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In this paper we investigate 1-dimensional foliations with a transverse hyperbolic structure. There is a danger of confusion here, because of different meanings of the word "hyperbolic". What is meant in this paper by a transverse hyperbolic structure for on a 1-dimensional foliation of a manifold of dimension n, is that the given foliation is given by local submersions into $\mathbf{H}^{n-1}$, the hyperbolic space of dimension $(n-1)$, and that different submersions differ from each other on their common domain by composition with an isometry of $\mathbf{H}^{n-1}$.

There is a theory of transverse Riemannian structures on foliations, of which the above is a very special case. The first paper in this area was by Reinhart [9]. Amongst other contributors to the subject are Fedida [5] and Conlon [3]. The reader is referred to the papers of Molino, especially [8], for an elegant general treatment. Molino's idea of replacing the model transverse manifold by its associated bundle of frames is particularly illuminating. Papers by Carrière and Molino in these proceedings contain further details and references.

In this paper, we have thought it advisable to repeat some proofs of results due variously to Thurston, Carrière and Molino, in order to make the reader's task easier. Also, some of these results are not easily available for reference.

Our main result is the following generalization of a theorem due to Thurston [11].

Main Theorem. *Let M be a closed n-manifold with a smooth 1-dimensional foliation Φ, with transverse hyperbolic structure. Then one of the following two possibilities must occur:*

1) Each leaf of Φ is a circle. The holonomy map $h:\pi_1 M \to \mathrm{Isom}\mathbf{H}^{n-1}$ has a discrete image and a non-trivial kernel. The holonomy associated to each leaf is finite (and is generically zero).

2) The manifold M has dimension $n=3$ or 4. The closure of each leaf is a torus of dimension 2 or 3 respectively. The holonomy representation is injective, and fixes a certain point on the boundary sphere of $\mathbf{H}^{n-1}$. Transferring to the upper halfspace model, by putting this point at infinity, we find a similarity S of the boundary of the upper halfspace, $\mathbf{R}^{n-2}$, which has a change of scale $\lambda<1$, such that the image of $\pi_1 M$ acts by elements of the form $z \to S^k z+b$. As we range over elements of $\pi_1 M$, k takes on all integral values, and the elements b form a dense subgroup of $\mathbf{R}^1$ or $\mathbf{R}^2$ respectively. This subgroup is invariant under S. M is a fibre bundle, with fibre either T^2 or T^3 and each fibre is foliated by the leaves of Φ. A leaf is always dense in its fibre. The base space of the bundle is a circle, and the monodromy of the bundle is an isomorphism $A:T^n \to T^n$, where $A \in GL(n,\mathbf{Z})$ and $n=2$ or 3. A has one expanding direction, with eigenvalue $\pm\lambda^{-n+1}$ and either one or two contracting directions, according as $n=2$ or $n=3$. If $n=3$, then the contracting directions have eigenvalues which have equal absolute values. The foliation Φ consists of lines on T^n, parallel to the expanding direction of A.

The manifold and foliation also have an algebraic description. M is the quotient of a simply connected solvable Lie group G by a group Γ of affine automorphisms of G, which acts freely on G. Γ has a subgroup Γ_0 of index at most two which is a uniform discrete subgroup of G acting on the right, and the foliation is given by a 1-parameter subgroup acting on the left. Elements of Γ, which are not in Γ_0, reverse either the orientation of the leaves in G, or the transverse orientation.

§1. Transverse Riemannian Structures.

Let G be a group of isometries of a Riemannian manifold. Let M be a smooth manifold and let F be a smooth foliation. We say that F is a (G,X)*-foliation*, if it is defined by smooth local submersions $f_i:U_i \to X$, where the $\{U_i\}$ are an open covering of M, and the leaves of F are given locally as connected components of the inverse images of points of X. There is assumed to be a locally constant map $\gamma_{ij}:U_i\cap U_j \to G$, such that $f_i(x)=\gamma_{ij}(x)f_j(x)$ for $x\in U_i\cap U_j$. Clearly, γ_{ij} is determined by f_i and f_j. The $\{f_i\}$ are called *admissible submersions*. We will suppose that the family $\{f_i\}$ is maximal, with the object of making the structure unique, as is usual in manifold theory.

We now define the holonomy homomorphism and the developing map. Suppose $\alpha:I\to M$ is a path, u is the germ of an admissible submersion defined near $\alpha(0)$ and v is the germ of an admissible submersion defined near $\alpha(1)$. We choose a partition $0=t_0\le t_1\le \cdots \le t_k=1$ and admissible submersions $f_i:U_i\to X$, where $\alpha[t_{i-1},t_i]\subseteq U_i$ $(1\le i\le k)$. Let $f_0=u$ and $f_{k+1}=v$, and let $t_{-1}=t_0=0$, $t_{k+1}=t_k=1$. Let $\gamma_{i,i-1}\in G$ be defined by $\gamma_{i,i-1}f_{i-1}=f_i$ near $\alpha(t_{i-1})$, for $1\le i\le k+1$. We define

$$h(\alpha,v,u)=\gamma_{k+1,k}\gamma_{k,k-1}\cdots\gamma_{2,1}\gamma_{1,0}.$$

We now have to show that $h(\alpha,u,v)$ is independent of the choices involved.

Step 1. If $0=t_0\le t_1\le\cdots\le t_k=1$ is fixed, and $f_i:U_i\to X$ is changed to $f'_i:U'_i\to X$, then $h(\alpha,v,u)$ is unaltered.

Clearly, we may assume that only one f_i is changed. Then $f'_i=\gamma f_i$ on $\alpha[t_{i-1},t_i]$, for some $\gamma\in G$. It follows that $\gamma'_{i,i-1}=\gamma\gamma_{i,i-1}$ and $\gamma'_{i+1,i}=\gamma_{i+1,i}\gamma^{-1}$, and so $h(\alpha,v,u)$ is unaltered.

It follows that the definition of $h(\alpha,v,u)$ depends at most on the partition $0=t_0\le t_1\le\cdots\le t_k=1$.

Step 2. Given a partition $0=t_0\le t_1\le\cdots\le t_k=1$, and a computation of $h(\alpha,v,u)$ using this partition, any finer partition gives the same answer.

We may assume that a single point is added, with $t_{i-1}\le s\le t_i$. The computation may be performed by associating each of the two intervals $[t_{i-1},s]$ and $[s,t_i]$ with the same submersion $f_i:U_i\to X$. The corresponding coordinate transformation in G, coming from the point s, is the identity, so that $h(\alpha,v,u)$ is unchanged.

>From these two steps, it follows immediately that $h(\alpha,v,u)$ is well-defined. We have the following easily verified properties:

1.1. $h(\alpha^{-1},u,v)=h(\alpha,v,u)^{-1}$.

1.2. If α and β are two paths and $\alpha(1)=\beta(0)$, then we define the path $\beta\alpha$ in the obvious way. Let w be a germ of an admissible submersion near $\beta(1)$. Then

$$h(\beta\alpha,w,u) = h(\beta,w,v)h(\alpha,v,u).$$

1.3. If $\gamma_0,\gamma_1\in G$, then $h(\alpha,\gamma_1 v,\gamma_0 u)=\gamma_1 h(\alpha,v,u)\gamma_0^{-1}$.

1.4. If α is homotopic to β, keeping endpoints fixed, then $h(\alpha,v,u)=h(\beta,v,u)$. The proof of this is that a very small movement of α can be dealt with without changing the $f_i:U_i\to X$.

1.5. Let u be a germ of an admissible submersion at $m_0\in M$. Then $h_u:\pi_1(M,m_0)\to G$, defined by $h_u(\alpha)=h(\alpha,u,u)$, is a homomorphism. This homomorphism is called the *holonomy homomorphism based on* u of the (G,X)-foliation.

