D. BURGHELEA

Rational homotopy theory, group actions and algebraic *K*-theory of topological spaces

Astérisque, tome 113-114 (1984), p. 60-86

<http://www.numdam.org/item?id=AST_1984__113-114__60_0>

Rational homotopy theory, group actions and algebraic K-theory of topological spaces

by D. Burghelea (†

The Ohio State University
Department of Mathematics
Columbus, Ohio 43210

The most remarkable thing about rational homotopy theory is the fact that it has good algebraic models like CDGA's, CDGC's, DGL's, (* etc., and, in the corresponding category of models, the homotopy type can be described by (unique) minimal models which are analogous to the nilpotent groups in the category of groups. Most of the minimal models are algebraic objects which can be effectively manipulated. Consequently,

1) some very subtle algebraic-arithmetic structures in homotopy theory have been noticed and

2) effective computational methods for problems reducible to rational homotopy theory have been discovered.

---

*) CDGA = commutative differential graded algebra
CDGC = commutative differential graded coalgebra
DGL = differential graded lie algebra

†) Supported by NSF-Grant No. MCS8102686

The best known illustration of 1) is the result that $\pi_0(\mathcal{H}(X))$ (resp. $\pi_0(\mathrm{Diff}\ M^n)$ resp. $\pi_0(\mathrm{Homeo}\ N^n)$) is commensurable to an arithmetic group when X (resp. M^n resp. N^n) is a 1-connected compact polyhedron (resp. differentiable manifold resp. topological manifold) (see [W] and [S]). The best known illustrations of 2) are the description of the homotopy groups of the topological monoid of self-homotopy equivalences $\mathcal{H}(X)$ of a 1-connected polyhedron X, due to D. Sullivan [S] and Haefliger's description of the Gelfand-Fuks cohomology [H].

In this report I will present two other illustrations of 1) and 2). First, I will discuss the homotopy theorey of actions of discrete groups (mostly infinite) on spaces and show how the algebraic structure of the automorphisms of the minimal models permits us to obtain some unexpected results about such actions (theorems C and D, section I). The material in this section is essentially chapter I of John Oprea's thesis [O].

In section II, I will discuss computations of the algebraic K-theory of a topological space tensored by Q, the field of rational numbers. This ultimately permits us to compute the rank of the homotopy groups of diffeomorphisms and homeomorphisms $[B]_1$, [M]. This section is based on the results of myself $[B]_1$, $[B]_2$, $[B]_3$; Dwyer-Hsiang-Staffeldt [DHS], and Hsiang-Staffeldt $[HS]_1$, $[HS]_2$. Both sections, aside from the specific questions they address, provide new situations where important results in algebraic and arithmetic groups have interesting consequences in homotopy theory or geometric topology. They also provide new relations between homotopy theory, algebraic groups, algebraic geometry and invariant theory.

Section I:

In this section, the models we will work with are CDGA's (any other models ,namely CDGC's or DGL's, will produce equivalent results).

A minimal CDGA $\mathcal{A}$ is a commutative differential graded algebra over $\mathbb{Q}$ which is free (i.e., $\mathcal{A} = S(V)$ where V is a graded vector space $V = V_1 \oplus V_2 \oplus \dots$) , decomposable (i.e., $d(\mathcal{A}_+) \subset \mathcal{A}_+ \cdot \mathcal{A}_+$ where $\mathcal{A}_+$ is the ideal generated by positive degree elements) and nilpotent (*) (i.e, there exists a totally ordered basis of V , $\{x_\alpha\}$ so that $dx_\alpha \in S(V_{<\alpha})$ where $V_{<\alpha}$ is the subspace of V generated by x_β , $\beta < \alpha$) .

The nilpotency is the fundamental reason for the behaviour which will be described below.

Unless otherwise mentioned, all our minimal CDGA's will be assumed of finite type. This means that either $\{x_\alpha\}$ is finite or $\dim H^*(\mathcal{A}) < \infty$. Any of the above mentioned restrictions imply that the group $\mathrm{Aut}(\mathcal{A})$ is an algebraic group over $\mathbb{Q}$ of finite dimension (**) .

It is easy to see that for a minimal CDGA of finite type, $\mathrm{Aut}_h(\mathcal{A})$ (resp. $\mathrm{Aut}_H(\mathcal{A})$) , the subgroup of automorphisms homotopic to the identity (resp. inducing the identity on cohomology), is an algebraic subgroup. The following facts were first noticed by D. Sullivan [S].

---

*) If $\mathcal{A}$ is 1-connected, free and decomposable $\Rightarrow$ nilpotent.

**) If one works with algebraic groups of ∞-dimension one can weaken the finiteness condition to the following: in each degree there are only finitely many elements of the set $\{x_\alpha\}$.

<u>Theorem 1.1.</u> 1) $Aut_h(\mathcal{A}) \subset Aut_H(\mathcal{A}) \subset R(Aut(\mathcal{A}))$ are algebraic normal subgroups contained in the unipotent radical of $Aut(\mathcal{A})$, $R(Aut(\mathcal{A}))$.

2) The surjections $Aut(\mathcal{A}) \to Aut^h(\mathcal{A}) \to Aut^H(\mathcal{A})$ with $Aut^{\cdots}(\mathcal{A}) = Aut(\mathcal{A})/Aut_{\cdots}(\mathcal{A})$ induce homorphisms on the reductive quotiens,

$$\frac{aut(\mathcal{A})}{R(Aut\ \mathcal{A})} \xrightarrow{p} \frac{aut^h(\mathcal{A})}{R(Aut^h(\mathcal{A}))} \xrightarrow{p'} \frac{Aut^H(\mathcal{A})}{R(Aut^H(\mathcal{A}))} .$$

Equivalently, the above mentioned surjections induce isomorphisms on the maximal reductive subgroups.

The reader should be reminded that for an algebraic group over Q , any reductive subgroup is contained in a maximal reductive subgroup, any two maximal reductive subgroups are conjugate, and if $R(G)$ denotes the unipotent radical, any maximal reductive subgroup is the image of a crossection homomorphism $G/R(G) \xrightarrow{i} G$ (i.e., $p \cdot i = id$, $p : G \to G/R(G)$. He should also be reminded that for a reductive algebraic group, any finite dimensional algebraic representation is semisimple (i.e., decomposes as a direct sum of irreducible algebraic representations).

An action, (resp. <u>action up to homotopy</u> resp. <u>action up to homology</u>) of a group Γ on a minimal CDGA, $\mathcal{A}$, is a homomorphism $\mu : \Gamma \to Aut(\mathcal{A})$ (resp. $\tilde{\mu} : \Gamma \to Aut^h(\mathcal{A})$ resp. $\hat{\mu} : \Gamma \to Aut^H(\mathcal{A})$) . An action (resp. action up to homotopy resp. an action up to homology) is called reductive if $\mu(\Gamma)$ (resp. $\tilde{\mu}(\Gamma)$ resp. $\hat{\mu}(\Gamma)$ lies inside a reductive (*) subgroup of $Aut(\mathcal{A})$ (resp. $Aut^h(\mathcal{A})$ resp. $Aut^H(\mathcal{A})$) . The

*) If Γ is finite, any action is reductive.

following theorem is a straightforward consequence of Theorem 1.1.

