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BIFURCATIONS OF GRADIENT VECTORFIELDS

by

Gert VEGTER

Introduction.

In this paper we consider the connection between unfoldings of gradientvectorfields and unfoldings *) of the corresponding potential functions. Our unfoldings will be within the world of all gradientvectorfields.
First consider a corank one singularity $f: (\mathbb{R}^n,0) \longrightarrow \mathbb{R}$ of finite codimension, i.e. with a finite number of parameters in its universal unfolding $\{f_\mu | \mu \in \mathbb{R}^k\}$. It has been proven in [11] that a universal unfolding of the singularity $X = \text{grad}_g f$ is the family $\{\text{grad}_g f_\mu\}$, whatever Riemannian metric g one takes (provided one restricts to a sufficiently small neighbourhood of $0 \in \mathbb{R}^n$).
This relation between unfoldings of (germs of) gradientvectorfields and the corresponding potential functions breaks down if the corank of the singularity is greater than one. In that case even the topological type of the gradient-singularity may change if the metric varies over all Riemannian metrics (c.f. [4],[7]).
In [2] John Guckenheimer considers (on a neighbourhood of $0 \in \mathbb{R}^2$) the potential function $f(x,y) = \frac{1}{3}(x^3 + y^3)$ and the standard Riemannian metric $g = dx \otimes dx + dy \otimes dy$. A universal unfolding of f is the well-known three parameter family $f_{u,v,w}(x,y) = \frac{1}{3}(x^3 + y^3) + wxy + ux + vy$. However, the three parameter family $\{\text{grad}_g f_{u,v,w}\}$ is not a universal unfolding of $\text{grad}_g f$.
The reason for this is the absence of gradientvectorfields with saddle connections in the family $\{\text{grad}_g f_{u,v,w}\}$, while on the other hand it is easy to perturb the singularity $\text{grad}_g f$ within the class of gradientvectorfields in such a way that one obtains a saddle connection in any arbitrarily small neighbourhood of $0 \in \mathbb{R}^2$.
According to Guckenheimer this example shows that it is not justified to assume - as René Thom did [10] - that one can pass from the bifurcation of gradient dynamical systems to the unfoldings of their potential functions in studying catastrophes. This is obvious in some cases, since stable functions (Morse functions with distinct critical levels) may give rise to gradientvectorfields exhibiting saddle connections. However, in these cases <u>global</u> conditions (transversality of stable and unstable manifolds) are not satisfied.
Guckenheimer's example intends to show that <u>local</u> conditions, that guarantee stability of the unfolding of the germ of the potential function, may not be sufficient to guarantee stability of the corresponding unfolding of the gradientvectorfield.

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*) A family $\{f_\mu | \mu \in \mathbb{R}^k\}$ is called an unfolding of f if $f_o = f$. In appendix A we give another definition.

However, if one considers the Riemannian metric
$g_z = dx \otimes dx + z\, dx \otimes dy + z\, dy \otimes dx + dy \otimes dy$, we shall prove (theorem II.1) that (for $|z_o|$ small but positive) the three parameterfamily $\{grad_{g_{z_o}} f_{u,v,w}\}$ (z_o fixed) is an almost universal unfolding of $grad_{g_{z_o}} f$.
Moreover, the four parameter family $\{grad_{g_z} f_{u,v,w}\}$ (with parameters u,v,w,z) is an almost universal unfolding of $grad_g f$. We conjecture that the unfolding is even universal in both cases.
Crucial for our attack of the problem is the study of unfoldings of gradient-vectorfields, possessing a so called generalized saddle connection. Using the method of blowing up we obtain a gradientfamily, which is simpler than the family $\{grad_{g_z} f_{u,v,w}\}$, due to the fact that its singularities have corank at most 1.
As was remarked earlier the local unfoldings of this type of singularity are well understood. But the family we obtain after blowing up possesses instabilities of another kind, namely non-transversal intersections (i.e. coincidence, in dimension two) of (strong) stable and (strong) unstable manifolds. In order to deal with these bifurcations we first study this phenomenon in a slightly more general setting in section one. A main tool for the determination of a universal unfolding of saddle connections is the concept of strong contact equivalence. This extension of the theory of "normal" contact equivalence is developped in appendix A.
In section II we state and prove the main theorem, partially based on the results of section I. We end this section with some questions.
Part of the proof of the theorem of section II consists of checking the genericity of the blown-up family. This is the subject of appendix B.

Acknowledgements

I would like to thank Floris Takens for many stimulating discussions and for his conjecture that the metric in [2] might be non-generic. I am also indebted to Henk Broer for making some useful remarks.

I. Unfoldings of generalized saddle connections.

As stated in the introduction we first consider vectorfields on a two dimensional manifold, which have singularities of corank not greater than one.

Definition I.1:

A singularity p of a vectorfield X is called quasi-hyperbolic of type (1,k) ($k \geq 2$) if there is a locally X-invariant, one dimensional C^r-manifold W^c ($r > k$) such that:

(i) T_pW^c is the kernel of the linear part DX_p of X at p

(ii) DX_p has no purely imaginary eigenvalues

(iii) There is a local C^r-coordinate x on W^c such that
$X|W^c = x^k F(x) \frac{\partial}{\partial x}$, with $F(0) \neq 0$.

The existence of a centermanifold W^c follows from conditions (i) and (ii), cf. [3]. Centermanifolds are not unique. We can arrange that r is arbitrarily high, taking W^c small enough. Since two C^r centermanifolds have contact of order r at p, the degree of degeneracy k, appearing in condition (iii), is well defined. If p is a quasi-hyperbolic singularity of X, there are X-invariant manifolds W^{ss} and W^{uu}, containing p, such that the real parts of the eigenvalues of $DX_p|T_pW^{uu}$ and $DX_p|T_pW^{ss}$ are positive and negative resp. These manifolds, the strong-unstable and strong-stable manifolds, are locally unique, cf. [3]. In our two dimensional case they are one dimensional. We shall refer to the components of $W^{ss}(p) \setminus \{p\}$ as the strong (un-)stable separatrices of p.

In this section we shall be concerned with a vectorfield X_o which has two quasi-hyperbolic singularities p and q of type (1,k) and (1,ℓ) resp., such that a strong unstable separatrix of p coincides with a strong stable separatrix of q. This separatrix γ of X_o will be called a generalized saddle connection (see fig. I.1)

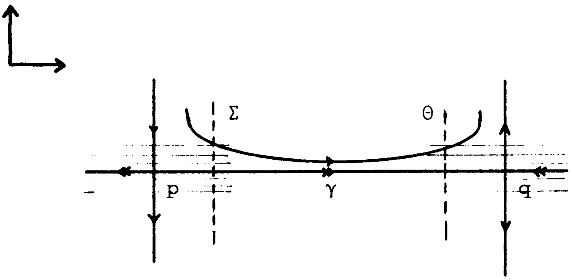

Fig.I.1
A generalized saddle connection

Remark.
Our considerations also apply - with minor modifications - to the cases where at least one of the singularities p and q is a hyperbolic saddle. Both singularities have a one dimensional stable and unstable manifold, whose tangent spaces at the singularity are the eigenspaces of the linear parts corresponding to the negative and positive eigenvalue, respectively. A saddle connection between p and q ~~its~~ is an orbit of X, which is contained in the unstable / stable manifold of p and the stable / unstable manifold of q. This situation occurs in generic one parameter families of vector fields, as we shall see presently (cf. fig. I.2.i).

The main result of this section deals with unfoldings of X_o in a neighbourhood of γ, i.e. a family $\{X_\mu | \mu \in \mathbb{R}^k\}$, such that $X_{\{\mu=0\}} = X_o$, and the mapping $(\mu,x) \to X_\mu(x)$ is defined on a neighbourhood of $\{0\} \times \gamma$ in $\mathbb{R}^k \times M$.
Since we want to relate properties like (uni-) versality of families of potential functions and of the corresponding gradient families we have to carry over these concepts to families of vectorfields (cf. [1]).
First recall that two families $\{X_\mu | \mu \in \mathbb{R}^k\}$ and $\{Y_\mu | \mu \in \mathbb{R}^k\}$, depending continuously on the parameter μ, are called topologically equivalent if there is a family $\{H_\mu\}$ of homeomorphisms, also depending continuously on μ, such that H_μ is a topological equivalence between X_μ and Y_μ. If we consider unfoldings of X_o along an orbit γ, the domain of the family $\{H_\mu\}$ should also be restricted to a neighbourhood of γ.
If h: $(\mathbb{R}^\ell,0) \longrightarrow (\mathbb{R}^k,0)$ is a continuous mapping and $\{X_\mu | \mu \in \mathbb{R}^k\}$ is a family of vectorfields, then we define the induced family h^*X to be the ℓ-parameter family with $(h^*X)_\nu = X_{h(\nu)}$. An unfolding $\{X_\mu\}$ of X_o is called versal if any other unfolding of X_o is equivalent to an unfolding induced by $\{X_\mu\}$, and universal if it is a versal family with a minimal number of parameters.
In the sequel we won't always succeed completely in proving universality for a given family. This lack of success will be camouflaged by introduction of the term almost universal; the adjective almost means that the topological equivalence between two families does only depend continuously on the parameter outside the origin of the parameter space.

