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LOCAL ALGEBRA AND RATIONAL HOMOTOPY

by

Luchezar L. Avramov

C'est de l'algèbre,
se dit d'une chose, à laquelle
on ne comprend rien.

PETIT LITTRÉ

This paper represents a largely expanded version of my talk at the conference. The expansion was aimed at two targets, namely: (a) giving precise formulations each time, including definitions when necessary, and (b) setting the results as much as possible against their historical background. However, from lecture to paper, the organization of the material has remained invariant, and is described by the following table of

Contents

The paper, as the talk, represents a mixture of research announcements and survey material. The basic object of investigation in Sections 1 through 6 is the structure of a noetherian local (i.e. with unique maximal ideal) commutative ring, as reflected by several invariants of a (co-)homological nature. There are two basically new points of view which are described in this paper (and which, in a still somewhat tentative version, have already been proposed in [16]). The first one consists in the realization that in order to study homology of local rings one has to work in a broader category of DG algebras. Thus, even starting with a homomorphism of rings, one has to introduce non-trivial DG algebras in order to obtain the relevant notion of homotopy fibre of the map. We have found that the category of DG algebras with divided powers is particularly suited to such investigations, since it offers (almost) equally good chances to the case of residual positive characteristic as to that of characteristic zero. Thus, in Section 1 we introduce for a DG Γ-algebra a graded Lie algebra which will have the significance of a homotopy algebra. The second section describes the homotopy exact sequence, whose properties have important implications in the theories of flat extensions of rings and ideals of finite projective dimension: we do not describe them in the text, but rather refer to [12], [5], [16], [19].

The second innovation is the introduction in local algebra of the technique of minimal models. This is done in Section 4, where the Lie product of π^* (or equivalently, the Yoneda product in Ext^*) is described in terms of the model. Given the enormous technical difficulty in computing with Yoneda products, such an interpretation represents considerable interest, and has already had several applications. One of them - the determination of the rate of growth of the Betti numbers of the ring - is presented in Section 6. Needless to say, that in our approach to minimal models we have been inspired by Sullivan's theory of rational homotopy types, for which we refer the reader to the original treatment [66], as well as to the expositions [22], [39], [41]. Details and proofs for the results of Sections 1, 2, 4, 6 are to be found in [19].

Since I was talking to an audience consisting mainly of non-algebraists, I felt some motivation might be required for the introduction of all this machinery in algebraic problems. This is done in Section 5, where I choose to introduce questions on homology in the most down-to-earth way, as questions concerning the properties of the sequence of numerical invariants given by the Betti numbers (hence, ultimately, by the numbers of solutions of systems of linear equations). All unexplained notions from commutative algebra can be found in the books by Matsumura [49] and Serre [62]. Also, let me recall that the two fields from the title of the talk first appeared side-by-side on page IV-52 of Serre's Lecture Notes.

Section 3 has a particular status. Its main purpose is not to formulate or comment theorems, but to establish a kind of correspondence between (co-)homological functors in algebra and topology, on the basis of evidence they have shown to behave in similar ways. This translation game provides some fun - while looking for the correct matching - but it has the much more important function of suggesting where and how to look for new developments.

Several examples of applications of this principle, going from topology to algebra, are scattered throughout the first sections. Paying a debt back, in the last section we give a theorem on the homotopy Lie algebra of formal manifolds (which by [26] include all 1-connected complex projective algebraic varieties). This is the topological counterpart to a result on local rings obtained by Levin and the author [46].

1 - The homotopy Lie algebra of a DG algebra with divided powers

In this paper the name DG algebra is reserved to associative, $\mathbb{Z}$-graded (with trivial components in negative degrees), strictly skew-commutative algebras, equipped with a differential d of degree -1. It should be stressed that no ground ring (different from the integers) is fixed, so in fact we work with DG rings, but prefer to stick to the more conventional terminology. As usual, commutative rings are treated as DG algebras concentrated in degree zero, with trivial differential. An augmentation is a surjective map of DG algebras, landing on a field. A DG algebra is said to have locally noetherian homology, if $H_o(F)$ is a noetherian ring, and $H_i(F)$ is a noetherian $H_o(F)$-module for each i.

We use the term Γ-algebra to denote a DG algebra with divided powers. Rather than give a formal definition - for which we refer to [24, Exposé 7] or to [37, Chapter I, §7] - we insert a few comments.

(i) Any DG algebra defined over the field $\mathbb{Q}$ of rational numbers has a unique structure of DG Γ-algebra, given by the formula $u^{(K)} = (K!)^{-1}u^K$ for each u of even positive degree (in the sequel we write $|u|$ for the degree of the homogeneous element u).

(ii) Any commutative ring has the obvious trivial structure of DG Γ-algebra.

(iii) Even in characteristic 2, we suppose the divided powers are defined only for (homogeneous) elements of even positive degree.

(iv) A homomorphism of Γ-algebras is always supposed to commute with the divided powers.

(v) A Hopf Γ-algebra is a graded algebra A with $A_o = k$ a field, for which the unit $\eta : k \to A$, augmentation $\varepsilon : A \to k$, and diagonal $\Delta : A \to A \otimes_k A$ are all homomorphisms of Γ-algebras, satisfying the usual identities: for details cf. [1].

The first step in the construction of a homotopy Lie algebra is the following.

THEOREM (1.1). Let $F \to k$ be an augmented DG Γ-algebra with locally noetherian homology. Then $\mathrm{Tor}^F(k,k)$ is a Hopf Γ-algebra which is locally (i.e. in each degree) finite-dimensional over k.

REMARKS (i) Tor^F is taken in the sense of Eilenberg and Moore's "differential graded homological algebra": cf. [52]; in particular, when $F = F_o$, one obtains the classical object.

(ii) In the case when $F = R$ is a local ring, the diagonal Δ on $\mathrm{Tor}^R(k,k)$ was introduced by Assumus [9], who showed it is counitary. It was not until Levin's thesis [43] that $\mathrm{Tor}^R(k,k)$ was proved to be a Hopf Γ-algebra. An independent proof is due to Schoeller [61], and a detailed account is available in [37, Chapter II].

(iii) For a general F the Hopf algebra structure was introduced - somewhat implicitly - in Moore's foundational paper [52], and explicited and used in [11]. The complete result requires some further elaborations, which are carried out in [19].

Now we introduce the object which in this paper will be called a graded Lie algebra.

DEFINITION. A graded Lie algebra L over k is a collection $\{L^i\}_{i\geq 1}$ of k-vector spaces together with a bilinear pairing

$$[\ ,\ ] : L^i \times L^j \to L^{i+j}$$

and a quadratic operator

$$q : L^{2i+1} \to L^{4i+2}$$

satisfying the following properties:

(1) $[a,b] = -(-1)^{|a||b|}[b,a]$;

($1\frac{1}{2}$) $[a,a] = 0$ for $|a|$ even;

(2) $(-1)^{|a||c|}[a,[b,c]] + (-1)^{|b||a|}[b,[c,a]] + (-1)^{|c||b|}[c,[a,b]] = 0$;

($2\frac{1}{3}$) $[a,[a,a]] = 0$ for $|a|$ odd;

(3) $q(ua) = u^2q(a)$ for $u \in k$ and $|a|$ odd;

(4) $[a,b] = q(a+b) - q(a) - q(b)$ for $|a| = |b|$ odd;

(5) $[a,[a,b]] = [q(a),b]$ for $|a|$ odd.