1.6. $h_{\gamma u}=\gamma h_u\gamma^{-1}$ for $\gamma\in G$.

Let $\tilde{M}$ be the universal covering of M. The *developing map* $D_u:\tilde{M}\to X$, is defined as follows. We regard elements of $\tilde{M}$ as homotopy classes of paths $\alpha:I,0\to M,m_0$. Using the notation above, we define

$$D_u(\alpha)=h(\alpha,v,u)^{-1}v(\alpha(1)).$$

1.7. This map is a well-defined submersion $D_u:\tilde{M}\to X$ and

$$D_{\gamma u}(\alpha)=\gamma D_u(\alpha)$$

1.8. If β is a loop based at m_0, 1.2 implies that

$$D(\alpha\beta,u)=h(\alpha\beta,v,u))^{-1}v(\alpha(1))$$

$$=h(\beta,u,u)^{-1}D(\alpha,u)$$

or $D_u(\tilde{m}\beta)=h_u(\beta)^{-1}D_u(\tilde{m})$

Thus, transferring the action of $\pi_1(M,m_0)$ to the left of $\tilde{M}$ by the definition $\beta.\alpha=\alpha\beta^{-1}$, we obtain the equation

1.9. $D_u(\beta\tilde{m})=h_u(\beta)D_u(\tilde{m})$.

A (G,X)-foliation is said to be *complete* if $D_u:\tilde{M}\to X$ is a locally trivial fibre bundle for some (and hence for any) u. The following result is due to Thurston [11]. A version of it was proved earlier by Ehresmann [4], under much stronger assumptions.

1.10. **Theorem.** *If M is a closed manifold, the foliation is complete. If X is simply connected, then each leaf of the foliation of $\tilde{M}$ is a fibre of $D_u:\tilde{M}\to X$ and conversely.*

Proof. Define a smooth field τ of planes on M, which are transverse to the foliation and have the same dimension as X. (For example, take any smooth metric on M and take the planes orthogonal to the foliation.) Use τ to construct a new Riemannian metric on M, which induces the same metric as before on each leaf, such that τ is orthogonal to the foliation, and such that each admissible local submersion f maps τ_x $(x\in U)$ by a linear isometry onto the tangent space to X at fx.

A path in M (or $\tilde{M}$) is said to be *horizontal* if it is tangent to the field τ (or the lifted field $\tilde{\tau}$ in $\tilde{M}$) at each point of the path.

1.11. **Lemma.** *If M is closed, then, given any path $\alpha:(a,b)\to X$ $(-\infty\le a<b\le\infty)$ and any c $(a<c<b)$, and an element $\tilde{\alpha}(c)\in\tilde{M}$, such that $D_u\tilde{\alpha}(c)=\alpha(c)$, there is a unique horizontal lifting $\tilde{\alpha}:(a,b)\to\tilde{M}$, such that $D_u\tilde{\alpha}=\alpha$.*

Proof. Paths can be lifted locally, using the differential equation $\frac{d\tilde{\alpha}}{dt}=\Theta(\tilde{\alpha}(t))\frac{d\alpha}{dt}$, where $\Theta(\tilde{m})$ is the linear isometry described above, from the tangent space to X at $D_u(\tilde{m})$ to $\tau_{\tilde{m}}$. Clearly, the lifting is unique and α and $\tilde{\alpha}$ have the same length. Let $(a',b')\subset(a,b)$ be a maximal subinterval over which $\tilde{\alpha}$ is defined. Since M is compact, $\tilde{M}$ is complete as a Riemannian manifold, and so, if $b'<b$, $\tilde{\alpha}(t)$ tends to a limit as t tends to b'. But then $\tilde{\alpha}$ can be extended beyond b'. This contradiction shows that $b'=b$. Similarly $a'=a$. This completes the proof of the lemma.

It follows that a ball in X with centre t_0 can be uniquely lifted into $\tilde{M}$ so that radii are horizontal, once the lifting of the centre is fixed. The lifting is smooth (with no singularity at the centre) because solutions to a differential equation depend smoothly on parameters. This gives a local product structure to the map $D_u:\tilde{M}\to X$ and completes the proof of Theorem 1.10, except for the special situation when X is simply connected, which will be dealt with in Lemma 1.12.

If F is a foliation on a connected manifold M with a transverse Riemannian structure, modelled on a Riemannian manifold X, then the developing map $D_u:\tilde{M}\to X$ maps into a single component of X. It follows that there is no loss of generality in supposing that X is connected. Since the transverse structure is locally defined, we may also replace X by its universal cover.

The following lemma is an immediate consequence of the homotopy exact sequence of the fibre bundle $D:\tilde{M}\to X$, using exactness at $\pi_1 F$, where F is the fibre.

1.12. **Lemma.** *Suppose X is simply connected and M is a closed manifold with a foliation modelled on X, then each fibre of $D_u:\tilde{M}\to X$ is connected.*

The last sentence of Theorem 1.10 now follows.

1.13. **Proposition.** *Let M be a closed manifold and let F be a foliation with a transverse Riemannian structure modelled on a connected Riemannian manifold X. Then there is a positive number l such that any ball of radius l in X is convex and is embedded. Consequently, X is complete.*

Proof. Let $C_1,\cdots,C_k$ be a finite covering of M by compact foliation charts. Let $f_i:C_i\to X$ be the associated submersion. Since $D:\tilde{M}\to X$ is surjective, we see that each point $x\in X$ is the image of some point of f_iC_i for some i, under an isometry of X. Since the radius of convexity has a positive lower bound on any compact set, the result follows.

1.14. **Proposition.** *Let M be a closed manifold with a foliation F, modelled on a simply connected manifold X. Let $h:\pi_1 M\to \mathrm{Isom}X$ be the holonomy homomorphism, let H be the closure of $h(\pi_1 M)$ in* Isom$X$, *and let H_0 be the component of the identity in H. Then the following conditions are equivalent:*

1. $H_0=\{\mathrm{id}\}$;

2. $h(\pi_1 M)$ is a discrete subgroup of Isom$X$;

3. $h(\pi_1 M)$ is a closed subgroup of Isom$X$.

4. $H=h(\pi_1 M)$;

5. Each leaf of F is compact.

Proof. To see that 1) implies 2), suppose $H_0=\{\mathrm{id}\}$. Then H is a discrete subgroup of Isom$X$, since Isom$X$ is a Lie group. Hence $h(\pi_1 M)$ is also discrete. The fact that 2) implies 3) is standard for subgroups of topological groups. It is immediate that 3) implies 4).

To see that 4) implies 5), note that $D:\tilde{M}\to X$ induces a map $M\to X/h(\pi_1 M)$. The orbit space of X under any closed subgroup of Isom$X$ is a Hausdorff space. Hence the inverse image of any point of $X/h(\pi_1 M)$ is a closed subset This means that each leaf of F is a closed, and hence compact, subset of M.

We show that 5) implies 1) by contradiction. We assume that $H_0\neq\{\mathrm{id}\}$. Let $\alpha_i\in\pi_1 M$, such that $h(\alpha_i)$ tends to the identity in Isom$X$. We may choose a point x_0, such that the points $h(\alpha_i)x_0$ are distinct. Let $\tilde{L}=D^{-1}(x_0)$ be the corresponding leaf of $\tilde{M}$. Then the leaves $\alpha_i\tilde{L}$ all project to the same leaf L. Locally $\tilde{M}$ and M are isomorphic. It follows that any foliation chart meeting L must contain an infinite number of plaques of L, and so L is not compact.

