3 <u>Remarks</u> <u>1.</u>) Incidentally, the proof of Theorem 1 also gives an existence result for the obstacle Plateau problem in locally pseudoconvex domains. We observe that in [M2] the obstacle ψ is of class C^1 in the interior of the domain. <u>2.</u>) We have supposed $u_\nu = 0$ on $\partial\Omega_\nu$ only for simplicity. As in [Ma], one could assume $u_\nu = \phi$ on $\partial\Omega_\nu$, at least for some $\phi \in C^2(\mathcal{O})$, such that $\phi > \psi$ in $\mathcal{O} \setminus \Omega_\nu$, for any ν. 3.) Similar results can be obtained in the more general case of hypersurfaces of prescribed mean curvature over obstacles (see [G]), provided a Serrin's type condition is assumed on each $\partial\Omega_\nu$. □

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