It should be remarked that as soon as 2 is invertible in k, one gets from (3) and (4) that $q(a) = \frac{1}{2}[a,a]$ ($|a|$ odd), and all the relations except ($2\frac{1}{3}$) are consequences of (1) and (2). If moreover 3 is invertible in k, then ($2\frac{1}{3}$) follows from (2). Hence in characteristic different from 2, 3, we have the usual gadget.

The typical example of a graded Lie algebra is given by the underlying vector space of an associative graded algebra (not necessarily commutative), with $[a,b] = ab - (-1)^{|a||b|}ba$, and $q(a) = a^2$. In fact, the usual construction provides any graded Lie algebra L with a graded universal enveloping algebra UL, which is a Hopf algebra for the map extending $a \to (a,a)$ ($a \in L$). The fundamental relation between Hopf Γ-algebras and graded Lie algebras is contained in the following result proved by Milnor and Moore (char k = 0: [51]); André (char k > 2: [1]), and Sjödin (char k = 2: [64]), of which we quote only a portion:

THEOREM (1.2). The graded vector space dual $A^\vee$ of a Hopf Γ-algebra A with $\dim_k A_i < \infty$ for all i is the universal enveloping algebra of a uniquely defined graded Lie algebra.

Combining theorems (1.2) and (1.2) we now see, that every augmented DG Γ-algebra $F > k$ with locally noetherian homology uniquely defines a graded Lie algebra over k, which we shall denote $\pi^*(F)$ and call the homotopy Lie algebra of F (with coefficients in k). In the sequel we shall consider only algebras with locally noetherian homology.

REMARKS. (i) This Lie algebra, in case F = R is a (noetherian) local ring with residue field k, is closely related to the cohomology of the cotangent complex of the R-algebra k with coefficients in k : $H^*(R,k;k)$: cf. [2], [55]. This cohomological functor (also called simplicial cohomology or André-Quillen cohomology) plays an important role in deformation theory, and there is a canonical homomorphism of Lie algebras $\varphi : H^*(R,k;k) \to \pi^*(R)$. Due to Quillen [55] (cf. also [2]), it is known that φ is an isomorphism if the characteristic of k is zero, or if it is considered in degrees $\leq 2p$ ($p = \text{char}(k) > 0$). On the other hand, André [4] has constructed, for any $p > 0$, a ring for which φ is not an isomorphism in degree $(2p+1)$.

We want to point out here that the "homotopical" point of view of this paper leads to the discovery of some unsuspected properties of the cotangent homology: for instance the results of the next section show that the connecting maps of some change of rings Jacobi-Zariski exact sequences (cf. [2]), are often trivial.

(ii) A survey of recent activities involving the use of the Lie algebra $\pi^*(R)$ (R-local) has been given by Roos [60].

2 - Functoriality and the homotopy exact sequence

A significant difference between the algebraic setup and the topological one occurs because of problems with the coefficients. In fact, in topology, (co-)homology and homotopy depend bifunctorially on the space and the ring of coefficients. In our algebraic problems the field of coefficients is intrinsically linked to the basic object, i.e. the ring under inspection. One way to circumvent this inconvenient is to consider only homomorphisms of rings augmented to a fixed field k, i.e. commutative triangles of ring homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & S \\ & \searrow \quad \swarrow & \\ & k & \end{array}$$

This was for instance the point of view of [13], and since it embraces all surjective maps, it is sufficient for many purposes, e.g. for handling many important cases of Golod homomorphisms, as defined by Levin [44]. However, it gives in general no hold on such an important class of homomorphisms as flat extensions, and sticking to it rules out the possibility of treating the basic problem of change of rings in its natural framework (cf. [17] for a preliminary version).

DEFINITION. A map of augmented DG Γ-algebras is a commutative square

$$\begin{array}{ccc} F & \xrightarrow{\ f\ } & F' \\ \downarrow & & \downarrow \\ k & \xrightarrow{\ \varphi\ } & k' \end{array}$$

of homomorphisms of DG Γ-algebras.

We write $(f,\varphi) : (F \to k) \to (F' \to k')$, or simply $f : F \to F'$, when no ambiguity arises.

The following is proved in [19]:

THEOREM (2.1). *The homomorphism* (f,φ) *of* DG Γ-*algebras defines a homomorphism*

$$f^* : \pi^*(F') \to k' \underset{\varphi}{\otimes} \pi^*(F)$$

of graded Lie algebras over k'. *If* $H(f)$ *is an isomorphism, then* f^* *is an isomorphism*.

If $(f',\varphi') = (F' \to k') \to (F'' \to k'')$ *is a map of* DG Γ-*algebras, then*

$$(f' \circ f)^* = (k'' \otimes_{\varphi'} f^*) \circ f'^* : \pi^*(F'') \to k'' \otimes_{\varphi'\varphi} \pi^*(F)$$

as homomorphisms of graded Lie algebras over k''.

In order to construct homotopy exact sequences we first introduce fibrations:

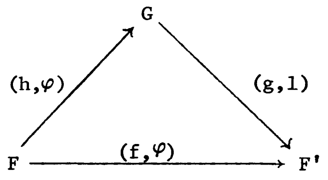
DEFINITION. A map $h : F \to G$ of DG Γ-algebras is called a fibration if, forgetting differentials, G is flat over F for the module structure induced by h. If h is a fibration, we call $k \otimes_F G$ the fibre of h.

The last object is in fact a DG Γ-algebra, and the natural map $p : G \to k \otimes_F G$ is compatible with this structure, since $\operatorname{Ker} p$ is the ideal of G generated by $h(\operatorname{Ker}(F \to k))$, and this is easily verified to be stable under the differential and divided powers of G.

Recall that a homomorphism is called a weak equivalence, if it induces an isomorphism in homology. Two DG Γ-algebras A et A' are said to be homotopically equivalent, if there exist DG Γ-algebras $A = B_0, B_1, \ldots, B_{m+1} = A'$ and weak equivalences of DG Γ-algebras $B_i \to B_{i+1}$ *or* $B_i \leftarrow B_{i+1}$ for $i = 0,1,\ldots,m$. Such a sequence is called a homotopy equivalence and according to (2.1) it induces a well defined isomorphism $\pi^*(A) \simeq \pi^*(A')$.

The next result is proved in [54], in a somewhat different situation, but its proof carries over to give:

PROPOSITION (2.2). *Let* $(f,\varphi) : (F \to k) \to (F' \to k')$ *be a homomorphism of* DG Γ-*algebras. Then there exists a* DG Γ-*algebra* $(G \to k')$ *and a commutative diagram of augmented* DG Γ-*algebras*:



such that h *is a fibration and* g *is surjective and a weak equivalence*.

Note that G is clearly uniquely defined up to homotopy equivalence, and it is not difficult to prove that so is the $k \otimes_F G$. Hence we can unambiguously speak of the homotopy fibre of the map f, and we shall denote it by T^f.

In characteristic zero, the homotopy exact sequence can easily be abstracted from Gulliksen's work in [37, Chapter III]:

THEOREM (2.3). Suppose k is a field of characteristic zero. Then for every map $(f,\varphi) : (F \to k) \to (F' \to k')$ of augmented DG algebras there is a natural long exact sequence:

$$
\begin{aligned}
0 &\longrightarrow \pi^1(T^f) \xrightarrow{q^1} \pi^1(F') \xrightarrow{f^1} k' \otimes_{\varphi} \pi^1(F) \\
&\xrightarrow{\partial^1} \pi^2(T^f) \longrightarrow \dots \longrightarrow k' \otimes_{\varphi} \pi^{i-1}(F) \\
&\xrightarrow{\partial^{i-1}} \pi^i(T^f) \xrightarrow{q^i} \pi^i(F') \xrightarrow{f^i} k' \otimes_{\varphi} \pi^i(F) \\
&\xrightarrow{\partial^i} \pi^{i+1}(T^f) \longrightarrow \dots,
\end{aligned}
$$

in which f^* is the homomorphism of graded Lie algebras given by (2.1), and q^* is the homomorphism of graded Lie algebras $(g^*)^{-1} \circ p^*$, with g as in (2.2) and $p : G \to k \otimes_F G$ the canonical projection.

