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THE NORMAL VARIATIONS TECHNIQUE FOR STUDYING THE SHAPE OF CAPILLARY SURFACES

by N. KOREVAAR (University of Kentucky)

A capillary surface S is the (equilibrium) interface between two adjacent fluids that are also contacting rigid walls. In this paper we study the case for which S is the interface between two fluids in a vertical capillary tube, in the presence of a downward pointing gravitational field. S is the graph S_u of a function u whose domain is the (arbitrary) horizontal cross section Ω of the tube. Because S_u is in equilibrium its mean curvature is proportional to its height above a fixed reference plane, and the contact angle between (the normals of) S_u and the tube $\partial\Omega \times \mathbb{R}$ is physically prescribed.

Specifically, for $\Omega \subset \mathbb{R}^n$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a classical solution to the capillary problem if (Figure 1):

$$\begin{aligned} &\operatorname{div} Tu = \kappa u \quad \text{in} \quad \Omega \\ (1) \quad &Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \qquad \kappa > 0 \\ &Tu \cdot n = \cos\gamma \quad \text{on} \quad \partial\Omega , \\ &0 < \gamma < \pi \ , \ n = \text{exterior normal to } \partial\Omega . \end{aligned}$$

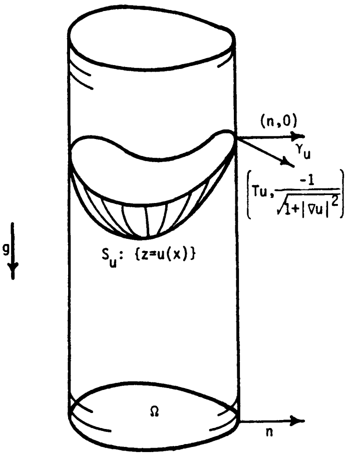

Figure 1: Configuration

We often write $H(x,u(x))$ for $\operatorname{div} Tu$. Geometrically $H(x,u(x))$ is n times the mean curvature of S_u, at the point

$(x,u(x))$.

The equations (1) arise from the energy functional

$$(2)\quad E(f) = \int_\Omega \left(\sqrt{1+|Df|^2} + \frac{\kappa}{2} f^2\right) - \int_{\partial\Omega} \mu f \ , \quad \mu = \cos\gamma \ .$$

The aim of this note is to indicate a natural way of studying the steepness of S_u. (Full details will appear elsewhere [4]). It is well understood how to estimate the height of S_u by constructing comparison surfaces; studying the steepness is the next logical step. (Actually, several impressive techniques for estimating $|\nabla u|$ are known, developed because of the important role that *a priori* gradient bounds play in regularity theory. The methods involved are fairly technical, however, and obscure the geometric content of the problem).

Intuitively, if a capillary surface is too steep it won't minimize (2) and will seek to reduce its energy by perturbing itself (perhaps as indicated by the shaded region in Figure 2). This intuition can be made rigorous: Perturbing S_u a small $(O(\varepsilon))$ amount along its (downward) normal produces a comparison surface S_v whose mean curvature can be estimated in terms of the original surface and the perturbation. If for appropriate perturbations and after raising S_v an amount also $O(\varepsilon)$, one can conclude that the lifted S_v lies above S_u (via a comparison principle), then one is in business: as $\varepsilon \to 0$ this yields a bound on $|\nabla u|$ (bottom of Figure 2).