§2. One-dimensional foliations.

In this section, we prove a result due to Thurston [11] in the transversely hyperbolic case, and to Carrière [1,2] in the form stated. We follow Thurston's method, with improvements due to Carrière.

2.1. Theorem. *Let M be a connected, closed manifold and let F be a 1-dimensional foliation with a transverse Riemannian structure, modelled on a simply connected manifold X. Let $h:\pi_1 M \to \mathrm{Isom}X$ be the holonomy homomorphism, and let H be the closure of $h(\pi_1 M)$. Let H_0 be the component of the identity of H. Then H_0 is abelian. Moreover if h is not injective, then the five equivalent conditions of Proposition 1.14 are satisfied and each leaf of F is a circle.*

Proof. If $h:\pi_1 M \to \mathrm{Isom}X$ is not injective, let $\alpha \in \pi_1 M$ be a non-trivial element in the kernel. From the equivariance of $D:\tilde{M} \to X$, we see that α preserves each fibre. Since it also acts fixed point free, the quotient of the fibre by α is a circle. This proves the last sentence of Theorem 2.1. By Proposition 1.14, $H_0 = \{\mathrm{id}\}$.

So we may now assume that h is injective. By Proposition 1.13, there is a leaf of F, which is a copy of R. But this means that $H_0 \neq \{\mathrm{id}\}$. The fibre bundle $D:\tilde{M} \to X$ has contractible fibres and is therefore trivial. So $\tilde{M} \cong X \times \mathbb{R}$ and D corresponds to projection onto the first factor.

We may assume without loss of generality that the foliation has oriented leaves. The reason is that going to the double cover, which results from orienting the leaves, replaces H by a subgroup of index at most two. But then the component of the identity of H is unaltered.

We impose on M the adapted metric used in the proof of Theorem 1.10. Let $l>0$ be chosen so that every loop in M of length at most l is contractible. We choose a left invariant Riemannian metric on $\mathrm{Isom}X$. If $\gamma \in \pi_1 M$ and $x \in X$, we write γx instead of $h(\gamma)x$. We define U_ϵ to be the open ball in $\mathrm{Isom}X$ of radius ϵ. For each $\epsilon>0$, let $\Gamma_\epsilon = U_\epsilon \cap h(\pi_1 M)$. Since h is injective, this is isomorphic to $h^{-1}(U_\epsilon)$. We know from Proposition 1.13 that there is an ϵ_0 such that each ball in X of radius $4\epsilon_0$ is convex, and such that $4\epsilon_0 < l$.

Given a compact connected subset K of X, there is an $\epsilon(K)$ with the property that if $g_1, g_2 \in U_{\epsilon(K)}$ and $x \in K$, then $d(g_1 x, g_2 x) < \epsilon_0$.

We now order the elements of $\Gamma_{\epsilon(K)}$. Given $x \in K$ and $\gamma_0, \gamma_1 \in \Gamma_{\epsilon(K)}$, we have $d(\gamma_0 x, \gamma_1 x) < \epsilon_0$. Let α be a path in X from $\gamma_0 x$ to $\gamma_1 x$ of length less than $4\epsilon_0$, and let $\tilde{x} \in D^{-1}(x) \subset \tilde{M}$. Let $\tilde{\alpha}$ be the horizontal lifting of α, with $\tilde{\alpha}(0) = \gamma_0 \tilde{x}$. Then $\gamma_1 \tilde{x}$ and $\tilde{\alpha}(1)$ both lie in the oriented real line $D^{-1}(\gamma_1 x)$. We define $\gamma_1 > \gamma_0$ if $\gamma_1 \tilde{x} > \tilde{\alpha}(1)$ and $\gamma_1 < \gamma_0$ otherwise. Note that we can not have $\gamma_1 \tilde{x} = \tilde{\alpha}(1)$, for otherwise $\tilde{\alpha}$ would represent a non-trivial path in M of length less that l, and this is impossible.

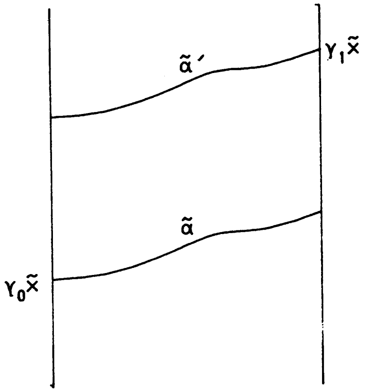
Since $\tilde{\alpha}$ depends continuously on $\tilde{x}$ and α, and since equality $\gamma_1 \tilde{x} = \tilde{\alpha}(1)$ is not possible, we see that the truth of the inequality $\gamma_1 > \gamma_0$ or $\gamma_1 < \gamma_0$ is independent of the homotopy class of α fixing the endpoints, provided the homotopy varies through paths of length less than $4\epsilon_0$. The homotopy class of α is equal to that of the short geodesic from $\gamma_0 x$ to $\gamma_1 x$ — a length-decreasing homotopy is given by taking the short geodesic from $\alpha(0)$ to $\alpha(t)$ and then the original path from $\alpha(t)$ to $\alpha(1)$. Also the inequality is independent of small movements of x. But since K is connected, the inequality $\gamma_1 > \gamma_0$ or $\gamma_1 < \gamma_0$ depends only on K. We will write $\gamma_1 >_K \gamma_0$ or $\gamma_0 <_K \gamma_1$ for $\gamma_0, \gamma_1 \in \Gamma_{\epsilon(K)}$. (In fact one can show that if K is large enough and ϵ small enough, the ordering is independent of K, but we will not bother to prove this.)

We now verify a number of properties of the ordering.

2.2.1. We have $\gamma_1 >_K \gamma_0$ if and only if $\gamma_0 <_K \gamma_1$.

Proof. This follows from Diagram 1, where $\tilde{\alpha}$ and $\tilde{\alpha}'$ are horizontal lifts of α.

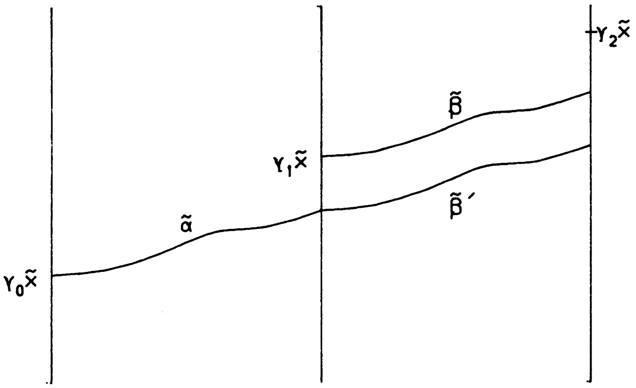
Diagram 1



2.2.2. If $\gamma_0 <_K \gamma_1$ and $\gamma_1 <_K \gamma_2$, then $\gamma_0 <_K \gamma_2$.

Proof. This follows from Diagram 2.

Diagram 2



2.2.3. If $K \subset L$ are compact sets, then $\Gamma_\epsilon(L) \subset \Gamma_{\epsilon(K)}$, and the two orderings agree on $\Gamma_{\epsilon(L)}$. It follows that the ordering is well-defined on the germ of $h(\pi_1 M)$ near the identity in Isom$X$.

2.2.4. Let K be a compact connected set which contains a ball of radius ϵ_0. Let $U = U^{-1}$ be a neighbourhood of the identity in Isom$X$, such that $U^2 \subset U_{\epsilon(K)}$. Let $\gamma, \gamma_0, \gamma_1 \in U$ and let $\gamma_0 <_K \gamma_1$. Let $\tilde{\alpha}$ be a horizontal path in $\tilde{M}$ from $\gamma_0\tilde{x}$ to $\tilde{\alpha}(1) < \gamma_1\tilde{x}$ and let $D\tilde{x} = x$.