Trying to drop the restriction on the characteristic, one soon comes across examples like this one:

EXAMPLE (2.5). Let $F = k\langle u\rangle$ be the exterior algebra on one generator of odd degree, $F' = F/uF = k$, with f the natural projection. The inclusion of F in the free Γ-algebra $G = k\langle u,v \mid dv = u\rangle$ fits into the diagram of (2.2), hence one can take $T^f = k\langle v; dv = 0\rangle$. Clearly $\pi^*(F') = 0$; $\pi^*(F)$ is concentrated in degree $|u| + 1$, and has dimension one. However, when $\mathrm{char}(k) = p > 0$, H. Cartan's computation [24] of the homology of the $K(\mathbb{Z}/p\mathbb{Z},n)$ shows that $\pi^i(T^f)$ is one-dimensional for $i = |v|p^n + 1$ and $i = |v|p^{n+1} + 2$ for all $n \geq 0$ (and is zero otherwise). Hence the sequence (2.4) is in this case (very much) inexact.

However, it turns out that things do come straight at times, and then Nature is really generous. First a useful notation, which extends one introduced by Hochster: for any DG module M over F set

$$\dim^F(k,M) = \sup\{j \mid \mathrm{Tor}_j^F(k,M) \neq 0\};$$

this is a nonnegative integer, $+\infty$, or $-\infty$, the last possibility occuring exactly when $\mathrm{Tor}_j^F(k,M) = 0$ for all j.

REMARKS (2.6). Note that this definition is very natural from the point of view of commutative algebra. In fact, let $R \to S$ be a local homomorphism of local noetherian rings, and let M be a finitely - generated S - module. Then $\dim^R(k,M) \geq 0$, with equality precisely when M is R-flat: this is one of the forms of Bourbaki's local criterion of flatness, cf. [21, Chapitre III, §5, Théorème 1]. On the other hand, when $R = S$, then $\dim^F(k,M) < \infty$ if and only if M admits a resolution of finite length by free R-modules of finite rank. In this case $\dim^F(k,M)$ equals the minimal length of such a resolution, and is called the projective dimension of M over R (denoted usually by $\mathrm{pd}_R M$).

THEOREM (2.7). Let $(f,\varphi) : (F \to k) \to (F' \to k')$ be a homomorphism of augmented DG Γ-algebras, such that $\dim^F(k,F') < \infty$. Then:

(i) the sequence (2.4) is exact;

(ii) $\partial^{2j+1} = 0$ for all $j \geq 0$;

(iii) $\sum \dim \operatorname{Im} \partial^j \leq \dim^F(k,F') + \dim \pi^1(T^f)$.

When F and F' are local (noetherian) rings, parts (i) and (ii) are proved by the author in [12] for the case of a flat homomorphism and in [18] in general. Part (iii) is established by André [5] for the flat case, but as noted in [18] the argument stretches to cover $\dim^F(k,F') < \infty$, as soon as (i) is available. In the general framework of DG Γ-algebras the theorem is proved in [19]. Finally, let us note that it has been conjectured in [15], [5] that $\partial^i = 0$ for f a flat homomorphism of rings, and $i \neq 1$, and that this is known to be true (even under the weaker assumption of finite $\dim^F$), when the characteristic of k is 2: cf. [3]. Finally, one can use Jacobsson's argument [40] to show that $\operatorname{Ker} \partial^*$ is contained in the centre of the Lie algebra $\pi^*(T^f)$.

3 Two dictionaries

I. In deference to Littré's authority, we try to understand our algebra by looking for analogies with topology. A first aspect of such an attempt is the shocking realization that all arrows seem to point in the wrong direction: the DG algebras which come up from cohomological functors in topology have differentials of degree +1, while the ones that are introduced in algebra come equipped with a derivation of degree −1; cohomology (resp. homotopy) give contravariant (resp. covariant) functors from topological spaces to skew-commutative graded algebras (resp. graded Lie algebras), while homology (resp. homotopy) go from rings to the same categories, but with the opposite variance; etc., etc., etc... . In order to give the correct functoriality to our intuition, we invoke a fact which is systematically fading out from the algebraist's attention, namely that commutative rings more often than not have come into algebra as sets of functions defined on some space. With this correction in mind we now turn to setting a correspondence between the notions arising in topology and in algebra:

Sullivan-De Rham complex of a 1-connected space X with rational homology of finite type	DG Γ-algebra F with $H_o(F) = k$ and locally noetherian homology.
$H^*(X,\mathbb{Q})$	$H(F)$
cup-product in $H^*(X,\mathbb{Q})$	homology product in $H(F)$
Massey products	Massey products
Pontryagin product in $H_*(\Omega X,\mathbb{Q}) = [H^*(\Omega X,\mathbb{Q})]^\vee$	Yoneda product in $\mathrm{Ext}_F(k,k) = [\mathrm{Tor}^F(k,k)]^\vee$
$H^*(\Omega X,\mathbb{Q})$ as a Hopf algebra	$\mathrm{Tor}^F(k,k)$ as a Hopf algebra
Eilenberg-Moore spectral sequence $E^2 = \mathrm{Tor}^{H^*(X)}(\mathbb{Q},\mathbb{Q}) \Rightarrow H^*(\Omega X)$	Eilenberg-Moore spectral sequence $E^2 = \mathrm{Tor}^{H(F)}(k,k) \Rightarrow \mathrm{Tor}^F(k,k)$
$\pi_*(\Omega X) \otimes \mathbb{Q}$ with the Samelson product	$\pi^*(F)$ with the commutator product induced from the Yoneda product in $[\mathrm{Tor}^F(k,k)]^\vee$.
Hurewicz homomorphism $\pi_*(\Omega X) \otimes \mathbb{Q} \to H_*(\Omega X,\mathbb{Q})$	inclusion $\pi^*(F) \to [\mathrm{Tor}^F(k,k)]^\vee$
homotopy exact sequence of a map	homotopy exact sequence of a map
cohomology suspension $H^*(X,\mathbb{Q}) \to H^{*-1}(\Omega X,\mathbb{Q})$	homology suspension $H_*(F) \to \mathrm{Tor}^F_{*+1}(k,k)$

The important circumstance, of course, is not the length of the parallel list - which can be extended substantially - but the striking similarity of properties displayed by the objects on each side. To quote but one instance: the splitting of the homotopy sequence in case the fibre has finite - dimensional homology, which on the right-hand side is asserted by Theorem (2.7), is given for the left-hand side by results of Halperin [38] for (ii) and Felix-Halperin[29] for (iii). Another example is given in Section 6.

