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FREE BOUNDARY PROBLEMS FOR THE NAVIER-STOKES EQUATIONS

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1. SOME EXAMPLES OF FREE BOUNDARIES IN VISCOUS FLUID FLOW.

Let us consider the following free boundary problem for the Navier-Stokes equations: a drop of a viscous, incompressible fluid with prescribed volume is given and an exterior force density f generates a flow inside the drop; we assume that the shape of the boundary is governed by surface tension. If f does not depend on time, this situation can be described by the following equations:

$$(1.1)\quad \begin{array}{l} -\nu\Delta v + \nabla p + (v\cdot\nabla)v = f \\ \nabla\cdot v = 0 \end{array} \quad \text{in } \Omega$$

$$(1.2)\quad v\cdot n = 0,\ \ \tau_k\cdot T\cdot n = 0 \ \text{ on } \Sigma,\ \ k = 1,2 \ .$$

$$(1.3)\quad n\cdot T\cdot n = 2\kappa H \ \text{ on } \Sigma,\ \ \text{meas}\ \Omega = \frac{4}{3}\pi \ .$$

As usual v and p denote the fluid's velocity and pressure, Ω is the unknown domain which is occupied by the fluid, and $\Sigma = \partial\Omega$ denotes its boundary. ν is the kinematic viscosity, n is the exterior normal, and τ_1, τ_2 span the tangent plane of Σ. The stress tensor is given by

$$(1.4)\quad T_{ij} = -p\delta_{ij} + \nu\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right) \ .$$

According to (2) the tangential component of the stress vector $T\cdot n$ vanishes on Σ. The normal component of $T\cdot n$ is proportional to the mean curvature H of Σ, hence equation (4) which determines the free boundary can be interpreted as an equilibrium condition for the normal stresses and the force due to surface tension. We may assume the surface tension κ normalized to be 1.

There are several extensions of the above problem which can be treated with the same approach. The drop Ω may be immersed in a (finite or infinite) reservoir G of another fluid with the same density but with different viscosity. In this case we have to solve in addition to (1) - (3) the Navier-Stokes equations in G for u and q with Dirichlet conditions $\underline{u} = 0$ on ∂G or rather $\underline{u}(x) \to 0$, as $|x| \to \infty$, if $G = \mathbb{R}^3 \setminus \overline{\Omega}$. The stress tensor in (2) and (3) on the interface Σ between the two fluids has to be replaced by the difference $[T] = T(v,p) - T(u,q)$.

Of particular interest is the motion of a drop Ω that falls down under its own weight in an unbounded reservoir of another fluid. In this case the motion is steady only with respect to a reference frame that is attached to the falling drop. Its speed of falling γ with respect to a fixed frame is an unknown of the problem, too, and γ will be determined by an equilibrium condition for the viscous forces acting on the drop and its weight. We now have the equations

$$(1.5)\qquad \begin{aligned} &-\mu\Delta v + \nabla p + \rho((v-\gamma)\cdot\nabla)v = \rho g \\ &\nabla\cdot v = 0 \end{aligned} \qquad \text{in } \Omega$$

$$(1.6)\qquad \begin{aligned} &-\mu^*\Delta u + \nabla q + ((u-\gamma)\cdot\nabla)u = \rho^* g \\ &\nabla\cdot u = 0 \end{aligned} \qquad \text{in } \mathfrak{E}$$

$$(1.7)\qquad u(x) \to 0\ ,\ \text{as}\ \ |x| \to 0$$

$$(1.8)\qquad \begin{aligned} &v - u \equiv 0\ ,\quad v\cdot n = u\cdot n = \gamma\cdot n && \text{on } \Sigma \\ &\tau_k\cdot[T]\cdot n = 0 && \text{on } \Sigma \end{aligned}$$

$$(1.9)\qquad n\cdot[T]\cdot n = 2\kappa H \ \text{ on } \Sigma\ ,\quad \text{meas}\ \Omega = \frac{4}{3}\pi$$

(1.10) $\int_\Sigma [T] \cdot n \, d\sigma = (\rho^* - \rho) \, g \text{ meas } \Omega$.

Here ρ, ρ^* and μ, μ^* are the densities and the viscosities of the two fluids, g is the gravitational force which we can assume to be of the form $(0,0,g)$.

This paper contains results on stationary problems only. Local existence and regularity theorems for the time-dependent versions of (1) - (3), and (5) - (10) can be found in [5].

2. RESULTS.

The following result is contained in [2].

