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by

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0. Introduction. Let R be a local noetherian ring with residue field k. The n-th deviation of R, $e_n(R)$, is the dimension of a functorially defined k-vector space $V_n(R)$ (cf. Gulliksen [8]). We have $e_1(R) = \text{emb.dim.}(R)$, and the equality for the Poincaré series of R;

$$P_R(z) = \sum_{j=0}^{\infty} \dim_k(\operatorname{Tor}_j^R(k,k))z^j = \prod_{j=1}^{\infty} \frac{(1+z^{2j-1})^{e_{2j-1}(R)}}{(1-z^{2j})^{e_{2j}(R)}} .$$

Let $A \to B$ be a local flat homomorphism with fibre $\overline{B}$, A and B having residue fields k and $\mathbb{1}$ respectively. T. Gulliksen [8] has shown that we then have a long exact sequence of $\mathbb{1}$-vector spaces, which L. Avramov [3] has shown splits into exact sequences of six terms:

$$0 \to V_{2n}(A)\otimes_k \mathbb{1} \to V_{2n}(B) \to V_{2n}(\overline{B}) \longrightarrow V_{2n-1}(A)\otimes_k \mathbb{1} \to V_{2n-1}(B) \to V_{2n-1}(\overline{B}) \to 0$$

$$V_{2n}(\overline{B}) \searrow I_{2n} \nearrow V_{2n-1}(A)\otimes_k \mathbb{1}, \qquad 0 \nearrow I_{2n} \searrow 0$$

Put $\delta_{2n} = \dim_{\mathbb{1}}(I_{2n})$ and $\delta_{2n}(\overline{B}) = \max\limits_{A,B} \delta_{2n}$.

It is easy to see that $\delta_2 = e_1(A) - e_1(B) + e_1(\overline{B})$ in some cases can be greater than zero, but for the higher δ-s M. André [2] and L. Avramov among others has put forward the following conjecture

CONJECTURE 1: For all local noetherian rings $\overline{B}$ we have $\delta_{2n}(\overline{B}) = 0$ for $n > 1$. In other words, for all local flat homomorphisms $A \to B$ with fibre $\overline{B}$, we have

$$P_A(z)\cdot P_{\overline{B}}(z) = P_B(z)\,\frac{(1+z)^{\delta_2}}{(1-z^2)^{\delta_2}} \quad \text{with} \quad \delta_2 = e_1(A) - e_1(B) + e_1(\overline{B}) .$$

The conjecture is obviously true if $\overline{B}$ is a complete intersection.

M. André [1] has proved the conjecture in the case where $\mathrm{char}(k) = 2$, and he has also shown [2] that all but a finite number of the δ-s are zero. More precisely, $\sum_{n=1}^{\infty} \delta_{2n}(\overline{B}) \leq e_1(\overline{B}) - \mathrm{depth}(\overline{B})$ with equality if and only if $\overline{B}$ is a complete intersection.

In this paper we show that the number $\delta_{2n}(\overline{B})$ is not greater than the dimension of the k-vector space of the <u>central</u> elements of degree $2n$ of the graded Lie algebra underlying the Yoneda Ext-algebra $\mathrm{Ext}^*_{\overline{B}}(k,k)$. Using this, we prove the conjecture for local rings $\overline{B}$ attached by a finite sequence of Golod epimorphisms to a regular ring, e.g. Golod rings and quotients of regular rings by ideals generated by monomials in the elements of some regular sequence.

1. Liftings and special variables

This section is a slight reformulation, suited to our purposes, of some parts of the paper [2] of M. André. Let the fibre $\overline{B}$ be a fixed local noetherian ring in the following.

Let $A \to B$ be a local flat homomorphism with fibre $\overline{B}$ as above, and let X be a minimal A-resolution of k. Then $X \otimes_A B$ becomes a minimal B-resolution of $\overline{B}$, so $X \otimes_A B \xrightarrow{\sim} \overline{B}$ ($\overline{B}$ in degree 0) induces isomorphism in the homology.