One objection to such a translation may be that the local rings - which are the objects of basic interest to us - will never fit in this scheme except for the trivial case $R = k$. However, for most purposes one can replace the ring by its Koszul complex. Recall that if $t_1,\dots,t_n$ form a minimal system of generators of the maximal ideal $\underline{m}$ of R, the Koszul complex K^R is the exterior algebra on the free module R^n with basis $T_1,\dots,T_n$ $(|T_i| = 1)$, furnished with the differential

$$d(T_{i_1} \dots T_{i_m}) = \Sigma(-1)^{j-1} t_{i_j} T_{i_1} \dots \hat{T}_{i_j} \dots T_{i_m}.$$

It is well known and easily verified that up to isomorphism K^R is independent of the choice of the generators of $\underline{m}$. Now one has:

PROPOSITION (3.1). <u>The inclusion</u> $i : R \to K^R$ <u>induces an exact sequence of graded Lie algebras</u>

$$(3.2) \qquad 0 \longrightarrow \pi^*(K^R) \xrightarrow{i^*} \pi^*(R) \longrightarrow s(\underline{m}/\underline{m}^2)^\vee \to 0$$

<u>where on the right hand side one has the vector space</u> $\pi^1(R)$, <u>considered as an abelian Lie algebra</u>.

<u>PROOF</u>. Indeed, i is a fibration with fibre the exterior algebra $\Lambda = \Lambda s(\underline{m}/\underline{m}^2)$ (s denotes the suspension operator, raising degrees by 1). Since $\mathrm{Tor}^\Lambda(k,k) = \Gamma s^2(\underline{m}/\underline{m}^2)$, with Γ denoting the free Γ-algebra functor, (2.7) shows that $i^* : \pi^j(K^R) \to \pi^j(R)$ is an isomorphism for $j > 2$, and gives the exact sequence

$$0 \to \pi^1(K^R) \xrightarrow{i^*} \pi^1(R) \xrightarrow{\partial^1}$$
$$s^2(\underline{m}/\underline{m}^2)^\vee \to \pi^2(K^R) \xrightarrow{i^*} \pi^2(R) \longrightarrow 0.$$

But in the Eilenberg-Moore spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{H(K)}_{p,q}(k,k) \Rightarrow \mathrm{Tor}^K_{p+q}(k,k)$$

one has $E^2_{p,q} = 0$ when $p + q = 1$ (since $H(K)$ is connected, hence $\mathrm{Tor}^K_1(k,k) = 0$ and $\pi^1(K^R) = 0$). Taking into account that $\pi^1(R) = s(\underline{m}/\underline{m}^2)^\vee$, one sees that ∂^1 is an isomorphism for dimensional reasons, hence i^* is an isomorphism in dimension 2 also.

Thus we can and shall identify $\pi^*(K^R)$ with the subalgebra $\pi^{\geq 2}(R)$ of the elements of $\pi^*(R)$ of degree ≥ 2. The following result settles the question of when the sequence (3.2) splits:

THEOREM (3.3) (Sjödin, [63]). The Lie algebra $\pi^*(R)$ is the semi-direct product of $\pi^{\geq 2}(R)$ and $s(\underline{m}/\underline{m}^2)^{\vee}$ if and only if $\dim_k(\underline{m}^2/\underline{m}^3) = \binom{n+1}{2}$ (which is the maximal possible value).

In general the action of $\pi^1(R)$ on $\pi^{\geq 2}(R)$ can be highly non-trivial. A thorough investigation of this phenomenon, in terms of Yoneda products in $\mathrm{Ext}_R(k,k)$, has been given by Roos [58, §1]. In the case when $\underline{m}^3 = 0$, the study of the Lie subalgebra of $\pi^*(R)$, generated by $\pi^1(R)$, is the main contents of Löfwall's thesis [47].

II. Another way in which objects studied in algebra relate to those studied by topologists involves the following ingredients:

- on the algebraic side: skew-commutative graded connected algebras over a field k of characteristic zero; the graded rings of commutative algebra are strictly contained in this class as the evenly graded objects, since a doubling of all degrees is harmless for the cohomological considerations.

- on the topological side: formal spaces; recall that X is formal if $H^*(X,k)$ has the same minimal model as the Sullivan-De Rham complex $A_k^*(X)$, and that this property does not depend on the choice of the field k (of characteristic zero).

The invonvenience from such an interpretation of the concepts (and results) comes from the fact that it is available on proper subclasses of objects in each of the categories involved (in particular, the double gradings on the cohomological and homotopical functors have no analogue for non-graded rings, resp. non-formal spaces). Its advantage stems from the observation, that through the isomorphisms

$$H^*(\Omega X,k) \simeq \mathrm{Tor}^{H^*(X)}(k,k) \qquad \text{(as Hopf algebras)}$$

$$\pi_*(\Omega X) \otimes k \simeq \pi^*(H^*(X)) \qquad \text{(as Lie algebras)}$$

every result obtained in one context has an immediate meaning in the other. One should, however, not forget, that under these isomorphisms an element of $\mathrm{Tor}_{p,q}^{H_*(X)}(k,k)$ of homological degree p and internal degree q corresponds to an element in $H^{q-p}(\Omega X,k)$.

We close this section by quoting two important results, obtained by using the "dictionary II":

THEOREM (3.4) (Roos [58]). For a field k of characteristic zero, the following are equivalent:

(i) $\sum_{i\geq 0} \dim_k \mathrm{Tor}_i(k,k)t^i$ is a rational function for all local rings R with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^3 = 0$ and $R/\mathfrak{m} = k$;

(ii) $\sum_{i\geq 0} \dim_k H^i(\Omega X,k)t^i$ is a rational function for all finite 1-connected CW-complexes X with $\dim X \leq 4$.

Thus, when some time later Anick [7], [8] constructed a CW-complex with the required properties, having an irrational Poincaré series, a corresponding example of a local ring was automatically exhibited, since the correspondence in Roos' theorem is obtained through an effective construction, using $H^*(X,k)$. For the story of the construction of non-rational series we refer to [59] and [49]. Both papers contain also an outline of an alternate construction (inspired from Anick's), due to Löfwall-Roos [48].

THEOREM (3.5) (Felix-Thomas [31]). Let R be a connected graded skew-commutative noetherian algebra over a field k of characteristic zero. Then the integers $\dim_k \mathrm{Tor}^R_{i,*}(k,k)$ form a non-decreasing sequence, which for $i >> 0$ has either a polynomial or an exponential growth.

The proof of this theorem heavily uses results of Felix-Halperin [29] on the growth of $\dim_{\mathbb{Q}} H^i(X,\mathbb{Q})$.

4 - Minimal models

Sullivan's equivalence of the rational homotopy category and the homotopy category of minimal $\mathbb{Q}$-algebras shows in particular that the graded Lie algebra $\pi^*(\Omega X) \otimes \mathbb{Q}$ (for the Samelson product) can be obtained from the minimal model of the 1-connected CW complex X of finite type. The explicit way in which this is done, using the quadratic part of the differential of the minimal model, is stated in [26] (cf. [6] for details). The purpose of this section is to introduce the concept of minimal model for (some) DG Γ-algebras F, and to state the two main results on these objects, namely an existence theorem, and a description of $\pi^*(F)$ in terms of a minimal model of F.

DEFINITION. A DG k-algebra V is called minimal if the following conditions are satisfied:

(i) $V_0 = k$;

(ii) as a graded algebra V is isomorphic to the free algebra ΛQ on a finite dimensional vector space Q;

(iii) $dV \subset (IV)^2$, where IV is the augmentation ideal $\mathrm{Ker}(V \to k)$.

Note that V is the tensor product of exterior algebras on the odd components of Q with symmetric algebras on its even components, hence in positive characteristic it is not a Γ-algebra (unless Q is concentrated in odd degrees).

DEFINITION. The minimal algebra V is called a minimal model of the DG Γ-algebra $F \to k$ if there exists a DG Γ-algebra $F' \to k$, which is homotopically equivalent to F, (cf. Section 2), and a weak equivalence $g : V \to F'$ of augmented DG-algebras.