Theorem A: 1) Any reductive action up to homotopy $\tilde{\mu}$ (resp. up to homology $\hat{\mu}$) has a lifting to a reductive action μ (i.e., $\tilde{\mu}$ (resp. $\hat{\mu}$) in the compontion of μ with the projection $\mathrm{Aut}(\mathcal{A}) \to \mathrm{Aut}^h(\mathcal{A})$ (resp. $\mathrm{Aut}(\mathcal{A}) \to \mathrm{Aut}^H(\mathcal{A})$) .

2) Two reductive actions $\mu_i : \Gamma \to \mathrm{Aut}(\mathcal{A})$ $i = 1, 2$ which are conjugate in $\mathrm{Aut}^H(\mathcal{A})$ are conjugate in $\mathrm{Aut}(\mathcal{A})$.

3) If $\tilde{\mu} : \Gamma \to \mathrm{Aut}^h(\mathcal{A})$ is a reductive action up to homotopy , g_1 and $g_2 \in \Gamma$ and $\tilde{\mu}(g_1)$ and $\tilde{\mu}(g_2)$ are equal in $\mathrm{Aut}^H(\mathcal{A})$ then they are equal in $\mathrm{Aut}^h(\mathcal{A})$.

Let G be an abstract group, $U \subset G$ be a normal subgroup and $\mu : \Gamma \to G$ be a group homomorphism. Let $\mathrm{ad} : \Gamma \to \mathrm{Aut}(G)$ be the adjoint action $\gamma \to h_\gamma$, $h_\gamma(g) = \mu(\gamma)g\,\mu(\gamma)^{-1}$, which also defines $\mathrm{ad} : \Gamma \to \mathrm{Aut}(U)$ since U is a normal subgroup. Let $H^1(\Gamma ; U)$ be the nonabelian cohomology of Γ with coefficients in U ; $H^1(\Gamma ; U)$ is a base-pointed set defined by $Z^1(\Gamma ; U)/\sim$ where $Z^1(\Gamma ; U) = \{\delta : \Gamma \to U \mid \delta(\gamma\gamma') = \delta(\gamma)\, h_\gamma(\delta(\gamma'))\}$ and the equivalence relation $\sim$ makes δ equivalent to δ^1 iff there exists $u \in U$ so that $\delta'(\gamma) = u^{-1}\delta(\gamma)\, h(_\gamma(u)$ for each $\gamma \in \Gamma$. Let $f \in G$ and assume that the set fU is invariant under $\mathrm{ad} : \Gamma \to \mathrm{Aut}(G)$; this means that $\mu(\gamma)f\,\mu(\gamma^{-1}) = f \cdot \delta^f(\gamma)$, where $\delta^f : \Gamma \to U$ is a cocycle in $Z^1(\Gamma ; U)$. The cohomology (equivalence)class of δ^f depends only on the set $[f\mathcal{U}]$ and not on the representative f .

Proposition 1.2 [O]. The action $ad \circ \mu : \Gamma \times [fU] \to [fU]$ has a fixed point iff the cohomology class $[\delta^f] \in H^1(\Gamma ; U)$ is the base-point.

Let us consider an action $\mu : \Gamma \to Aut(\mathcal{A})$ and let U be $Aut_h(\mathcal{A})$ (resp. $Aut_H(\mathcal{A})$) . Let ${}_\Gamma Aut(\mathcal{A})$ (resp. ${}_\Gamma Aut^h(\mathcal{A})$ resp. ${}_\Gamma Aut^H(\mathcal{A})$) be the subgroup of $Aut(\mathcal{A})$ (resp. $Aut^h(\mathcal{A})$ resp. $Aut^H(\mathcal{A})$) consisting of those elements which commute with the action μ (resp. $\hat{\mu} = p \circ \mu$ resp. $\mu = = p' \circ p \circ \mu$) . The construction described above produces the maps ${}_\Gamma Aut^h(\mathcal{A}) \xrightarrow{[\delta^{\cdots}]} H^1(\Gamma ; Aut_h(\mathcal{A}))$ and ${}_\Gamma Aut^H(\mathcal{A}) \xrightarrow{[\delta^{\cdots}]} H^1(\Gamma ; Aut_H(\mathcal{A}))$.

Theorem B [O]. The following sequences are exact

$$Aut_\Gamma(\mathcal{A}) \to Aut^h_\Gamma(\mathcal{A}) \xrightarrow{[\delta^{\cdots}]} H^1(\Gamma ; Aut_h(\mathcal{A}))$$

$$Aut_\Gamma(\mathcal{A}) \to Aut^H_\Gamma(\mathcal{A}) \xrightarrow{[\delta^{\cdots}]} H^1(\Gamma ; Aut_H(\mathcal{A}))$$

(this means that an automorphism $f \in Aut\, \mathcal{A}$ which is homotopy (resp. homology equivariant) is homotopic to an equivariant one iff $[\delta^f] = x$.

Here are some situations when the set $H^1(\Gamma ; U)$ consists of only one point. The exact sequence of nonabelian cohomology (see [Se]) combined with the nilpotency of $U = Aut_h(\mathcal{A})$ or $Aut_H(\mathcal{A})$ implies

Proposition 1.3. If $H^1(\Gamma ; \{V\}) = 0$ for any irreducible Γ-representation of finite dimension $\{V\}$, then $H^1(\Gamma ; U)$ reduces to $*$.

Examples: The hypothesis $H^1(\Gamma ; \{V\}) = *$ in Proposition 1.3 is satisfied for a) finite groups, b) any arithmetic subgroup of $SL_n(Q)$ $n \geq 3$ (in particular, $SL_n(Z)$) and any arithmetic subgroup of $Sp_n(Q)$ $n \geq 2$ (in particular, $Sp_n(Z)$). The hypothesis is probably satisfied for c) any arithmetic subgroup of a connected semisimple Q-algebraic group G of $\text{rank}_Q G \geq 2$ (see [BMS]).

Comments: Theorem B claims that if Γ belongs to the above mentioned class of groups, any isomorphism $f \in \text{Aut}(\mathcal{A})$ which is homotopically (resp. homologically) equivariant is homotopic (resp. homologically equivalent) to an equivariant one. The most natural way to prove this fact is to consider $\hat{\Gamma}$ = Zariski closure of $\mu(\Gamma)$ in $\text{Aut}(\mathcal{A})$, the unipotent normal subgroup $U = \text{Aut}_h(\mathcal{A})$ or $\text{Aut}_H(\mathcal{A})$ and the action by conjugation of $\hat{\Gamma}$ or $\text{Aut}(\mathcal{A})$; we have to show that whenever the subvariety fU is invariant under the adjoint action of $\hat{\Gamma}$, then it has a fixed point. Our conclusion should probably follow from some general fixed point theorems about algebraic action of reductive groups on some type of affine varieties. I do not know any result in this direction although something can be said (for instance, any linear representation of a reductive group decomposes as a direct sum of irreducible representations). The dual situation, algebraic actions of nilpotent algebraic groups on projective varieties is better known (see M. Rosentlich--Some basic theorems on algebraic groups--Annali di Math., 1957). It is not hard to supply a proof along these lines when Γ if finite.