2.2.4.1. We have $\gamma\gamma_0 <_K \gamma\gamma_1$. To see this, note that $\gamma\tilde{\alpha}$ is a horizontal path from $\gamma\gamma_0\tilde{x}$ to $\gamma\tilde{\alpha}(1) < \gamma\gamma_1\tilde{x}$.

2.2.4.2. We have $\gamma_0\gamma <_K \gamma_1\gamma$. To see this, we suppose that the point x defined above is the centre of a ball in K of radius ϵ_0. Then $\gamma x \epsilon K$ and, writing $\tilde{y} = \gamma^{-1}\tilde{x}$, we see that $\tilde{\alpha}$ from $\gamma_0\gamma\tilde{y}$ to $\gamma_1\gamma\tilde{y}$ can be used to show that $\gamma_0\gamma <_K \gamma_1\gamma$.

Let $\Gamma_\epsilon^+ = \{\gamma \epsilon \Gamma_\epsilon : \gamma >_K \mathrm{id}\}$, where $\epsilon < \epsilon(K)$. We call this the set of *positive elements* in Γ_ϵ.

2.2.5. By 2.2.4, we see that if ϵ is small enough, then $\gamma >_K \mathrm{id}$ if and only if $\mathrm{id} >_K \gamma^{-1}$.

2.3. **Lemma.** *If K is a given compact connected set, then the ordering $<_K$ on Γ_ϵ^+, the set of positive elements of Γ_ϵ, has the same order type as the positive integers.*

Proof.

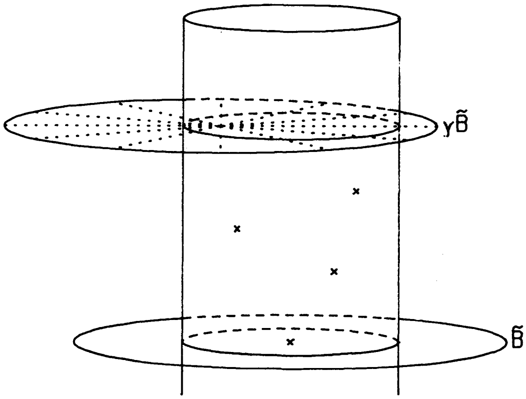


Diagram 3

Since $4\epsilon_0 < l$, we see that for any ball $\tilde{B}$ in $\tilde{M}$ of radius $2\epsilon_0$, and any $\gamma \neq \mathrm{id}$ in $\pi_1 M$, $\tilde{B} \cap \gamma\tilde{B} = \emptyset$. We fix our attention on a certain $x_0 \epsilon K$ and let $B(r)$ be the ball in X of radius r and centre x_0. Then $D^{-1}B \cong B \times \mathrm{R}$ for $r < 4\epsilon_0$. Let $\tilde{x}_0 \epsilon D^{-1}x_0$. Let $\tilde{B}(r)$ be the lift of B, whose centre is at $\tilde{x}_0$, and such that each radius is horizontal. If $\gamma \epsilon \Gamma_\epsilon$ then $d(x_0, \gamma x_0) < \epsilon_0$, because $\epsilon < \epsilon(K)$. Therefore $B_\gamma = \gamma\tilde{B}(2\epsilon_0) \cap D^{-1}B(\epsilon_0)$ is a disk mapped diffeomorphically by D onto $\gamma B(2\epsilon_0) \cap B(\epsilon_0) = B(\epsilon_0)$. Therefore B_γ separates $D^{-1}B(\epsilon_0) = B(\epsilon_0) \times \mathrm{R}$ into two cylinders. Note that, if $\gamma(1) \neq \gamma(2) \epsilon \Gamma_\epsilon$, then $B_{\gamma(1)} \cap B_{\gamma(2)} = \emptyset$. Therefore the space between $B_{\gamma(1)}$ and $B_{\gamma(2)}$ in $D^{-1}B(\epsilon_0)$ is a relatively compact subset, which contains at most a finite number of translates of $\tilde{x}_0$ under Γ_ϵ, since $\pi_1(M)$ acts properly discontinuously on $\tilde{M}$. Since the set of translates is countable, the lemma follows.

2.4. **Lemma.** *For any sufficiently small $\eta > 0$, the group generated by Γ_η is equal to $\Gamma = H_0 \cap h(\pi_1 M)$ and is dense in H_0. (See Theorem 2.1 for the definition of H_0.)*

Proof. If η is small enough, $U_\eta \cap H = U_\eta \cap H_0$. Therefore $\Gamma_\eta = U_\eta \cap h(\pi_1 M)$ is dense in $U_\eta \cap H_0$. Now $U_\eta \cap H_0$ generates H_0, since H_0 is connected. Hence, given $g \in H_0$, we can find a $k > 0$ and $\gamma_1, \cdots, \gamma_k \in \Gamma_\eta$, such that $g(\gamma_1 \cdots \gamma_k)^{-1} \in U_\eta$. If $g \in \Gamma$, then $g(\gamma_1 \cdots \gamma_k)^{-1} \in \Gamma_\eta$. This proves the result.

We can now prove Theorem 2.1. Let K be any compact connected set in X containing a ball of radius ϵ_0. Let U_ϵ be an open neighbourhood of the identity in $\mathrm{Isom}X$, such that $\epsilon < \epsilon(K)$, where ϵ is small enough for 2.2.4 and 2.2.5 to be valid, whenever they are invoked in the rest of the proof.

We will prove by contradiction that all elements of Γ_ϵ^+ are in the centre of Γ. Having proved this, we will know by 2.2.5 that all elements of Γ_ϵ commute with each other. By Lemma 2.4, Γ and hence H_0 will then be abelian. Let $\gamma\in\Gamma_\epsilon^+$ be the smallest element which is not in the centre of Γ. Since U_ϵ is open and $\gamma\in U_\epsilon$, there is a very small neighbourhood U of the identity in Isom$K$, such that $U\subset U_\epsilon$, $U=U^{-1}$ and $U\gamma\subseteq U_\epsilon$. Let $\eta>0$ be small enough so that $U_\eta\subset U$. By our choice of γ, and by Lemma 2.4, there is an element $\alpha\in\Gamma_\eta$, such that $\alpha\gamma\alpha^{-1}\neq\gamma$.

There are two possibilities: $\gamma<_K \alpha\gamma\alpha^{-1}$ and $\alpha\gamma\alpha^{-1}>_K\gamma$. In fact we may restrict to the second possibility, if we replace α by α^{-1}. By 2.2.4, $\gamma>_K$ id implies that $\alpha\gamma\alpha^{-1}>_K$ id. By the definition of α this means that $\alpha\gamma\alpha^{-1}$ commutes with all elements of Γ. Therefore Γ commutes with all elements of Γ. This contradiction shows that H_0 is abelian.

There are some general results which now apply. For example, one can prove that the closure of any leaf is a torus, and one can say a great deal about the foliation structure. For this we refer the reader to the articles by Carrière and Molino in these proceedings. We will discuss only the transversely hyperbolic situation, which is more special.

§3. Abelian groups of isometries.

3.1. Lemma. *Let G be an abelian group of Euclidean isometries of affine Euclidean space of dimension k. Then we can choose an origin, making the space into a vector space isomorphic to $\mathbb{R}^k$, such that each $\phi\in G$ has the form $\phi(x)=T_\phi x+b_\phi$, $T_\phi\in O(k)$, $b_\phi\in\mathbb{R}^k$ and $T_\phi b_\phi=b_\phi$. Moreover the union of all minimal G-invariant affine subspaces is*

$$M=\{x: T_\phi x=x \text{ for all } \phi\in G\}$$

and these subspaces are disjoint and have the same dimension.

