Astérisque

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Astérisque, tome 113-114 (1984), p. 109-117

<http://www.numdam.org/item?id=AST_1984__113-114__109_0>

THE STRUCTURE OF $\pi_*(\Omega S)$

by

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1 . INTRODUCTION : In this lecture S will always denote a simply connected CW complex with finitely many cells in each dimension. Associated with S are the two algebraic invariants :

(i) Its cohomology, $H^*(S)$,

and

(ii) The homotopy of the loop space, $\pi_*(\Omega S)$.

These are both graded groups each of which carries additional structure : $H^*(S)$ is a graded commutative (associative) algebra and $\pi_*(\Omega S)$ is a graded Lie algebra, the homotopy Lie algebra for S .

These two invariants are Eckmann-Hilton dual to each other, and play symmetric roles in the two major approaches to rational homotopy theory. At a deeper level, however, the duality breaks down. A simple instance of this is the enormous difference between free graded commutative associative algebras and free graded Lie algebras ; the latter have a very much richer product structure. This can be seen, in particular, from the fact that for graded Lie algebras the subobject of a free object is again free. There is no analogous result in the other category.

Let me recall how the Lie algebra $\pi_*(\Omega S)$ is defined. Of course $\pi_p(\Omega S) \cong \pi_{p+1}(S)$ is the group of homotopy classes of base point preserving continuous maps $S^p \to \Omega S$, with the standard addition. If $f : S^p \to \Omega S$, $g : S^q \to \Omega S$ then the map $S^p \times S^q \to \Omega S$ given by

$$(x,y) \longmapsto f(x)g(y)f(x)^{-1}g(y)^{-1}$$

is null homotopic on $S^p \vee S^q$ and hence defines a map

$$[f,g] \ : S^{p+q} = S^p \times S^q/_{S^p \vee S^q} \longrightarrow \Omega S$$

This is the Lie bracket.

A theorem of Serre guarantees that $\pi_p(S)$ is a finitely generated (abelian) group for each p . Hence $\pi_*(\Omega S) \otimes \mathbb{Q}$ is a graded connected rational Lie algebra of finite type (finite dimensional in each degree). It is the rational homotopy Lie algebra of S . One the first results in rational homotopy theory was the remarkable theorem of Quillen [Q] : every graded connected Lie algebra over $\mathbb{Q}$ of finite type arises in this way.

Here I will be concerned with the following question, and variations thereof.

PROBLEM 1 : What conditions are imposed on the rational homotopy Lie algebra of S if S is a finite complex.

This may be regarded as an analogue of the well known

PROBLEM 1' : What conditions are imposed on a discrete group G if $K(G,1)$ is a finite complex ?

Now let me restate problem 1, with its variations.

PROBLEM : What conditions are imposed on the rational homotopy Lie algebra of S if

1. S is a finite complex.

or

2. dim $H^*(S ; \mathbb{Q}) < \infty$.

or

3. S is a closed manifold.

or

4. S has finite rational category : $\mathrm{cat}_o(S) < \infty$.

The restrictions on S in problems 1 and 2 are equivalent (for this problem), the restriction in 3 is stronger while that in 4 is weaker.

I include problem 4 because almost all the results we have up to now are answers to it (which then apply to the other problems) ; shortly I will attempt to explain why.

As far as problem 3 is concerned, it is known that with the exception of the spheres a manifold cannot have a free rational homotopy Lie algebra. I am unaware of any other restrictions which do not also hold for finite complexes.

As to problems 1 and 4 we have available the beautiful

<u>Conjecture</u> (Avramov-Felix). If $\mathrm{cat}_o(S) < \infty$ then $\pi_*(\Omega S) \otimes \mathbb{Q}$ contains a free Lie algebra with at least two generators.

Henceforth I shall always assume $\mathrm{cat}_o(S) < \infty$, and attempt to survey known results on $\pi_*(\Omega S) \otimes \mathbb{Q}$. Let us denote the integers dim $\pi_p(S) \otimes \mathbb{Q}$ by $\rho_p(S)$ and call them the <u>Hurewicz numbers for</u> S . Results fall into three classes :

(i) Restrictions on the $\rho_p(S)$.

(ii) Restrictions on the Lie structure.

(iii) Spaces of low category.

Before beginning the survey, however, it seems reasonable to recall the definition of $cat_o(S)$ and explain its role here.

2 . <u>THE ROLE OF RATIONAL CATEGORY.</u> The <u>rational category</u> of S is the Lusternik-Schnirelmann category of the localization $S_{\mathbb{Q}}$, normalized so that $cat_o(\text{point}) = 0$. It is majorized by the L-S category of S and by the largest n such that $H^n(S : \mathbb{Q}) \neq 0$.

