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ALGEBRAIC CATEGORIES AND THE HOMOTOPY THEORY OF SOME C.W. COMPLEXES

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Here we present an equivalence between the homotopy category of certain C.W. complexes of finite type and algebraic homotopy categories of commutative differentail graded algebras over the integers and differential graded Lie algebras over the integers.

This result adds to the number of homotopy categories of C.W. complexes of finite type for which there is a known algebraic equivalent [1], [10], [11].

Differential forms provide the bridge between algebra and geometry.

For graded rings, H^*, satisfying the C-r condition (see definition below), the result leads to a parameterization of the set of all homotopy types of 1-connected finite C.W. complexes with integer cohomology ring isomorphic to H^*. Here ; our techniques are motivated by analogous results in rational homotopy theory [5], [6], [7].

For the remainder of this paper, r denotes an integer , $r \geq 3$.

<u>DEFINITION</u> : A graded group H^* (with grading indicated by upper index) is said to satisfy the C-r condition if multiplication by p defines an automorphism on H^{r+k} for all pairs of positive integers (p,k) with $2p - 3 \leq k+1$.

<u>DEFINITION</u> : A graded group H_* (with grading indicated by lower index) is said to satisfy the H-r condition if H_r has no elements of order 2 and multiplication by p defines an automorphism on H_{r+k} for all pairs of positive integers (p,k) with $2p - 3 \leq k+2$.

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REMARK : For X, an r-1 connected C.W. complex of finite type, $H^*(X:Z)$ satisfies the C-r condition iff $H_*(X:Z)$ satisfies the H-r condition.

DEFINITION : A space X is called a C-r space if X is (r-1) connected and is homotopy equivalent to a C.W. complex of finite type whose cohomology groups, $H^*(X:Z)$, satisfy the C-r condition.

PROPOSITION : For each integer $r \geq 3$ the homotopy category of C-r spaces is equivalent to

a) an algebraic homotopy category of (r-1) connected commutative differential graded algebras of finite type over the integers whose cohomology groups satisfy the C-r condition ;
and to.

b) an algebraic homotopy category of (r-1) reduced differential graded Lie algebras, L, of finite type over the integers with the property that the homology groups of $L/[L,L]$ satisfy the H-(r-1) condition.

The proof has two main parts. First, the filtered algebra of differential forms introduced in [2] is used to obtain equivalences between the tame homotopy theory of Dwyer [4] and algebraic homotopy theories of commutative differential graded algebras and differential graded Lie algebras [3].
Secondly, it is shown that tame localization restricted to the category of C-r spaces is an equivalence onto its image.

EXAMPLE : For N a product of primes greater than 3 the set of homotopy types of simply connected C.W. complexes X with

$$H^*(X:Z) = \left\{ \begin{array}{ll} Z & * = 0,3 \\ Z/NZ & * = 4,8 \\ 0 & \text{otherwise} \end{array} \right\}$$

and trivial cup product is in 1-1 correspondence with the orbits of the group of matrices $\begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}$ with a_1, a_3 units in Z/NZ and a_2 an arbitrary element in Z/NZ acting by multiplication on the left on the set of two by one matrices with entries in Z/NZ.

Proof : For k a nonnegative integer, S_k, denotes the smallest subring of the rationals containing $\frac{1}{p}$ for each prime p with $2p - 3 \leq k$.

For $r \geq 3$ an (r-1) connected space X is called <u>tame</u> if π_{r+k} is an S_k

module for eadh $k \geq 0$. If each $\pi_{r+k}(X)$ is a finitely generated S_k module, we say that X is a tame space of finite type. Tame spaces were introduced by Dwyer in [4] who proved that the homotopy category of tame $(r-1)$ connected spaces $(r \geq 3)$ is equivalent to an algebraic homotopy category of tame $(r-1)$ reduced differential graded Lie algebras over Z.

We list some properties of tame spaces :

(i) Given an $(r-1)$ connected space X, there is a tame space X_T and a map $X \to X_T$ inducing isomorphisms $\pi_{r+k}(X) \otimes S_k \simeq \pi_{r+k}(X_T)$ for $k \geq 0$.

(ii) Given a map $f : X \to Y$ of $(r-1)$ connected spaces there is a map $f_T : X_T \to Y_T$, unique up to homotopy, such that the diagram below homotopy commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_T & \xrightarrow{f_T} & Y_T \end{array}$$

We call X_T the <u>tame localization</u> of X.

iii) The homotopy category of $(r-1)$ connected tame spaces of finite type is equivalent to an algebraic homotopy category of $(r-1)$ connected commutative differential graded algebras of finite type over the integers [3].

The results of [3] can be used to prove

iv) The homotopy category of $(r-1)$ connected tame spaces of finite type is equivalent to an algebraic homotopy category of $(r-1)$ reduced differential graded Lie algebras of finite type over the integers.