Proof. We can write each ϕ in the form $\phi(x)=T_\phi x+b_\phi$. Suppose first that each T_ϕ is the identity. Then the result is clear. So suppose that for some ϕ, $T_\phi\neq$id. Let $M_1=\{x: T_\phi x=x\}$ and let M_2 be the orthogonal complement. Let $b_\phi=b_1+b_2$ with $b_1\in M_1$ and $b_2\in M_2$. We solve for $x_\phi\in M_2$, such that $T_\phi x_\phi=x_\phi-b_2$. Then $\phi(x+x_\phi)=T_\phi x+b_2+x_\phi$. Changing the origin to x_ϕ, we obtain

$$\phi(x)=T_\phi x+b_1 \text{ with } T_\phi b_1=b_1.$$

So ϕ induces an orthogonal transformation of $\mathbb{R}^k/M_1$, which can be thought of as $T_\phi|M_2$. The only fixed point of ϕ acting on $\mathbb{R}^k/M_1$ is the origin.

Any ϕ-invariant affine subspace of $\mathbb{R}^k/M_1$ must contain the origin, otherwise the nearest point to the origin would be fixed. Clearly, M_1 is the union of ϕ-invariant affine subspaces which are either all of dimension zero or all of dimension one. Hence M_1 is the union of all minimal ϕ-invariant affine subspaces of $\mathbb{R}^k$ and $\dim M_1<k$. Moreover M_1 is G-invariant.

The lemma now follows by induction on k, since every G-invariant subspace meets M_1, and hence every minimal G-invariant affine subspace is contained in M_1.

3.2. Proposition. *Let G be an abelian group of isometries of $\mathbb{H}^n$ $(n\geq2)$. Then all minimal G-invariant hyperbolic subspaces have the same dimension and they are disjoint. Their union S is a hyperbolic subspace.*

3.2.1. If the minimal invariant subspaces have dimension zero, then each element of G is elliptic and contains S in its fixed point set. We have $\dim S<n$.

3.2.2. *If the minimal invariant subspaces have dimension one, then there is only one minimal invariant subspace. This case occurs if and only if G contains a hyperbolic element ϕ, and then S is the axis of ϕ. G may contain elliptic elements, but it contains no parabolic elements.*

3.2.3. *If the minimal invariant subspaces have dimension at least two, then we can choose upper half space coordinates such that each $\phi \in G$ has the form $\phi(x)=T_\phi x+b_\phi$, where $T_\phi \in O(n-1)$, $b_\phi \in \mathbb{R}^{n-1}$ and $T_\phi b_\phi=b_\phi$. The space*

$$M = \{x \in \mathbb{R}^{n-1} : T_\phi x = x \text{ for all } \phi \in G\}$$

is an affine space which is the boundary of the hyperbolic subspace S. G may contain elliptic elements ($b_\phi=0$), but it contains no hyperbolic elements. At least one element of G is parabolic.

Proof. Suppose there is an element $\phi \in G$, such that the union Y of all minimal ϕ-invariant hyperbolic subspaces is a hyperbolic subspace with $\dim Y<n$. Y is clearly G-invariant. Any G-invariant subspace is ϕ-invariant, and therefore meets Y. So any minimal G-invariant subspace must be contained in Y.

By induction, we may suppose that Proposition 3.2 holds for $G \mid Y$. If 3.2.1 or 3.2.2 apply to $G \mid Y$, they will apply to G as well.

If 3.2.3 applies to $G \mid Y$, then G contains no hyperbolic element, and it contains at least one parabolic element. Since a point at infinity in $\bar{Y}$ is fixed, we can choose an upper half space model so that every element $\psi \in G$ has the form $\psi(x)=T_\psi x+b_\psi$ with $T_\psi \in O(n-1)$ and $b_\psi \in \mathbb{R}^{n-1}$. We can now apply Lemma 3.1. Since G contains a parabolic element, any invariant hyperbolic subspace is a vertical halfplane. Case 3.2.3 follows, provided a ϕ exists as in the first paragraph of this proof.

To complete the proof, we need to look at the case where, for every non-trivial isometry $\phi \in G$, the space Y, defined above, is the entire space. It follows that no element of the group is hyperbolic, no element is elliptic, and every parabolic element is a pure translation (see Lemma 3.1). This is case 3.2.3.

§4. Proof of main theorem.

We suppose now that M is a closed n-manifold with a transversely hyperbolic foliation whose leaves have dimension 1. We have already dealt with the first case of the theorem, so we may assume that not every leaf is a circle. We have defined $D:\tilde{M} \to \mathbb{H}^{n-1}$, the holonomy homomorphism $h:\pi_1 M \to \mathrm{Isom}\,\mathbb{H}^{n-1}$, the closure H of $h(\pi_1 M)$ and H_0, the component of the identity of H. By Theorem 2.1, we know that H_0 is abelian. By Proposition 1.14, we know that $H_0 \neq \{\mathrm{id}\}$.

4.1. Lemma. *The union of minimal H_0-invariant hyperbolic subspaces of $\mathbb{H}^{n-1}$ is equal to $\mathbb{H}^{n-1}$ and we have Case 3.2.3 of Proposition 3.2.*

Proof. Let S be the union of minimal H_0-invariant subspaces. It is invariant under H, since H_0 is a normal subgroup of H. If S is not equal to $\mathbb{H}^{n-1}$, let $f(x)$ be the distance from S to x. Then f is an unbounded H-invariant function. In particular, it is $\pi_1 M$-invariant. Hence $f\tilde{D}:\tilde{M} \to \mathbb{R}$ is an unbounded $\pi_1 M$-invariant function. But this defines a continuous unbounded function on M, which is impossible. The lemma follows.

4.2. **Lemma.** *H_0 consists of all translations of the upper halfspace — i.e. $H_0 \cong \mathbb{R}^{n-2}$ and H_0 consists of all transformations of the form $x \to x+b$ $(b \in \mathbb{R}^{n-2})$.*

Proof. H_0 is connected and, by Proposition 3.2.3 and Lemma 4.1, consists of transformations of the form $x \to x+b$. Therefore H_0 consists of all transformations of the form $x \to x+b$, with $b \in V$, a vector subspace of $\mathbb{R}^{n-2}$. We want to show that $V=\mathbb{R}^{n-2}$. Since H_0 fixes a unique point at infinity, and H_0 is normal in H, every element of H also preserves this point. Hence every element ϕ of H has the form $x \to \lambda_\phi T_\phi x + c_\phi$ with $\lambda_\phi > 0$, $T_\phi \in O(n-2)$, $c_\phi \in \mathbb{R}^{n-2}$. Moreover V must be T-invariant. Since $\pi_1 M \subseteq H$, we have a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{D} & \mathbb{H}^{n-1} \\ \downarrow & & \downarrow \\ M & \to & \mathbb{H}^{n-1}/H \end{array}$$

with D surjective. It follows that $\mathbb{H}^{n-1}/H$ is compact. H can not be a subgroup of the group C of all Euclidean isometries of $\mathbb{R}^{n-2}$, since $\mathbb{H}^{n-1}/C \cong (0,\infty)$. Hence there must be an element $\phi \in H$, with $\lambda_\phi > 1$. Such a ϕ is hyperbolic. We choose coordinates so that the axis of ϕ passes through the origin.