Next we associate a pair of k-parameter families of real valued functions with any k-parameter unfolding $\{X_\mu | \mu \in \mathbb{R}^k\}$ of X_o.
These functions will be introduced in terms of a kind of normal form coordinates for X_o in a neighbourhood of the singularities p and q. This pair of functions will determine the equivalence class of the unfolding $\{X_\mu\}$ completely.
According to [8] there are local C^q coordinates (q arbitrarily high) $\mu_1,\ldots,\mu_r,x,y$ on a neighbourhood U of (0,p) in $\mathbb{R}^r \times M$ and $\mu_1,\ldots,\mu_r,\bar{x},\bar{y}$ on a neighbourhood V of (0,q) in $\mathbb{R}^r \times M$ such that:

(i) $X|U = X_1(\mu_1,\ldots,\mu_r;\, x)\,\frac{\partial}{\partial x} + X_2(\mu_1,\ldots,\mu_r;\, x)\; y\,\frac{\partial}{\partial y}$, where X_1 is regular of order k *) at $x = 0$, $\mu_1 = \ldots = \mu_r = 0$ and $X_2(0,\ldots,0;0) > 0$.

(ii) $X|V = Y_1(\mu_1,\ldots,\mu_r,\bar{x})\frac{\partial}{\partial\bar{x}} + Y_2(\mu_1,\ldots,\mu_r;\bar{x})\,\bar{y}\frac{\partial}{\partial\bar{y}}$,

where Y_1 is regular of order ℓ at $\bar{x} = 0, \mu_1 = \ldots = \mu_r = 0$ and $Y_2(0,\ldots,0;0) < 0$.

Remark:
In general we cannot take $q = \infty$, cf. [6].

Let X be the vectorfield on $\mathbb{R}^r \times M$ defined by $X(\mu,m) = X_\mu(m)$. As local C^q centermanifolds for X in $\mathbb{R}^r \times M$ we take:

$$W^c(0,p) = U \cap \{y=0\} \text{ and } W^c(0,q) = V \cap \{\bar{y} = 0\}.$$

Note that $W^c_{\bar{\mu}}(0,p) := W^c(0,p) \cap \{\mu = \bar{\mu}\}$ is an invariant manifold for $X_{\bar{\mu}}$, containing all singularities of $X_{\bar{\mu}}$ in $U_{\bar{\mu}}$. Let $\Sigma \subset U$ be a smooth transversal section for X, such that $\gamma \cap \Sigma \neq \emptyset$. Taking Σ smaller if necessary, we may assume that there is a positive real number σ such that $\Theta := D_{X,\sigma}(\Sigma)$ is again a transversal section for X, contained in V. **)
Note that U and V are foliated by the leaves $\{x = \text{const.}, \mu = \text{const.}\}$ and $\{\bar{x} = \text{const.}, \mu = \text{const.}\}$ resp. These partial foliations are locally X-invariant. Projection along their leaves yield locally defined diffeomorphisms $i: W^c(0,p) \longrightarrow \Sigma$ and $\pi: \Theta \longrightarrow W^c(0,q)$. These mappings may be considered as r-parameter families of one dimensional diffeomorphisms. The diffeomorphism $P := \pi \circ D_{X,\sigma} \circ i: W^c(0,p) \longrightarrow W^c(0,q)$ is defined on a neighbourhood of $(0,p)$ in $W^c(0,p)$. Observe that P is of the form $P(\mu,x) = (\mu,P_\mu(x))$ and $P(0,p) = (0,q)$. With the aid of P we introduce the C^q mappings $f,g: W^c(0,p) \longrightarrow \mathbb{R}$, defined by

$$f(\mu_1,\ldots,\mu_r,x) = X_1(\mu_1,\ldots,\mu_r,x)$$
$$g(\mu_1,\ldots,\mu_r,x) = Y_1(P(\mu_1,\ldots,\mu_r,x)).$$

The methods used in [8] allow us to assume that for any unfolding $\{X_\mu\}$ of X_o the coordinates x,y and $\bar{x},\bar{y}$ are such that

$$\begin{aligned} X_1(0,x) &= x^k.F(x) \quad , \quad F(0) \neq 0 \\ X_2(0,x) &= F_1(x) \quad . \quad F_1(0) > 0 \\ \text{and}\quad Y_1(0,\bar{x}) &= \bar{x}^\ell.\bar{G}(\bar{x}) \quad , \quad \bar{G}(0) \neq 0 \\ Y_2(0,\bar{x}) &= \bar{G}_1(\bar{x}) \quad , \quad \bar{G}_1(0) < 0. \end{aligned}$$

*) I.e. $X_1(0,\ldots,0;0) = \frac{\partial X_1}{\partial x}(0,\ldots,0;0) = \ldots = \frac{\partial^{k-1} X_1}{\partial x^{k-1}}(0,\ldots,0;0) = 0;\ \frac{\partial^k X_1}{\partial x^k}(0,\ldots,0;0) \neq 0.$

**) Here $D_{X,\sigma}$ denotes the time-σ-map of X.

As a consequence, the r parameter unfoldings $f(\mu,x)$ and $g(\mu,x)$ are unfoldings of $f(0,x) = x^k \cdot F(x)$ and $g(0,x)$, which is of the form $x^\ell \cdot G(x)$ with $G(0) \neq 0$. (f,g) will be called a reduced pair of the family $\{X_\mu\}$. Conversely, using standard suspension arguments it is easy to associate with any pair of r-parameter unfolding $(\bar{f}(\mu,x), \bar{g}(\mu,x))$ of $(f(0,x), g(0,x))$ an r-parameter unfolding $\{X_\mu\}$ of X_o.

Remark:

Suppose p is a hyperbolic saddle of X_o. In that case we may assume that there are local coordinates $\mu_1,\ldots,\mu_r,x,y$ and a positive function ψ on a neighbourhood U of $(0,p)$ in $\mathbb{R} \times M$ such that

$$\psi.X|U = X_1(\mu_1,\ldots,\mu_r,x)\frac{\partial}{\partial x} + X_2(\mu_1,\ldots,\mu_r,x,y)\frac{\partial}{\partial y},$$

where $X_1(\underline{0};x) = x\cdot F(x)$, $F(0) < 0$

$$X_2(\underline{0};0,0) = 0, \quad \frac{\partial X_2}{\partial y}(\underline{0};0,0) > 0.$$

Since we are classifying modulo topological equivalence we may and do assume $\psi \equiv 1$.

Consequently, the following theorem also holds in case $k = 1$ (i.e. p is a hyperbolic saddle) or $\ell = 1$ (i.e. q is a hyperbolic saddle).

Theorem I.3.

A universal unfolding of the vectorfield X_o along γ is given by the $(k+\ell-1)$-parameter family $X = \{X_\mu \mid \mu \in \mathbb{R}^{k+\ell-1}\}$ with reduced pair $(f(\mu,x), g(\mu,x))$ given by

$$f(\mu,x) = (x^k+\mu_1 x^{k-1}+\mu_2 x^{k-2}+\ldots+\mu_{k-1}\cdot x + \mu_k)\cdot F(x)$$

$$g(\mu,x) = (x^\ell \qquad +\mu_{k+1}\cdot x^{\ell-2} +\ldots+\mu_{k+\ell-2}\cdot x + \mu_{k+\ell-1})\cdot G(x)$$

(Assume $\ell \geq k \geq 1$)

Corollary I.4.

The catastrophe set of a universal unfolding of X_o is locally C^1-diffeomorphic *) to the following semi-algebraic subset K of $\mathbb{R}^{k+\ell-1}$ (assume $F(0)>0, G(0)>0$ for $k \geq 2, \ell \geq 2$):

$\mu \in K$ iff there is an $x \in \mathbb{R}$ such that at least one of the following cases occurs:

(i) $Q_1(\mu,x) = 0$, $\frac{\partial Q_1}{\partial x}(\mu,x) = 0$ (X_μ has a quasi-hyperbolic singularity at $(x,0) \in U$)

(ii) $Q_2(\mu,x) = 0$, $\frac{\partial Q_2}{\partial x}(\mu,x) = 0$ (X_μ has a quasi-hyperbolic singularity at $(P_\mu(x),0) \in V$)

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*) Two subsets K_1 and K_2 are called C^1-diffeomorphic if there is an ambient C^1-diffeomorphism φ such that $\varphi(K_1) = K_2$.

(iii) $Q_1(\mu,x) = 0$, $\frac{\partial Q_1}{\partial x}(\mu,x) \leq 0$ (X_μ has a (generalized) saddle connection, going from $(x,0) \in U$ to $(P_\mu(x\ ,0) \in V)$

$Q_2(\mu,x) = 0$, $\frac{\partial Q_2}{\partial x}(\mu,x) \geq 0$

Here $Q_1(\mu,x) = \pm x^k + \mu_1 x^{k-1} + \ldots + \mu_{k-1}x + \mu_k$ (minus sign in case k=1)

and $Q_2(\mu,x) = x^\ell + \mu_{k+1}x^{\ell-2} + \ldots + \mu_{k+\ell-2}x + \mu_{k+\ell-1}$ (=x, if ℓ=1)

Remark:
A parameter value $\mu \in \mathbb{R}^{k+\ell-1}$ belongs to the catastropheset of the family $\{X_\mu\}$ if the vectorfield X_μ is not structurally stable.

Before we give the proof of the theorem we shall consider some special cases; these will return in section II.

Example I.5

(i) k=1, ℓ=1, so k+ℓ-1 = 1

The universal unfolding contains one parameter. The catastrophe set is $\{0\} \subset \mathbb{R}$.

$Q_1(\mu,x) = -x+\mu$

$Q_2(\mu,x) = x$

For $\mu = 0$ we have a saddle connection between two hyperbolic saddles. This case was also studied by Sotomayor in [5]

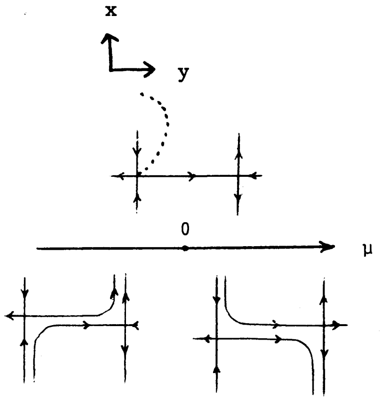


Fig. I.2.i.