When we start to construct a minimal $\overline{B}$-resolution of k by adjoining a variable T_1 to kill a cycle t_1, we can lift this cycle to a cycle $\tilde{t}_1$ of $X \otimes_A B$, and adjoin a variable $\tilde{T}_1$ to kill $\tilde{t}_1$. The mapping $X \otimes_A B \langle \tilde{T}_1 \rangle \xrightarrow{\sim} \overline{B} \langle T_1 \rangle$ then induces isomorphism in the homology. If we continue in this way to lift successively cycles t_i to cycles $\tilde{t}_i$, and to lift variables T_i to variables $\tilde{T}_i$, then all the mappings $X \otimes_A B \langle \tilde{T}_1, \ldots, \tilde{T}_i \rangle \xrightarrow{\sim} \overline{B} \langle T_1, \ldots, T_i \rangle$ will induce isomorphisms in the homology. The resulting complex $X \otimes_A B \langle \tilde{T}_1, \ldots, \tilde{T}_n, \ldots \rangle$ will be a B-resolution of k, which is not necessarily minimal.

<u>Definition</u>: A cycle $\tilde{t}_n$ of degree $2j-1$ in $X \otimes_A B \langle \tilde{T}_1, \ldots, \tilde{T}_{n-1} \rangle$ is called a <u>special cycle</u>, and the variable $\tilde{T}_n$ of degree $2j$, $d\tilde{T}_n = \tilde{t}_n$, a <u>special variable</u>,

if there exists a derivation $\tilde{j}$ on $X \otimes_A B\langle \tilde{T}_1,\ldots,\tilde{T}_{n-1}\rangle$ such that $\tilde{j}(\tilde{t}_n) = 1$ and $\tilde{j}(X \otimes_A B) \subseteq X \otimes_A B$.

The special variables occur exactly when the B-resolution above is not minimal. We need the following two important results due to M. André, concerning special variables.

THEOREM A: For any local noetherian ring $\overline{B}$, the number $\delta_{2j}(\overline{B})$ is less than or equal to the number of variables T_n of degree $2j$ in a $\overline{B}$-resolution of $\mathbb{1}$ that can be lifted to special variables $\tilde{T}_n$ for some A and B as above. The total number of such variables is less than or equal to $e_1(\overline{B}) - \operatorname{depth}(\overline{B})$, with equality precisely when $\overline{B}$ is a complete intersection.

Consequently, Conjecture 1 can be proved by showing that only variables of degree two can be lifted to special variables.

THEOREM B: Let $\tilde{T}_n$ be a special variable. We can then modify the cycles $\tilde{t}_i$ $i > n$ with boundaries, in such a way that we can adjoin all the variables $\tilde{T}_i$ $i \neq n$ before having adjoined $\tilde{T}_n$. Having done so, we have $X \otimes_A B\langle \tilde{T}_1,\ldots,\tilde{T}_{n-1},\tilde{T}_{n+1},\ldots\rangle =$ $= \Omega \oplus \Omega\tilde{t}_n$ where Ω is an acyclic differential subalgebra containing all $\tilde{t}_i$ and $\tilde{T}_i$, excluding of course $\tilde{t}_n$ and $\tilde{T}_n$.

2. The Yoneda Ext-algebra of the fibre $\overline{B}$

Using the Eilenberg-Moore spectral sequence for Hopf algebras (cf. Avramov [5])

$$E_2^{p,q} = \operatorname{Ext}^{p,q}_{H(Y)}(\mathbb{1},\mathbb{1}) \Rightarrow \operatorname{Ext}^{p+q}_Y(\mathbb{1},\mathbb{1})$$

with $Y = X \otimes_A B$ and consequently $H(Y) = \overline{B}$ (in degree $q = 0$ only), we see that $\operatorname{Ext}^*_{\overline{B}}(\mathbb{1},\mathbb{1}) \cong \operatorname{Ext}^*_{X \otimes_A B}(\mathbb{1},\mathbb{1})$. We can thus choose any lifting $X \otimes_A B$ of $\overline{B}$, as above, to study the Yoneda Ext-algebra of $\overline{B}$.

We are now able to state the main result of this paper.

THEOREM 1: Let $\overline{B}$ be a local noetherian ring with residue field $\mathbb{1}$. A variable T_n, in a $\overline{B}$-resolution of $\mathbb{1}$, that can be lifted to a special variable $\tilde{T}_n$, corresponds to a central element of the graded Lie algebra underlying the Yoneda Ext-algebra $Ext^*_{\overline{B}}(\mathbf{1},\mathbf{1})$.

Conjecture 1 will thus follow from the conjecture below.

CONJECTURE 2: For any local noetherian ring $\overline{B}$ with residue field $\mathbf{1}$, the centre of the graded Lie algebra underlying $Ext^*_{\overline{B}}(\mathbf{1},\mathbf{1})$ is finite-dimensional and concentrated in degrees one and two.

