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M.J. Hopkins

The Lusternik-Schnirelmann category of a space X is usually defined to be one less than the minimum number of open subsets U_i it takes to cover X such that the inclusions $U_i \to X$ are nullhomotopic. As a homotopy invariant it is usually associated with cohomological nilpotence. It was Ganea [3] who first gave a formulation of category to which one could apply Eckmann-Hilton duality. He was then able to define a notion of "cocategory" which bore a similar relation to homotopical nilpotence.

To define category Ganea first constructed a sequence of spaces B_iX inductively, by starting with $B_oX = PX \to X$ and obtaining $B_{i+i}X$ as the mapping cone $B_iX \cup CF_iX$, where $F_iX \to B_iX$ is the inclusion of the homotopy fibre of $B_iX \to X$. With care he was able to piece these into a sequence filtering the homotopy type of X.

$$\begin{array}{ccccc}
\Omega X & & \Omega X * \Omega X & & \Omega X * \Omega X * \Omega X \\
\downarrow & & \downarrow & & \downarrow \\
B_0X = PX & \to & PX/\Omega X = B_iX \cong \Sigma\Omega X & \to & B_1X/\Omega X * \Omega X \to \ldots \\
 & & \downarrow & & \\
 & & X & &
\end{array}$$

($B_nX \to X$ has the homotopy type of the $(n+1)$-fold fibre join of the universal fibration over X, and the above sequence is the natural sequence of iterated fibre joins). Ganea then defined cat$X \leq n$ if $B_n X \to X$ admits a section. He was able to show that this definition agrees with the original definition in suitable cases (when X is path connected, paracompact, and say, locally contractible).

Ganea's formulation of cocategory [3,4] was then as one would expect. Beginning with $X \to CX = B^o X$, and having defined $X \to B^i X$, one obtains $B^{i+i}X$ as the homotopy fibre of $B^i X \to B^i X/X$.

$$\begin{tikzcd} & B^{i+1}X \arrow[d] & \\ X \arrow[ur, dashed] \arrow[r] & B^i X \arrow[r] & B^i X/X \end{tikzcd}$$

With care, the $B^i X$ fit naturally together to form a tower. Ganea defined cocat$X \leq n$ if $B^n X$ retracts back to X.

One unsatisfactory feature of this formulation is its lack of resemblance to the original definition of category. Consequently, whatever it is that cocategory measures about a space is somewhat obscure.

In this note I will present new formulations of category and cocategory closer in spirit to the original definition of category. Hopefully this will make the meaning of cocategory less obscure. One byproduct of these formulations is a new characterization of iterated loop spaces and a dual characterization of iterated suspensions. Along with these characterizations come spectral sequences whose edge homomorphisms are the iterated homotopy and homology suspensions. I have confined myself to statements of most results. Details will appear elsewhere.

1 Lusternik-Schnirelmann Category

The original definition of Lusternik-Schnirelmann category was based on decomposing a space as a union of subspaces. As Eckmann-Hilton duality can only be applied to functors and the transformations between them we must phrase this decomposition in category theoretic terms.

Let $X = U_1 \cup U_2 \cup U_3$. This can be represented by the diagram

$$\begin{tikzcd}
& & U_0 & & \\
& U_{01} \arrow[ur] \arrow[dl] & U_{012} \arrow[l] \arrow[r] \arrow[d] & U_{02} \arrow[ul] \arrow[dr] & \\
U_1 & & U_{12} \arrow[ll] \arrow[rr] & & U_2
\end{tikzcd}$$

where $U_{01} = U_0 \cap U_1$ etc., and the maps are the inclusions. Such a diagram is a (contravariant) functor from the category of non-empty subsets of $\{0,1,2\}$ to the category of topological spaces, and the condition that X be the union of the U_i translates into X being the direct limit of this functor.

For a set S let C_S denote the category of non-empty subsets of S and inclusions. With the above as motivation we make the following

<u>Definition</u>: A <u>homotopy covering</u> of a space X is a contravariant functor $F: C_S \to$ Spaces for some set S, together with maps $p: \underrightarrow{\mathrm{holim}}\, F \to X$ and $s: X \to \underrightarrow{\mathrm{holim}}\, F$ such that $p \circ s$ is homotopic to the identity map of X. ($\underrightarrow{\mathrm{holim}}\, F$ denotes the homotopy direct limit of F [2, 13]).