If F is itself a k-algebra, and $H_0(F) = k$, then a weak equivalence $g : V \to F$ can be constructed by mimicking Sullivan's arguments. In general, the problem with coefficients, already invoked at the beginning of Section 2, clearly rules out such a possibility (since F_0 may contain no field). However we have

THEOREM (4.1). Suppose F_0 is a noetherian ring. Then $F \to k$ has a minimal model if (and only if) $H_0(F) = k$ and $H_i(F)$ is a finite-dimensional k-vector space for each i.

For the minimal algebra $V = \Lambda Q$ we now fix a homogeneous basis $\{x_\alpha\}_{\alpha \in N}$ of Q, ordered in such a way that $|x_\alpha| < |x_\beta|$ implies $\alpha < \beta$. By the definition of V there exist uniquely defined constants $c^\gamma_{\alpha\beta} \in k, c^\gamma_\alpha \in k$, such that

$$dx_\gamma - \sum_{\substack{\alpha<\beta \\ |x_\alpha|+|x_\beta|+1=|x_\gamma|}} c^\gamma_{\alpha\beta} x_\alpha x_\beta - \sum_{\substack{\alpha \\ 2|x_\alpha|+1=|x_\gamma|}} c^\gamma_\alpha x_\alpha^2 \in (IV)^3$$

(for α, β, γ not satisfying the constraints above we set $c^\gamma_{\alpha\beta} = 0 = c^\gamma_\alpha$). We identify Q with $IV/(IV)^2$ and set $L = (sQ)^\vee$; so that $L^i = \mathrm{Hom}_k(Q_{i-1}, k)$. Finally, let $\{y_\beta\}_{\beta \in N}$ be a (homogeneous) basis of L, dual to the basis $\{sx_\alpha\}$ of sQ:

$$\langle y_\beta, sx_\alpha \rangle = \delta_{\alpha\beta}.$$

In this notation we have:

THEOREM (4.2). Define on L a bracket $[\ ,\ ]$ and a quadratic operator q by extending the operations given on the basis $\{y_\beta\}$ by means of the following formulas:

$$[y_\beta, y_\alpha] = \sum_{\substack{\gamma \\ |x_\gamma| = |x_\alpha| + |x_\beta| + 1}} (-1)^{|y_\alpha|} c^\gamma_{\alpha\beta} y_\gamma$$

$$= -(-1)^{|y_\beta||y_\alpha|}[y_\alpha, y_\beta], \qquad \underline{\text{for}}\ \beta > \alpha;$$

$$[y_\alpha, y_\alpha] = 0 \qquad \underline{\text{for}}\ |y_\alpha|\ \underline{\text{even}}$$

$$[y_\alpha, y_\alpha] = 2q(y_\alpha) \qquad \underline{\text{for}}\ |y_\alpha|\ \underline{\text{odd}};$$

$$q(y_\alpha) = -\sum_{\substack{\gamma \\ |x_\gamma| = 2|x_\alpha| + 1}} c^\gamma_\alpha y_\gamma \qquad \underline{\text{for}}\ |y_\alpha|\ \underline{\text{odd}};$$

Then L becomes a graded Lie algebra over k, such that $\pi^*(F) \simeq L$ as graded Lie algebras.

Both theorems are proved in [19] (and announced, in a slightly different form, in [16]). We apply below Theorem (4.2) to a very simple example, where the results obtained have been known for a long time, but which will be necessary for the discussion in the next section.

EXAMPLE (4.3). Suppose R is a local noetherian ring whose Koszul complex (cf. (3.1)) satisfies $H(K^R) = \Lambda H_1(K^R)$. Let V be a minimal model for K^R, which exists by (4.1). Since $H(V) \simeq \Lambda H_1(K^R)$ a generator of V of minimal degree $m > 1$ will have to be a cycle, hence to be killed by a generator of degree $m + 1$, thus contradicting the minimality of V: one sees that $V \simeq \Lambda H_1(K^R)$ with the trivial differential. By (4.2) we conclude that $\pi^*(K^R)$ is the abelian Lie algebra $s(H_1(K^R))^\vee$, concentrated in degree 2, hence the exact sequence (3.2) becomes in this case:

$$0 \to s(H_1(K^R))^\vee \to \pi^*(R) \to s(\underline{m}/\underline{m}^2)^\vee \to 0.$$

In particular, $\pi^*(R)$ is concentrated in degrees 1 and 2, and its graded Lie algebra structure is completely determined by the quadratic operator

$$q : s(\underline{m}/\underline{m}^2)^\vee \to s(H_1(K^R))^\vee. \tag{4.4}$$

The local rings whose Koszul homology is the exterior algebra on its one-dimensional elements have been characterized by Assmus.

In order to state his theorem we recall two standard definitions from commutative algebra. A sequence of elements $x_1, \ldots, x_n \in R$ is called regular, if x_i is a non-zero divisor in $R/(x_1, \ldots, x_{i-1})R$ for $i = 1, \ldots, n$ (and $x_0 = 0$), and $(x_1, \ldots, x_n)R \neq R$. A local ring is called regular if its maximal ideal can be generated by a regular sequence. Finally, a deep result of I. S. Cohen says that every local ring, which is complete in the linear topology defined by choosing the powers of the maximal ideal for a basis of open neighbourhoods of 0, is the homomorphic image of a regular local ring R.

THEOREM (4.5) (Assmus [9]). The following conditions on a local noetherian ring R are equivalent:

(i) its completion (for the powers of its maximal ideal) is isomorphic to the quotient of a regular local ring R by an ideal generated by a regular sequence;

(ii) $H(K^R) \simeq \Lambda H_1(K^R)$;

(iii) $H_2(K^R) = [H_1(K^R)]^2$;

(iv) $\pi^3(R) = 0$;

(v) $\pi^i(R) = 0$ for $i \geq 3$.

(Note that (i) implies the inequality $\dim_k H_1(K^R) \leq \dim_k \underline{m}/\underline{m}^2$).

The rings satisfying the conditions of (4.5) are called (local) complete intersections, and have first been considered in connection with questions arising in algebraic geometry. From the point of view of the homological theory of local rings they have an exceptional status, which will be discussed in the next section. By now their homotopy Lie algebra is completely understood, and a result of Sjödin [63] shows, that any quadratic operator q from an n-dimension k-vector space to a space of dimension $r \leq n$ can be realized as the structure map (4.4) of $\pi^*(R)$ for a suitable complete intersection R.

5 - The sequence of Betti numbers of a module over a local ring

We now turn to the problems which have motivated the study of the homological invariants of commutative rings discussed above. In this section R denotes a (noetherian) local ring with maximal ideal $\underline{m}$ and residue field $k = R/\underline{m}$, and M is a finitely-generated R-module. It is an easy consequence of the Nakayama lemma that M has a resolution

$$0 \leftarrow M \leftarrow L_0 \leftarrow L_1 \leftarrow \cdots \leftarrow L_{i-1} \xleftarrow{d_i} L_i \leftarrow \cdots$$

by free modules of finite rank, such that $d_i(L_i) \subset \underline{m}L_{i-1}$ for $i \geq 1$; equivalently, the entries of the matrix of d_i in some (resp. any) bases of L_i and L_{i-1}, are all contained in $\underline{m}$. Moreover, it is classically known that such a resolution is unique up to a (non-canonical) isomorphism of complexes, hence one can define the i-th Betti number of M over R by the equality

$$b_i(M) = \operatorname{rank}_R L_i .$$

Note that by their very definition the Betti numbers are associated with a problem of linear algebra over R : $b_i(M)$ is the minimal number of generators for the module of solutions of the homogeneous linear system of equations

$$A_{i-1}\begin{pmatrix} X_1 \\ \vdots \\ X_{b_{i-1}} \end{pmatrix} = 0,$$

$A_{i=1}$ being the matrix of d_{i-1} $(i \geq 2)$.