This conjecture - if true - would correspond to results of Y. Felix, S. Halperin and J.-C. Thomas [7] on the centre of the homotopy Lie algebra $\pi_*(\Omega S) \otimes \mathbf{Q}$ of a finite, simply connected CW complex S . The conjecture would also generalise a result of L. Avramov [4], i.e. if $Ext^*_{\overline{B}}(\mathbf{1},\mathbf{1})$ is abelian, then $\overline{B}$ must be a complete intersection.

Proof of Theorem 1: The set of variables $\{T_i\}$ is in a one-to-one correspondence with a vector space basis of the graded Lie algebra underlying $Ext^*_{\overline{B}}(\mathbf{1},\mathbf{1})$. The Lie algebra structure is given by the action of the derivations associated with the variables T_i . Suppose T_n can be lifted to a special variable $\widetilde{T}_n$, starting with $X \otimes_A B \xrightarrow{\sim} \overline{B}$ as in Section 1 above. Since we have seen that $Ext^*_{\overline{B}}(\mathbf{1},\mathbf{1}) \cong Ext^*_{X\otimes_A B}(\mathbf{1},\mathbf{1})$, it is enough to study the derivation $j_{\widetilde{T}_n}$ associated with $\widetilde{T}_n$ (cf. L. Avramov [5]).

This derivation is defined by $j_{\widetilde{T}_n}(X \otimes_A B) = j_{\widetilde{T}_n}(\widetilde{T}_i) = 0 \quad i < n,\ j_{\widetilde{T}_n}(\widetilde{T}_n) = 1$ and is then extended to all higher $\widetilde{T}_i$-s. If $j_{\widetilde{T}_n}(\widetilde{t}_i) = s_i$, $ds_i = 0$, then we define $j_{\widetilde{T}_n}(\widetilde{T}_i) = S_i$ with $dS_i = s_i$. This is always possible to do since $X \otimes_A B\langle\widetilde{T}_1,..,\widetilde{T}_n,..\rangle$ augmented to $\mathbf{1}$ is acyclic, and since T_n can not have unit coefficient in t_i , neither can $\widetilde{T}_n$ have unit coefficient in $\widetilde{t}_i$.

When $\widetilde{T}_n$ is a special variable, Theorem B gives us that $\widetilde{T}_n$ does not occur in any of the cycles $\widetilde{t}_i$. Thus, we have $j_{\widetilde{T}_n}(\widetilde{t}_i) = 0$ for low degree $\widetilde{t}_i \quad i \neq n$, and using induction we can define $j_{\widetilde{T}_n}(\widetilde{T}_i) = 0 \quad i \neq n$ and as before $j_{\widetilde{T}_n}(\widetilde{T}_n) = 1$.

Let $\widetilde{T}_m$ be some other variable. The associated derivation, having $j_{\widetilde{T}_n}(X \otimes_A B) = j_{\widetilde{T}_m}(\widetilde{T}_i) = 0$ for $i < m$, $j_{\widetilde{T}_m}(\widetilde{T}_m) = 1$ is to be extended to all $\widetilde{T}_i$-s. Using Theorem B above, we first adjoin all the variables except $\widetilde{T}_n$ $(n \neq m)$, to get

$X \otimes_A B\langle \tilde{T}_1,\dots,\tilde{T}_{n-1},\tilde{T}_{n+1},\dots\rangle \cong \Omega \oplus \Omega\tilde{t}_n$. Since Ω is an acyclic algebra containing all $\tilde{t}_i$ and $\tilde{T}_i$ $i \neq n$, inductively we have $j_{\tilde{T}_m}(\tilde{t}_i) = s_i \in \Omega$ and we can define $j_{\tilde{T}_m}(\tilde{T}_i) = S_i \in \Omega$ for all $i \neq n$. But since $j_{\tilde{T}_m}$ has negative degree, $j_{\tilde{T}_m}(\tilde{t}_n) = s_n \in \Omega$ and we can also choose $j_{\tilde{T}_m}(\tilde{T}_n) = S_n \in \Omega$. Thus, we see that we can define $j_{\tilde{T}_m}$ in such a way that $\tilde{T}_n$ does not occur in any $j_{\tilde{T}_m}(\tilde{T}_i)$.