Its usefulness stems from the result of Felix-Halperin [F - H] that if $\varphi : S \to T$ induces an injection $\pi_*(S) \otimes \mathbb{Q} \xrightarrow{\varphi_\#} \pi_*(T) \otimes \mathbb{Q}$ then $cat_o(S) \leq cat_o(T)$. This implies in particular that in any fibration $S_F \longrightarrow S \xrightarrow{p} S_B$ in which $p_\#$ is surjective, $cat_o(S_F) \leq cat_o(S)$.

<u>CONJECTURE</u> : If $2 \leqq cat_o(S) < \infty$ then there exists such a fibration with

$$1 \leq cat_o(S_F) < cat\ (S) \quad .$$

This conjecture implies the Avramov-Felix conjecture.

<u>REMARK</u> : An unpublished result of Felix-Halperin-Thomas asserts the existence (if $\dim \pi_*(S) \otimes \mathbb{Q} = \infty$ and $\dim H^*(S ; \mathbb{Q}) < \infty$) of a Postnikov decomposition $S_F \to S \to S_B$ in which $\dim H^*(S_F ; \mathbb{Q}) = \infty$!

3 . <u>RATIONALLY ELLIPTIC SPACES</u> : There is a profound difference in the behaviour of S of finite rational category depending on whether $\dim \pi_*(S) \otimes \mathbb{Q}$ is finite or infinite. In the first case S is called <u>rationally elliptic</u> and according to [F-H]

$$\dim H^*(S;\mathbb{Q}) < \infty \quad \text{and} \quad cat_o(S) \geqq \dim \pi_{odd}(S) \otimes \mathbb{Q} \quad .$$

Furthermore [H] , the algebra $H^*(S ; \mathbb{Q})$ must satisfy Poincaré duality, and the degree n , of the fundamental class is given by

$$n = \sum_{p\ odd} p\, \rho_p - \sum_{p\ even} (p-1)\rho_p \quad .$$

Friedlander and Halperin [Fr-H] have completely solved the problem of characterizing the Hurewicz numbers of rationally elliptic spaces. Indeed let $f(t) = \sum_{i=1}^{r} t^{2a_i} + \sum_{j=1}^{q} t^{2b_j-1}$ be any polynomial with non negative integral coefficients and zero constant and linear terms. Then $f(t) = \sum \rho_p(S)t^p$ for rationally elliptic S if and only if for each s and each $i_1 < \dots < i_s \le r$ there exist $j_1 < \dots < j_s \le q$ and $k_{\nu\mu} \in \mathbb{Z}$ such that

$$k_{\nu\mu} \geq 0 \quad , \quad \sum_{\mu=1}^{s} k_{\nu\mu} \geq 2 \quad , \quad \nu = 1,\dots,s, \quad \text{and}$$

$$b_{j_\nu} = \sum_{\mu=1}^{s} k_{\nu\mu}\, a_{i_\mu} \;, \quad \nu = 1,\dots,s \;.$$

In particular, setting $s = r$ one sees that

$$\chi_\pi \stackrel{\text{def}}{=} \sum (-1)^p \rho_p = r-q \leq 0 \;.$$

They also deduce the relations

$$\dim \pi_*(S) \otimes \mathbb{Q} \leq \sum_{p \text{ even}} \rho_p \cdot p + |\chi_\pi| \leq n$$

and

$$\sum_{p \text{ odd}} \rho_p(p+1) \leq 2n \;.$$

Since [H] the largest p for which $\rho_p \neq 0$ is odd it follows that

$$\rho_p = 0 \;, \quad p \leq 2n \quad \text{and} \quad \sum_{p=n}^{2n-1} \rho_p \leq 1 \;.$$

Finally let me mention the inequality

$$\dim H^*(S) \leq 2^n$$

As to the Lie structure, one sees trivially that the Lie algebra is nilpotent because $\dim \pi_*(S) \otimes \mathbb{Q} < \infty$. It can in fact easily be abelian, and there does not seem to be any reasonable structure theorem.

4 . <u>RATIONALLY HYPERBOLIC SPACES</u>. If $\mathrm{cat}_0(S) < \infty$ and $\dim \pi_*(S) \otimes \mathbb{Q}$ is infinite, S is called <u>rationally hyperbolic</u>. The justification for this is the result of Felix-Halperin-Thomas [F-H-T] .