Theorem 2.1. *Let* $f \in C^{0+\alpha}$ *be small. Then the free boundary problem* (1.1) - (1.3) *admits a unique solution* $v \in C^{2+\alpha}(\bar{\Omega})$, $p \in C^{1+\alpha}(\bar{\Omega})$, $\Sigma \in C^{3+\alpha}$, *provided* f *satisfies the equilibrium condition*

(2.1) $\int_\Omega f \, dx = 0$.

Remark 2.2. As (2.1) contains the free boundary, one cannot check a priori whether f satisfies (2.1). In some physically interesting situations we can state sufficient conditions on f such that (2.1) is satisfied, cf. also [4]. If f is a function of $r = \sqrt{(x^1)^2 + (x^2)^2}$ and x^3 with $f(r,x^3) = f(r,-x^3)$, then (2.1) holds.

On the other hand, if the flow in Ω is generated by a rigid sphere $B_\varepsilon(0)$ which is located in Ω and rotates with constant angular velocity then the equilibrium condition (2.1) is again satisfied. Now Ω is bounded by $\partial B_\varepsilon(0)$ and by the free boundary Σ and the extension of the nonhomogeneous Dirichlet data on $\partial B_\varepsilon(0)$ lead to a right-hand side in (1.1) that fulfills (2.1).

Theorem 2.3. *If the difference* $|\rho - \rho^*|$ *of the densities is small, problem* (1.5) - (1.10) *admits a unique solution* $v \in C^{2+\alpha}(\bar{\Omega})$, $p \in C^{1+\alpha}(\bar{\Omega})$, $u \in C^{2+\alpha}(\bar{\mathbb{E}})$, $q \in C^{1+\alpha}(\bar{\mathbb{E}})$, $\Sigma \in C^{3+\alpha}$, *and* $\gamma \in \mathbb{R}$. *The free surface* Σ *and the functions*

v,p,u,q *are rotationally symmetric with respect to the* x^3*-axis.*

This result is contained in [3]. It is natural to expect the solutions to be more regular if more regularity is assumed for f. However, the usual bootstrap argument does not apply to these free boundary problems, and hence we get regularity theorems only for small solutions.

Theorem 2.4. *Let* f *be of class* $C^{k+\alpha}$, *and* v,p,Σ *the solution to* (1.1) - (1.3). *Under the assumption that* v *is small in* $C^{o+\alpha}$*-norm the solution is regular:* $v \in C^{k+2+\alpha}(\bar{\Omega})$, $p \in C^{k+1+\alpha}(\bar{\Omega})$, $\Sigma \in C^{k+3+\alpha}$. *The same result holds for a small solution* (v,p,u,q,Σ,γ) *of* (1.5) - (1.10).

That the solution (v,p,Σ) of (1.1) - (1.3) is analytic if the exterior force is analytic was proved by A. Friedman and the author in [7].

Theorem 2.5. *Let* f *be an analytic force density. Then the solution* (v,p,Σ) *to* (1.1) - (1.3) *is analytic, if* $\|v\|_{C^{o+\alpha}}$ *is small.*

The assumption that $\|v\|_{C^{o+\alpha}}$ has to be small is connected with the approximation procedure that yields the solutions to (1.1) - (1.3) and (1.5) - (1.10), resp.

3. AN APPROXIMATION SCHEME FOR THE FREE BOUNDARY PROBLEMS.

In the absence of an exterior force f the solution to (1.1) - (1.3) would be a spherical drop with no flow inside: $v_o \equiv 0$, $p_o = \text{const}$, $\Sigma_o = \partial B_1(0)$. Therefore we seek a solution to $f \equiv 0$ in the neighborhood of this rest-solution (v_o,p_o,Σ_o). Free surfaces in the neighborhood of $S \equiv \partial B_1(o)$ can be represented as graphs over S:

$$(3.1) \qquad \Sigma = \{(\xi,u(\xi)) : u : S \to \mathbb{R},\ \xi \in S\} \quad ,$$

and (1.3) can be written in terms of u:

$$(3.2) \qquad \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial\xi^i}\,\frac{\sqrt{g}\,g^{ij}u_{\xi^j}}{\sqrt{1+|\mathfrak{D}u|^2}} - \frac{\partial}{\partial u}\left(\sqrt{g}\sqrt{1+|\mathfrak{D}u|^2}\right)\right\} = n\cdot T\cdot n \quad ,$$