The homological theory of modules over a local ring splits into two distinct parts. One one hand are the modules of finite projective dimension (cf. (2.6)). Since a minimal resolution factors out as a direct summand from any free resolution of M over R, this condition is equivalent to the equality $b_{i+1}(M) = 0$ for some i (resp. $b_j(M) = 0$ for all $j > i$ for some i). The smallest i with this property is called the projective dimension $\operatorname{pd}_R M$ of M, and equals the $\dim^R(k,M)$ of (2.6). Extensive qualitative and quantitative results are available on such modules. Let us quote three of the most famous:

(5.1) $$\operatorname{pd}_R(M) + \operatorname{depth}(M) = \operatorname{depth}(R),$$

where $\operatorname{depth}(M)$ denotes the length of a maximal sequence of elements, which is regular on M (compare (4.3)): this is the Auslander-Buchsbaum equality [10]. In the same paper is proved the inequality

(5.2) $$\textstyle\sum_{i\geq 0} (-1)^i b_i(M) \geq 0,$$

the equality holding if and only if M is annihilated by a non-zero divisor of R. Finally, setting $\operatorname{pd}_R M = d$, one has

(5.3) $$b_i(M) \geq 2i + 1 \quad \text{for} \quad 0 \leq i \leq d - 2,$$

$$b_{d-1}(M) \geq d - 1 + b_d(M):$$

this is the very recent Syzygy theorem of Evans and Griffith [28], which has yet to be extended to cover local rings of unequal characteristic; note also, that by the result of Bruns [23], the lower bounds in the Syzygy theorem can be simultaneous attained, over any local ring.

In order to explain in part the interest in the properties of modules of finite projective dimension, recall the fundamental Auslander-Buchsbaum-Serre theorem, which really started everything:

THEOREM (5.4). <u>The following conditions are equivalent</u>:

(i) R <u>is regular</u>;

(ii) $\mathrm{pd}_R M < \infty$ <u>for every finitely-generated module</u> M;

(iii) $\mathrm{pd}_R k < \infty$.

The regular local rings arise in algebraic (and analytic) geometry as the rings of germs of functions defined in some neighbourhood of a non-singular point. In the singular case the previous theorem focuses attention on the homological properties of the R-module k. Since a minimal resolution tensored with k is a complex with trivial differentials, one has that

$$b_i(M) = \dim_k (L_i \underset{R}{\otimes} k) = \dim_k \mathrm{Tor}_i^R(M,k).$$

In particular, with $M = k$ we see the dimensions of the graded components of the Hopf algebra $\mathrm{Tor}^R(k,k)$, dual to the universal envolope of $\pi^*(R)$, acquire a new significance. We write b_i for $b_i(k)$, e_i for $\dim_k \pi^i(R)$, and note the equality of formal power series

$$P_R(t) \overset{\mathrm{def}}{=\!=} \sum_{i\geq 0} b_i t^i = \frac{(1+t)^{e_1}(1+t^3)^{e_3}\cdots}{(1-t^2)^{e_2}(1-t^4)^{e_4}\cdots}, \tag{5.5}$$

which is a consequence of the Poincaré-Birkhoff-Witt theorem. Due to the work of Gulliksen we have rather good information on the properties of the Betti numbers of modules over a complete intersection (defined in (4.5)):

THEOREM (5.6) (Gulliksen [34], [35], [36]). <u>Let</u> R <u>be a local ring with</u> $\dim_k H_1(K^R) = r$. <u>The following are equivalent</u>:

(i) R <u>is a complete intersection</u>;

(ii) <u>for every finitely-generated</u> R-<u>module</u> M, $\sum_{i\geq 0} b_i(M)t^i = \dfrac{P_M(t)}{(1-t^2)^r}$ <u>for some polynomial</u> $P_M(t)$ <u>with integer coefficients</u>;

(iii) <u>for every</u> M <u>as in</u> (ii) <u>there is a polynomial</u> $q_M(t)$ <u>such that</u> $b_i(M) \leq q_M(i)$ <u>for</u> $i \geq 0$;

(iv) <u>there is a polynomial</u> $q_k(t)$ <u>such that</u> $b_i(k) \leq q_k(i)$ <u>for all</u> $i \geq 0$;

(v) $\pi^{2i}(R) = 0$ *for all sufficiently large* i;

(vi) $\dim_k \pi^*(R) < \infty$.

The implications (ii) ⇒ (iii) ⇒ (iv) are trivial. The implications (iv) ⇒ (v) ⇒ (vi) are proved in [33]. That (vi) implies (i) is established in [34]. Finally (i) ⇒ (ii) is a consequence of the much more general result of [35]. We now give a direct proof of this, inspired by Levin's proof of [45, Theorem 6.1].

Let P denote the universal enveloping algebra of $\pi^{\geq 2}(R)$. According to (4.4), since R is a complete intersection, this is the commutative polynomial algebra on r generators of degree 2. Thus (ii) is implied by (i) because of the classical result of Hilbert on the Hilbert (sic) series of graded modules over polynomial rings, applied via the following:

LEMMA (5.7). *Let* R *be a complete intersection. For a finitely-generated* M, $\mathrm{Ext}_R(M,k)$ $(\simeq \mathrm{Tor}^R(M,k)^\vee)$ *is a finitely-generated* P*-module, for the action induced by the inclusion* $P \subset \mathrm{Ext}_R(k,k)$ *and the Yoneda product*:

$$\mathrm{Ext}^j_R(M,k) \times \mathrm{Ext}^i_R(k,k) \to \mathrm{Ext}^{i+j}_R(M,k).$$

PROOF: For $M = k$ this is true since by the Poincaré-Birkhoff-Witt theorem $\mathrm{Ext}_R(k,k) \simeq P \otimes_k \Lambda\pi^1(R)$ as graded P-modules. For a module of finite length this follows by induction on $\ell(M)$ from the exact sequence of $\mathrm{Ext}_R(k,k)$-modules induced by applying $\mathrm{Ext}_R(-,k)$ to the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

with a submodule $N \neq 0,M$. Finally, for an arbitrary M this follows from the case of finite length, and the surjectivity of the map of $\mathrm{Ext}_R(k,k)$-modules:

$$\mathrm{Ext}_R(M/\underline{m}^s M,k) \to \mathrm{Ext}_R(M,k),$$

where s is a sufficiently large integer whose existence is guaranteed by the result of [13, Appendix].

The subject of minimal resolutions of modules over complete intersections has been taken up by Eisenbud in [27], where by an alternate approach he also recovers some of Gulliksen's results. One of the main theorems of his paper is

THEOREM (5.8) (Eisenbud [27]). *Let* R *be a complete intersection with* $n = \dim_k \underline{m}/\underline{m}^2$ *and* $r = \dim_k H_1(K^R)$. *Then if the finitely-generated* R*-module* M *has a bounded sequence of Betti numbers* $b_i(M)$, *its minimal free resolution becomes periodic of period* 2 *after at most* $m = n - r + 1$ *steps, i.e. for* $i \geq m$ *one has* $L_i = L_{i+2}$ *and* $d_i = d_{i+2}$.