<u>THEOREM</u> : If S is rationally hyperbolic there exists an infinite sequence $p_1, p_2, \ldots$ with $p_{i+1} = \ell_i p_i - 1$ (ℓ_i an integer in $[2, \mathrm{cat}_0(S)+1]$) and there is a constant $C > 1$ such that

$$\rho_{p_i}(S) \geq C^{p_i} .$$

Let R_S denote the radius of convergence of the series $\Sigma \rho_i(S) t^i$:

$$\frac{1}{R_S} = \limsup_{p \to \infty} \rho_p^{1/p}$$

This theorem then implies that $R_S < 1$. Indeed, if $m = \mathrm{cat}_0(S)$ and $e = \left(\frac{1}{2(m+1)}\right)^{m+1}$ it follows from [F H T] that

$$\frac{1}{R_S} \geq (e\rho_p)^{1/p} \quad , \quad \text{all } p .$$

Suppose now that $H^p(S ; \mathbb{Q}) = 0$, $p > n$. A result of Babenko [B] shows that R_S is the radius of convergence of the Poincaré series $\sum \dim H^p(\Omega S;\mathbb{Q})t^p$ for ΩS . It can also be shown that there is a constant $C_n > 1$, depending only on n such that

$$\frac{1}{R_S} \geq C_n .$$

Finally in [F-T] Felix and Thomas give a lower bound for $\frac{1}{R_S}$ for a large class of spaces S , including all formal spaces with $\dim H^*(S;\mathbb{Q}) < \infty$: $R_S \leq r$ where r is the least modulus of the roots of $\sum \dim H^p(S;\mathbb{Q})t^p = 0$.

5 . <u>LIE STRUCTURE FOR RATIONALLY HYPERBOLIC SPACES.</u> Suppose S is rationally hyperbolic. As we have just seen this implies that the integers $\dim \pi_{2k}(\Omega S) \otimes \mathbb{Q}$ are unbounded. Thus the following theorem of Felix-Halperin-Thomas [F-H-T] guarantees the existence of enormous numbers of non zero brackets in the rational homotopy Lie algebra.

<u>THEOREM</u> : Suppose $\mathrm{cat}_o(S) = m$ and $\dim \pi_*(S) \otimes \mathbb{Q} = \infty$. If $\alpha_1, \ldots, \alpha_m \in \pi_{2k}(\Omega S) \otimes \mathbb{Q}$ are linearly independent then either the α_i generate an infinite dimensional sub lie algebra, or for some $\beta \in \pi_*(\Omega S) \otimes \mathbb{Q}$ and some i , $1 \leq i \leq m$, $(\mathrm{ad}\,\alpha_i)^q \beta \neq 0$, for all q .

<u>COROLLARY</u> : A space of finite category and finite cocategory is rationally elliptic.

For any (graded) Lie algebra L , its upper central series is the increasing sequence $Z^{(i)}$ of ideals in L in wich $Z^{(o)} = 0$ and $Z^{(i+1)}$ projects to the centre of $L/Z^{(i)}$. Put $\tilde{Z} = \bigcup_i \tilde{Z}^{(i)}$. The théorem above implies the

<u>COROLLARY</u> : if $Z(S) = {}_k\tilde{Z}_k(S)$ is associated with the Lie algebra $\pi_*(\Omega S) \otimes \mathbb{Q}$ where $\mathrm{cat}_o(S) = m$ and $\dim \pi_*(S) \otimes \mathbb{Q} = \infty$ then

$$\dim \tilde{Z}_{2k}(S) < m \quad , \text{ all } k \quad .$$

If S is π-formal it then follows that $\dim \tilde{Z}_{even}(S) \leqq m$ and $\dim \tilde{Z}(S) < \infty$; it seems reasonable to make the

<u>CONJECTURE</u> : If $\dim \pi_*(S) \otimes \mathbb{Q} = \infty$ and $\mathrm{cat}_o(S) < \infty$ then

$$\dim \tilde{Z}(S) < \infty \quad .$$

Finally, from F H T we have the

THEOREM : If $cat_o(S) < \infty$ and $\dim \pi_*(S) \otimes \mathbb{Q} = \infty$, then the Lie algebra $\pi_*(\Omega S)$ $\mathbb{Q}$ is not solvable.

6 SPACES OF LOW CATEGORY : A well known result going back to Toomer T asserts that $cat_o(S) = 1$ if and only if $\pi_*(\Omega S) \otimes \mathbb{Q}$ is a free graded Lie algebra. One possible attack on the conjectures is thus by induction on $cat_o(S)$

In fact by a collection of ad hoc techniques the Avramov-Felix conjecture has been established when $cat_o(S) = 2$ and S is not π-formal (F-H-T') . It is unclear how to proceed when $cat_o(S) = 3$.

7 . QUANTITATIVE RESULTS : When $H^p(S ; Q) = 0$, $p > n$ it should be possible to obtain estimates in terms of n for the size of the ρ_p and for the location of non-trivial Lie brackets. For instance it is shown in [F-H] that for some N ,

$$\sum_{p=k+1}^{k+n} \rho_p \geq 1 \quad , \text{ if } k \geq N$$

when S is rationally hyperbolic .

Felix has conjectured that this should be true for all $N \geq n$. It can in fact be shown that for rationally hyperbolic S

$$\sum_{p=k+1}^{nk} \rho_p \geq 1 \quad , \ k \geq 1 \ .$$

and it is this fact which gives the estimate $1/R_S \geq C_n > 1$ referred to in sec.4.

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