(ii) k=1, ℓ=2

$Q_1(\mu,x) = -x + \mu_1$

$Q_2(\mu,x) = x^2 + \mu_2$

In this case (μ_1,μ_2) belongs to the catastrophe set iff

$$\mu_2 = 0 \quad \text{or} \quad \begin{cases} -x + \mu_1 = 0 \\ x^2 + \mu_2 = 0, \ x \geqq 0 \end{cases}$$

So $K = \{(\mu_1,\mu_2) \in \mathbb{R}^2 \mid \mu_2 = 0 \text{ or } \mu_1^2 + \mu_2 = 0, \ \mu_1 \geqq 0\}$

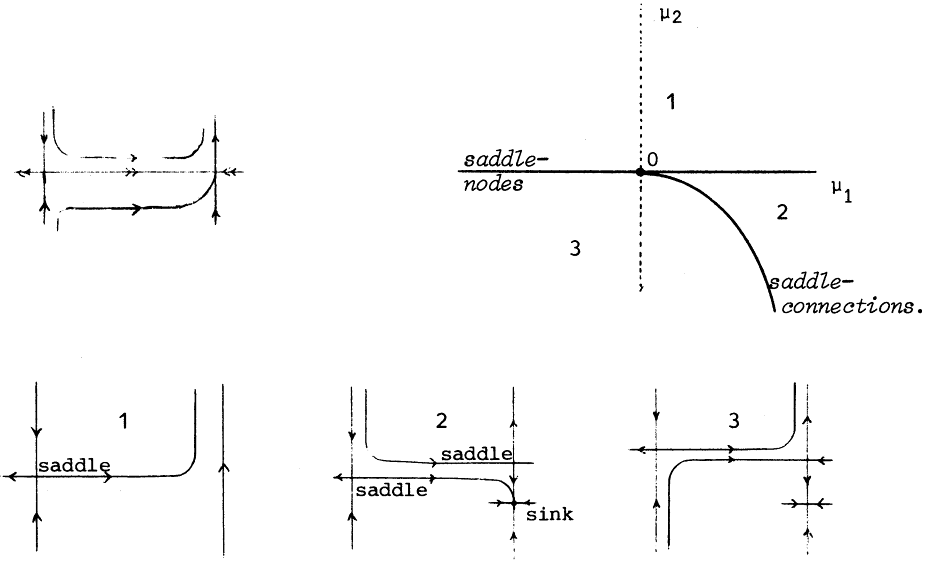


Fig. I.2.(ii)

(iii) <u>k=ℓ=2</u>

Now $Q_1(\mu_1,\mu_2,\mu_3,x) = x^2 + \mu_1 x + \mu_2$

$Q(\mu_1,\mu_2,\mu_3,x) = x^2 + \mu_3$

Bifurcations occur if:

(i) $\mu_1^2 - 4\mu_2 = 0$ or

(ii) $\mu_3 = 0$ or

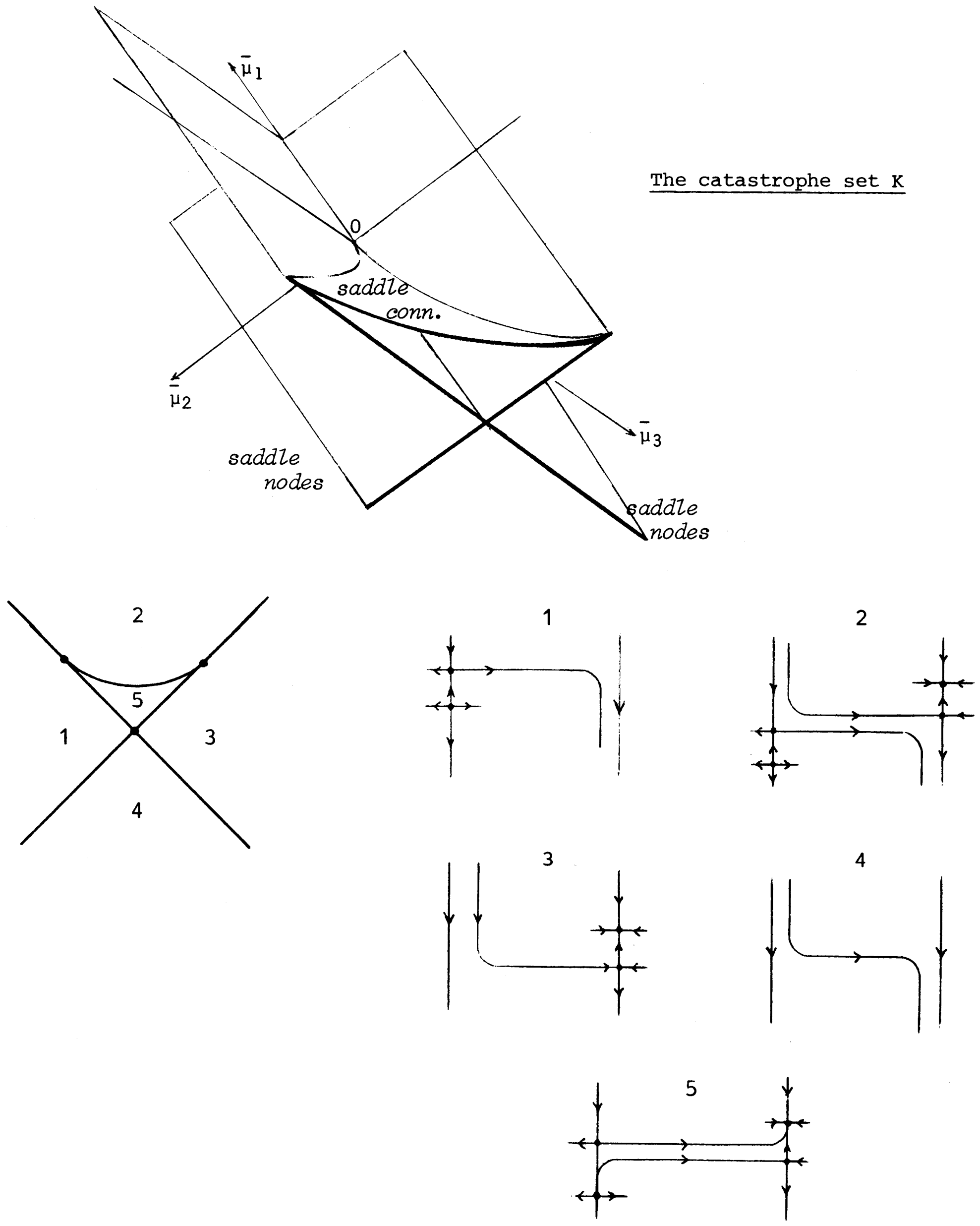
(iii) $\begin{cases} x^2 + \mu_1 x + \mu_2 = 0, & 2x + \mu_1 \leqq 0 \\ x^2 + \mu_3 = 0, & 2x \geqq 0 \end{cases}$

Having performed the following change of parameters:

$$\begin{cases} \bar{\mu}_1 = \mu_1 \\ \bar{\mu}_2 = \mu_2 - \frac{1}{4}\mu_1^2 \\ \bar{\mu}_3 = \mu_3 \end{cases}$$

we obtain:

$$K = \{(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3) \in \mathbb{R}^3 \mid \bar{\mu}_3 = 0 \text{ or } \bar{\mu}_2 = 0 \text{ or } [\bar{\mu}_2 + (\tfrac{1}{2}\bar{\mu}_1 + \sqrt{-\bar{\mu}_3})^2 = 0,\ \bar{\mu}_3 \leqq 0 \ \& \ \bar{\mu}_1 + 2\sqrt{-\bar{\mu}_3} \leqq 0]\}$$


The catastrophe set K
$\bar{\mu}_1$
0
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$\bar{\mu}_2$
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saddle
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saddle
nodes
2
5
1
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4
1
2
3
4
5

Fig. I.2.iii

Proof of theorem I.3.

Let U,V,Σ and Θ be associated with the family X like indicated in fig. I.1. Suppose $\{\tilde{X}_\nu | \nu \in \mathbb{R}^t\}$ is a t-parameter unfolding of X_o along γ. For this family we choose neighbourhoods $\tilde{U}$ of (0,p) and $\tilde{V}$ of (0,q) in $\mathbb{R}^t \times M$ and t-parameter unfoldings $\tilde{f}(\nu,x)$ and $\tilde{g}(\nu,x)$ of f(0,x) and g(0,x) resp. in the way described above.

According to corollary A.6 of appendix A there are

- a C^1-mapping H_1: $\tilde{W}^c(0,p) \longrightarrow W^c(0,p)$ of the form $H_1(\nu,x) = (h(\nu), H_\nu(x))$, with H_ν a C^1-diffeomorphism on a neighbourhood of x=0 in $\tilde{W}^c(0,p)$
- two positive functions $\tilde{N}_1$, $\tilde{N}_2$: $\tilde{W}^c(0,p) \longrightarrow \mathbb{R}$,

such that:

$$\tilde{f}(\nu,x) = \tilde{N}_1(\nu,x) . f(H_1(\nu,x)) \tag{I.1}$$

$$\tilde{g}(\nu,x) = \tilde{N}_2(\nu,x) . g(H_1(\nu,x))$$

Note that $\tilde{W}^c(0,p)$ is a centermanifold for $\tilde{X}(\nu,x) := (\nu,\tilde{X}_\nu(x))$, containing (0,p); moreover, in order to apply corollary A.6 the degree of differentiability of $\tilde{f}$ and $\tilde{g}$ should be sufficiently high. This can be arranged by taking $\tilde{W}^c(0,p)$ small enough.