In [27] one also finds the following:

CONJECTURE: If the finitely-generated module M over an arbitrary local ring R has a bounded sequence of Betti numbers, then its minimal free resolution becomes eventually periodic of period 2.

Since a resolution of period 2 has a stationary sequence of Betti numbers, this brings us to questions concerning the asymptotic properties of the $b_i(M)$. The principal one seems to be the

PROBLEM. Is the sequence $b_i(M)$ eventually non-decreasing for any finitely-generated module M over the local ring R?

In what follows we assemble the information available by now.

(5.9). The problem has a negative answer if one drops the word "eventually": in [27, §3] Eisenbud constructs - over any artinian complete intersection with $n = \dim_k \geq 2$ a finitely generated module M_h for every positive integer h, whose sequence of Betti numbers strictly decreases the first h steps.

(5.10). Ramras [56], and Gover with Ramras [33], have produced examples of rings over which the Betti numbers (strictly) increase, starting with b_1, for every finitely-generated module M. In [57] Ramras considers the following question a positive answer to which will follow from positive solutions to the conjecture and problem above:

QUESTION: Is it true that for an arbitrary finitely-generated module over a local ring R, there are only two possibilities: either the sequence $b_i(M)$ is eventually constant, or $\lim_i b_i(M) = \infty$?

(5.11). When R is a complete intersection, elementary manipulations with the formula (5.6 ii) show that the sequences $b_{2j+1}(M)$ and $b_{2j}(M)$ are each one given, for j sifficiently large, by a polynomial in j. Hence the question above has a positive answer, and more generally the odd and the even Betti sequences are each one eventually non-decreasing. However, insufficient information on the $P_M(t)$ of (5.6 ii) prevents us to compare two consecutive Betti numbers. The only exception is for $\dim_k H_1(K) = 1$, when it is known that the resolution of each module becomes periodic (of period 2) after n steps [27].

(5.12) Ghione and Gulliksen have shown [32] that over a Golod ring R the Poincaré series $\sum b_i(M)t^i$ of any module is of the form (integer polynomial in t).

$(1 = \sum \dim_k H_i(K^R)t^{i+1})^{-1}$; (cf. [37], [11], [44], [60] for the definition and many properties of Golod rings). This is essentially the only class of rings - besides the complete intersections - over which the rationality of the Poincaré series of an arbitrary module is known. The expression for this series strongly suggests a positive answer to the problem above, but not much is known about its numerator in order to make some conclusive steps.

(5.13). In contrast to the generally very incomplete picture, the asymptotic properties of the sequence $b_i(k)$ can be described very precisely, over an arbitrary R. This is the contents of the next section, and one of the main results announced in this paper.

6 - The homotopy Lie algebra and the Betti numbers of a local ring

By the usual abuse of language, we refer to the integers $b_i = b_i(k)$ as the Betti numbers of R. Recall that $e_1 = \dim_R \pi^1(R) = \dim_k \underline{m}/\underline{m}^2$, $e_2 = \dim_k H_1(K^R)$, and that the equality $e_2 = 0$ characterizes the regular local rings (defined in (4.3)): details can be found e.g. in [37]. A glance at the expression (5.5) for the Poincaré series of R now shows that:

(6.1). The sequence of Betti numbers $\{b_i\}$ is non-decreasing for $i \geq n + 1$ and for every local ring R; if furthermore R is not regular, then the sequence is non-decreasing for $i \geq 0$, and strictly increasing when $n > 1$.

The question of determining the rate of growth of this sequence was first raised in [14], where it was conjectured that it can be either polynomial or exponential. That this is in fact the case is shown by

THEOREM (6.2). For the sequence $\{b_i = b_i(k)\}$ over the local ring R the following dichotomy holds:

- either b_i is a polynomial in i, and this is possible if and only if R is a complete intersection,

- or there exist an integer N and real numbers $D > C > 1$ such that for each $i \geq N$ the double inequality

$$C^i < b_i \leq D^i$$

is satisfied.

An immediate corollary is the following result, conjectured independently by Golod and Gulliksen: cf. [14], [15]:

COROLLARY (6.3). <u>Let</u> r_R <u>denote the radius of convergence of the Poincaré series</u> $P_R(t) = \sum b_i t^i$. <u>Then the following possibilities occur</u>:

- $r_R = \infty$ <u>if and only if</u> R <u>is regular</u>;
- $r_R = 1$ <u>if and only if</u> R <u>is a non-regular complete intersection</u>;
- $0 < r_R < 1$ <u>in all other cases</u>.

REMARKS: In case R is the localization of a connected graded algebra over a field k of characteristic zero, the theorem (and its corollary) are immediate consequences of the result of Felix and Thomas, reproduced in (3.5). A proof for all local rings which contain the rational numbers (which works also in case the positive characteristic is "large" compared to the number of generators of $\underline{m}$), was then found by the author [16]. The details of the final answer reported above are contained in [19]. They require the establishment of certain algebraic properties of the minimal model of K^R (cf. Section 4). After that one can use the scheme of proof of the corresponding result for topological spaces (cf. the dictionary I in Section 3), which was recently obtained by Felix-Halperin-Thomas [30].

Let us now briefly consider the Lie algebra structure of $\pi^*(R)$. One of the first results on the subject was "negative": Roos [58] produced an example of R (with $\underline{m}^3 = 0$), whose homotopy Lie algebra is <u>not</u> finitely-generated. It should be noted that this construction was obtained as an algebraic parallel to Lemaire's example of a finite CW-complex X with $\pi_*(X) \otimes \mathbb{Q}$ not finitely-generated, which in turn was inspired by an example of Stallings in group theory... . In contrast to the extremely simple structure of $\pi^*(R)$ when R is a complete intersection, cf. Section 4, in [14] we advanced the following:

CONJECTURE: When R is <u>not</u> a complete intersection $\pi^*(R)$ contains a free graded Lie subalgebra on two generators.

(Note that an affirmative solution would imply Theorem (6.2).)

A main result of [15] (announced in [14]) is

THEOREM (6.4). <u>The conjecture holds for all rings with</u> $\dim_k \underline{m}/\underline{m}^2$ - depth $R \leq 3$ <u>and for Gorenstein rings with</u> $\dim_k \underline{m}/\underline{m}^2$ - depth $R = 4$.

In general, the best approximation to the conjecture is the fact that $\pi^*(R)$ cannot be nilpotent when R is not a complete intersection: cf. the note by the author and Halperin in these proceedings.

An old question in the homology theory of local rings asks whether $\pi^i(R) \neq 0$ for all $i \geq 1$ when R is not a complete intersection (cf. [37, p. 154] and

[15, Conjecture 3]).

New results have been reported at the conference. In the joint paper with Halperin (loc. cit.) we prove the answer is "yes for i sufficiently large" (in the graded, characteristic zero case this is already in [31]), and it is positive for all $i \geq 0$ when R is as in (6.4). On the other hand, Löfwall (these Proceedings) obtains a positive answer under the assumption that $\underline{m}^3 = 0$.

7 - Attaching the top cell to a formal manifold

In this section we exploit the relation between formal spaces and skew-commutative graded algebras, outlined in Section 3.II, in order to express the rationalized homotopy Lie algebra of a formal n-dimensional manifold M in terms of the homotopy Lie algebra of the manifold with boundary $M \setminus \mathring{D}^n$, obtained by cutting out from M a (sufficiently small) n-dimensional open disc.

The example of formal manifolds of prime geometric interest is given by

(7.1) any 1-connected compact Kähler manifold is formal [26].