where $g_{ij}(\xi,r)$ is the metric on $\partial B_r(o)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$, $g = \det\{g_{ij}\}$, $|\mathcal{D}u|^2 = g^{k\ell}\, u_{\xi k} u_{\xi\ell}$. Now we are able to formulate the approximation scheme. Start with $\Sigma_o = S$ and solve in Ω_o , the domain bounded by Σ_o , the Navier-Stokes equations (1.1) - (1.2) with the given f. Call the solution (v_1,p_1) and set $h_1(\xi) = n(\xi,u_o(\xi)) \cdot T(v_1(\xi,u_o(\xi)),p_1(\xi,u_o(\xi))) \cdot n(\xi,u_o(\xi))$. Solve (3.2) for the righthand side h_1 and call its solution u_1. This function u_1 determines a new boundary Σ_1 , and hence a new domain Ω_1 , in which we can solve again (1.1) - (1.2). In this way we obtain a sequence $\{(v_n,p_n,u_n)\}$ which converges to (v,p,u) , the solution to (1.1) - (1.3), if we can show the following properties: (i) there exists a function space $\mathfrak{X}$, such that $(v_{n+1},p_{n+1},u_{n+1}) \in \mathfrak{X}$ if the previous approximation is in $\mathfrak{X}$; (ii) $\{(v_n,p_n,u_n)\}$ is a Cauchy sequence with respect to the norm of $\mathfrak{X}$.

Let G be a domain and $\phi : \overline{B_1(o)} \to \overline{G}$ be a diffeomorphism. Solenoidal vector fields $\omega : \overline{G} \to \mathbb{R}^3$ can be mapped onto solenoidal vector fields $\omega^* : \overline{B_1(o)} \to \mathbb{R}^3$ by the transformation

$$(3.3) \qquad \omega^*(X) := \frac{\partial\phi}{\partial x} \cdot \omega(\phi(X)) \ , \quad X \in B_1(0) \quad .$$

Hence we can think of v_n and p_n to be defined on a fixed domain $\overline{B} = \overline{B_1(o)}$ instead of Ω_{n-1}. For the function space $\mathfrak{X}$ we choose:

$$(3.4) \qquad v \in C^{2+\alpha}(B),\ p \in C^{1+\alpha}(B),\ u \in C^{3+\alpha}(S) \quad .$$

A solution v,p to (1.1) - (1.2) can be estimated in the $C^{2+\alpha}\times C^{1+\alpha}$ -norm

$$(3.5) \qquad \|v\|_{C^{2+\alpha}(\overline{B})} + \|p\|_{C^{1+\alpha}(\overline{B})} \le C\Big(\|f\|_{C^{o+\alpha}(\overline{B})}, \|u\|_{C^{3+\alpha}(S)}\Big) \ ,$$

and that $\|u\|_{C^{3+\alpha}(S)}$ cannot be replaced by a weaker norm is an immediate consequence of the fact that the Navier-Stokes equations are elliptic in the sense of Agmon-Douglis-Nirenberg and that (1.2) satisfies the Complementing Boundary Condition. Now $v_n \in C^{2+\alpha}$, $p_n \in C^{1+\alpha}$ implies $T_n \in C^{1+\alpha}$, and therefore

$u_{n+1} \in C^{3+\alpha}$, as (3.2) is an elliptic equation of second order. This shows that the approximation scheme can be carried out in $\mathfrak{X}$.

The convergence of $\{(v_n, p_n, u_n)\}$ follows from the estimates

$$(3.6) \qquad \|v_{n+1} - v_n\|_{C^{2+\alpha}} + \|p_{n+1} - p_n\|_{C^{1+\alpha}} \leq c(\nu)\,\|u_n - u_{n-1}\|_{C^{3+\alpha}}$$

$$(3.7) \qquad \|u_n - u_{n-1}\|_{C^{3+\alpha}} \leq c^* \left\{ \|v_n - v_{n-1}\|_{C^{2+\alpha}} + \|p_n - p_{n-1}\|_{C^{1+\alpha}} \right\}$$

together with $cc^* < 1$, which can be achieved by making ν^{-1} small.

Remark 3.1. If the fluid body is very large, surface tension can be neglected and we get a related boundary problem consisting of (1.1), (1.2) and

$$(3.8) \qquad n \cdot T \cdot n = 0 \quad \text{on } \Sigma ,$$

which replaces (1.3). In this case one looses two derivatives in every approximation step, and the above approximation scheme cannot be continued for all $n \in \mathbb{N}$. In the corresponding non-stationary problem V.A. Solonnikov [10] introduced Lagrangian coordinates, such that the boundary condition for $v \cdot n$ is automatically fulfilled on the free boundary. This reduces the free boundary problem to a Neumann problem on a fixed domain. The same transformation was used by J.T. Beale [1] and by the author [6] for this type of free boundary problem.