Now we will discuss a few geometric applications of Theorems A and B. Let us first recall the following definitions: If Γ is a discrete group, then a Γ-space (X , μ) , or an action, is an homomorphism $\mu : \Gamma \to \mathrm{Homeo}(X)$ (or equivalently a continuous map $\mu : G \times X \to X$ satisfying $\mu(g_1 g_2 , x) = \mu(g_1 , \mu(g_2 , x))$ and $\mu(e , x) = x)$. An action up to homotopy on X is an homorphism $\tilde{\mu} : \Gamma \to \pi_0(\mathcal{H}(X))$ (or equivalently a continuous map $\tilde{\tilde{\mu}} : \Gamma \times X \to X$ which satisfies $\mu(g_1 g_2 ; \ldots)$ is homotopic to $\mu(g_1 , \mu(g_2 , \ldots)$ and $\mu(e , \ldots)$ is homotopic to $id_X)$.

Years ago D. Sullivan noticed the following:

<u>Theorem:</u> Given a compact riemannian manifold M^n which is 1-connected (or more generally a nilpotent space) any two isometries which induce the same isomorphism for cohomology are rationally homotopic.

The conclusion above is definitely false if h and g are only diffeomorphisms (and probably even affine maps. Here is a substantially stronger statement.

<u>Theorem C</u>: If Γ is finite or an arithmetic subgroup of the algebraic groups $SL_n(Q)$ $n \geq 3$ or $Sp_n(Q)$ $n \geq Z$, (and probably any simple algebraic group over Q ; G , with $\mathrm{rank}_Q\ G \geq 2)$ and $\tilde{\mu}$ is an action up to homotopy of Γ on the nilpotent space of finite type [*] X ,

*) X is of finite type if either $\dim \pi_*(X) \otimes Q < \infty$ or $\dim H_*(X) \otimes Q < \infty$.

then for any two elements g_1 , $g_2 \in \Gamma$ so that $\mu(g_1)$ and $\mu(g_2)$ induce the same isomorphism for cohomology $\mu(g_1)$ and $\mu(g_2)$ are rationally homotopic.

If M^n is a compact riemannian manifold, $\pi_o(\mathrm{Iso}(M^n))$ acts up to homotopy on M^n ; Sullivan's conclusion follows from Theorem C.

Proof: Let $\mathcal{A}$ be the minimal model of X . The action $\tilde{\mu}$ induces the action up to homotopy $\tilde{\mu} : \Gamma \to \mathrm{Aut}^h(\mathcal{A})$. Let g_1 , g_2 be two elements of Γ so that $\tilde{\mu}(g_1)$ and $\tilde{\mu}(G_2)$ induce the same isomorphism in cohomology. To verify Theorem C, we have to show that $\tilde{\mu}(g_1 g_2^{-1}) = e$; for this purpose we consider Γ' , the subgroup of Γ generated by $g_1 g_2^{-1}$. We will show that there exists a finite index subgroup $\Gamma_o \subset \Gamma$ so that $\tilde{\mu} : \Gamma_o \to \mathrm{Aut}^h(\mathcal{A})$ is reductive. If so, $\Gamma_o \cap \Gamma'$ is of finite index in Γ' and, since $\tilde{\mu} : \Gamma_o \cap \Gamma' \to \mathrm{Aut}^h(\mathcal{A})$ is reductive and trivial after composing with $\mathrm{Aut}^h(\mathcal{A}) \xrightarrow{p'} \mathrm{Aut}^H(\mathcal{A})$, then $\tilde{\mu} : \Gamma_o \cap \Gamma' \to \mathrm{Aut}^h(\mathcal{A})$ is trivial. Therefore, $\tilde{\mu}$ factors through $\tilde{\tilde{\mu}} : \Gamma'/\Gamma_o \cap f' \to \mathrm{Aut}^h(\mathcal{A})$ which is trivial after composing with p' and $\Gamma'/\Gamma_o \cap \Gamma'$ is finite. The same arguments as above show that $\tilde{\mu}$ is trivial. To check that $\tilde{\mu}$ restricted to a finite index subgroup is reductive, we apply the main results of [BMS] and conclude that there exists an algebraic morphism $\bar{\mu} : G \to \mathrm{Aut}^h(\mathcal{A})$ so that $\bar{\mu}$ restricted to a subgroup Γ_o of infinite index in Γ is exactly $\tilde{\mu}$. Clearly, $\bar{\mu}(G)$ is reductive since G is a connected simple algebraic group.

To better understand the contents of Theorem D, a few more definitions and considerations are necessary.

Two Γ-spaces (X_i , μ_i) $i = 1 , 2$ are called equivalent (resp. rationally equivalent) iff there exists a Γ-space (Y , η) and equivariant maps $f_i : X_i \to Y$ which are homotopy (resp. rational homotopy) equivalences.

An action up to homotopy $\tilde{\mu} : \Gamma \to \pi_0(\mathcal{H}(X))$ is said to be realizable) iff there exists an action $\mu : \Gamma \times Y \to Y$ together with a homotopy (resp. rational homotopy) equivalence $f : X \to Y$ so that the following diagram is homotopy commutative.

$$\begin{array}{ccc} \Gamma \times X & \xrightarrow{\tilde{\mu}} & X \\ \downarrow id \quad \downarrow f & & \downarrow f \\ \Gamma \times Y & \xrightarrow{\mu} & Y \end{array}$$

In this case (Y , μ) is called a realization of $\tilde{\mu}$ (resp. a rational realization of $\tilde{\mu}$) . Given a Γ-space (X , μ) and $f : X \to X$ a homotopy equivariant map f is said to be realizable (resp. rationally realizable) by an equivariant map iff there exists two Γ-spaces (Y , μ') , (X' , μ') , equivariant maps $f : X' \to X'$, $\alpha : X' \to Y$, $\beta ; X \to Y$ with α and β homotopy (resp. rational homotopy) equivlences and the following diagram homotopy commutative $(f' : X' \to X'$ is called a realization of $f : X \to X)$.

$$\begin{array}{ccccc} X & \xrightarrow{\beta} & Y & \xleftarrow{\alpha} & X' \\ \uparrow f & & & & \uparrow f' \\ X & \xrightarrow{\beta} & Y & \xleftarrow{\alpha} & X' \end{array}$$

Let us remind the reader of the homotopy theoretic context in which the above definitions occur.

Given a Γ-space (X, μ) one associates a fibration (actually, a bundle) $\psi(\mu) : X \to E \to K(G, 1)$ with fibre X. For any fibration $\xi : X \to E \to K(G, 1)$ one can associate an action up to homotopy $\xi : G \times X \to X$ and clearly $\tilde{\Psi}(\mu) = p \circ \mu$ where $p : \mathrm{Homeo}(X) \to \pi_0(\mathcal{H}(X))$.

