Astérisque

RONALD BROWN

Non-abelian cohomology and the homotopy classification of maps

Astérisque, tome 113-114 (1984), p. 167-172
<http://www.numdam.org/item?id=AST_1984__113-114__167_0>

© Société mathématique de France, 1984, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

NON-ABELIAN COHOMOLOGY AND THE HOMOTOPY CLASSIFICATION OF MAPS (*)

by Ronald Brown

To a filtered space

$$X : X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$$

we can associate the *homotopy crossed complex* πX, which consists for $n = 1$ of the fundamental groupoid $\pi_1 \underset{\sim}{X} = \pi_1(X_1, X_0)$, and for $n \geq 2$ of the family $\pi_n \underset{\sim}{X}$ of relative homotopy groups $\pi_n(X_n, X_{n-1}, v)$, $v \in X_0$, with the usual boundaries $\delta : \pi_n \underset{\sim}{X} \to \pi_{n-1} \underset{\sim}{X}$ and action of $\pi_1 \underset{\sim}{X}$ on $\pi_n \underset{\sim}{X}$. The formal properties satisfied by $\pi \underset{\sim}{X}$ define the notion of *crossed complex*, and we have a category XC of crossed complexes. Note that crossed complexes generalise chain complexes C (with $C_i = 0$ for $i < 1$), and they also generalise groups, groupoids, and crossed modules. A brief survey of their use in topology and algebra is given in [6]. See also [4, 5, 7].

The category XC of crossed complexes has a convenient notion of homotopy [10, 6, 7]. So for crossed complexes D, C we can define the set

$$[D, C]$$

of homotopy classes of morphisms $D \to C$.

The objetc of this talk is to advertise the definition (suggested in §.5 of [6])

$$H^0(X ; C) = [\pi \underset{\sim}{X}, C]$$

for CW-complex X with skeletal filtration $\underset{\sim}{X}$, and for a crossed complex C. That is, we take $[\pi X, C]$ as the *cohomology of* X *with coefficients in* C.

The definition makes sense, because πX is a homotopy invariant of X. The proof of this is not entirely trivial. One proof is given by J.H.C. Whitehead in [10] another is given in [7]. (Here wemean $X \simeq Y$ implies $\pi \underset{\sim}{X} \simeq \pi \underset{\sim}{Y}$).

The point of the definition is that we expect cohomology to have something to do with the sets [X,Y] of homotopy classes of maps of spaces. From [7] we take :

(*)
U.C.N.W. Pure Maths. Preprint 82.6
May, 1982

Prepared for presentation to the conference "Méthodes d'algèbre homotopique en topologie" at the C.I.R.M., Marseille-Luminy, 1-5 June, 1982.

Theorem 1. *There is a functor* $B : XC \to Top$ *assigning to a crossed complex* C *a CW-complex* BC *with the property that there is a natural bijection*

$$[X, BC] \cong H^0(X ; C)$$

for CW-complexes X.

Two special cases are of interest :

(i) If C is a group G in dimension n (where G is abelian if $n \geq 2$) and zero otherwise, then $BC = K(G,n)$, and Theorem 1 generalise a classical result of Eilenberg-MacLane. Note that the non-abelian case $n = 1$ is also included.

(ii) If C_1 is a group G, C_n is a G-module M, $C_i = 0$ for $i \neq 1$, n and all boundaries are zero then $H^0(X ; C)$ is a kind of twisted cohomology of X with coefficients in the G-module M, and so we have a twisted homotopy classification theorem.

There are three obvious questions about Theorem 1,

Q1. How do you prove it ?
Q2. What use is it in tackling the *general* problem of listing the elements of the set $[X, Y]$ of homotopy classes of maps $X \to Y$?
Q3. How do you compute $H^0(X ; C)$?

All these have interesting answers which we can only outline here. More details are given in [4,5,7].