Next we extend H_1 to a C^1-diffeomorphism on a full neighbourhood of γ in $\mathbb{R}^t \times M$ in such a way that the partial foliations of $\tilde{U}$ and U are H_1-invariant (i.e. H_1 maps leaves on to leaves) and such that H_ν is the identity outside a small neighbourhood of $\tilde{U}$. Moreover we require that this extended diffeomorphism is again of the form

$$H_1(\nu,m) = (h(\nu), H_\nu(m)) , m \in M.$$

Using this diffeomorphism we can arrange that the t-parameter family h*X, defined by $(h^*X)_\nu = X_{h(\nu)}$, is topologically equivalent to a t-parameter family $\{\bar{X}_\nu\}$, such that the reduced pair $(\bar{f},\bar{g})$ of $\{X_\nu\}$ is a ν-parameter unfolding of the pair (f(0,x), g(0,x)) and satisfies:

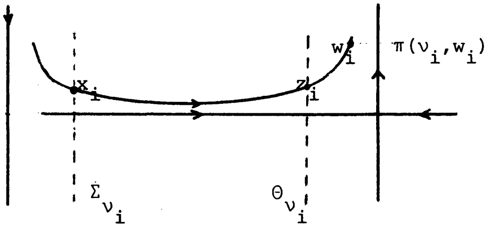
$$\tilde{f}(\nu,x) = \bar{N}_1(\nu,x) . \bar{f}(\nu,x) \tag{I.2}$$

$$\tilde{g}(\nu,x) = \bar{N}_2(\nu,x) . \bar{g}(\nu,x) .$$

Here $\bar{N}_1$ and $\bar{N}_2$ are positive functions on $W^c(0,p)$. Moreover, the invariant partial foliation is invariant for both $\tilde{X}$ and $\bar{X}$. Note that multiplication of a vectorfield with a positive function does not affect its orbits: only their parametrization might be changed. Since we are classifying modulo topological equivalence it will be clear from (I.2) that we may even assume: $\tilde{f}(\nu,x) = \bar{f}(\nu,x)$. To this end we extend $\bar{N}_1$ to a positive function on a full neighbourhood of γ in $\mathbb{R}^t \times M$, which is constant along the leaves of the invariant partial

foliation of $\tilde{U}$ and assumes the value 1 on a neighbourhood of $\tilde{V}$.
In this way we achieve that the partial foliation of $\tilde{U}$ is still invariant for $\frac{1}{\overline{N}_1} \cdot \overline{X}$, while the functions $\overline{g}$ of $\overline{X}$ is not affected.

The rest of the proof deals with the construction of a <u>positive function φ and a conjugacy H_ν between $\tilde{X}_\nu$ and $\varphi_\nu . \overline{X}_\nu$</u>, both defined on a neighbourhood of γ in $\mathbb{R}^t \times M$.



<u>Fig.I.3</u>

We define H_ν to be the identity on the section Σ_ν (fig. I.3). Then for any positive, continuous function φ, defined on a full neighbourhood of γ in $\mathbb{R}^t \times M$, we can extend H_ν to a conjugacy between $\tilde{X}_\nu$ and $\varphi_\nu . \overline{X}_\nu$ on the saturated set of Σ_ν. However, we have to impose additional conditions on φ_ν in order to extend H_ν continuously to a full neighbourhood of γ. Note that we may take $\varphi_\nu \equiv 1$ on $\tilde{U}_\nu$, since the $\overline{X}_\nu$- and $\tilde{X}_\nu$- invariant foliations of $\tilde{U}_\nu$ coincide and $\overline{f}_\nu(x) = \tilde{f}_\nu(x)$. So we can extend H_ν to $\tilde{W}^c_\nu(0,p)$ continuously.

The second condition we impose on φ_ν is that it should satisfy:

$$D_{\varphi_\nu . \overline{X}_\nu, \sigma}(\Sigma_\nu) \subset \overline{\Theta}_\nu \qquad \text{(I.3)}$$

Then the conjugacy H_ν maps $\tilde{\Theta}_\nu$ onto $\overline{\Theta}_\nu$.

Finally we have to check which additional conditions φ_ν should satisfy in order to extend it to $\tilde{W}^c_\nu(0,q)$ continuously. Suppose we have chosen φ. Let $\{(\nu_i, {}_i)\}_{i=1}^{\infty}$ be a sequence in $\mathbb{R}^t \times M$ such that (cf. fig. I.3)

- $\lim_{i \to \infty} (\nu_i, w_i) = (\nu_o, w_o) \in \overline{W}^c_{\nu_o}(0,q)$
- the backward orbit of w_i passes $\overline{\Theta}_{\nu_i}$ before it leaves $\overline{V}_{\nu_i}$.

So there is a $T_i > 0$ and a $z_i \in \overline{\Theta}_{\nu_i}$ such that

$D_{\varphi \overline{X}, T_i}(z_i) = w_i$ and $D_{\varphi \overline{X}, t}(z_i) \in \overline{V}_{\nu_i}$ for $0 \leq t \leq T_i$.

Set $x_i = D_{\varphi \overline{X}, -\sigma}(z_i)$, then $x_i \in \Sigma_{\nu_i}$, in view of (I.3)

Since H_ν is (partially) defined by $D_{\widetilde{X}_{\nu_i}, t} \circ H_\nu = H_\nu \circ D_{\varphi_\nu \overline{X}_\nu, t}$,

we have $H(\nu_i, z_i) = (\nu_i, D_{\widetilde{X}_{\nu_i}, \sigma}(x_i))$

and $H(\nu_i, w_i) = (\nu_i, D_{\widetilde{X}_{\nu_i}, T_i}(H_{\nu_i}(z_i)))$

We shall determine φ such that $\lim_{i \to \infty} H_{\nu_i}(w_i)$ exists and such that φ is constant along the leaves of the $\overline{X}$-invariant foliation of $\overline{V}$; so this foliation is also invariant for $\varphi \overline{X}$.

So suppose

$$\varphi \overline{X} | \overline{V} = \varphi(\nu, \overline{x}) \, . \, Y_1(\nu, \overline{x}) \frac{\partial}{\partial \overline{x}} + Y_2(\nu, \overline{x}, \overline{y}) \frac{\partial}{\partial \overline{y}} \, .$$

We also have

$$\widetilde{X} | \widetilde{V} = \widetilde{Y}_1(\nu, \widetilde{x}) \frac{\partial}{\partial \widetilde{x}} + \widetilde{Y}_2(\nu, \widetilde{x}, \widetilde{y}) \frac{\partial}{\partial \widetilde{y}}.$$

Let $\pi : \overline{V} \longrightarrow \overline{W}^c(0,q)$ and $\widetilde{\pi} : \widetilde{V} \longrightarrow \widetilde{W}^c(0,q)$ be the projections along leaves of the invariant foliations.

Then:

$$T_i = \int_{\pi(\nu_i, z_i)}^{\pi(\nu_i, w_i)} \frac{d\xi}{\varphi(\nu_i, \xi) \, . \, Y_1(\nu_i, \xi)} \quad \ldots . (I_1)$$

and also

$$T_i = \int_{\widetilde{\pi}(H(\nu_i, w_i))}^{\widetilde{\pi}(H(\nu_i, w_i))} \frac{d\xi}{\widetilde{Y}_1(\nu_i, \xi)} \quad \ldots . (I_2).$$

Assume $\lim_{i \to \infty} T_i = \infty$, otherwise the convergence of $\{H_{\nu_i}(w_i)\}$ is obvious. So the sequence $\{z_i\}$ tends to a point z_o on the stable separatrix of a (possibly degenerate) saddle $\pi(\nu_o, z_o)$, so $Y_1(\pi(\nu_o, z_o)) = 0$. Take $x_o := D_{\varphi \overline{X}, -\sigma}(\nu_o, z_o)$, then $\lim x_i = x_o$.

The mappings $P_1 : \Sigma \longrightarrow \overline{W}^c(0,q)$ and $\widetilde{P}_1 : \Sigma \longrightarrow \widetilde{W}^c(0,q)$ are defined by $P_1 = \pi \circ D_{\varphi \overline{X}, \sigma}$ and $\widetilde{P}_1 = \widetilde{\pi} \circ D_{\widetilde{X}, \sigma}$, so:

$$Y_1(\nu_o, P_1(\nu_o, x_o)) = 0 \text{ and } \widetilde{Y}_1(\nu_o, \widetilde{P}_1(\nu_o, x_o)) = 0.$$

(Note that we sometimes also use the symbol P_1 for the mapping $(\nu, x) \longrightarrow (\nu, P_1(\nu, x))$. This will be done without mentioning it). Note that the lower boundaries in I_1 and I_2 satisfy:

$$\pi(\nu_i, z_i) = P_1(\nu_i, x_i) \text{ and } \widetilde{\pi}(H(\nu_i, w_i)) = \widetilde{P}_1(\nu_i, x_i).$$