Given a homotopy covering F of X we give precedence to the spaces $F\{\alpha\}$ $\alpha \in S$ over their intersections by saying that the $F\{\alpha\}$ homotopy cover X via the maps $F\{\alpha\} \to \underrightarrow{\mathrm{holim}}\, F \xrightarrow{p} X$. We will also say that X can be homotopy covered by $\{p_\alpha\colon U_\alpha \to X \mid \alpha \in S\}$ if there exists a homotopy covering $F\colon C_S \to$ Spaces of X together with homotopy equivalences $F\{\alpha\} \to U_\alpha$ such that the diagrams

$$\begin{array}{ccc} F\{\alpha\} & \to & \underrightarrow{\mathrm{holim}}\, F \\ \downarrow & & \downarrow \\ U_\alpha & \xrightarrow{p_\alpha} & X \end{array}$$

commute up to homotopy.

Example: A connected space X is a co h-space if and only if it can be homotopy covered by two points. Indeed, consider the functor $F\colon C_{\{0,1\}} \to$ Spaces given by the diagram $* \leftarrow \Omega X \to *$. The homotopy direct limit of this diagram is $\Sigma\Omega X$ and it is well-known that a connected space is a co h-space if and only if it is dominated by the suspension of its loop space.

We can now define category.

Definition: Let X be a connected space. Then $\mathrm{cat}\, X \le n$ if X can be homotopy covered by $n+1$ points.

We will see at the end of this section that this definition agrees with that of Ganea [3,4].

To actually make computations it is impossible to go through the whole category of spaces looking for intersections. One's natural instinct is to ask about universal intersections. They do indeed exist and are provided by the homotopy inverse limit [2, 13].

More precisely, given $\{p_\alpha\colon U_\alpha \to X \mid \alpha \in S\}$ we construct a functor

$p : C_S \to$ Spaces by setting $FA = \underleftarrow{\text{holim}}\ \{p_\alpha : U_\alpha \to X \mid \alpha \in A\}$ for $A \subset S$.

Proposition 1: A space X can be homotopy covered by $\{p_\alpha : U_\alpha \to X \mid \alpha \in S\}$ if and only if X is dominated by $\underrightarrow{\text{holim}}\ F$ in such a way that the composition

$$U_\alpha = F\{\alpha\} \to \underrightarrow{\text{holim}}\ F \to X$$

is homotopic to p_i.

Remarks: 1) There is an obvious generalization of proposition 1 in which one is allowed to specify various of the intersections. The statement involves beginning with a functor defined on a full subcategory of C_S together with a natural transformation to the functor which is constant at X, and then using homotopy inverse limits to "fill in" the missing intersections thereby extending functor and transformation to all of C_S.

2) In proposition 1 (and its generalization) it is possible to obtain a canonical map $p : \underrightarrow{\text{holim}}\ F \to X$ with the property that X can be homotopy covered by $\{p_i : U_i \to X\}$ iff p admits a section up to homotopy. To construct p one must either convert all of the $p_i : U_i \to X$ into fibrations or else be willing to replace X by the function space $(X \times \Delta_S)^{\Delta_S}$ where Δ_S is the simplex with $|S|$ vertices.

3) Proposition 1 and its generalization influence the theory of homotopy coverings in two ways. Not only do they convert questions about homotopy coverings into questions about certain maps admitting sections, but they transform questions like "how many points does it take to homotopy cover X?" into questions about certain filtrations of X. We shall see in §3 that the filtrations arising in this way are themselves of some

interest.

Example: Consider the following situation: $p : E \to B$ is a fibration and we want to know how many open subsets U_i it takes to cover B such that each inclusion $U_i \to B$ lifts through p. The universal example of a lift through p is $p : E \to B$ itself so in the present terminology we would ask how many copies of $p : E \to B$ it takes to homotopy cover B. Proposition 1 suggests an approach. For example, to see if two copies suffice we must consider the functor $F : C_{\{0,1\}} \to$ Spaces given by the diagram

$$E \leftarrow E \times_B E \to E \ .$$

In this case $\underrightarrow{\text{holim}}\, F$ is the fibre join of E with itself over B. In general one finds that B can be homotopy covered by n copies of $p : E \to B$ iff the n fold fibre join of p admits a section. The filtration of B we obtain is the filtration by iterated fibre joins of p. (This should be compared with the work of A.S. Švarc [7 ,12] who first pointed out the relationship between sections of the iterated fibre joins of a fibration and coverings of the base by open subsets over which the fibration admits a section.)

The above example with $E \to B$ the standard contractible fibration shows that $\operatorname{cat} B \leq n$ iff the $(n+1)$-fold fibre join of the universal fibration over B admits a section. It follows that our definition of category agrees with Ganea's.