The construction of the "classifying space" BC is done dubically. So we construct a cubical complex NC, the *nerve* of C, by setting

$$(NC)_n = XC(\pi \underset{\sim}{I}^n, C)$$

where I^n is the standard skeletalfiltration of the n-cube. We then set $BC = |NC|$, the geometric realisation of the cubical complex NC. (There is also a simplicial, and homotopy equivalent, version $B^{\Delta}C$; see the Introduction to 3 , which includes the relevant theses [1,8].)

The first part of the proof of Theorem 1 is to note that it is sufficient to restrict to the case when X is the realisation $|K|$ of a cubical complex K, and then to use an equivalence of homotopy categories to obtain

$$[|K|, BC] \cong [K, NC] .$$

For this we need to know NC is a Kan complex. In fact, NC has a lot of extra structure, since it turns out to be an example of an ω-*groupoid*, which is a complicated algebraic structure defined in [4]. Any ω-groupoid is a Kan complex, and hence NC is a Kan complex. We write (as in [4,5]) λC for NC with its structure of ω-groupoid.

Because λC is an ω-groupoid, we have a bijection

$$[K,\ NC] \cong [\rho K,\ \lambda C]$$

where the latter set of homotopy classes is taken in the category of ω-groupoids, and ρK denotes the *free ω-groupoid* on K. But it also turns out that there is an equivalence, of categories with homotopy, between ω-groupoids and crossed complexes, and that this equivalence takes ρK to $\pi|\underset{\sim}{K}|$, and λC to C. So

$$[\rho K,\ \lambda C] \cong [\pi|\underset{\sim}{K}|,\ C]$$

and we are done.

Unfortunately, the details of the above are strenuous. However, the pattern of argument parallels the case $BC = K(G,n)$ $(n \geq 2)$, which uses the simplicial abelian group structure on $K(G,n)$. We are using ω-groupoid structures instead, and this is what allows for non-abelian results.

Something needs to be said about the homotopy type of BC. For convenience we restrict to the reduced cas, i.e. when C_0 is a point. Then $\pi_1(BC,v)$ is the quotient group $G = C_2/\delta C_1$, while for $n \geq 2$ $\pi_n(BC,v)$ is the homology of C, i.e. $\mathrm{Ker}\delta/\mathrm{Im}\delta$, together with the action of G. Further, there is a fibration $BC \to K(G,1)$ whose fibre is 1-connected and is of the homotopy type of a product of Eilenberg-MacLane spaces. (This observation is due to J.L. Loday. I am not too clear about the classification of such non-principal fibrations.)

Now let Y be a reduced CW-complex with cellular filtration $\underset{\sim}{Y}$. We can form the homtopy crossed complex $\pi\underset{\sim}{Y}$ and the classifying space $B\pi\underset{\sim}{Y}$. In this case $\pi_1(B\pi\underset{\sim}{Y},\ v) \cong \pi_1(Y,v)$ and for $n \geq 2$ $\pi_n(B\pi\underset{\sim}{Y},\ v)$ is isomorphic to $H_n(\tilde{Y})$, the homology of the universal cover $\tilde{Y}$ of Y. Further there is a map $q : Y \to B\pi\underset{\sim}{Y}$ which induces, on homotopy groups π_n, an isomorphism for $n = 1$, and for $n \geq 2$ a morphism equivalent to the Hurewicz $\pi_n(Y,v) \xrightarrow{\omega} H_n(\tilde{Y})$.

These facts are deducible from results of §.8, 9 of [5], but are not explicit there, so it should prove useful to exeplain the procedure.