<u>Observation 1:</u> (* a) There exist actions up to homotopy (on 1-connected rational spaces) which do not come from any fibration over $K(G, 1)$.

b) An action up to homotopy comes from a fibration iff it is realizable.

c) There exist fibrations $X \to E_1 \to K(G, 1)$ and $X \to E_2 \to K(G, 1)$ whose associated actions up to homotopy are equal but where E_1 and E_2 have different rational homotopy types.

c) Two Γ-spaces (X_1, μ_2) and (X_2, μ_2) are equivalent (resp. rationally equivalent) iff the associated fibrations $\psi(\mu_1)$ and $\psi(\mu_2)$ are fibre homotopy (resp. rational homotopy) equivalent.

If $\xi : X \to E \to K(G, 1)$ is a given fibration and $\alpha : E \to E$ a fibre homotopy equivalence which induces $\alpha' : X \to X$, then clearly, α' commutes with the associated action up to homotopy.

<u>Observations 2</u>: (* a) There exists $\alpha' : X \to X$ which commutes up to homotopy with a given action, but for which there is no fibre

(* The homotopy theory behind these observations is due to Dweyer-Dror-Kan [D.D.K.].

homotopy equivalence α of the fibration $\psi(\mu)$ inducing α' .

b) A necessary and sufficient condition for α' to be induced from a fibre homotopy equivalence is that α' is realizable in the sense described above.

Definition: An action up to homotopy $\mu : \Gamma \to \pi_0(\mathcal{H}(X))$ is rationally reductive iff $\mu : \Gamma \to \pi_0(\mathrm{Aut}^h(\mathcal{M}))$ is reductive, where $\mathcal{M}$ is the minimal model of X . (This definition makes sense only when X is of finite type) .

An action $\mu : \Gamma \to \mathrm{Homeo}(X)$ is called rationally reductive iff there exists a reductive action $\mu' : \Gamma \to \mathrm{Aut}(\mathcal{M})$ ($\mathcal{M}$ being a minimal CDGA) so that μ and the spatial realization $|\mu'| : \Gamma \to \mathrm{Homeo}(|\mathcal{M}|)$ are rationally equivalent. In the light of Observations 1 and 2, the following theorem is interesting.

Theorem D [O]: a) Any action up to homotopy $\mu : \Gamma \to \pi_0(\mathcal{A}(X))$, X nilpotent and of finite type, which is rationally reductive has a realization as a reductive action.

b) Any two rationally reductive actions (X_i , μ_i) $i = 1 , 2$ which are conjugate up to homotopy are rationally equivalent.

c) If (X , μ) is a reductive action and $f : X \to X$ a homotopy equivalence which is homotopy equivariant, then $f : X \to X$ has a rational realization as an equivariant map iff the associated obstruction $[\delta^f] \in H^1(\Gamma ; \mathrm{Aut}^h(\mathcal{M}))$ is the base point (this is always the case when G is finite or an arithmetic subgroup of $SL_n(Q)$ $n \geq 3$ or $Sp_n(Q)$ $n \geq 2$) . Here $\mathcal{M}$ is the minimal CDGA associated to X (the unique minimal model) on which the action μ provides a unique reductive action.

Section II:

Let us first recall that for an associative ring with unit R, the algebraic K-groups in positive dimension are defined as the homotopy groups of the space $K(R) = BGL(R)^+ \times K_0(R)$ here "+" denotes the Quillen "+"-construction. In principle, it is very difficult to compute these groups even in the simplest cases like $R = Z$ the ring of integers. However, because $K(R)$ is an H-space (in fact, an ∞- loop space), $\pi_*(K(R)) \otimes Q$ identifies to the primitive part of $H_*(BGL(R)^+ ; Q)$ which as a graded vector space is the symmetric algebra generated by $\pi_*(K(R)) \otimes Q$. The methods of differential geometry have permitted the calculation of $H_*(BGL(Z) ; Q)$, [BO], and as a consequence, we know that $\dim K_i(Z) \otimes Q = \begin{cases} 1 & \text{if } i = 0 \text{ and } i \equiv 1 \bmod 4 \\ 0 & \text{otherwise.} \end{cases}$

Waldhausen [W] has introduced the algebraic K-theory $K(R.)$, of a topological or simplicial ring $R.$, and he has defined $\underline{K}(X)$, the algebraic K-theory of connected base pointed space X, as the algebraic K-theory of the simplicial ring $Z(\Omega X)$, where ΩX is regarded as a simplicial group and $Z(\Omega X)$ is the simplicial group ring of ΩX. For a topological or simplicial ring $R.$ the construction of $K(R.)$ goes as follows: instead of the group $GL_n(R)$ one considers the topological associative monoid $\widetilde{GL}_n(R.)$ of $n \times n$ matrices with coefficients in $R.$ which are homotopy invertible; a matrix $\|a_{ij}\|$ is homotopy invertible if the matrix $\|\pi\, a_{ij}\|$, with $\pi : R. \to \pi_0(R.)$ denoting the projection on the ring of connected components is invertible. The topology of $\widetilde{GL}_n(R.)$ is the one induced from $M_n(R.) = R.^{n^2}$.

Matrix composition provides a structure of topological monoid on $\widetilde{GL}_n(R.)$ and hence on $\widetilde{GL}(R.) = \lim \widetilde{GL}_n(R.)$; this monoid has

$\pi_0(\widetilde{GL}(R.)) = GL(\pi_0(R.))$ and, therefore, one can define $K(R.) = B\widetilde{GL}(R.)^+ \times K_0(\pi_0(R))$. Clearly, if R is a discrete ring, when regarded as a topological ring with the discrete topology, K(R.) is the same as the usual algebraic K-theory. The functor K has the following properties: If $\varphi : R^1_. \to R^2_.$ is a continuous, resp. simplicial unit preserving homomorphism which induces an isomorphism for π_0 and is k-connected modulo a class of abelian groups, then $K(\varphi)$ is $(k + 1)$-connected modulo the same class. As a consequence, if $f : X \to Y$ is a base point preserving map of connected spaces which induces an isomorphism for π_1 and is k-connected modulo a class of abelian groups, so is K(f).

The calculation of $\pi_*(K(R.)) \otimes Q$ and in particular of $\pi_*(\underline{K}(X)) \otimes Q$ is very important for geometric topology. Here we will outline some results about these calculations for a class of topological (simplicial) rings called $\mathfrak{F}$. These results can be regarded as applications of rational homotopy theory. It is important to mention that the class $\mathfrak{F}$ contains $Z(\Omega X)$ for X 1-connected.