Now take $\eta_o \in \Sigma_{\nu_o}$ in such a way that η_o is on the same side of Σ_{ν_o} as $\{x_i\}$, while P_1 and $\widetilde{P}_1$ are both defined on $[x_o, \eta_o] \subset \Sigma_{\nu_o}$. In the integral (I.1) we perform the change of coordinates $(\nu_i, \xi) = P_1(\nu_i, \eta)$ and in $I_2: (\nu_i, \xi) = \widetilde{P}_1(\nu_i, \eta)$. This yields:

$$T_i = \int_{\eta=x_i}^{\eta_o} \frac{\frac{\partial P_1}{\partial \eta}(\nu_i, \eta)}{\varphi \circ P_1(\nu_i, \eta) \cdot Y_1 \circ P_1(\nu_i, \eta)} d\eta + \int_{\xi=P_1(\nu_i, \eta_o)}^{\pi(\nu_i, w_i)} \frac{d\xi}{\varphi(\nu_i, \xi) \cdot Y_1(\nu_i, \xi)}$$

$$= \int_{\eta=x_i}^{\eta_o} \frac{\frac{\partial \widetilde{P}_1}{\partial \eta}(\nu_i, \eta)}{\widetilde{Y}_1 \circ \widetilde{P}_1(\nu_i, \eta)} d\eta + \int_{\xi=\widetilde{P}_1(\nu_i, \eta_o)}^{\widetilde{\pi}(H(\nu_i, w_i))} \frac{d\xi}{\widetilde{Y}_1(\nu_i, \xi)} \quad \ldots \tag{I.4}$$

Recall that we have the relation (I.2):

$$\widetilde{Y}_1 \circ \widetilde{P}_1(\nu, x) = \overline{N}_2(\nu, x) \cdot Y_1 \circ P_1(\nu, x)$$

So if we define φ on $\widetilde{V}$ in such a way that

$$\varphi \circ P_1(\nu, x) = \frac{\partial P_1}{\partial x}(\nu, x) \cdot \left\{ \frac{\partial \widetilde{P}_1}{\partial x}(\nu, x) \cdot \overline{N}_2(\nu, x) \right\}^{-1},$$

then φ is a positive function, and from (I.4) we obtain:

$$\int_{\xi=\widetilde{P}_1(\nu_i, \eta_o)}^{\widetilde{\pi}(H(\nu_i, w_i))} \frac{d\xi}{\widetilde{Y}_1(\nu_i, \xi)} = \int_{\xi=P_1(\nu_i, \eta_o)}^{\pi(\nu_i, w_i)} \frac{d\xi}{\varphi(\nu_i, \xi) \cdot Y_1(\nu_i, \xi)}$$

Since both integrands are regular on the integration interval, it is easy to see that with this choice of φ we have achieved that $\{H_{\nu_i}(w_i)\}$ converges.

Q.E.D.

II. Deformations of the gradient singularity $x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$

Statement of the result.

In this section we shall prove the results, announced in the introduction. We first recall some well known facts concerning Riemannian metrics, gradients, etc. If g is a Riemannian metric on a manifold M, then any diffeomorphism $\varphi: M \longrightarrow M$ associates with g a metric $\varphi^* g$ defined by:

$$(\varphi^* g)_x \ (v, w) := g_{\varphi(x)} (d\varphi_x(v), d\varphi_x(w) \qquad (x \in M; v, w \in T_x M).$$

We also obtain the metric $\varphi_* g := (\varphi^{-1})^* g$.
For any real valued function f on M we define the function $\varphi_*(f)$ to be $f.\varphi^{-1}$.
Recall that the gradientvectorfield X, corresponding to a potential function f and a Riemannian metric g on M, is uniquely determined by the relation:

$$g(X,Y) = df(Y), \text{ for all } C^\infty \text{ vectorfields } Y \text{ on } M \quad ..... \tag{II.0}$$

This relation yields the transformation rule

$$\varphi_* (\text{grad}_g f) = \text{grad}_{\varphi_* g}\, \varphi_* f. \tag{II.1}$$

From now on we shall only be dealing with gradient vectorfields defined on a small neighbourhood of $0 \in \mathbb{R}^2$. Let x,y be local coordinates on such a neighbourhood, then the metric g is completely determined by its four components $g_{11}(x,y)$, $g_{12}(x,y) = g_{21}(x,y)$ and $g_{22}(x,y)$. Let $G(x,y) = (g^{ij}(x,y))$ be the inverse of the matrix $(g_{ij}(x,y))$. For any potential function f, defined on a neihgbourhood of $0 \in \mathbb{R}^2$, we obtain from (II.0) the following relation for the components X_1, X_2 of the vectorfield $\text{grad}_g f$:

$$\begin{pmatrix} X_1(x,y) \\ X_2(x,y) \end{pmatrix} = G(x,y) . \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

The definition of <u>unfolding</u> should be slightly adapted to make it suitable for the category of gradient vector fields. A k-parameter unfolding of a gradient vector field $X_o = \text{grad}_g f$ is a smooth family $X_\mu = \{\text{grad}_{g_\mu} f_\mu \mid \mu \in \mathbb{R}^k\}$, where $\{g_\mu\}$ and $\{f_\mu\}$ are smooth k-parameter families of metrics and potential functions respectively, such that $g_o = g$ and $f_o = f$.

In the remaining part of this paper we consider pairs (g,f) which - in suitable coordinates - have the following form:

(*)

$f(x,y) = \frac{1}{3}(x^3 + y^3)$

g(x,y) has components $g_{ij}(x,y)$ such that
$g_{11}(0,0) = \alpha$, $g_{22}(0,0) = \alpha\beta$, $g_{12}(0,0) = g_{21}(0,0) = \alpha z_o$,
where
$\alpha > 0$, β and z_o are real numbers sufficiently close to 1 and 0 respectively, and of course such that $\beta - z_o^2 > 0$.

The vector field $\text{grad}_g f$, with quadratic part

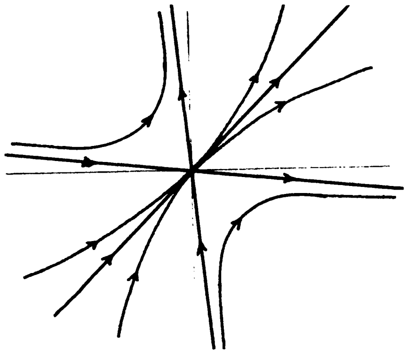
$$Q(x,y) = \alpha[(x^2 + z_o y^2)\frac{\partial}{\partial x} + (z_o x^2 + \beta y^2)\frac{\partial}{\partial y}]$$

has a degenerate singularity at $(x,y) = (0,0)$, with two hyperbolic and two parabolic sectors. The quadratic part Q has three invariant lines. If $z_o \neq 0$ these have equation $y = mx$, where m satisfies

$$p_{\beta,z_o}(m) = \lim_{x\to 0} \frac{1}{x^2} \text{Det}[Q(x,mx), \binom{1}{m}] = 0,$$

i.e.: $p_{\beta,z_o}(m) = \alpha(z_o m^3 - \beta m^2 + m - z_o) = 0.$

If $z_o = 0$, this quadratic part has three invariant lines: $x = 0$, $y = 0$ and $y = \frac{1}{\beta}x$.



Note that the existence of three invariant lines is guaranteed by the fact that $\beta \approx 1$ and $z_o \approx 0$.

In fact our results hold for any pair (β, z_o) such that p_{β,z_o} has three real zeroes or such that $z_o = 0$.

It is well known that the 3-parameter family $f_{u,v,w}(x,y) = \frac{1}{3}(x^3 + y^3) + wxy + ux + vy$ is a universal unfolding of f, cf. [10]. The next result indicates the relation between this universal unfolding of the potential function f and the gradient vector field $\text{grad}_g f$.

Theorem II.1.

a. If $z_o \neq 0$ the three parameter gradient family $X_{u,v,w} = \text{grad}_g f_{u,v,w}$ is an almost universal unfolding of the vector field $\mathrm{grad}_g f$.

b. If $z_o = 0$ the four parameter family $X_{u,v,w,z} = \mathrm{grad}_{g_z} f_{u,v,w}$ is an almost unfolding of $\mathrm{grad}_g f$, where g_z is the one parameter family of metrics given by $g_z^{11} \equiv g^{11}$, $g_z^{22} \equiv g^{22}$, $g_z^{12} = g_z^{21} \equiv g^{12} + z$.

Remark:
In view of the Tarski - Seidenberg theorem (cf. [9]) the set of pairs (g,f), admitting local coordinates in which they have the form (*) with $z_o \neq 0$ ($z_o = 0$ resp.), is a closed semi-algebraic set of codimension 3 (4 resp.) in the set of all pairs (g,f)(also cf. [10]).
Sometimes the codimension of an object is defined as the number of parameters contained in a universal unfolding (within a suitable category). In our context these concepts of codimension coincide for this example, contrary to the claim of Guckenheimer (cf. [2]).

In the proof of theorem II.1. we show that for $z_o \neq 0$ the family $X_{u,v,w}$ is almost topologically equivalent to the quadratic 3-parameter family $Q_{u,v,w} = \mathrm{grad}_{g_{z_o}} f_{u,v,w}$, where g_{z_o} is the constant metric $g_{z_o} \equiv g(0,0)$.

Hence : $$Q_{u,v,w}(x,y) = \begin{pmatrix} 1 & z_o \\ z_o & \beta \end{pmatrix} \begin{pmatrix} x^2 + wy + u \\ y^2 + wx + v \end{pmatrix}$$

If $z_o = 0$ we can prove similarly that $X_{u,v,w,z}$ is almost equivalent to

$$Q_{u,v,w,z}(x,y) = \begin{pmatrix} 1 & z \\ z & \beta \end{pmatrix} \begin{pmatrix} x^2 + wy + u \\ y^2 + wx + v \end{pmatrix}.$$

Before proceeding with the proof of these results we investigate for which values of the parameters bifurcation occurs in the afore mentioned quadratic families.

<u>The catastrophe set of the family $Q_{u,v,w,z}$</u>$(x,y) = \begin{pmatrix} 1 & z \\ z & \beta \end{pmatrix} \begin{pmatrix} x^2 + wy + u \\ y^2 + wx + v \end{pmatrix}$

We first observe that $= (u,v,w,z)$ is a bifurcation value for the family $\{Q_\mu\}$ if at least one of the following situations occurs:

(i) X_μ has at least one degenerate singularity.