For any filtered space Y there are cubical complexes and maps

$$\begin{array}{ccc} R\underset{\sim}{Y} & \xrightarrow{\ i\ } & KY \\ {\scriptstyle p}\big\downarrow & & \\ \rho\underset{\sim}{X} & & \end{array}$$

where KY is the cubical singular complex of Y, and i is the inclusion of the *filtered singular complex* $R\underset{\sim}{Y}$ of $\underset{\sim}{Y}$; that is $R\underset{\sim}{Y}$ consists in dimension n of all filtered maps $\underset{\sim}{I}^n \to \underset{\sim}{Y}$. The mapping p is a quotient mapping. It identifies two filtered maps $\underset{\sim}{I}^n \to \underset{\sim}{Y}$ if and only if they are homotopic, relative to the vertices of $\underset{\sim}{I}^n$, and through filtered maps. (This definition is not exactly the same as that given in [5], but the two definitions agree if $\pi_0 Y_0 = Y_0$, which is sufficient for our purposes.)

The cubical complex $\rho\underset{\sim}{Y}$ has the structure of ω-groupoid, and its associated crossed complex is $\pi\underset{\sim}{Y}$. That is, $\rho\underset{\sim}{Y}$ is isomorphic as ω-groupoid to $\lambda\pi\underset{\sim}{Y}$.

In [5] it was shown that $p : R\underset{\sim}{Y} \to \rho\underset{\sim}{Y}$ is a fibration in the sense of Kan. This result was found to be an important technical tool in the proofs of the main results of [5], since it helped in proving $\rho\underset{\sim}{Y} \cong \lambda\pi$, and in establishing a crucial property of "thin elements" in $\rho\underset{\sim}{Y}$. We can now give this fibration property of p another rôle.

The cubical complexes $R\underset{\sim}{Y}$ and KY are known to be Kan complexes. (The corresponding property for $\rho\underset{\sim}{Y}$ is not so simple to prove.) The inclusion $i : R\underset{\sim}{Y} \to KY$ is a homotopy equivalence if the functions induced by inclusion $\pi_0 Y_r \to \pi_0 Y$ are surjective for $r = 0$ and bijective for $r > 0$, and the based pairs (Y, Y_m, v) are m-connected for all $m \geqslant 1$ and $v \in Y_0$. In particular, i is a homotopy equivalence if Y is the skeletal filtration of a CW-complex Y. For such a Y, the realisation $|KY|$ has the same homotopy type as Y, and in this way we obtain the map $q : Y \to B\pi\underset{\sim}{Y}$ with the properties set out above.

Let X be a CW-complex. We have an induced function

$$q_* : [X,Y] \to [X, B\pi\underset{\sim}{Y}].$$

This function is bujective if $\dim X \leqslant m$ and $q : Y \to B\pi\underset{\sim}{Y}$ has m-connected homotopy fibre. This will be true if, for example, $\pi_i Y = 0$ for $1 < i < m$. In these circumstances we obtain a bijection

$$[X,Y] \to H^0(X ; \pi\underset{\sim}{Y}).$$

So we can see the relevance of this non-abelian cohomology to some general homotopy classification problems, particularly in the non-simply connected case.

How do we compute $H^0(X ; C)$? For this we generalise some ideas of Whitehead in [10].

For simplicity, we restrict to the reduced case. Let GC_* be the category with objects the triples (K,G,v) in which G is a group, K is a chain complex of G-modules (with $K_i = 0$ for $i < 0$), and K_0 is a free G-module with basis the element $v \in K_0$. The morphisms of GC_* are to be pairs $(f,\theta) : (K,G,v) \to (K',G',v')$ where $\theta : G \to G'$ is a morphism of groups, $f : K \to K'$ is a chain map and an operator morphism over θ, and $f(v) = v'$.

Let XC_* be the category of reduced crossed complexes. There is a functor $\Delta : XC_* \to GC_*$ in which if $(K,G,v) = \Delta C$, then $G = C_1/\delta C_2$; $K_n = C_n$ as a G-module for $n \geq 3$; K_2 is C_2 made abelian ; K_1 is the C-module induced from the augmentation ideal IC_1 by the quotient morphism $C_1 \to G$; and K_0 is the free G-module on the element $v \in C_0$. (This construction is given in [7] and extends a construction given in [10] for the case C_1 is free. A further result proved in [7] is that Δ has a right adjoint, and so preserves colimits.) This functor Δ transforms homotopies to homotopies, for a suitable definition of homotopy in GC_*. So for reduced crossed complexes C,D we have a function