4. REGULARITY OF THE SOLUTIONS.

It is well known that if (v,p) solves (1.1) - (1.2) in a given domain the solution is as regular as the data allow. The same holds for any solution u of (3.2) with a given right-hand side. Using a bootstrap argument one can separate the question of regularity of solutions from that of existence. This kind of linearization does not yield any improvement of $(v,p,u) \in \mathfrak{X}$ for the free boundary problem. Therefore we obtain higher regularity $v \in C^{k+2+\alpha}$, $p \in C^{k+1+\alpha}$, $u \in C^{k+3+\alpha}$ if $f \in C^{k+\alpha}$ by proving the existence of such a solution in the same way which we described in 3. The estimates (3.6), (3.7) hold with $C^{i+\alpha}$, $i=0,..,3$, replaced by $C^{k+i+\alpha}$ for any $k>0$, and hence $CC^* < 1$ can be obtained again by assuming ν^{-1} to be small. As the constants C and C^* generally depend on k this requires smaller and smaller data in every step. To avoid this restriction we give improved Schauder estimates:

$$(4.1)\qquad \|v_{n+1}-v_n\|_{C^{k+2+\alpha}} + \|p_{n+1}-p_n\|_{C^{k+1+\alpha}} < C(\nu)\,\|u_n-u_{n-1}\|_{C^{k+3+\alpha}} +$$

$$+\sum_{i=o}^{k-1} c(i)\left\{\|u_n-u_{n-1}\|_{C^{i+3+\alpha}} + \|v_{n+1}-v_n\|_{C^{i+2+\alpha}}\|p_{n+1}-p_n\|_{C^{i+1+\alpha}}\right\}$$

and similarly for $\|u_n-u_{n-1}\|_{C^{k+3+\alpha}}$, where $C(\nu)$ does not depend on k.

The approximation scheme of 3 can be combined with A. Friedman's method for proving analyticity of solutions of non-linear systems, cf. [8]. For the proof of Theorem 2.5 we need

Lemma 4.1. *Let* $\|\cdot\|_{\alpha,\delta}$ *denote the* $C^{o+\alpha}$*-norm over a ball of radius* δ.
If

$$(4.2)\qquad \|D_\xi^{m+1}\,u\|_{\alpha,\delta} \leq c_o\left(\frac{A}{\delta}\right)^{m-2}(m-2)!$$

and

$$(4.3)\qquad \begin{aligned} &\|D_\xi^{m}\,D_x^{1-i}\,v\|_{\alpha,\delta} \leq \left(\frac{A}{\delta}\right)^{m-1-i}(m-1-i)!\,, \quad i=0,1\,, \\ &\|D_\xi^{m}\,p\|_{\alpha,\delta} \leq \left(\frac{A}{\delta}\right)^{m-1}(m-1)! \end{aligned}$$

for all $m\leq s$, $0<\delta<R$, then

$$(4.4)\qquad \|D_\xi^{s+2}\,u\|_{\alpha,\delta} < c_o\left(\frac{A}{\delta}\right)^{s-1}(s-1)!\,,$$

provided $\|v\|_{C^{o+\alpha}}$ is small.

Lemma 4.2. *Under the hypotheses of Lemma 4.1 there holds*

$$
\|D_\xi^s D_x^2 v\|_{\alpha,\delta} < \left(\frac{A}{\delta}\right)^s s!
$$

(4.5)

$$
\|D_\xi^s D_x p\|_{\alpha,\delta} < \left(\frac{A}{\delta}\right)^s s!
$$

Here D_ξ^i and D_x^i denote partial differentiations of order i with respect to the variables ξ^1 and ξ^2 or x^1, x^2, and x^3.

The estimates (4.4) and (4.5) imply that u is analytic and that some derivatives of v and p have the right growth. But once the boundary u is analytic, the analyticity of v and p follows from well known results.

We finally remark that there exists a counterexample due to T.A. McCready [9] which shows that Dirichlet's integral of a solution v to the Navier-Stokes equations in a fixed domain cannot be bounded a priori if the boundary data are other than Dirichlet. This suggests that global solutions to free boundary problems like (1.1) - (1.3) do not exist.

R E F E R E N C E S

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