The calculation of $\pi_*(K(R.)) \otimes Q$ has to be at least as complicated as that of $\pi_*(K(\pi_0(R.) \otimes Q$ but the class $\mathfrak{F}$ contains topological rings R. with $\pi_0(R.) = Z$, and as noted, $\pi_*(K(Z)) \otimes Q$ is known. The extra complexity in the structure of $\pi_*(K(R.)) \otimes Q$ versus that of $\pi_*(K(Z)) \otimes Q$ comes from the homotopy type of R. . Fortunately, this is treatable by the techniques of rational homotopy theory combined with important results from classical invariant theory. The reader will not see too much invariant theory since it is hidden in Theorem 2.3 below (due to Hsiang I. Staffeldt) and in the proof of

Theorem 2.7. From "rational homotopy theory" we use: Quillen's results about the equivalence of the homotopy categories of various algebraic models for rational homotopy theory, the existence, uniqueness and the structure of minimal models for commutative differential graded coalgebras (as well as the existence of Γ-minimal models for Γ-commutative differential graded co-algebras on which the discrete group Γ acts reductively) and for differential graded chain algebras [B-L].

We begin with the description of the class $\mathfrak{F}$. Let us first call rationally equivalent two topological rings $R'_.$ and $R''_.$ for which one can find the topological rings $R^1_.$, $R^2_.$, ..., $R^{2s-1}_.$, and the continuous unit preserving homomorphism f^1, f^2, ..., f^{2s}, $R'_. \xleftarrow{f^1} R^1 \xrightarrow{f^2} R^2 \xleftarrow{f^3} \dots R^{2s-1} \xrightarrow{f^{2s}} R''_.$ with all f^i's inducing an isomorphism for π_0 and $\pi_i(\dots) \otimes Q$ $i \geq 1$. This insures that $K(R'_.)$ and $K(R''_.)$ are rationally homotopy equivalent. We say that a simplicial ring $R.$ is special if it satisfies properties A) B) C) below and we say that a simplicial resp. topological ring belongs to the class $\mathfrak{F}$ if it is rationally equivalent to a special ring resp. to a geometric realization of a special ring.

Property A): There exists a continuous (simplicial) ring homomorphism $i : \pi_0(R.) \to R.$ so that $\pi \cdot i = id$.

Property B): $\pi_0(R.) = Z$ and $R.$ is complete with respect to some filtration (* which begins with $R^{(1)}_. = Ker(\pi : R \to \pi_0(R.)) = \bar{R}.$;

*) A filtration is a sequence of ideals $R^{(1)}_. \supseteq R^{(2)}_. \supseteq .$ so that $R^{(i)}_. \cdot R^{(j)}_. \subseteq R^{(i+j)}$.

this means that $R. \to \varprojlim R./R.^{(i)}$ is an isomorphism .

Property C): $\mathring{R}$ has only one zero dimensional simplex and is a Q-algebra.

Proposition 2.1 1) Given two special rings R' and R'' there exists a new special ring $R' \boxplus R''$, the pull back of $R' \times R'' \to Z \times Z$ via the diagonal $Z \to Z \times Z$ with $R' \boxplus R'' = \mathring{R}' \times \mathring{R}''$.

2) Given a special simplicial ring $R.$, for each integer k , there exists a diagram $(k)R. \xrightarrow{i} R. \xrightarrow{p} R.(k)$ with all terms special simplicial rings, $i : (k)\mathring{R}. \to \mathring{R}.$, a k-connected covering and $p : \mathring{R} \to \mathring{R}.(K)$ a k-Postnikov fibration.

3) If X is 1-connected simplicial complex, $Z(\Omega X)$ is in $\mathfrak{F}$ and $\pi_i(Z(\Omega X) \otimes Q = \pi_i(Q(\Omega X)) = H_i(\Omega X ; Q)$, $i \geq 1$.

Let $M_n(\mathring{R}.)$ *) be the simplicial monoid of $n \times n$ matrices with coefficients in the ideal $\mathring{R}.$ equipped with the operation $A * B = A + B + A \circ B$. If the ring $R.$ has Property A, $GL_n(\pi_0(R.))$ acts on $M_n(R.)$ by conjugation and, therefore, on $BM_n(\mathring{R}.)$; we have the following commutative diagram.

$$\begin{array}{ccccc} GL_n(\pi_0(R.)) & \times & BM_n(\mathring{R}.) & \to & BM_n(\mathring{R}.) \\ \downarrow & & \downarrow & & \downarrow \\ GL_{n+1}(\pi_0(R.)) & \times & BM_{n+1}(\mathring{R}.) & \to & BM_{n+1}(\mathring{R}.) \end{array}$$

In order to prove geometric properties which make it possible to calculate the algebraic K-theory of special simplicial rings, it is convenient to use as models the category of commutative differential

*) If $R.$ satisfies Proberty B, then $M_n(\mathring{R}.)$ is a simplicial group.

graded coalgebras (CDGC) (whose duals are commutative differential graded algebras). One can consider actions of discrete groups on CDGC's, and for a fixed discrete group Γ we will denote by Γ-CDGC, the category whose objects are pairs consisting from a object $\mathcal{A} \in$ ob CDGC and action $\Gamma \times \mathcal{A} \to \mathcal{A}$ and whose morphisms are morphisms of CDGCs which are equivariant. There is an evident concept of minimal model (dual to the one for CDGA's) made explicit by Neisendorfer and Bauer-Lemaire and for any $\mathcal{A} \in$ ob CDGC one can prove the existence and uniqueness (up to a noncanonical isomorphism) of a minimal model $\mathcal{M}(\mathcal{A})$ and of a homotopy class of quasi-isomorphisms $\psi : \mathcal{A} \to \mathcal{M}(\mathcal{A})$ [N] [B-L]. Moreover, if $\mathcal{A} \in$ ob Γ-CDGC and in each degree n the representation of Γ on the n-th component of $\mathcal{A}$ is a direct sum of irreducible representations, one can construct an action Γ on $\mathcal{M}(\mathcal{A})$ and one can choose ψ to be equivariant. In fact, $\mathcal{M}(\mathcal{A})$ is still unique up to a noncanonical isomorphism as an object of Γ-CDGC. In [Q], Quillen has defined a spatial realization functor $| \; | :$ CDGC $\to$ Top$_*$. The same construction defines a spatial realization functor $| \; | : \Gamma$-CDGC $\to \Gamma$-Top$_*$ which associates with each action $\mu : \Gamma \times \mathcal{A} \to \mathcal{A}$ an action $|\mu| : \Gamma \times |\mathcal{A}| \to |\mathcal{A}|$ of the group Γ on the topological space $|\mathcal{A}|$
If $\mathcal{A}$ is 1-connected CDGC, $\mathcal{A}$ is minimal iff $\mathcal{A} = S(V)$ with V a graded vector space $V = \sum_{i=2}^{\infty} V_i$, $S(V)$ denotes the symmetric coalgebra generated by V , and the induced differential $d \cdot \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ (*) is zero. Notice that as a graded vector space the symmetric co-

*) $\mathcal{P}(\mathcal{A})$ denotes the graded vector space of primitive elements.

algebra generated by V is isomorphic to the symmetric algebra generated by V .

If $\mu : \Gamma \times X \to X$ is an action of Γ on the space X , let us denote by $X // \Gamma$ the total space of the fibration $\psi(\mu)$ defined in section I.