The equivalences in iii) and iv) restricted to tame spaces of the form X_T with X a C-r space give :

v) The homotopy category of tame spaces of the form X_T for X a C-r space is equivalent to

a) an algebraic homotopy category of $(r-1)$ connected commutative differential graded algebras of finite type over the integers whose cohomology groups satisfy the c-r condition ;

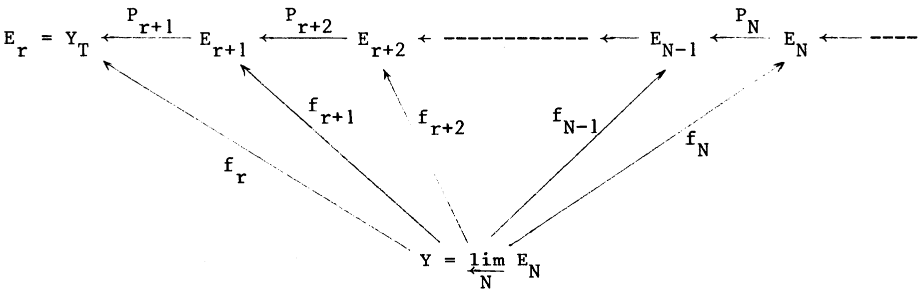
b) an algebraic homotopy category of $(r-1)$ reduced differential graded Lie algebras, L, of finite type over the integers such that the homology groups of $L/[L,L]$ satisfy the H-$(r-1)$ condition.

The proof of the Proposition is completed by proving :

LEMMA : For $r \geq 3$, tame localization induces an equivalence between the homotopy category of C-r spaces and the homotopy category of tame spaces of the form X_T for some C-r space X.

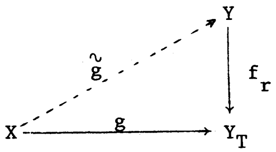
Proof : It suffices to show that for any two C-r spaces X and Y tame localization induces a bijection between $[X,Y]$ and $[X_T,Y_T]$. From ii) above it follows that $[X_T,Y_T]$ can be replaced by $[X,Y_T]$ and hence it is enough to show that the map $Y \to Y_T$ induces a bijection $[X,Y] \to [X,Y_T]$.

From the general theory of Moore-Postnikov systems and facorizations of maps [8] it follows that up to a homotopy equivalence Y is the inverse limit of a sequence of fibrations



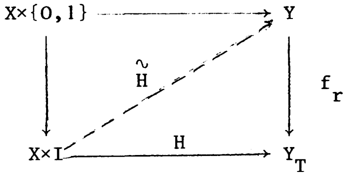
where $f_r : Y \longrightarrow Y_T$ is tame localization and $E_{N-1} \xleftarrow{P_N} E_N$ is the pullback of the path space fibration over a map from E_{N-1} to the Eilenberg-MacLane space, $K(\pi_N(Y_T,Y),N)$.

Hence the obstructions to the existence of a map $\tilde{g}$ corresponding to the dotted arrow in the solid arrow diagram



lie in the groups $H^N(X : \pi_N(Y_T,Y))$.

The obstructions to the existence of a map $\tilde{H}$ corresponding to the dotted arrow in the solid arrow diagram



lie in the groups $H^N((X\times I,\ X\times\{0,1\} : \pi_N(Y_T,Y)) \simeq H^{N-1}(X:\pi_N(Y_T,Y))$.

The proof of the lemma is completed by showing :

<u>CLAIM</u> : For any two C-r spaces X and Y, $H^{\ell}(X:\pi_N(Y_T Y)) = 0$ for $\ell \geq N-1 \geq r-1$.

Proof of claim : For $k \geq 0$, the long exact homotopy sequence of the pair (Y_T,Y) remains exact when tensored with S_k. Since $\pi_{r+k}(Y) \otimes S_k \to \pi_{r+k}(Y_T) \otimes S_k$ is an isomorphism it follows that $\pi_{r+k}(Y_T,Y) \otimes S_k = 0$ for $k \geq 0$. For $k = 0$, $S_k = Z$ and hence $\pi_r(Y_T,Y) = 0$.

To prove that $\pi_{r+1}(Y_T,Y) = 0$ note that since Y is a C-r space, Y has the same rational homotopy type as a (possibly empty) wedge of r-dimensional spheres. The minimal model [9] of a wedge of r-spheres does not have any generators in dimension r+1, (recall $r \geq 3$) and hence $\pi_{r+1}(Y)$ is a finite abelian group. It follows that $\pi_{r+1}(Y) \to \pi_{r+1}(Y_T) \simeq \pi_{r+1}(Y) \otimes S_1$ is an epimorphism and hence $\pi_{r+1}(Y_T,Y) = 0$.

At this point we have shown $H^{\ell}(X:\pi_N(Y_T,Y)) = 0$ for $N = r$, $N = r+1$.

Assume $\ell \geq N-1 > r$. Write $\ell = r+k$ and $N = r+j+1$ with $k \geq j \geq 1$. $\pi_N(Y_T,Y) \otimes S_j = 0$ implies that $\pi_N(Y_T,Y)$ is a torsion group each of whose elements has order some product of primes $p_1 \times p_2 \ldots \times p_t$ with $2p_i - 3 \leq j+1$ for $1 \leq i \leq t$. On the other hand, the condition that X be a C-r space implies that $H^{\ell}(X:Z) = H^{r+k}(X:Z)$ is an S_{k+1} module and a finitely generated abelian group. Hence, $H^{\ell}(X:Z)$ is a finite abelian group each of whose elements has order some product of primes $p_1 \times \ldots \times p_t$ with $2p_i - 3 > k+1$ for $1 \leq i \leq t$. Since $k \geq j$ it follows that $H^{\ell}(X:Z) \otimes \pi_N(Y_T,Y) = 0$ and $\mathrm{Tor}(H^{\ell}(X:Z),\ \pi_N(Y_T,Y)) = 0$ for $\ell \geq N-1 > r$. Hence $H^{\ell}(X:\pi_N(Y_T,Y)) = 0$ for $\ell \geq N-1 > r$ by the universal coefficient theorem for cohomology.

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