(ii) X_μ exhibits a (generalized) saddle connection (cf. section I).

Note that the first case occurs iff μ is in the catastrophe set of the potential function $f_\mu = \frac{1}{3}(x^3 + y^3) + wxy + ux + vy$, i.e. iff:

$$\begin{aligned} x^2 + wy + u &= 0 \\ y^2 + wx + v &= 0 \\ 4xy - w^2 &= 0 \end{aligned}$$

This catastrophe set is well-known, c.f. [10]. (see figure II.1.).

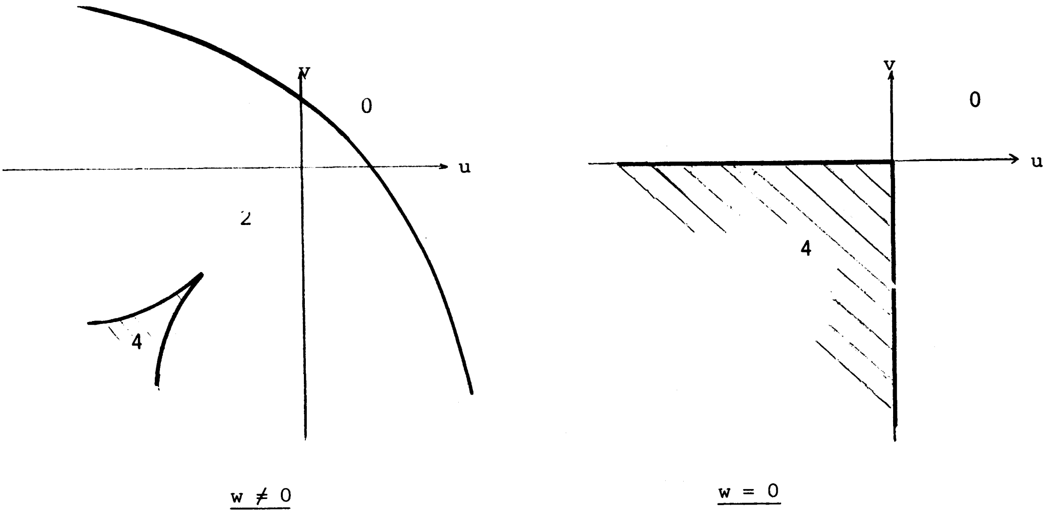


Fig. II.1. The catastrophe set of the family $f_{u,v,w}$

In order to determine the parameter values μ for which X has a saddle connection, the following result will be useful.

Lemma II.2.:

If $X = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}$ is a quadratic gradient vectorfield on $\mathbb{R}^2$ with saddles S_1 and S_2 and a saddle connection γ between S_1 and S_2, then γ is a straight line (with respect to the coordinates x,y).

For a proof of this result we refer to [13].

As a consequence we have to determine for which values of a,b,u,v,w and z the quadratic form (in x,y):

$$q_{a,b,u,v,w,z}(x,y) = aQ_1(u,v,w,z,x,y) + bQ_2(u,v,w,z,x,y)$$

contains a factor $\varphi(x,y) = ax + by + c$, for some real number c.

A straightforward calculation yields:

$$\left.\begin{aligned} q(x,y) &= Ax^2 + By^2 + Bwx + Awy + Au + Bv \\ &= A(x+\frac{Bw}{2A})^2 + B(y+\frac{Aw}{2B})^2 + R_{A,B} \\ \text{with}\quad & A = a + zb\ ,\ B = za + \beta b \\ & R_{A,B} = Au + Bv - \frac{1}{4}w^2\,(\frac{B^2}{A}+\frac{A^2}{B}) \end{aligned}\right\} \quad \ldots\ (II.2)$$

So the zero set of q consists of two lines with equation

$$y + \frac{Aw}{2B} = \pm\sqrt{-\frac{A}{B}}\ .\ (x+\frac{Bw}{2A})$$

if and only if $R_{A,B} = 0$ and $AB < 0$. (II.3)

If condition (II.3) holds, we can find a real number c such that $\varphi(x,y) = ax + by + c$ is a factor of q iff.

$$\frac{a}{b} = \mp\sqrt{-\frac{A}{B}} \qquad (II.4)$$

Setting $\xi = \frac{a}{b}$, it is clear from (II.2) and (II.4) that ξ should be a zero of $P_{z,\beta}(\xi) := z\xi^3 + \beta\xi^2 + \xi + z$ .... (II.5)

This polynomial has three real zeroes ξ_1, ξ_2 and ξ_3 if $|z|$ is small and positive. Application of the implicit function theorem yields:

$$\left.\begin{aligned} \xi_1(\beta,z) &= -z - \beta z^2 + O(z^3) \\ \xi_2(\beta,z) &= -\frac{\beta}{z} + \frac{1}{\beta} + O(z) \\ \xi_3(\beta,z) &= -\frac{1}{\beta} + O(z) \end{aligned}\right\} \text{ as } z \longrightarrow 0. \qquad (II.6)$$

Furthermore it is obvious that in order to have a saddle connection for $Q_{u,v,w,z}$, (u,v,w) should lie in the closure of the shaded region of figure II.1.

If (u,v,w) is in this region, then the point (x,y) is a (generalized) saddle of $Q_{u,v,w,z}$ iff:

$$x^2 + wy + u = 0$$
$$y^2 + wx + v = 0$$
$$4xy - w^2 \leq 0$$

Note that, modulo some positive factor, $4xy - w^2$ is the Jacobian determinant of the linear part of $Q_{u,v,w,z}$ at the singularity (x,y). It is clear from figure II.3 that in case we have two saddles, the slope of the line joining them is negative.

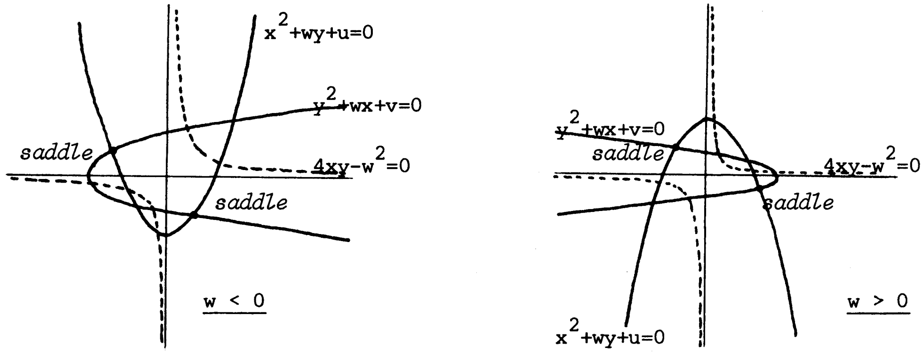


Fig.II.3

With this observation in mind we conclude that for $z \geq 0$ no saddle connections occurr, while for $z < 0$ they actually do. In the latter case the values of $\frac{a}{b}$, corresponding to ξ_1 and ξ_2, yield a saddle connection since condition (II.3) is satisfied for $|z|$ small enough. This fact will be clear from (II.6) and the equality

$$\frac{A}{B} = \frac{\xi + z}{z\xi + \beta} \qquad \text{(II.7)}$$

So for $z < 0$ and $|z|$ small enough, the catastrophe set also contains two halflines, namely those parts of the lines with equations:

$$R_{A_i, B_i} = A_i u + B_i v - \frac{1}{4} w^2 \left(\frac{A_i^3 + B_i^3}{A_i B_i} \right) = 0 \quad (i=1,2)$$

that ly in the shaded regions of figure II.1.

For the slopes of these lines with respect to the horizontal and vertical directions we obtain resp. using (II.6) and (II.7)

$$\left.\begin{aligned} -\frac{A_1}{B_1} &= -\frac{\xi_1 + z}{z\xi_1 + \beta} = z^2 + o(z^3) \\ -\frac{B_2}{A_2} &= -\frac{z\xi_2 + \beta}{\xi_2 + z} = \frac{z^2}{\beta^2} + o(z^3) \end{aligned}\right\} \text{ as } z \uparrow 0$$

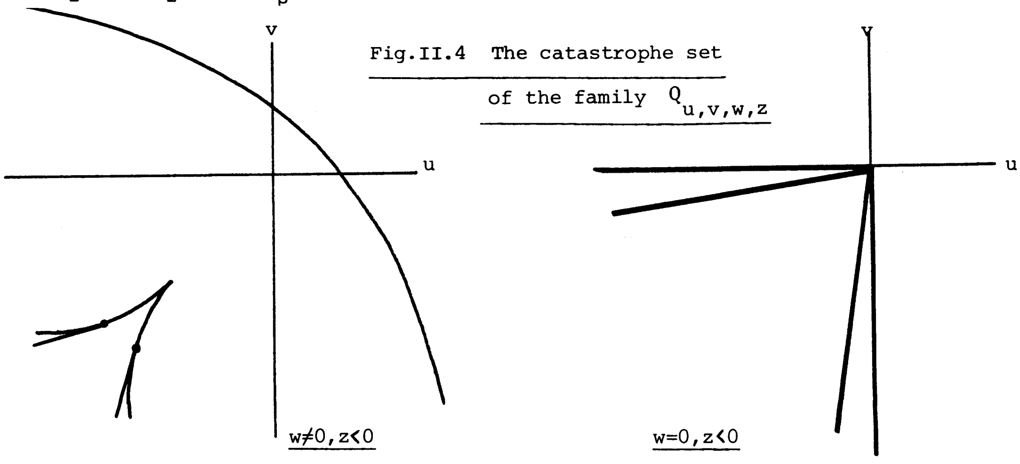


Fig.II.4 The catastrophe set of the family $Q_{u,v,w,z}$

A parametrization for the curve in the plane $\{w=w_o\}$, $w_o \neq 0$, of figure II.4 is easily seen to be

$$t \longrightarrow \left(\frac{1}{4} w_o^2\left(-\frac{2}{t} - t^2\right), \frac{1}{4} w_o^2\left(-\frac{1}{t^2} - 2t\right)\right).$$

Using this parametrization one easily checks that the lines $\{R_{A_i,B_i} = 0\}$, corresponding to this value of w, are indeed <u>tangent</u> to this curve. Note that at the point of tangency the situation is locally like that of figure I.2.(ii).