2 Formulations of Cocategory

We now present the dual of §1.

Definition: A homotopy co-covering of a space X is a covariant functor $F : C_S \to$ Spaces for some set S, together with maps $i : X \to \underleftarrow{\mathrm{holim}}\, F$ and $r : \underleftarrow{\mathrm{holim}}\, F \to X$ such that $r \circ i$ is homotopic to the identity map of X. X is said to be <u>homotopy co-covered</u> by $F\{\alpha\}$ $\alpha \in S$ via

$$X \xrightarrow{i} \underleftarrow{\mathrm{holim}}\, F \to F\{\alpha\}.$$

The geometric situation out of which such a functor arises occurs when one space is written as an intersection of others. For instance $X = U^0 \cap U^1 \cap U^2$ can be interpreted as saying that X is the inverse limit of

$$\begin{tikzcd}
 & & U^0 \arrow[dl] \arrow[dr] \arrow[d, dotted] & & \\
 & U^{01} \arrow[r] & U^{012} & U^{02} \arrow[l] & \\
U^1 \arrow[ur] \arrow[rr] & & U^{12} \arrow[u] & & U^2 \arrow[ll] \arrow[ul]
\end{tikzcd}$$

where $U^{01} = U^0 \cup U^1$ etc. It follows that the notion of cocategory should be based on writing a space as an intersection of other spaces in the same way that category is based on writing a space as a union of subspaces.

There is one important property of unions for which there is not a clear cut dual. Namely, suppose $X = U_1 \cup U_2$ and $U_1 = V_1 \cup \dots \cup V_n$, $U_2 = W_1 \cup \dots \cup W_m$. Then it is hard to deny that $X = V_1 \cup \dots \cup V_n \cup W_1 \cup \dots \cup W_m$. However in the dual situation $X = U_1 \cap U_2$, $U_1 = V_1 \cap \dots \cap V_n$, $U_2 = W_1 \cap \dots \cap W_m$, one needs to do something artificial to even form $V_1 \cap \dots \cap V_n \cap W_1 \cap \dots \cap W_m$. For this reason it is possible to formulate many notions of cocategory. We single out the two most important.

Definition: A space X has symmetric cocategory $\leq n$ (abbr. sym cocat $X \leq n$) if it can be homotopy co-covered by $n + 1$ points.

Definition: A space X has inductive cocategory $\leq n$ (abbr. ind cocat $X \leq n$) if it can be homotopy co-covered by a point and a space X' with ind cocat $X' \leq n - 1$.

When one wishes to know whether a collection of maps $\{i_\alpha : X \to U|\alpha \in S\}$ suffices to homotopy co-cover X, universal unions are provided by the homotopy direct limit. Thus given $\{i_\alpha : X \to U|\alpha \in S\}$ define a covariant functor $F : C_S \to$ Spaces by $FA = \underrightarrow{\mathrm{holim}}\ \{i_\alpha : X \to U_\alpha | \alpha \in A \quad S\}$. We then have the dual of proposition 1.

Proposition 1': X can be homotopy co-covered by $\{i_\alpha : X \to U_\alpha|\alpha \in S\}$ iff X is dominated by $\underleftarrow{\mathrm{holim}}\ F$ in such a way that the composition

$$X \to \underleftarrow{\mathrm{holim}}\ F \to F\{\alpha\} = U_\alpha$$

is homotopic to i_α.

Remark: The remarks after proposition 1 dualize.

Proposition 1' translates the question of computing sym cocat and ind cocat into a question about certain towers of fibrations. It is easy to see that the tower used to compute inductive cocategory is the same as Ganea's tower so that his notion of cocategory is our notion of inductive cocategory.

Symmetric and inductive cocategory certainly appear to be distinct notions though at present I can prove neither that they coincide nor that they differ. The following sums up most of the present state of affairs:

1) ind cocat $\leq$ sym cocat.

2) ind cocat $X = 0 \Leftrightarrow$ sym cocat $X = 0 \Leftrightarrow X$ is contractible.

3) ind cocat $X = 1 \Leftrightarrow$ sym cocat $X = 1 \Leftrightarrow \Omega\Sigma X$ dominates $X \Leftrightarrow X$ is an h-space.

4) ind cocat $K(\pi,1)$ = sym cocat $K(\pi,1)$ = nilpotency class of π.

5) If there is a non-trivial n-fold Whitehead product in $\pi_* X$ then both ind cocat X and sym cocat X are $\geq n$.

6) If cat $X \leq n$ then both ind cocat and sym cocat of the space $(Y,*)^{(X,*)}$ of base point preserving maps are $\leq n$.