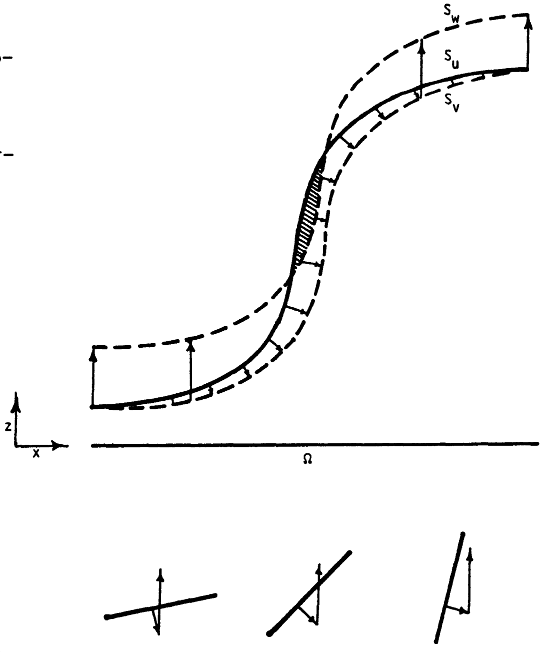

Figure 2: Intuition

The formula relating the mean

curvatures of S_u and S_v can be derived in a straightforward way. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be the perturbation function, ν the downward normal to S_u. Perturb $(x,u(x)) \in S_u$ by the vector $\varepsilon f \nu$. For small ε there is the correspondence (Figure 3):

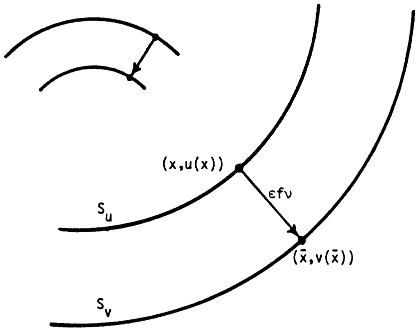

Figure 3: Normal perturbations

$$(3) \quad (\bar{x},v(\bar{x})) = (x,u(x)) + \varepsilon f \nu(x,u(x)) \ .$$

Using (3), the inverse function theorem (for $\partial x/\partial \bar{x}$) and the chain rule one can differentiate v. Using tangential coordinates yields

$$(4) \quad H(\bar{x},v(\bar{x}))$$

$$= H(x,u(x)) - \varepsilon(f\|\mathcal{D}^2u\|^2 + \Delta_T f - f_\nu H(x,u(x)) + O(\varepsilon^2)$$

$\mathcal{D}^2u$ = "tangential Hessian" = Hessian of the function parameterizing S_u above its tangential plane

$\Delta_T f$ = "tangential Laplacian" = $\Delta f - f_{\nu\nu}$

$f_\nu = \nabla f \cdot \nu$.

Lifting S_v an amount $O(\varepsilon)$ gives a surface S_w which will be compared to S_u via the well known:

Comparison Principle (C.P.): *Let* $\mathcal{O} \subset \Omega$, n = *exterior normal to* $\partial\mathcal{O}$, $u,w \in C^2(\overline{\mathcal{O}})$. *Suppose*

(i) $\forall x \in \mathcal{O}$ s.t. $w(x) < u(x)$, $H(x,w(x)) < H(x,u(x))$

(ii) $\forall x \in \partial\mathcal{O}$ s.t. $w(x) \leq u(x)$, $\gamma_w \leq \gamma_u$ (i.e. $Tw \cdot n \geq Tu \cdot n$) .

Then, in fact $w(x) \geq u(x)$ $\forall x \in \overline{\mathcal{O}}$.

The easiest gradient estimate is the *a priori* interior bound for the capillary problem:

<u>Theorem 1</u>. *Let* u *satisfy* $H(x,u(x)) = \kappa u$ *in* $B_R = \{x \in \mathbb{R}^n \,\big|\, |x| < R\}$, $\kappa > 0$. *Then* $\exists C = C(R,\kappa)$ s.t. $|\nabla u(x)| \leq \dfrac{C}{R - |x|}$.

<u>Proof</u>. Pick $f(x,z) = f(x)$ as indicated in Figure 4 and so that $|f|_\infty \leq 1$. Perturb S_u to S_v. From (4), $\exists$ M s.t.

(5) $H(\bar{x},v(\bar{x})) \leq H(x,u(x)) + \varepsilon M$.