Given a linear representation $\xi : \Gamma \times V \to V$ one denotes by $Cov(V) = V/\hat{V}$ where $\hat{V}$ is the intersection of all subrepresentation V' so that V/V' is trivial. $Cov(V)$ is a trivial representation and the canonical projection $V \to Cov(V)$ is equivariant. If the representation ξ is semisimple the composition $Inv(V) \subseteq V \to Cov(V)$ is an isomorphism. If $\Gamma \times \mathcal{A} \to \mathcal{A}$ is a Γ-CDGC then $Cov(\mathcal{A})$ is a CDGC.

<u>Theorem E:</u> 1) If $R.$ satisfies Property A, then $B\widetilde{GL}_n(R.) = BM_n(R.) // GL_n(\pi_0(R.))$ for any n finite and $B\widetilde{GL}^+(R.) = (\mathbf{T}(R.) \times BGL^+(\pi_0(R.))$ with $\mathbf{T}(R.)$ an ∞-loop space.

2) If $R.$ is special, there exists a $GL_n(Z)$ - CDGC map $\tilde{\mu}_n : GL_n(Z) \times \mathcal{A}_n \to \mathcal{A}_n$, $n \leq \infty$, and a commutative diagram

$$\begin{array}{ccccc} GL_n(Z) & \times & \mathcal{A}_n & \to & \mathcal{A}_n \\ \downarrow & & \downarrow & & \downarrow \\ GL_{n+1}(Z) & \times & \mathcal{A}_{n+1} & \to & \mathcal{A}_{n+1} \end{array}$$

so that $\mathcal{A} = \mathcal{A}_\infty = \varinjlim \mathcal{A}_n$, and $|\tilde{\tilde{\mu}}_n| : GL_n(Z) \times |\mathcal{A}_n| \to |\mathcal{A}_n|$ is rationally equivalent to $\mu_n : GL_n(Z) \times BM_n(\hat{R}) \to BM_n(\hat{R}.)$. Moreover, for each n

the representation of $GL_n(Z)$ on each component of $\mathcal{A}_n$ is semisimple *).

3) If $\mathcal{M}(\mathcal{A}_n)$ is a $GL_n(Z)$-minimal model for $\mathcal{A}_n$ one has the sequence $\ldots \to Cov(\mathcal{M}(\mathcal{A}_n)) \to Cov(\mathcal{M}(\mathcal{A}_{n+1})) \to \ldots Cov(\mathcal{M}(\mathcal{A}))$ which for each component stabilizes, i.e., $(Cov\, \mathcal{M}(\mathcal{A}_n))_p = Cov(\mathcal{M}(\mathcal{A}_{n+1}))_p = \ldots = Cov(\mathcal{M}(\mathcal{A}))_p$ if $n \gg p$. Moreover, $\mathbf{T}(R.)$ has the same rational homotopy type as $|Cov(\mathcal{M}(\mathcal{A}))|$ and $Cov(\mathcal{M}(\mathcal{A}))$ is 1-connected, free commutative differential graded coalgebra (not necessary minimal).

Proof (sketch): The first part requires a careful description of the fibration $Fibre\ (\tilde{\pi}) \to B\widetilde{GL}_n(R.) \overset{\tilde{\pi}}{\to} BCL_n(\pi_0(R.))$, the observation that this fibration is exactly $\psi(\mu_n)$ as well as the proof (similar to the case of discrete rings) that $B\widetilde{GL}^+(R)$ is an ∞-loop space.

To prove 2, one needs the full chain of equivalences between the homotopy categories of various algebraic models which Quillen uses in [Q]. The proof goes as follows: first we notice (using [Q]) that the action μ_n is rationally equivalent to the one obtained by the Quillen spatial realization from the action $\tilde{\mu}_n : GL_n(Z) \times \mathcal{C}NM_n(\mathring{R}.) \to \mathcal{C}NM_n(\mathring{R}.)$. To describe $\mathcal{C}NM_n(\mathring{R}.)$ one begins with the Q-simplical algebra $\mathring{R}$; $M_n(\mathring{R}.)$ is the simplicial lie algebra of $n \times n$ matrices with coefficients in $\mathring{R}.$ and $NM_n(\mathring{R}.)$ is the differential graded lie algebra obtained from $M_n(\mathring{R}.)$ by normalization. $\mathcal{C}$ is the Koszul-Quillen functor which assocites with a differential graded algebra a commutative differential graded coalgebra. It is immediate that

*) semisimple representation = direct sum of irreducible representations

$\mathcal{C}NM_n(\mathring{R}.) = \mathcal{C}M_n(N(\mathring{R}.))$. $GL_n(Z)$ acts by conjugation on $M_n(N(\mathring{R}))$ and then on $\mathcal{C}(M_n(N(\mathring{R}))$; clearly, any component of $\mathcal{C}(M_n(N(\mathring{R}))$ regarded as a $GL_n(Z)$ representation is a direct sum of tensor products of symmetric and exterior powers of the adjoint representation $\rho_n : GL_n(\mathbf{Z}) \times M_n(Q) \to M_n(Q)$ $(\rho_n(A ; M) = AMA^{-1})$ and is, therefore, semisimple.

3) is a consequence of Borel's theorems about the vanishing of the cohomology (in dimension $\ll n$) of $GL_n(Z)$ with coefficients in a nontrivial irreducible representation. Applying the theorems one concludes that $BM(\mathring{R}.) // GL(Z)$ has the same rational cohomology as $|\mathrm{Cov}\ \mathcal{M}(\mathcal{A}_\infty)| \times BGL(Z)$. To check that $\mathrm{Cov}\ \mathcal{M}(\mathcal{A}_\infty)$ is a free commutative graded coalgebra one observes that one can produce a ring $R.$ so that $\mathrm{Cov}(\mathcal{M}(\mathcal{A}_\infty))$ is the homology coalgebra of $\mathbf{T}(R.)$.

It is clear that replacing $N(\mathring{R}.)$, the augmentation ideal of the chain differential graded algebra $N(\mathring{R}.)$, by the augmentation ideal of any other homotopy equivalent *) chain differential graded algebra one obtains the same result.