$$\Delta_* : [D,C] \to [\Delta D, \Delta C].$$

Now Whitehead proves (but does not state) that if C_1 and D_1 are free groups and D_2 is a free crossed D_1-module, then Δ_* is a bijection. Also, he notes that if $\underset{\sim}{X}$ is the skeletal filtration of a reduced CW-complex X, then $\Delta\pi\underset{\sim}{X}$ consists of the cellular chains $C_*(\tilde{X})$ of the universal cover $\tilde{X}$ of X, these chains being taken as modules over the fundamental group of X. That is, we have a bijection

$$H^0(X \; ; \; C) \cong [C_*(\tilde{X}), \Delta C].$$

This gives a reasonable computational description of $H^0(X \; ; \; C)$, and so of $[X,BC]$. For example, it leads to the homotopy classification of maps from a surface to the projective plane [2].

Consider again the bijection

$$[X,Y] \cong [C_*(\tilde{X}), C_*(\tilde{Y})]$$

given when $\dim X \leq m$ and $\pi_i Y = 0$ for $1 < i < m$. If also $\pi_1 Y = 0$, then $\tilde{Y} = Y$ and the definition of morphism and chain homotopy in GC_* implies that

$$[C_*(\tilde{X}), \; C_*(\tilde{Y})] \cong [C_*(X), \; C_*(Y)]$$

where $C_*(X)$ is the usual cellular chain complex of X. Since $C_*(Y)$ is a chain complex of free abelian groups there is a chain map $\phi : C_*(Y) \to H_*(Y)$ (where the latter has zero differential) inducing an isomorphism in homology. So we obtain

$$\begin{aligned}[X,Y] &\cong [C_*(X), \; H_*(Y)] \\ &\cong H^0(X \; ; \; H_*(Y)) \\ &\cong H^m(X \; ; \; H_m(Y)).\end{aligned}$$

This result includes the Hopf classification theorem (which is the case $Y = S^m$). Thus the non-abelian results reduce to classical abelian results.

All these results give point to a remark of Whitehead in the Introduction to [10], which reads in our terminology :

The crossed complex $\pi\underline{X}$ *appears to be more useful than the chain complex* $C_*(\tilde{X})$ *in problems concerning geometric realisability. On the other hand, the chain complex* $C_*(\tilde{X})$ *is useful in studying concrete problems.*

References

1. N.K. Ashley, *Crossed complexes and* T-*complexes*, Ph.D. Thesis, University of Wales, (1978).

2. R. Brown, 'The homotopy classification of maps from a surface to the projective plane', Ba,gor Preprint 82.5, (1982).

3. R. Brown, (ed.), *Simplicial* T-*complexes*, Esquisses Math. (to appear).

4. R. Brown and P.J. Higgins, 'The algebra of cubes', J. Pure Appl. Alg. 21 (1981), 233-260.

5. R. Brown and P.J. Higgins, 'Colimit theorems for relative homotopy groups', J. Pure Appl. Alg. 22 (1981), 11-41.

6. R. Brown and P.J. Higgins, 'Crossed complexes and non-abelian extensions', *Proc. Int. Conf. on Category Theory, Gummersbach, 1981* (ed. H. Kamps, D. Pumplum, M. Tholen) Springer L.N.M. (to appear).

7. R. Brown and P.J. Higgins, 'Crossed complexes and chain complexes with operators', (in preparation).

8. M.K. Dakin, *Kan complexes and multiple groupoids*, Ph.D. Thesis, University of Wales, (1977).

9. G. Segal, 'Classifying spaces and spectral sequences', Publ. Math. I.H.E.S. 34 (1968), 105-112.

10. J.H.C. Whitehead, 'Combinatorial homotopy II', Bull. Amer. Math. Soc. 55 (1949) 453-496.