7) If $F \to E \to B$ is a fibration then ind cocat $F \leq$ ind cocat $E + 1$.

Property 7) actually characterizes inductive cocategory [3,4]. Could it be established for symmetric cocategory it would follow that the two notions agree.

Finally, I have remarked that proposition 1' gives us towers of fibrations for computing ind cocat X and sym cocat X. Taking homotopy groups of the tower results in an exact couple and hence a spectral sequence. In good cases (when X is connected and can be written as an inverse limit of nilpotent spaces) these spectral sequences converge to $\pi_* X$.

The spectral sequence arising from sym cocat is particularly interesting as its E^2 term can be computed by applying the cobar construction to ΣX (in the homotopic category), taking homotopy groups, and then taking cohomology of the resulting complex. An easy consequence of the Hilton-Milnor theorem [6,8] is that in a range of about three times the connectivity of X the spectral sequence collapses to an exact sequence

$$\pi_n(\Sigma X) \to \pi_n(\Sigma X \wedge X) \to \pi_{n-2}(X) \to \pi_{n-1}(\Sigma X)$$

which is the EHP sequence.

This spectral sequence is the dual of the Rothenberg-Steenrod spectral sequence [9] and was originally obtained by M.G. Barratt in a slightly different setting [1]. One advantage of the above formulation is that it is made clear that the higher differentials are the obstructions to maps $S^n \to X$ being co h-maps compatible with higher associativity. A second advantage is that it generalizes to n-fold suspensions.

3 The Iterated Suspension

We can summarize the preceding sections with a slightly different emphasis. Let C_n denote the category of non-empty subsets of $\{0,\ldots,n\}$ and let $P_0,\ldots,P_n : * \to X$ be $n + 1$ copies of the inclusion of some point in the connected space X. To decide whether X can be homotopy covered by $n + 1$ points one is led by proposition 1 to consider the homotopy direct limit of the (contravariant) functor $F_n : C_n \to$ Spaces given by $F_nA = \underleftarrow{\text{holim}}$ $\{p_i : * \to X | i \in A\}$. The inclusions $\{0,\ldots,n-1\} \to \{0,\ldots,n\}$ define natural transformations $\to F_{n-1} \to F_n \to F_{n+1} \to \ldots$ hence one obtains a sequence of cofibrations

$$\ldots \to \underrightarrow{\text{holim}}\, F_{n-1} \to \underrightarrow{\text{holim}}\, F_n \to \underrightarrow{\text{holim}}\, F_{n+1} \to \ldots$$

We have remarked that this is just the sequence of iterated fibre joins of the universal fibration over X, so that in the limit one recovers the homotopy type of X.

The spaces $\underrightarrow{\text{holim}}\, F_n$ are built out of the "universal intersections" F_nA and various maps between them. It is not hard to see that the F_nA all have the homotopy type of a product of copies of ΩX, and furthermore, that the only maps between these products which arise are the projections and the loop multiplication. It follows that the sequence of functors $\ldots \to F_{n-1} \to F_n \to F_{n+1} \to \ldots$ stores the algebraic information needed to recover a connected space from its loop space.

Dually, to decide whether a space X can be homotopy co-covered by n + 1 points one is led by proposition 1' to consider the (covariant) functor $F^n : C_n \to$ Spaces given by

$$F^nA = \underrightarrow{\mathrm{holim}}\ \{i_\alpha : X \to * \mid \alpha \in A\} \qquad A \subset \{0,\ldots,n\}$$

where $i_0, \ldots, i_n$ are n + 1 copies of the unique map $X \to *$. The inclusions $\{0,\ldots,n-1\} \to \{0,\ldots,n\}$ define a sequence of transformations $F^{n-1} \to F^n \to F^{n+1}$ and hence a tower of fibrations

$$\ldots \leftarrow \underleftarrow{\mathrm{holim}}\ F^{n-1} \leftarrow \underleftarrow{\mathrm{holim}}\ F^n \leftarrow \underleftarrow{\mathrm{holim}}\ F^{n+1} \leftarrow \ldots$$

In good cases (when X is connected and can be written as an inverse limit of nilpotent spaces) the inverse limit of this tower recovers the homotopy type of X.

The spaces $\underleftarrow{\mathrm{holim}}\ F^n$ are built out of wedges of ΣX and use only inclusions of factors and the co-multiplication. It follows that the sequence of functors $\ldots \to F^{n-1} \to F^n \to F^{n+1} \to \ldots$ stores the algebraic information needed to recover a space from its suspension.