We return to this fact later on.

<u>Proof of theorem II.a.</u>

Let $\{Y_\nu | \nu \in \mathbb{R}^\ell\}$ be an unfolding of $X_o = \mathrm{grad}_{g_{z_o}} f_o$, within the class of gradient-vectorfields,i.e. $Y_\nu = \mathrm{grad}_{g_\nu} f_\nu$, with $g_o = g_{z_o}$ en $f_o(x,y) = \frac{1}{3}(x^3+y^3)$.

<u>Step 1</u>. First we shall be concerned with the case where $\{f_\nu | \nu \in \mathbb{R}^\ell\}$ is a <u>versal</u> unfolding of f_o. From the theory of C^∞-singularities of functions it is known that there is a submersion h: $(\mathbb{R}^\ell,0) \longrightarrow (\mathbb{R}^3,0)$ such that f_ν is right-equivalent to $F_{h(\nu)}$, the equivalence depending smoothly on the parameter ν.

Here $F_\mu(x,y) = \frac{1}{3}(x^3+y^3) + \mu_3 xy + \mu_1 x + \mu_2 y$.

We may assume that h is of the form $h(\nu_1,\ldots,\nu_\ell) = (\nu_1,\nu_2,\nu_3)$.

In view of (II.1) we may assume that

$$\begin{pmatrix} Y_\nu^1(x,y) \\ Y_\nu^2(x,y) \end{pmatrix} = G_Y(\nu,x,y). \begin{pmatrix} x^2 + \nu_3 y + \nu_1 \\ y^2 + \nu_3 x + \nu_2 \end{pmatrix} \qquad \text{(II.8)}$$

where:

$$G_Y(0;0,0) = \begin{pmatrix} 1 & z_o \\ z_o & \beta \end{pmatrix} = G_Q(x,y) \ , \ z_o \neq 0$$

We shall proof that the family (II.8) is almost topologically equivalent to the family

$$\begin{pmatrix} Q_\nu^1(x,y) \\ Q_\nu^2(x,y) \end{pmatrix} = G_Q(x,y) \begin{pmatrix} x^2 + \nu_3 y + \nu_1 \\ y^2 + \nu_3 x + \nu_2 \end{pmatrix} \qquad \text{(II.9)}$$

It is easy to check that the 0-jet of G_Y is unique, although the coordinates (x,y) are not unique in general.

This will establish the proof for this case. To this end we shall use a version of the method of "blowing up" ("rescaling"), introduced by Takens [11].

Let Φ: $S^2 x\ [0,\infty)\ x\ \mathbb{R}^{\ell-3} x\ \mathbb{R}^2 \longrightarrow \mathbb{R}^{\ell} x\ \mathbb{R}^2$ be the mapping defined by $\Phi((\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3),\lambda,\nu_4,\ldots,\nu_\ell,x,y) = (\lambda^2\bar{\nu}_1,\lambda^2\bar{\nu}_2,\lambda\bar{\nu}_3,\nu_4,\ldots,\nu_\ell,\lambda x,\lambda y)$ where $\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3$, with $\bar{\nu}_1^2 + \bar{\nu}_2^2 + \bar{\nu}_3^2 = 1$, are coordinates on S^2.
Considering the families (II.8) and (II.9) as vectorfields on $\mathbb{R}^{\ell} x\ \mathbb{R}^2$, we obtain vectorfields $\tilde{Q}$ and $\tilde{Y}$ on $S^2 x\ [0,\infty)\ x\ \mathbb{R}^{\ell-3} x\ \mathbb{R}^2$, such that $\Phi_*(\tilde{Q}) = Q$ and $\Phi_*(\tilde{Y}) = Y$.
Setting $\bar{Q} = \lambda^{-1}\tilde{Q}$ and $\bar{Y} = \lambda^{-1}\tilde{Y}$ yields the ℓ-parameter families

$$\bar{Q}(\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3,\lambda\ \nu_4,\ldots,\nu_\ell,x,y) = G_Q(\lambda x,\lambda y)\begin{pmatrix} x^2 + \bar{\nu}_3 y + \bar{\nu}_1 \\ y^2 + \bar{\nu}_3 x + \bar{\nu}_2 \end{pmatrix} \quad \ldots. \quad (II.10)$$

and:

$$\bar{Y}(\bar{\nu}_1,\bar{\nu}_2,\bar{\nu}_3,\lambda,\nu_4,\ldots,\nu_\ell,x,y) = G_Y(\lambda^2\nu_1,\lambda^2\nu_2,\lambda\nu_3,\nu_4,\ldots,\nu_\ell,x,y)\begin{pmatrix} x^2 + \bar{\nu}_3 y + \bar{\nu}_1 \\ y^2 + \bar{\nu}_3 x + \bar{\nu}_2 \end{pmatrix} (II.1$$

In appendix B we prove that the two parameter family
$\bar{Q}|\ S^2 x\ \{\lambda=0\}\ x\ \{\nu_4=\ldots=\nu_\ell=0\} = \bar{Y}|S^2 x\ \{\lambda=0\}\ x\ \{\nu_4=\ldots=\nu_\ell=0\}$ is generic.
In fact the proof implies that this family is transversal to some stratified submanifold Σ of the set of all gradientvectorfields, defined on a neighbourhood of 0 in $\mathbb{R}^2$. (For similar constructions of codimension 1 submanifolds we refer to [5]). The inverse image of this subset Σ is the catastrophe set $C(z_o)$ of the family $\bar{Q}|\ S^2 x\ \{\lambda=0\}\ x\ \{\nu_4=\ldots=\nu_\ell=0\}$. In view of the preceding paragraph we obtain the following picture for $C(z_o)$ after stereographic projection from the point $(\bar{\nu}_1=\bar{\nu}_2= \frac{1}{2}\sqrt{2},\bar{\nu}_3 = 0)$.

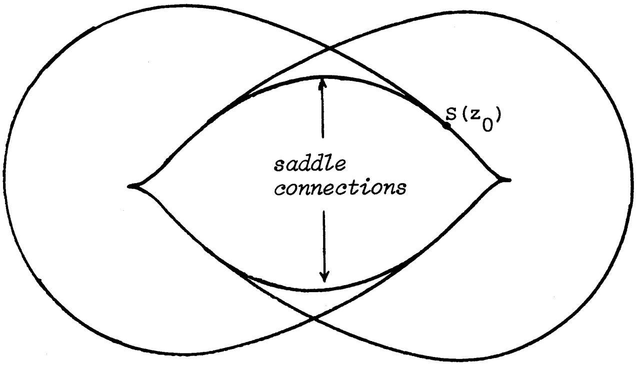


Fig.II.5 The catastrophe set $C(z_0)$ in case $z_0 < 0$

Using the fact that the germs of the ℓ-parameter families Q and Y at any point of S^2 are versal, one can prove the existence of a homeomorphism $\overline{h}$: $S^2 \times [0,\delta) \times U \longrightarrow S^2 \times [0,\delta') \times U$ (δ,δ' are small positive numbers, U,U' are neighbourhoods of 0 in $\mathbb{R}^{\ell-3}$), mapping the catastrophe set of the family $\overline{Y}$ onto $C(z_o) \times [0,\delta') \times U'$. Since $G_Q(x,y)$ contains no parameters, the latter is the catastrophe set of the family $\overline{Q}$. Note that in case $z_o > 0$ no saddle connections can occurr, so we can take for $\overline{h}$ the identity mapping. Using $\overline{h}$, we can finish the construction of an equivalence between $\overline{Q}$ and $\overline{Y}$, defined on a neighbourhood of $0 \in \mathbb{R}^2$, in a straightforward way. We fix the topological equivalence imposing the conditions that it should:

(1) map the level curves of the potential function $f_{\overline{\nu}}$ onto those of $f_{\overline{h}(\overline{\nu})}$.

(2) map the singularities of $\overline{Y}_{\overline{\nu}}$ onto the corresponding singularities of $\overline{Q}_{\overline{h}(\overline{\nu})}$.

(3) map (strong) separatrices of $\overline{Y}_{\overline{\nu}}$ onto corresponding (strong) separatrices of $\overline{Q}_{\overline{h}(\overline{\nu})}$.

Note that these conditions do not determine the topological equivalence completely. Observe that the three conditions above can be satisfied, since $\overline{Y}_{\overline{\nu}}$ and $\overline{Q}_{\overline{h}(\overline{\nu})}$ have the same topological type and the objects occurring in (1) to (3) vary continuously with $\overline{\nu}$.

We won't go into more details, since the rest of the construction is fairly standard.

Blowing down again by means of Φ yields a topological equivalence between the families $\{Y_\nu | \nu \in (\mathbb{R}^\ell \setminus 0)\}$ and $\{Q_{h(\nu)} | \nu \in (\mathbb{R}^\ell \setminus 0)\}$. Here h: $= \Phi \circ \overline{h} \circ \Phi^{-1}$ outside $\nu=0$, and $h(0)=0$. Hence $\{Y_\nu\}$ and $\{Q_{h(\nu)}\}$ are almost equivalent.

