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POINCARÉ DUALITY ALGEBRAS AND THE RATIONAL CLASSIFICATION OF DIFFERENTIABLE MANIFOLDS

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1. Poincaré duality algebras. Let H be a connected P.d.a. For such an algebra, we shall focus our attention on two natural invariants, namely: n=c-dim H (the formal dimension) and $C_H=H^+/H^+.H^+$ (the graded vector space of indecomposable generators).

In order to state the result, we fix n and C and introduce a few notations: let $A=A=\Lambda(C)$ be the free commutative graded algebra on C, let G(C) be the group of automorphisms of A, and denote by $H^{reg}(C,n)\subset \mathrm{Hom}(A^n,Q)\setminus\{0\}$, the regular functionals L, defined by the property:

$$\{b\in A^p \mid L(a.b)=0, \forall a\in A^{n-p}\}\subset (A^+.A^+)^p, \quad \forall p\leqslant n$$

Let us note that the natural action of G(C)xGL(1) on Hom (A^n,Q) restricts to the regular functionals.

Given $L\in H^{reg}(C,n)$, we can construct an ideal $I_L\subset A$ by:

(i)
$$I_L^p=\{b\in A^p \mid L(axb)=0, \forall a\in A^{n-p}\} \text{ for } p\leqslant n, \text{ and}$$
$$I_L^p=A^p, \text{ for } p>n.$$

We then associate to L the graded algebra H_L by:

(ii) $$H_L=\Lambda(C)/I_L$$

Theorem 1. The isomorphism classes of P.d.a.'s H, having n and C as invariants, are in bijection with the orbit space

$$H^{reg}(C,n)/G(C)\times GL(1)$$

by the correspondence described in (i) and (ii).

Sketch of proof: Fixing C and n means dealing with algebras of the form $H=A/J$, where the ideal J satisfies: $J \subset A^+.A^+$ and $J^p=A^p$ for $p > n$. The algebras H and H' are isomorphic precisely when the corresponding ideals are conjugate by an element of G(C). It finally turns out that the Poincaré duality requirement for A/J is strong enough to determine J by the formulae (i), where L mod GL(1) is given by: ker $L=J^n$. The details may be found in [6].

As an illustration, let us consider one of the simplest cases, namely when the group G(C) reduces to a linear group, i.e. homogenously generated algebras H (that is H is generated as an algebra by some homogenous component H^d - see also [5] for a geometric interpretation of this condition).

The invariants n and C reduce to: m (the number of generators), d (the degree of the generators) and c (the length of the product of H). We define the regular forms $q \in H^c_{d,reg}(m)$ to be those degree c homogenous polynomials (exterior forms), according to the parity of d, in m variables, with the property that the elements $\frac{\partial q}{\partial x_1}, \ldots, \frac{\partial q}{\partial x_m}$ are linearly independent.

Given such a form q, we write it as: $q=\sum_{|\alpha|=c} q_\alpha \cdot x^\alpha$ and we define a linear functional $L_q \in Hom(A^n,Q)$ by;

(iii) $$L_q(x^\alpha)=(\alpha!/c!)q_\alpha \quad , \text{ for } |\alpha|=c$$

Corollary: The isomorphism classification of homogenously generated P.d.a's having d,m and c as invariants coincides with the linear classification of regular forms ([6]).

In particular this shows that, even in this simple case, we are still left with a very difficult (classical) classification problem.

2. The rational homotopy types of closed manifolds

Sullivan's results from [9] suggest the following approach to the rational classification problem for closed manifolds: first classify the Poincaré duality algebras, then determine those which come from manifolds and then try to describe the Q-types within such a cohomology algebra. As far as the second step is concerned, one has the following "differentiability test":

Theorem ([9], [1]). Let H be a simple connected P.d.a. of formal dimension n. The necessary and sufficient condition for H in order to be the cohomology algebra of a closed manifold is: either $n \neq 0 \pmod 4$ or $n=4k$ and there exist an orientation $\omega \in \mathrm{Hom}(H^{4k},Q)\setminus\{0\}$ and a Pontrjagin class $p=\sum p_i \in H^{4i}$ such that:

(D1) the quadratic form $H^{2k} \otimes H^{2k} \xrightarrow{\cdot} H^{4k} \xrightarrow{\omega} Q$ is a sum of squares.

(D2) the numbers $\{\omega(p^I) \mid I \text{ a partition of } k\}$ are the Pontrjagin numbers of a closed manifold.

(D3) the Hirzebruch formula: $\omega(L_k(p))$=signature of the quadratic form $H^{2k} \otimes H^{2k} \to Q$.

In order to avoid the complications arising in general at the third step of the classification, we have restricted our attention to the case of intrinsically formal algebras, i.e. those which contain exactly one Q-homotopy type (see [3]).

Proposition. If H is an homogenously generated P.d.a. 1-connected and if $c \leq d+1$, then H is intrinsically formal ([8]).

If H is a (k-1)-connected P.d.a of formal dimension $\leq 4k-2$, then H is intrinsically formal ([4]).

It comes out that 1-connected homogenously generated P.d.a.'s, with $c \leq 3$, respectively the simply connected arbitrary P.d.a.'s of formal dimension $n \leq 6$, are intrinsically formal. It is worth mentioning that there are related examples of closed manifolds whose cohomology algebras are not intrinsically formal: for c=4 and d=2

$$(P^2C \# ((P^1C \times P^1C)) \times P^2C \quad ([8]) \text{ and for } n=7 \; (S^2 \times S^5) \# (S^2 \times S^5) \qquad ([7])$$

The rational classification of homologically 1-connected closed manifolds M (i.e. $H^1(M;Q)=0$) with homogenously generated cohomology and $c \leq 3$ is given by the theorem below.

Theorem 2. I) c=1: M has the Q-type of a sphere S^d.

II) c=2: if d=2k+1, M has the Q=type of a connected sum of p copies of $S^{2k+1} x S^{2k+1}$;

If d=2k, M has the Q-type of a complex of the form:

$(\underbrace{S^d v ... v S^d}_{m_+} v \underbrace{S^d v ... v S^d}_{m_-}) \bigcup_f e^{2d}$, where

$[f] \in \pi_{2d-1}(S^d v ... v S^d)$ is given by:

$[f] = \sum_{i=1}^{m_+} [S_i^d, S_i^d] - \sum_{j=1}^{m_-} [S_j^d, S_j^d]$; the invariants are: the dimension n=4k, the rank m ($m=m_+ + m_-$) and the signature σ ($\sigma = m_+ - m_-$) subject to a single restriction: σ is a multiple of a certain number σ_k, depending only on k.

III) c=3: for fixed d and m, the classification coincides with the linear classification of the regular forms in $H^3_{d,reg}(m)$; if d is odd we must have: m=3 or m$\geq$4, see [6].

The proof is given in [7] essentially by using the conditions (D1)-(D3) of Sullivan, which are nonvacuos only in the case c=2, d=2k. In this case, (D1) gives the normal form and (D2), (D3) reduce to the divisibility condition for the signature (compare with [10]). The numbers σ_k are computable; for example, using results from [2] it can be shown that:

$$\sigma_k = (2^{2k-1} - 1) \text{ numerator } (B_k/k) . \frac{2^{2k-1}}{2^{a_k}} \quad , \text{ for k odd,}$$

where B_k stands for the k-th Bernoulli number and 2^{a_k} denotes the greatest power of 2 which divides (2k)! (see [7]).

A similar classification for homologically 1-connected n-manifolds (for n$\leq$6) may also be found in [7].

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