Without going into too many details, the question "can X be homotopy covered by n + 1 points in such a way that the homotopy intersection of any j points, $j \leq k$, is again a point?" gives rise to a sequence of functors storing the algebraic information needed to recover the homotopy type of a space from its k-fold loop space.

The dual remark goes for k-fold suspensions.

To extract recognition principles from the above it is convenient to use the language of simplicial and cosimplicial spaces. Let Δ denote the category of finite ordered sets and let $[n] \in \mathrm{Ob}\,\Delta$ be the set $\{0 < \ldots < n\}$. The category of simplicial spaces is the category of contravariant functors $\Delta \to$ Spaces and natural transformations. There is a free functor F : Spaces $\to$ Simplicial Spaces given by $F\,X[n] = X^{\Delta_n}$ - the space of maps from the standard n-simplex into X. This functor has a right adjoint $X \to |X|$ which is the geometric realization [10].

Analogously, the category of cosimplicial spaces is the category of covariant functors $\Delta \to$ Spaces. There is a (co-)free functor $\tilde{F}$: Spaces $\to$ Cosimplicial Spaces given by $\tilde{F}\,X[n] = X \times \Delta_n$. This functor has a right adjoint $X \to \mathrm{Tot}\,X$ which is a kind of geometric realization [2].

<u>Theorem 1</u>: Let $[n] \to X_n$ be a simplicial space such that

i) $X_0, \ldots, X_{k-1}$ are contractible

ii) $P_n = \prod_{j=1}^{\binom{n}{k}} \theta_j^* : X_n \to \prod_{J=1}^{\binom{n}{k}} X_k$ is a homotopy equivalence,

where θ_j runs through the injections $[k] \to [n]$ satisfying $\theta_j(0) = 0$.

Then the map $X_k \to \Omega^k|X|$ adjoint to the inclusion of the "k-skeleton" is a homotopy equivalence.

<u>Theorem 1'</u>: Let $[n] \to X^n$ be a (pointed) cosimplicial space such that

i) $X^0, \ldots, X^{k-1}$ are contractible; X^k is k-connected

ii) $i_n = \bigvee_{j=1}^{\binom{n}{k}} \theta_{j*} : \bigvee_{j=1}^{\binom{n}{k}} X_k \to X_n$ is a homotopy equivalence.

Then the map $\Sigma^k \operatorname{Tot} X \to X^k$ adjoint to the projection onto the "k-coskeleton" ($\operatorname{Tot} X \to \operatorname{Tot}_k X$ in [2] p.271) is a homotopy equivalence.

Remarks: 1) Theorems 1 and 1' should be compared to [11] Proposition 1.5.

2) The functors $[n] \to (X,*)^{(\Delta_n, \Delta_n^{(k-1)})}$ and $[n] \to X \wedge \Delta_n/\Delta_n^{(k-1)}$ satisfy the conditions of Theorems 1 and 1' with $X_k = \Omega^k X$ and $X^k = \Sigma^k X$ respectively. It follows that we can deloop any n-fold loop space and desuspend any n-fold suspension using Theorems 1 and 1' respectively.

3) The geometric realizations in theorems 1 and 1' need to be modified slightly in order to land in the right homotopy type. The appropriate discussion can be found in the appendix to [11].

4) The proof of Theorem 1 is by induction on k using the simplicial path space PX, where $(PX)_n = X_{n+1}$, and the observation that the fibres of $d_{n+1} : (PX)_n \to X_n$ form a simplicial space satisfying the conditions of the theorem with k replaced by k - 1. Theorem 1' is an easy consequence of the work of Z. Wojtkowiak on the homology spectral sequence of a cosimplicial space [14].

5) The homology spectral sequence for simplicial spaces [10] applied to $[n] \to (X,*)^{(\Delta_n, \Delta_n^{(k-1)})}$ gives a spectral sequence starting from $H_* (\Pi \Omega^k X)$ and converging (by Theorem 1 - provided X is connected) to $H_* X$. The homotopy spectral sequence for cosimplicial spaces [2] applied to $[n] \to X \wedge \Delta_n/\Delta_n^{(k-1)}$ gives a spectral sequence starting from $\pi_* (\Sigma^k X \vee \dots \vee \Sigma^k X)$ and converging

(by Theorem 1' - provided X is connected and can be written as an inverse limit of nilpotent spaces) to $\pi_* X$. The edge homomorphisms of these spectral sequences are the iterated homotopy and homology suspensions respectively. They generalize the Rothenberg-Steenrod spectral sequence and the spectral sequence mentioned in §2.

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