Let H_1 be the subgroup of H consisting of ψ such that $\lambda_\psi = 1$. Clearly, H_1 is closed and normal in H and is contained in the group of Euclidean isometries of $\mathbb{R}^{n-2}$. Choose x_0 on the axis of ϕ. Since H_0 is normal, the orbit of x_0 under H consists of a discrete set of horizontal horospherical subspaces. The orbit of x_0 under H_1 consists of all such subspaces at the same Euclidean height as x_0. Let $x_1 \in H_1 x_0$ be a nearest point to x_0, not in $H_0 x_0$, if such a point exists. Let $x_1 = \psi x_0$ with $\psi \in H_1$. Now $\phi x_0 = \lambda_\phi x_0$ and therefore

$$d(x_0, \phi^{-1}\psi\phi x_0) = d(\phi x_0, \psi\phi x_0) = \frac{d(x_0, \psi x_0)}{\lambda_\phi}$$

by the definition of the hyperbolic metric in upper halfspace. By the definition of ψ, we then would have $\phi^{-1}\psi\phi x_0 = x_0$, or $\psi\phi x_0 = \phi x_0$. Since ψ is a Euclidean isometry, this means that $\psi x_0 = x_0$ which is impossible. It follows that the H_1-orbit of x_0 is the same as the H_0-orbit.

Let $\psi_1(x) = \lambda_1 T_1 x + c_1$ and $\psi_2(x) = \lambda_2 T_2 x + c_2$ with $\psi_1, \psi_2 \in H$. Then

$$\begin{aligned}[\psi_1, \psi_2] &= \psi_1\psi_2\psi_1^{-1}\psi_2^{-1}(x) \\ &= T_1T_2T_1^{-1}T_2^{-1}(x) - T_1T_2T_1^{-1}(c_2) - \lambda_2 T_1T_2T_1^{-1}(c_1) + \lambda_1 T_1(c_2) + c_1.\end{aligned}$$

We take ϕ as above, $\lambda = \lambda_\phi > 1$, $T = T_\phi$, $c_\phi = 0$, and we define $\psi_2 = \phi^k$, for some k. Then, since $T_1, T_2 \in O(n-2)$ and x_0 lies above the origin, $T_1 x_0 = T_2 x_0 = x_0$. Also $c_2 = 0$, $T_2 = T^k$ and $\lambda_2 = \lambda^k$. Therefore

$$[\psi_1, \psi_2](x_0) = x_0 - \lambda^k T_1 T^k T_1^{-1} c_1 + c_1.$$

and $[\psi_1, \psi_2] \in H_1$. By taking k large and negative, we see that $c_1 \in V$. We have observed at the beginning of this proof that T_1 must preserve V. Since $\psi_1 \in H$ is arbitrary, we see that, in terms of the action of H on $\mathbb{R}^{n-2}$, V is invariant under H. It follows that the hyperbolic subspace S whose boundary is V, is invariant under H. As before, the distance from S gives an unbounded function on M, and hence a contradiction, unless $V = \mathbb{R}^{n-2}$. This proves the lemma.

We have a homomorphism $H \to \mathbb{R}^*$, sending ϕ to $\lambda_\phi > 0$. The image of this homomorphism is discrete, for suppose $\phi_n \in H$, $\phi_n(x) = \lambda_n T_n x + b_n$ and λ_n converges through distinct values to 1.

Since $V=\mathbf{R}^{n-2}$, we may assume that $b_n=0$. We may also assume that T_n converges to a limit T, since $O(n-2)$ is compact. Then ϕ_n converges to $\phi(x)=Tx$ and $\phi\in H_1$. But this contradicts the fact that H/H_1 is discrete.

Let $\phi\in h(\pi_1 M)$ be such that $\lambda=\lambda_\phi>1$ is minimal. Let x_0 be a point on the axis of ϕ and let E_t be the horosphere at a hyperbolic distance t above x_0. Then $\phi(E_u)=E_{t+\log\lambda}$. We define

$$f:\mathbf{H}^{n-1}\to S^1=\mathbf{R}/\mathbf{Z} \text{ by } f(x)=t/\log\lambda \text{ if } x\in E_t.$$

Then $f(\phi x)=f(x)+1$. Since H_0 and H_1 preserve each E_t , f is H-invariant. We obtain a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{D} & \mathbf{H}^{n-1} \\ \downarrow & & \downarrow f \\ M & \xrightarrow{\pi} & S^1. \end{array}$$

Since D is a locally trivial fibre bundle, it is easy to see that π is a locally trivial fibre bundle, whose fibre is equal to $\mathbf{R}^{n-2}\times\mathbf{R}$ modulo the action of $\Gamma_1=h^{-1}(H_1)$. Later we will show that $H_1=H_0$, so that $\Gamma_0=h^{-1}(H_0)=\Gamma_1$. We also write $\Gamma=\pi_1 M$.

The following result is due to Fried [6].

4.4. Theorem. *Let M be a compact manifold, possibly with boundary, with an oriented 1-dimensional foliation. Let $p:\hat{M}\to M$ be an infinite cyclic covering such that the leaves of the induced foliation form the fibres of a smooth locally trivial fibre bundle with fibre a copy of* R. *Then there is a section of the fibre bundle which projects injectively to a section of the foliation on M.*

The following result is now obvious.

4.5. Corollary. Under the hypotheses of Lemma 4.4, M and its foliation are given by the mapping torus of a diffeomorphism of B with itself, where B is the quotient of $\hat{M}$ identifying each leaf to a point.

Let $\mathbf{R}^{n-1}$ act on itself by left translation.

The author learnt of the following result, due to A.Verjovsky, by reading Carrière [1].

4.6. Theorem. (A.Verjovsky). *Let M be a closed manifold with a 1-dimensional oriented foliation and a transverse $(\mathbf{R}^{n-1},\mathbf{R}^{n-1})$-structure. Suppose M has a dense leaf. Then one can impose a metric on M, which is adapted to the transverse structure as in the proof of Theorem 1.10, and in this metric M is the flat torus and the foliation is linear.*

Proof. We may suppose that $n>1$. We are suspending for the moment the hypothesis that the transverse structure is hyperbolic. Let H and H_0 be defined as in 1.14. Then there is a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{D} & \mathbf{R}^{n-1} \\ \downarrow & & \downarrow \\ M & \to & \mathbf{R}^{n-1}/H \end{array}$$

such that every leaf in M is mapped to a single point. Since there is a dense leaf, we see that $H=\mathbf{R}^{n-1}=H_0$. >From Proposition 1.14 we know that $h:\pi_1 M\to\mathbf{R}^{n-1}$ is injective. Therefore $\pi_1 M$ is free abelian. We choose a maximal set of generators whose images in $\mathbf{R}^{n-1}$ are linearly independent over R. Since $h(\pi_1 M)$ is dense, this set has exactly $(n-1)$ elements. Let Γ_1 be the subgroup of $\pi_1 M$ generated by these elements, and let $\hat{M} = \tilde{M}/\Gamma_1$ We may assume that $\pi_1 M/\Gamma_1$ is torsionfree. We have a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \to & \hat{M} \\ D\downarrow & & \downarrow \\ \mathbb{R}^{n-1} & \to & \mathbb{R}^{n-1}/\Gamma_1 = T^{n-1} \end{array}$$

and the map $\hat{M} \to T^{n-1}$ is a fibre bundle with fibre R. Hence $\hat{M} \cong T^{n-1} \times \mathbb{R}$. Now $\hat{M}$ has two ends and $\pi_1 M/\Gamma_1$ is a free abelian group acting as a group of covering translations of $\hat{M}$ with compact quotient. Therefore $\pi_1 M/\Gamma_1$ has two ends and must be cyclic infinite. Thus Fried's Theorem (4.5) and Corollary 4.6 apply, and there is an embedding of T^{n-1} in M, which is transverse to the foliation and consistent with the transverse Euclidean structure. It follows that, as a foliated manifold, M is the mapping torus of a diffeomorphism $T^{n-1} \to T^{n-1}$, and this diffeomorphism is left translation by an element γ of the group T^{n-1}. (The element γ is a generator of the image of $\pi_1 M/\Gamma_1$ in $\mathbb{R}^{n-1}/\pi_1 M_1$ under the holonomy homomorphism.)