Besides, one also has multiple occurrences of formality from:

(7.2) any p-connected compact manifold of dimension $\leq 4p + 2$ is formal [50];

(7.3) products (obvious) and connected sums [29] of formal manifolds are formal;

(7.4) 1-connected manifolds whose rational cohomology algebra is generated by a single element are formal.

In view of our main result the last case requires a separate consideration. The very easy arguments show that two cases occur:

(7.4a) M is rationally a homotopy sphere S^n $(n \geq 2)$. Then $M \setminus \mathring{D}^n$ has the rational homotopy of a point, and $\pi_*(M) \otimes Q$ is the free Lie algebra on the single generator $\beta : S^{n-1} \to \Omega S^n$ given by the adjunction map.

(7.4b) Let $P_{2m,r}$ denote a space whose rational cohomology ring is isomorphic to $k[u]/(u^{r+1})$, where $|u| = 2m > 0$ and r is an integer ≥ 2. In particular, $P_{2m,r}$ has the cohomology of a manifold of dimension $n = 2mr \geq 4$, and $P_{2,r}$ can be the projective space $\mathbb{C}\mathbb{P}^r$. Such a space is intrinsically formal, in the sense that any space with an isomorphic cohomology algebra is rationally homotopy equivalent to it.

In particular, $P_{2m,r} \setminus \mathring{D}^{2mr}$ is rationally homotopy equivalent to $P_{2m,r-1}$ when $r > 2$, and to S^{2m} when $r = 2$. In both cases $\pi^*(\Omega P_{2m,r}) \otimes \mathbb{Q}$ is the abelian Lie algebra on 2 generators v, w with $|v| = 2m - 1$ and $|w| = 2m(r+1) - 2$.

In the formulation of the main result we use the notation $\overset{b}{\vee} S^n$ to denote the wedge of b n-spheres.

THEOREM (7.5). Let M be a compact oriented (1-connected) formal manifold of dimension n. For a disc $D^n \subset M$ denote by i the inclusion of $M \setminus \overset{\circ}{D}{}^n$ into M, by T^i the corresponding mapping fibre, and by $j : T^i \to M \setminus \overset{\circ}{D}{}^n$ the canonical map.

When M is rationally homotopy equivalent to S^n or $P_{2m,r}$ for some n, $2m, r \geq 2$, the homotopy of the inclusion i is given by the remarks (7.4a) and (7.4b).

Otherwise $n \geq 4$, and the following hold:

(1) $M \setminus \overset{\circ}{D}{}^n$ is a formal space;

(2) T^i has the rational homotopy type of the wedge of spheres

$$\bigvee_{i \geq 0} \left[\overset{b_i}{\vee} S^{i+n-1} \right]$$

where $b_i = \dim_{\mathbb{Q}} H^i(\Omega M, \mathbb{Q})$ for $i \geq 0$;

in particular, $\pi_*(\Omega T^i) \otimes \mathbb{Q}$ can be identified with the free graded Lie algebra on the (n-2)-fold suspension of $H_*(\Omega M, \mathbb{Q})$;

(3) there is an exact sequence of graded Lie algebras

$$0 \to \pi_*(\Omega T^i) \otimes \mathbb{Q} \xrightarrow{j_*} \pi_*(\Omega(M \setminus \overset{\circ}{D}{}^n)) \otimes \mathbb{Q} \xrightarrow{i_*} \pi_*(\Omega M) \otimes \mathbb{Q} \to 0;$$

(4) the image of j_* is the ideal of $\pi_*(\Omega(M \setminus \overset{\circ}{D}{}^n)) \otimes \mathbb{Q}$, generated by the class of

$$S^{n-2} \xrightarrow{\beta} \Omega S^{n-1} \xrightarrow{\Omega(i|_{\partial D^n})} \Omega(M \setminus \overset{\circ}{D}{}^n);$$

(5) there is an equality of formal power series

$$P_{\Omega M}(t)^{-1} = P_{\Omega(M \setminus \overset{\circ}{D}{}^n)}(t)^{-1} + t^{n-2}$$

where $P_{\Omega X}(t) = \sum_{i \geq 0} \dim H^i(\Omega X, \mathbb{Q}) t^i$.

REMARKS: (1) At the conference I learned about the existence of Stasheff's paper [65], whose main result asserts that, even without formality hypotheses, the rational homotopy type of M is determined by that of $M \setminus \overset{\circ}{D}{}^n$. Thus, the main contents of the previous theorem can be viewed as an explicit description of how does that determination occur for a formal M, at the level of the rational homotopy Lie algebras.

(2) The proof of the theorem will be published elsewhere. It involves a reduction to an algebraic problem similar to the one treated by Levin and the author in [46]; some of the algebraic computations of that paper are then used.

As an <u>example</u> of application of the theorem we recover - through immediate specialization - the main results of [53] and [20] on the rational homotopy of complete intersections. Indeed, by (7.1) the theorem is applicable to complex projective algebraic varieties. When M moreover is a complete intersection of complex dimension $r \geq 2$, it is simply connected and the Lefschetz theorem explicits the structure of the algebra $H^*(M,\mathbb{Q})$. Namely, $H^*(M,\mathbb{Q}) \simeq k[u]/(u^{r+1}) \oplus V$ as vector spaces, where u is the Kähler form ($|u| = 2$), and V is a d-dimensional vector space, concentrated in degree r; the multiplication is described by letting the powers of u multiply in the usual way, $uV = Vu = 0$, and $v_1v_2 = \phi(v_1,v_2)u^r$ for some non-degenerate bilinear form ϕ (which is symmetric or alternating depending on the parity of r). Clearly $M\setminus \mathring{D}^{2r}$ has the cohomology, hence by (7.5.1) also the rational homotopy Lie algebra, of the wedge $(\overset{d}{\vee} S^r) \vee \mathbb{C}\mathbb{P}^{r-1}$ Since the rational homotopy Lie algebra of a wedge is the wedge (= free product = coproduct) of the respective Lie algebras, and the Hilbert series of the universal envelope of $L_1 \vee L_2$ is given by the formula $P^{-1}_{U(L_1 \vee L_2)} = P^{-1}_{UL_1} + P^{-1}_{UL_2} - 1$ (cf. [25]), we readily get (in the nontrivial case when $d \neq 0$, otherwise cf. (7.4)):

COROLLARY (7.6) (Neissendorfer [53]). <u>Let</u> L_1 <u>denote either the free Lie algebra on one generator of degree</u> 1, <u>in case</u> $r = 2$, <u>or the abelian Lie algebra on two generators of degrees</u> 1 <u>and</u> $2(r-1)$, <u>when</u> $r > 2$. <u>Let</u> L_2 denote the free Lie <u>algebra on</u> d <u>generators of degree</u> $r - 1$. <u>Then</u> $\pi_*(\Omega M) \otimes \mathbb{Q} \simeq (L_1 \vee L_2)/I$, <u>where</u> I <u>is an ideal generated by a single element of degree</u> $2(r-1)$.

(In fact, an explicit generator for I is given in [53], and it coincides with the one described in (7.5.4)).

COROLLARY (7.7) (Babenko [20]).

$$P_{\Omega M}(t)^{-1} = \sum_{i=0}^{2(r-1)}(-1)^i t^i - dt^{r-1}.$$

PROOF: Use $P_{\Omega \overset{d}{\vee} S^r}(t) = (1 - dt^{r-1})^{-1}$, $P_{\Omega\mathbb{C}\mathbb{P}^{r-1}}(t) = (1+t)\cdot(1 - t^{2(r-1)})^{-1}$, the previous discussion, and (7.5.5).

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