(We used the well-known height estimates that imply u is bounded in B_R so that κu is too). Writing $w(\bar{x}) = v(\bar{x}) + \varepsilon(1 + (M/\kappa))$ one verifies that $H(\bar{x},w(\bar{x})) \leq \kappa w(\bar{x})$, from which condition (i) of C.P. follows. Condition (ii) is true trivially. Hence $w \geq u$ in $\bar{B}_R$:

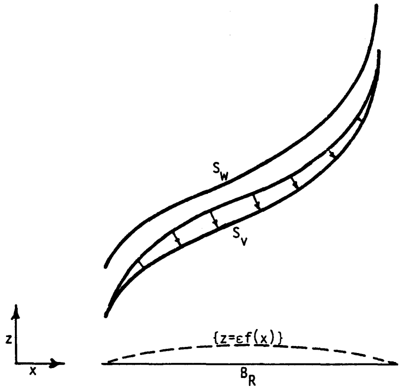

Figure 4: Interior gradient bound in gravity

$$w(\bar{x}) \geq u(\bar{x})$$

$$u(x) - \frac{\varepsilon f}{\sqrt{1 + |\nabla u|^2}} + \varepsilon\left(1 + \frac{M}{\kappa}\right) > u(x) + \varepsilon f \nabla u \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} + O(\varepsilon^2)$$

$$1 + \frac{M}{\kappa} > f \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} \quad .$$

q.e.d.

For the general prescribed mean curvature equation [1, 2, 6, 9] $f(x,z)$ must be chosen more carefully. After lifting S_v a large enough $O(\varepsilon)$ to get S_w , the only places where (i) must be checked are where $|\nabla u|$ is large. Hence the tangent plane is almost vertical. By introducing exponential growth in $f(x,z)$ in the z direction (and making other modifications too) one can make f_{zz} and hence $\Delta_T f$ large enough to dominate the other terms that arise in comparing $H(x,w(x))$ to $H(x,u(x))$.

To estimate $|\nabla u|$ near $\partial\Omega$ f must be non-zero there. Hence S_v may not lie above Ω (Figure 5). The fact that $Tu \cdot n$ is prescribed, however, forces S_v to lie (within $O(\varepsilon^2)$ of being) above Ω_ε, the domain whose boundary $\partial\Omega_\varepsilon$ is gotten by perturbing $\partial\Omega$ an amount $\varepsilon f \cos\gamma$ along n. Hence by following the normal perturbation with a change of x-coordinates one can return S_v to $S_{\tilde{v}}$, a surface above Ω. If one chooses f and the return transformation carefully the contact angle condition (ii) of C.P. can be controlled. In order to keep $H(x,\tilde{v}(x)) \leq H(x,u(x)) + \varepsilon M$ for condition (i), the return transformation must be chosen carefully (Figure 6). (For global estimates f is never zero and one can just shove everything back along n). One gets:

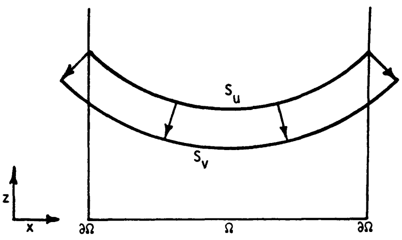

Figure 5: Normal perturbations near $\partial\Omega$

<u>Theorem 2</u> [3, 7, 8, 10]. *Let* $u \in C^2(\overline{\mathcal{O} \cap \Omega})$, $\partial\Omega \cap \mathcal{O}$ *smooth* (C^3), u *satisfying* (1) *in* $\overline{\mathcal{O} \cap \Omega}$ *and* $x \in \mathcal{O} \cap \overline{\Omega}$. *Then there is an a priori bound* $|\nabla u(x)| \leq C(x, \partial\Omega \cap \mathcal{O}, \kappa, \gamma)$.

<u>Remark</u>. G. Lieberman has recently been able to use maximum principle ideas to get global (but not local) gradient estimates for a class of operators and boundary conditions generalizing (1) [5].

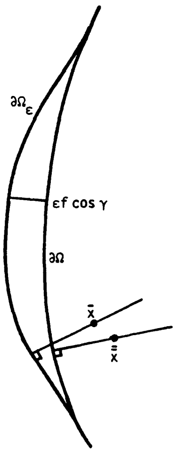

Figure 6: Returning Ω_ε to Ω

There are more applications of the normal variations technique for studying ∇u. Monotonicity results can be proven in domains whose shape leads one to expect monotonicity. Lipschitz continuity of solutions in singular domains of the correct geometry can be shown. Mean curvature problems other than those in capillarity can be studied.

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