It follows that M can be described as the quotient of $\mathbb{R}^{n-1} \times \mathbb{R}$ by the group generated by Γ_1 and by $(-\gamma_1, +1)$, where $+\gamma_1$ represents $\gamma \in \mathbb{R}^{n-1}/\Gamma_1$. This proves Verjovsky's theorem. The transverse structure is given by projection onto $\mathbb{R}^{n-1}$ in a direction parallel to $(-\gamma_1, 1)$.

4.7. Lemma. *Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be a torus foliated linearly by a dense line parallel to a non-zero vector $v \in \mathbb{R}^n$. The group of diffeomorphisms of T^n respecting the foliation can be deformation retracted to the group of transformations of the form $x \to Ax + b$, where $b \in T^n$, $A \in GL(n, \mathbb{Z})$ and v is an eigenvector of A. The deformation f_t of a diffeomorphism f has the property that for any leaf L, $f_t(L) = f(L)$, for each time t.*

Proof. The foliation represents a class in $H_1(T^n; \mathbb{R})$, defined up to multiplication by a non-zero real number, and this homology class is an invariant of the foliation. This class was first defined by Schwartzmann [10]. One takes a long piece of leaf and closes it up to a loop with a short path in T^n. One then normalizes by dividing by the length of the loop. Finally, one takes the limit as the length tends to infinity. The class in $H_1(T^n;\mathbb{R})$ is $[v]$, corresponding to $v \in \mathbb{R}^n$ under the isomorphism $H_1(T^n;\mathbb{R}) \cong \mathbb{R}^n$. Let $f: T^n \to T^n$ be a diffeomorphism respecting the foliation. Let $A \in GL(n, \mathbb{Z})$ be the map $f_*: H_1(T^n;\mathbb{Z}) \to H_1(T^n;\mathbb{Z})$. Then v is an eigenvector for A. Let $0 \in T^n$ be the identity element in the group $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Let $f(0) = b$. Then f is homotopic to g, defined by $g(x) = Ax + b$, by a homotopy which keeps 0 at b (but which may not respect the foliation). The diffeomorphism $h = f^{-1}g: T^n \to T^n$ preserves the foliation and the point 0. Hence it sends the leaf through 0 to itself.

Consider the lifting of h, $\tilde{h}: \mathbb{R}^n \to \mathbb{R}^n$, such that $\tilde{h}(0) = 0$. Then $\tilde{h} | \mathbb{Z}^n$ is the identity. Hence every leaf through a point of $\mathbb{Z}^n$ is sent to itself. Since this set of leaves is dense, $\tilde{h}$ sends every leaf to itself. It follows that f must preserve the transverse $\mathbb{R}^{n-1}$ structure. (This result is a special case of a general result due to Molino [7,8]. However, Molino's method requires f to be C^∞, whereas here the method works even if f is only a homeomorphism.)

Let $\tilde{g}, \tilde{f}: \mathbb{R}^n \to \mathbb{R}^n$ be liftings of g, f, such that $\tilde{f}(0) = \tilde{g}(0)$. Then for any leaf L of $\mathbb{R}^n$, $\tilde{g}(L) = \tilde{f}(L)$, as we have just seen. Since g is homotopic to f, keeping 0 fixed, $\tilde{f}$ and $\tilde{g}$ will be equal on $\mathbb{Z}^n$. Therefore either $\tilde{f}$ and $\tilde{g}$ both preserve the orientation of leaves, or $\tilde{f}$ and $\tilde{g}$ both reverse the orientation of leaves. Hence one can construct a linear homotopy from $\tilde{f}$ to $\tilde{g}$, and this homotopy will be $\mathbb{Z}^n$-equivariant and will not move any leaf out of its image leaf. This completes the proof of the lemma.

4.8. **Lemma.** *Let $f:T^n \to T^n$ preserve a linear foliation by dense lines, and suppose f has finite order. Then f induces the identity on $H_1(T^n;\mathbf{Z})$. For some $b \in \mathbf{R}^n$, we have a commutative diagram*

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{\tilde{f}} & \mathbf{R}^n \\ D\downarrow & & \downarrow D \\ \mathbf{R}^{n-1} & \xrightarrow{+b} & \mathbf{R}^{n-1} \end{array}$$

where D is the developing map.

Proof. Let $A \in GL(n,\mathbf{Z})$ be the map induced on $H_1(T^n;\mathbf{Z})$. Then $Av = \lambda v$, where $v \in \mathbf{R}^n$ is the direction of the foliation and $\lambda \in \mathbf{R}$. Then $\lambda = \pm 1$. Let $\mathbf{R}^n = V \oplus W$, where V is the λ-eigenspace of A and W is an A-invariant complementary subspace. Since λ is an integer, V and W can be defined over the rationals. In other words, we can find $v_1, \cdots, v_k, w_1, \cdots, w_{n-k} \in \mathbf{Z}^n$, such that the $\{v_i\}$ form a basis for V and the $\{w_i\}$ form a basis for W. But $v \in V$ and v generates a line which is dense in T^n. Hence $W=0$.

By the Lefschetz fixed point theorem, $\lambda = -1$ is impossible. This proves the result.

We can now go back to the transversely hyperbolic situation of the main theorem of this paper. We refer the reader to the definitions of H_0 and H at the beginning of §4, to the definition of H_1 in the proof of Lemma 4.2, and to the definitions of Γ_0, Γ_1 and Γ just before 4.4. We recall that we have shown that H consists of transformations of the form $x \to \lambda Tx + b$ in the upper half space model, with $\lambda > 0$, $T \in O(n-2)$ and $b \in \mathbf{R}^{n-2}$, and that H_0 consists of *all* transformations of the form $x \to x + b$.

4.9. **Lemma.** *$H_0 = H_1$, and the fibre bundle $\pi: M \to S^1$ defined above has for its fibre a torus T^{n-1}, which is foliated linearly by dense lines. This foliation is locally constant.*

Proof. We have the developing map $D:\tilde{M} \to \mathbf{H}^n$. Let $\tilde{N}$ be the inverse image in $\tilde{M}$ of a horosphere in $\mathbf{H}^n$, which is preserved by H_0. Then the image N of $\tilde{N}$ in M is a closed submanifold, namely a fibre of the fibre bundle $\pi: M \to S^1$ described above.

Let Γ_2 be the subgroup of Γ_0 of index at most two, preserving the orientation of the fibres of $D:\tilde{M} \to \mathbf{H}^n$. Now $N = \tilde{N}/\Gamma_1$, and the covering with fundamental group Γ_2 has finite index. Now $\tilde{N}/\Gamma_2$ is a torus of dimension $(n-1)$ with a linear foliation by dense leavres. The group Γ_1/Γ_2 acts on this torus. By Lemma 4.8, the elements of Γ_1/Γ_2 preserve the orientation of the foliation, and so $\Gamma_0 = \Gamma_2$. According to Lemma 4.8, the elements of Γ_1/Γ_0 are all translations. So this means that $\Gamma_1 = \Gamma_0$.