Let $\Sigma a_i\tilde{T}_n^{(i)}$, where $\tilde{T}_n$ does not occur in any a_i , be an element of $X \otimes_A B\langle \tilde{T}_1,\dots,\tilde{T}_n,\dots\rangle$. We have

$$j_{\tilde{T}_m} \circ j_{\tilde{T}_n}(\Sigma a_i\tilde{T}_n^{(i)}) = j_{\tilde{T}_m}(\Sigma a_i\tilde{T}_n^{(i-1)}) =$$
$$= \Sigma (j_{\tilde{T}_m}(a_i)\tilde{T}_n^{(i-1)} + a_i j_{\tilde{T}_m}(\tilde{T}_n)\tilde{T}_n^{(i-2)}) .$$

On the other hand we have

$$j_{\tilde{T}_n} \circ j_{\tilde{T}_m}(\Sigma a_i\tilde{T}_n^{(i)}) = j_{\tilde{T}_n}(\Sigma(j_{\tilde{T}_m}(a_i)\tilde{T}_n^{(i)} + a_i j_{\tilde{T}_m}(\tilde{T}_n)\tilde{T}_n^{(i-1)})) =$$
$$= \Sigma (j_{\tilde{T}_m}(a_i)\tilde{T}_n^{(i-1)} + a_i j_{\tilde{T}_m}(\tilde{T}_n)\tilde{T}_n^{(i-2)}) .$$

This shows that $\tilde{T}_n$ corresponds to a central element of the graded Lie algebra underlying $\mathrm{Ext}^*_{X\otimes_A B}(\mathbf{1},\mathbf{1})$, proving the theorem.

3. A class of local rings where the conjectures are valid

Let $R \to S$ be a Golod epimorphism of local rings. Let g_R and g_S be the Lie algebras underlying the Ext-algebras of R and S respectively. Then we have an extension of graded Lie algebras (cf. Löfwall [10], Avramov [5])

$$0 \longrightarrow L(W) \longrightarrow g_S \longrightarrow g_R \longrightarrow 0 ,$$

where $L(W)$ is the free Lie algebra on $W = s^{-1}(\mathrm{Ext}_R^{>0}(S,\mathbf{1}))$, s^{-1} changes the degree by $+1$ and $\mathbf{1}$ is the residue field of R . (This can serve as a definition of a Golod epimorphism; for other definitions we refer to L. Avramov [5] and G. Levin [9].) If g_R has no central element of degree greater than two, then such an element in g_S must be contained in $L(W)$. But $L(W)$ is free, so that W must be one-dimensional. Then W must also lie in degree two, since otherwise $\mathrm{Ext}^1_R(S,\mathbb{1}) = 0$, S is a free R-module and $W = 0$. The case where W is one-dimensional occurs exactly when $S = R/_{(r)}$, r being a non-zero-divisor of R belonging to the square of the maximal ideal of R. Consequently, g_S does not have a central element of degree greater than two, and we have proved

THEOREM 2: Let $R \longrightarrow S$ be a Golod epimorphism of local noetherian rings. If Conjecture 1 and 2 hold for R, then they also hold for S.

This theorem immediately gives the following corollary.

COROLLARY 1: Conjecture 1 and 2 both hold for a local noetherian ring $\overline{B}$, which can be attached to a regular ring by a finite sequence of Golod epimorphisms, e.g. if $\overline{B}$ is a Golod ring, or if $\overline{B}$ is a quotient of a regular ring by an ideal generated by a set of monomials in the elements of some regular seuqence.

We can convince ourselves that such a "monomial" ring is Golod-attached to a regular ring (cf. J. Backelin [6]), by using a theorem of G. Levin [9]. The theorem asserts that $R \longrightarrow R/_{rI}$ is a Golod map if r is neither unit nor zero-divisor of R and if rI is contained in the square of the maximal ideal.

If $\overline{B}$ is the quotient of the regular ring R_0 by an ideal generated by monomials in the R_0-sequence $x_1,\ldots,x_n$, we start by taking away the group of monomials divisible by x_1. From the remaining monomials, we then take away those divisible by x_2, and so on. Starting with R_0 and dividing out by the ideals generated by these groups of monomials, one group at a time, we of course end up with $\overline{B}$. But by reversing the order of the groups, all these maps will be of the form $R \longrightarrow R/_{x_i I}$, x_i not a zero-divisor of R, and will thus all be Golod maps.

Remark: Recently C. Löfwall [11] has proved that Conjecture 2, and thus also Conjecture 1, is valid for local rings $\overline{B}$ having $\underline{m}^3 = 0$ for the maximal ideal $\underline{m}$, with the possible exception for such rings with gl.dim. $\mathrm{Ext}^*_{\overline{B}}(\mathbf{1},\mathbf{1}) = 2$.

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