For $K.$ a 1-connected chain differential graded algebra over Q , let $\overline{K}$ be its augmentation ideal and $\overline{K}_{ab}$ its abelianization (precisely one regards $\overline{K}$ as a differential graded lie algebra with $[x , y] = xy \pm yx$ and one take $\overline{K}_{ab} = \overline{K}/[\overline{K} , \overline{K}]$ regarded as a com-

---

*) Two chain differential graded algebras K' and K'' are homotopy equivalent if there exists a chain of differential graded algebras $K_1 , K_2 , \ldots , K_{2s+1}$ and the quasi-isomorphims f^i's , $K' \xleftarrow{f^1} K_1 \xrightarrow{f^2} K_2 \xleftarrow{f^3} \ldots K_{2s+1} \xrightarrow{f^{2s+1}} K''$.

mutative differential graded algebra). Then "trace" induces an homomorphism of differential graded lie algebras $Tr : M(\overline{K}.) \to \overline{K}_{ab}$. Applying the functor $\mathcal{C}$ one obtains the following homotopy commutative diagram

$$\begin{array}{ccccc} & \mathrm{Cov}(\mathcal{M}(\mathcal{C}(M(\overline{K}.)))) & \longleftarrow & \mathrm{Cov}(\mathcal{C}(M(\overline{K}.))) & \\ \swarrow & & \nearrow & & \searrow \mathcal{L} \\ \mathcal{M}\ \mathrm{Cov}(\mathcal{M}(\mathcal{C}(M(\overline{K}.)))) & \longleftarrow\text{-----} & \mathcal{C}(M(\overline{K}.)) & \xrightarrow{\mathcal{C}(T_r)} & \mathcal{C}(\overline{K}_{ab}) \end{array}$$

<u>Theorem 2.3</u>.(Hsiang-Staffeldt) $[H\text{-}S]_1$. If K is a tensor algebra, then $\mathcal{L}$ is a quasi-isomorphism, consequently, $\mathrm{Cov}(\mathcal{C}(M(\overline{K}.)))$ and $\mathcal{C}(\overline{K}_{ab})$ have isomorphic minimal models.

<u>Observation:</u> If $\mathrm{Cov}(\mathcal{M}(\mathcal{C}(M(\overline{K}.)))$ describes the rational homotopy type of $\mathbf{T}(R.)$ then the minimal model of $\mathcal{C}(\overline{K}_{ab})$ which is the same as of $\mathrm{Cov}(\mathcal{M}(\mathcal{C}(M(\overline{K}.)))$ is exactly $(H_*(\mathcal{C}(\overline{K}_{ab}))\ , d = 0)$, hence $\pi_i(\mathbf{T}(R.)) = H_{i-1}(\overline{K}_{ab})$.

<u>Exact Formulae</u>: It is known that each 1-connected chain differential graded algebra has a minimal model which is a tensor algebra of a graded vector space [B-L]. So, in particular, one can take such a model for $N(\overset{o}{R}) \oplus Z$ and denote it by K . The Hsiang-Staffeldt theorem implies that $\pi_{*+1}(\mathbf{T}(R.)) = H_*(\overline{K}_{ab})$. If $R. = Z(\Omega X)$ with X 1-connected the model K is Chen's minimal model of X . There are two cases when $H_*(\overline{K}_{ab})$ (for K the Chen's minimal model) can be relatively easily calculated, namely when X is a 1-connected space

rationally homotopy equivalent to either $\bigvee_\alpha S^{n_\alpha}$ $n_\alpha \geq 2$ or $\prod_\alpha K(V_\alpha)$ $n_\alpha \geq 2$. In the first case one obtains.

Theorem 2.4 ($[H-S]_1$). If X is a 1-connected space rationally homotopy equivalent to a bouquet of spheres of dimension ≥ 2 then $K_i(X) \otimes Q = K_i(Z) \otimes Q + (\overline{H}_*(\Omega X : Q)_{ab})_{i-1}$.

This is a straightforward application of Theorem E and Theorem 2.3. As a consequence, one has $\dim K_i(S^{2n+1}) \otimes Q$ resp. $\dim K_i(S^{2n}) \otimes Q$ is the i-th term of the series $1 + t^5(1 - t^4)^{-1} + t^{2n+1}(1 - t^{2n})^{-1}$ resp. $1 + t^5(1 - t^4)^{-1} + t^{2n}(1 - t^{4n-2})^{-1}$.

For the second case, a few more considerations are necessary. Let V be a graded vector space $V = \sum_{i=1}^{\infty} V_i$ regarded as a commutative differential graded Lie algebra with $d = 0$ and let $(\mathcal{U}(V), d = 0)$ be the differential graded chain algebra obtained by the universal enveloping algebra construction $(\mathcal{U}(V), d = 0) = (S(V), d = 0)$. Let $(T(V'), d)$ *) be its minimal model ; it is unique up to an isomorphism and characterized by the following two properties: i) there exists a quasi-isomorphis $\psi : (T(V'), d) \to (\mathcal{U}(V), d = 0)$, ii) d followed by the projection $\overline{T}(V') \to \overline{T}(V')/\overline{T}(V') \cdot \overline{T}(V')$ is zero.

*) $T(V')$ denotes the free associative graded algebra generated by V' .

Theorem 2.5 $[H\text{-}S]_2$. If X is a 1-connected space rationally homotopy equivalent to $\prod_\alpha K(G_\alpha, n_\alpha)$, $n_\alpha \geq 2$, and $V = \pi_*(\Omega X) \otimes Q$, then $K_i(X) \otimes Q = K_i(Z) \otimes Q + H_{i-1}(\overline{T}(V')_{ab})$.

If V has an homogeneous base $\{u_i, v_j\}$ with $|u_i| = \deg u_i$ even and $|v_j| = \deg v_j$ odd then $t_i = \dim H_{i-1}(\overline{T}(V')_{ab})$ is the i-th coefficient of the serie $\prod_i (1 - t^{|u_i|-1})^{-1} \cdot \prod_j (1 - t^{|v_j|+1})^{-1} \cdot$
$\cdot\, t(1+t)^{-1}(\chi(X, t) - 1)$ where $\chi(X, t) = \prod_i (1 + t^{|u_i|}(1+t)) \cdot$
$\cdot \prod_j (1 + t^{|v_j|}(1+t))$.

If the set of even degree resp. odd degree generators are empty the product $\prod_i(\ldots)$ are considered equal to 1.

The first part is a straightforward consequence of Theorem E and Theorem 2.3. The second part follows from the first; if V has a homogeneous base with either elements of even degree or odd degree the formula above is not too hard to verify. The general case is much more complicated. Notice that always $V' = s^{-1}\,\widetilde{H}_*(X : Q)$ (**, hence $V' = s^{-1}\,S(sV)$. For example, if V' = the graded vector space with a base $\{v_i\}$ $\deg v_i = |v_i| = 2ni - 1$, then it is easy to see that one can take $dv_i = \sum_{r+s=i} v_r v_s$ and, therefore

$$H_r(T(V)_{ab} = \begin{cases} Q & \text{if } r = 2ni - 1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{so} \quad \dim K_i(K(Z, 2n)) \otimes Q =$$

(** If V is a graded vector spaces s^rV is the graded vector space defined by $(s^rV)_i = V_{i-r}$.

$\dim K_i(Z) \otimes Q + \begin{cases} 1 & \text{if } i = 2nr \\ 0 & \text{elsewhere} \end{cases}$. This result was proved first in $[B]_2$. In general, it is not easy to produce models $\overline{K}$ which are tensor algebras and to calculate $H(\overline{K}_{ab})$. The following result, which follows from Theorem E and the general properties of minimal models provides upper bound estimates for $\dim K_i(R.) \otimes Q$ resp. $\dim K_i(X) \otimes Q$ in terms of $r_{i+1}(R.) = \dim \pi_i(\overset{o}{R}.) \otimes Q$ resp. $s_i(X) = \dim H_i(X ; Q)$.