We have now proved that the fibre bundle $M \to S^1$ has for its fibre a torus with a linear foliation having dense leaves. It follows that $\Gamma_1/\Gamma_0 \cong H_1/H_0$ is cyclic infinite. However, we have not yet proved that the foliation is locally constant. This can be seen by improving Carrière and Verjovsky's theorem (4.6) to the situation where the transverse structure is an $(\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \times B)$-structure, where B is a ball on which $\mathbf{R}^{n-1}$ acts trivially. We further suppose that the manifold is fibred by tori, one for each $x \in B$, and that each torus has a dense leaf. Going through the proof, simply multiplying by B at appropriate points, we see that we have $B \times T^{n-1}$, where the foliation on T^{n-1} is constant. An alternative approach is to follow Molino [7,8] (but then, as already remarked, one is limited to the C^∞-case, whereas the above approach can be generalized).

Any fibre bundle $\pi:M\to S^1$ is determined by the monodromy, which is a diffeomorphism of the fibre to itself. In the situation under consideration, Lemma 4.7 shows that the monodromy diffeomorphism $\mu:T^{n-1}\to T^{n-1}$ can be chosen to be of the form $\mu(x)=Ax+b$.

It follows that $\pi_1 M\cong\Gamma$ is an extension of the free abelian group Γ_0 of rank $(n-1)$, by a cyclic infinite group, and that conjugation by a generator of the cyclic infinite group is equal to A. We consider Γ_0 and Γ as groups of transformations of the upper halfspace of the form

$$x\to\lambda Tx+b \quad (b\in\mathbb{R}^{n-2},\ \lambda>0,\ T\in O(n-2))$$

Let $x\to\lambda_0T_0x$ be a generator in Γ of the infinite cyclic group H_1/H_0, with $\lambda_0>1$. Then conjugation by this generator induces λ_0T_0 on H_0. We obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^{n-1} & \xrightarrow{u} & \mathbb{R}^{n-2} \\ A\downarrow & & \downarrow\lambda_0T_0 \\ \mathbb{R}^{n-1} & \xrightarrow{u} & \mathbb{R}^{n-2} \end{array}$$

where u is an R-linear surjection, induced by the holonomy homomorphism $h:\Gamma_1=\mathbb{Z}^{n-1}\to\mathbb{R}^{n-2}$. Hence $n-2$ of the eigenvalues of A are equal to λ_0 in absolute value, since $A\in GL(n-1,\mathbb{Z})$, and the exceptional eigenvalue is $\mu=\pm\lambda_0^{-n+2}$.

The eigendirection for the exceptional eigenvector is v, the direction of the foliation, since $u:\mathbb{R}^{n-1}\to\mathbb{R}^{n-2}$ can be identified with the restriction of the developing map to the inverse image of a horizontal horosphere. We now claim that the characteristic polynomial of A is irreducible over the integers. For suppose f is a polynomial over the integers, and suppose the exceptional eigenvalue is a root of f. Then $f(A)v=0$. Since the foliation parallel to v is dense, the induced map $f(A):T^{n-1}\to T^{n-1}$ is identically zero. Hence $f(A):\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ is equal to a constant, which must also be zero. It follows that the minimum polynomial of A is irreducible. Since the exceptional eigenvalue is a simple root, the minimum polynomial is equal to the characteristic polynomial.

§5. Algebraic number theory.

I would like to thank Simon Norton for the proof of the next theorem.

5.1. **Theorem.** *Let $a_0+a_1+\cdots+a_kx^k$ be an irreducible polynomial with integral coefficients and roots $\alpha_1,\cdots,\alpha_k$. Let $|\alpha_1|=\cdots|\alpha_{k-1}|\neq|\alpha_k|$. Then $k\leq3$.*

Proof Suppose $k\geq4$. Let $r=|\alpha_1|$. If α is a root, then so is $\bar{\alpha}$ and $|\alpha|=|\bar{\alpha}|$. Since $k\geq4$, we may assume that $\bar{\alpha}_1$ and α_1 are distinct roots. We may further assume that $\alpha_2\neq\bar{\alpha}_1$. It follows that $\bar{\alpha}_2\neq\alpha_1$. Futhermore, from the definition, $\alpha_2\neq\alpha_1$. We have the equation $\alpha_1\bar{\alpha}_1=\alpha_2\bar{\alpha}_2$. Let θ be an automorphism of the splitting field for the polynomial, taking α_1 to α_k. Then $|\theta\alpha_i|=r$ for $1<i\leq k$, and, in particular, $|\theta\alpha_2|=|\theta\bar{\alpha}_1|=|\theta\bar{\alpha}_2|=r$. We have

$$\theta\alpha_1.\theta\bar{\alpha}_1=\theta\alpha_2.\theta\bar{\alpha}_2$$

and so $|\alpha_k|r=r^2$, which is a contradiction.

It follows from the concluding paragraph of §4 that $n-1\leq3$, which means $n\leq4$. To obtain examples for $n=3$ and $n=4$, we can take

$$A=\begin{pmatrix}2&1\\1&1\end{pmatrix} \text{ and } A=\begin{pmatrix}0&0&1\\1&0&-1\\0&1&0\end{pmatrix}$$

In the second case, the characteristic equation is $\lambda^3+\lambda+1$, which has one real root μ and two complex roots $\lambda_0 e^{-i\theta}$ and $\lambda_0 e^{i\theta}$, with $\lambda_0>0$.

All examples of closed manifolds with transversely hyperbolic 1-dimensional foliations can be described explicitly. We describe the 4-dimensional case with a notation that is consistent with the example just looked at, and leave to the reader the trivial task of decreasing appropriate integers by 1 for the 3-dimensional case. Let $A \in GL(3,\mathbf{Z})$ have one real eigenvalue $\mu>1$ and two other eigenvalues of equal absolute value $\lambda_0<1$. (These will be complex conjugate in the transversely oriented case. In the transversely non-oriented case they are real with opposite signs. They can not be real with the same sign, since A acts irreducibly, as we saw at the end of §4.) Let $B=A^2$ if either $\mu<0$ or if the other two eigenvalues are real, equal and opposite in sign. Otherwise, let $B=A$. Let G be the simply connected 4-dimensional solvable Lie group, which is a split extension

$$0 \to \mathbf{R}^3 \to G \to \mathbf{R} \to 0$$

where the element 1 in $\mathbf{R}$ acts on $\mathbf{R}^3$ via B. G is diffeomorphic to $\mathbf{R}^4$. Let Γ be the split extension

$$0 \to \mathbf{Z}^3 \to \Gamma \to \mathbf{Z} \to 0,$$

where the generator of the quotient acts via A. Let Γ_0 (not the same Γ_0 as was defined previously) be the subgroup of index at most two, defined in the same way as Γ, but with B acting instead of A. Γ acts on G through right multiplication by A on the quotient copy of the reals (B is thought of as the number 1, and A as the number 1/2), and by the action of A as a matrix on $\mathbf{R}^3$. The action of Γ_0 on G is by right translation, but not the action of Γ, since A is not in G. There is a surjective homomorphism $G \to G_1$, obtained by taking the quotient of $\mathbf{R}^3<G$ by the normal subgroup which is the μ-eigendirection of A. Then G_1 is a split extension

$$0 \to \mathbf{R}^2 \to G_1 \to \mathbf{R} \to 0$$

and G_1 can be thought of as a simply transitive group of isometries of upper halfspace $\mathbf{H}^3$, which keep ∞ fixed. The subgroup $\mathbf{R}^2$ consists of parabolic translations, and the element 1 of $\mathbf{R}$ acts by a hyperbolic transformation, whose translation distance is $\log\lambda_0$ or $2\log\lambda_0$, depending on whether $A=B$ or $A^2=B$. G_1 is diffeomorphic to $\mathbf{H}^3$. The homomorphism $G \to G_1$ is the developing map. The manifold M is G/Γ.

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