<u>Theorem 2.7.</u> There exists the polynomials $\mathcal{R}_i(X_2 , X_3 ,\ldots, X_i)$ resp. $\mathcal{S}_i(X_2 , X_3 ,\ldots, X_i)$ whose monomials $X_2^{n_2} X_3^{n_3} \ldots X_i^{n_i}$ satisfy $n_i\, i + n_{i-1}(i - 1) + n_{i-2}(i - 2) + \ldots n_2\, 2 \leq i$ so that

a) For any topological (simplicial) ring R. in the class $\mathcal{F}$ resp. 1-connected topological space X ,

$$\left.\begin{array}{l}\dim K_i(R.) \otimes Q \\ \\ \dim K_i(X) \otimes Q\end{array}\right\} \leq \dim K_i(Z) \otimes Q + \begin{cases} R_i(r_2 , r_3 ,\ldots, r_i) \\ \\ \mathcal{S}_i(s_2 , s_3 ,\ldots, s_i) \end{cases}$$

b) These inequalities are sharp; this means they are equalities for some topological rings in $\mathcal{F}$ (for instance, those whose ideal $\overset{o}{R}.$ is a product $\prod_i \overset{o}{R}{}^i.$ where $\overset{o}{R}{}^i \approx K(Q^{r_i} , i))$ resp. for some 1-connected spaces (for instance, those which are of the form $X = \bigvee_{i=2}^{\infty} X_i$ with $X_i = \underbrace{S^i \vee S^i \vee \ldots S^i}_{s_i}$); they provide the best upper bound polynomial approximations for fixed $r_2 , r_3 \cdots r_n \cdots$ resp. $s_2 , s_3 \cdots s_n \cdots$.

The polynomials $\mathcal{S}_i$ can be calculated using Theorem 2.4. The calculation of $\mathcal{R}_i$ is much more elaborate $[B]_3$. The first eleven polynomials $\mathcal{R}_i(r_2 ,\dots, r_i)$ are listed below.

$$R_2 = r_2$$

$$R_3 = r_3$$

$$R_4 = r_4 + \frac{1}{2} r_2^2 + \frac{1}{2} r_2$$

$$R_5 = r_5 + r_2 r_3$$

$$R_6 = r_6 + r_2 r_4 + \frac{1}{2} r_3^2 + \frac{1}{3} r_2^2 - \frac{1}{2} r_3 + \frac{2}{3} r_2$$

$$R_7 = r_7 + r_2 r_5 + r_3 r_4 + r_2^2 r_3$$

$$R_8 = r_8 + r_2 r_6 + r_3 r_5 + \frac{1}{2} r_4^2 + r_2^2 r_4 + r_3^2 r_2 + \frac{1}{4} r_2^4 + \frac{1}{4} r_2^2 + \frac{1}{2} r_2 + \frac{1}{2} r_4$$

$$R_9 = r_9 + r_2 r_7 + r_3 r_6 + r_4 r_5 + \frac{1}{3} r_3^3 + r_2^2 r_5 + 2 r_2 r_3 r_4 + r_2^3 r_3 + \frac{2}{3} r_3$$

$$R_{10} = r_{10} + r_2 r_8 + r_3 r_7 + r_4 r_6 + \frac{1}{2} r_5^2 + r_2^2 r^6 + r_3^2 r_4 + r_4^2 r_2 + 2 r_2 r_3 r_5 + {}$$
$$+ r_2^3 r_4 + \frac{3}{2} r_2^2 r_3^2 + \frac{1}{5} r_2^5 - \frac{1}{2} r_2 r_3 - \frac{1}{2} r_5 - \frac{4}{5} r_2$$

$$R_{11} = r_{11} + r_2 r_9 + r_3 r_8 + r_4 r_7 + r_5 r_6 + 2 r_2 r_3 r_6 + 2 r_2 r_4 r_5 + r_2^2 r_7 + {}$$
$$+ r_3^2 r_5 + r_2^3 r_5 + 2 r_2^2 r_3 r_4 + r_4^2 r_3 + r_3^3 r_2 - r_2^4 r_3$$

References

[BMS] - H. Bass - J. Milnor - J.P. Serre - Solution of the congruence subgroup problem for SL_n $n \geq 3$ and SP_n $n \geq 2$. Publ. I.H.E.S. n° 33 .

$[B]_1$ - D. Burghelea - The rational homotopy groups of Diff(M) and Homeo(M) in stability range - Proc. Conf. Alg. Topology (Arahus 1978) Lecture Notes in Math., Vol. 763, Springer Verlag, pp. 604-626.

$[B]_2$ - D. Burghelea - some rational computations of the Waldhausen algebraic K-theory - Comment. Math Helv. 54 (1979), pp. 185-198.

$[B]_3$ - D. Burghelea - Algebraic K-theory of topological rings I, (preprint O.S.U.).

[B-L] - H.J. Baues and J.M. Lemaire - Minimal models in homotopy theory-Math. Annalen 225 (1977), pp. 219-242.

[BO] - A. Borel - Stable real cohomology of arithmetic groups, Ann. Sci. Ec. Norm. Sup., Vol. 7 (1974), pp. 217-240.

[B-W] - A. Borel and N. Wallach - Continuous cohomology, discrete sub-groups and representations of reductive groups, Ann. of Math. Studies, Vol. 94.

[DHS] - W. Dwyer, W.C. Hsiang, R. Staffeldt, Pseudo isotopy and invariant theory I - Topology, Vol. 19, pp. 367-385 (1980) and II - Proc. Siegen Conf. on algebraic topology - Lecture Notes in Math. - Springer Verlag, Vol. , pp. 418-441.

[DDK] - E. Dror, W. Dweyer, D. Kan - P.A.M.S., Vol. 80, pp. 491-94, 670-72.

[H] - André Haefliger - Sur la cohomology de l'algebre de lie de champs de vecturs - Ann. Sci. Ec. Norm. Sup. - t9, 1976, pp. 503-532.

$[H-S]_1$ - W.C. Hsiang and R. Staffeldt - A model for computing rational algebraic K-theory of simply connected spaces - preprint.

$[H-S]_2$ - W.C. Hsiang and R. Staffeldt - Rational algebraic K-theory of a product of Eilenberg-MacLane spaces - preprint.

[M] - W. Meyer - Some topological properties of Kahler manifolds and homogeneous spaces - preprint.

[N] - J. Neisendorfer - Lie algebras, coalgebras and rational homotopy theory for nilpotent spaces. Pacific J. Math. 74 (1978), pp. 429-460.

[O] - John Oprea - Contirbutions to rational homotopy theory - thesis - OSU, 1982.

[Q] - D. Quillen - Rational homotopy theory - Ann. of Math., Vol. 20, No. 2, 1969, pp. 205-295.

[S] - D. Sullivan - In finitesimal computations in Topology, Publ. I.H.E.S., n. 47.

[W.] - Clarence Wilkerson - Applications of minimal simplicial groups Topology 15, (1976), pp. 111-130.

[W_o] - F. Waldhausen - Algebraic K-theory of topological spaces I - Proc. Symp. in Pure Math., Vol. 32, pp. 35-60.