From the preceding result we immediately obtain that $\{X_{u,v,w}\}$ and $\{Q_{u,v,w}\}$ are almost topologically equivalent if $z_o \neq 0$. Hence $\{X_{h(\nu)}\}$ and $\{Y_\nu\}$ are almost equivalent, which proves the result for this case.

Step 2. If $\{f_\nu | \nu \in \mathbb{R}^\ell\}$ is not versal, we extend it to a versal family $\{f_\sigma | \sigma \in \mathbb{R}^k\}$ $(k \leq \ell+3)$. Set $Y_{(\nu_1,\ldots,\nu_\ell,\nu_{\ell+1},\ldots,\nu_k)} = \text{grad}_{g_{\nu_1,\ldots,\nu_\ell}} f_{\nu_1,\ldots,\nu_k}$.

According to step 1 there is a continuous mapping $\tilde{h}$: $(\mathbb{R}^k,0) \to (\mathbb{R}^3,0)$ such that Y is topologically equivalent to $X_{\tilde{h}(\sigma)}$, the equivalence depending continuously on σ if $\sigma \in \mathbb{R}^k \setminus 0$. Restriction to $\mathbb{R}^\ell \subset \mathbb{R}^k$ yields the desired result,

Q.E.D.

Remarks.

1. We conjecture that the topological equivalence H_ν between Y_ν and $X_{h(\nu)}$

($\nu \in \mathbb{R}^{\ell} \setminus 0$) can be extended continuously to a topological equivalence for ν in a full neighbourhood of $0 \in \mathbb{R}^{\ell}$. This is not clear from the preceeding construction. Moreover, one can show that in some cases the obvious choice $H_o = \mathrm{id}: (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ doesn't work.

2. The proof of theorem II.b is completely analogous to that of II.a.

Appendix A. Strong contact equivalence

In part I we used a result concerning a pair (f,g) of functions $(\mathbb{R},0) \to (\mathbb{R},0)$ and their k-parameter deformations[1] $F,G : (\mathbb{R}^k \times \mathbb{R},0) \to (\mathbb{R},0)$.
With these deformations we associate the following sets:

$$N_\ell(F) = \{(u,x) \in \mathbb{R}^k \times \mathbb{R} \mid F|\{u\}\times\mathbb{R} \text{ is } \ell\text{-regular}^{2)} \text{ at } x\}$$

In the same way we associate $N_\ell(G)$ with G.

When dealing with bifurcation problems the following question arises (cf.section I): How do $N_\ell(F)$ and $N_\ell(G)$ change simultaneously if we consider another pair of deformations of (f,g) ?

A slight extension of the concept contact equivalence yields a method for treating these problems. We shall put these questions in a slightly more general framework, dealing with m-tuples of mappings.
The definitions and results bear a strong resemblance to those appearing in the theory of 'normal' contact equivalence. The concepts appear in a C^q-setting ($q \leq \infty$), since we shall apply them to the study of families of vectorfields, restricted to centermanifolds. For more motivation and other applications we refer to [11].

For $i = 1,\dots,m$ $f_i : (\mathbb{R}^n,0) \to (\mathbb{R}^{k(i)},0)$ will be C^q-germs in $0 \in \mathbb{R}^n$, and $F_i, G_i : (\mathbb{R}^k \times \mathbb{R}^n,0) \to (\mathbb{R}^{k(i)},0)$ will be k-parameter C^q deformations of f_i. We shall consider the m-tuples $\underline{F} := (F_1,\dots,F_m)$ and $\underline{G} := (G_1,\dots,G_m)$ as mappings $(\mathbb{R}^k \times \mathbb{R}^n,0) \to (\mathbb{R}^K,0)$, where $K := k(1) + \dots + k(m)$.

Definition A1

(i) Two m-tuples $\underline{F}$ and $\underline{G}$ of k-parameter C^q deformations of $\underline{f} = (f_1,\dots,f_m)$ are called strongly-K^s-equivalent ($s \leq q$) if there is a k-parameter C^s-unfolding $I : (\mathbb{R}^k\times\mathbb{R}^n,0) \to (\mathbb{R}^k\times\mathbb{R}^n,0)$ of the identity mapping on $\mathbb{R}^n$ and a C^s-germ $A : (\mathbb{R}^k\times\mathbb{R}^n,0) \to G\ell_{(k(1),\dots,k(m))}(K;\mathbb{R})$ such that:
$\underline{F}(u,x) = A(u,x).\underline{G}(u,x)$ (matrix multiplication).
Here $G\ell_{(k(1),\dots,k(m))}(K;\mathbb{R})$ is the group consisting of all real $K\times K$ matrices of the form

---

1) In this appendix we shall distinguish between unfoldings and deformations.
A deformation of $f: \mathbb{R}^n \to \mathbb{R}^m$ is a family $\{f_\mu \mid \mu \in \mathbb{R}^k\}$ such that $f_0 = f$.
The corresponding unfolding is the fiber preserving mapping $F: \mathbb{R}^k\times\mathbb{R}^n \to \mathbb{R}^k\times\mathbb{R}^m$ defined by $F(\mu,x) = (\mu,f_\mu(x))$.

2) i.e. $F_u(x) = \dots = \dfrac{d^{\ell-1}F_u}{d\xi^{\ell-1}}(x) = 0$, $\dfrac{d^\ell F_u}{d\xi^\ell}(x) \neq 0$.

$$M = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_m \end{bmatrix}, \quad \text{where} \quad M_i \in G\ell(k(i), \mathbb{R})$$

(ii) Let $F_i : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^{k(i)}, 0)$ and $G_i : (\mathbb{R}^\ell \times \mathbb{R}^n, 0) \to (\mathbb{R}^{k(i)}, 0)$ be two C^q deformations of a C^q germ $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k(i)}, 0)$ $(i=1,\ldots,m)$

A C^s-morphism from the m-tuple $\underline{G}$ to the m-tuple $\underline{F}$ is a triple (h,I,A), where :

- $h: (\mathbb{R}^\ell, 0) \to (\mathbb{R}^k, 0)$ is a C^s germ
- $I: (\mathbb{R}^\ell \times \mathbb{R}^n, 0) \to (\mathbb{R}^\ell \times \mathbb{R}^n, 0)$ is an ℓ-paramèter C^s-unfolding of the identity mapping on $\mathbb{R}^n$.
- $A: (\mathbb{R}^\ell \times \mathbb{R}^n, 0) \to G\ell_{(k(1),\ldots,k(m))}(K; \mathbb{R})$ is a C^s germ satisfying $A(0,x) = Id$

such that $h^*\underline{F}$ and $\underline{G}$ are strongly-K^s-equivalent via the pair (A,I) in the sense of A.1.(i). The m-tuple $h^*\underline{F}$ is the ℓ-parameter deformation of $\underline{f}$ defined by $h^*\underline{F}(v,x) = \underline{F}(h(v),x)$.

If $k=\ell$ and h is a germ of a diffeomorphism, then (h,I,A) is called a C^s-isomorphism.

(iii) Let $F_i : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^{k(i)}, 0)$ be a C^q-deformation of the germ $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k(i)}, 0)$ for $i = 1,\ldots,m$.

The m-tuple $\underline{F} = (F_1,\ldots,F_m)$ is called strongly-(K^s,t)-versal if with Any m-tuple $\underline{G} = (G_1,\ldots,G_m)$ of C^t-deformations of $\underline{f} = (f_1,\ldots,f_m)$ we can associate a C^s-morphism (h,I,A) from $\underline{G}$ to $\underline{F}$.

$\underline{F}$ is called strongly-(K^s,t)-universal if moreover the number of parameters is minimal with respect to the property of being versal.

(iv) Let $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k(i)}, 0)$ $(i=1,\ldots,m)$ be C^q germs.

Set $K := k(1) + \ldots + k(m)$.

The K^s-tangentspace to the m-tuple $\underline{f} = (f_1,\ldots,f_m)$ is the following submodule of $(E_n^s)^K$:

$$T^s_{\underline{f}} = E^s_n \left\{ \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}, \ldots, \begin{pmatrix} \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_m}{\partial x_n} \end{pmatrix} \right\} + f_1^* n^s_{k(1)} . E^s_n\{e_1,\ldots,e_{k(1)}\} + \ldots$$

$$+ f_m^* n^s_{k(m)} . E^s_n\{e_{k(1)+\ldots+k(m-1)+1},\ldots,e_{k(1)+\ldots+k(m)}\}$$

Here $e_j : \mathbb{R}^n \to \mathbb{R}^K$ ($j = 1,\dots,K$) is the germ in $0 \in \mathbb{R}^n$ of the constant mapping, which has the j^{th} basisvector of $\mathbb{R}^K$ as its image.
A germ $\varphi \in (\mathcal{E}_n^s)^K$ belongs to $f_j^* n^s_{k(j)} . \mathcal{E}_n^s\{e_{k(1)+\dots k(j-1)+1},\dots,e_{k(1)+\dots+k(j)}\}$ iff there is a C^s germ $A : \mathbb{R}^n \to \mathrm{End}(K;\mathbb{R})$ such that $\varphi(x) = A(x).\underline{f}(x)$, where $\underline{f}(x) = (f_1(x),\dots,f_m(x))^t \in \mathbb{R}^K$ and $A(x)$ is a real $K\times K$ matrix of the form

$$A(x) = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & A_j(x) & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$
$$\longleftarrow k(j) \longrightarrow$$

(v) A k-parameter m-tuple $\underline{F} = (F_1,\dots,F_m)$ as in A.1.(iii) is called <u>strongly-$(\mathcal{K}^s,t)$-transversal</u> if $t \leqq q-1$ and

$$(\mathcal{E}_n^t)^K \subset T^s_{\underline{f}} + \mathbb{R}\{\dot{\underline{F}}_{(1)},\dots,\dot{\underline{F}}_{(k)}\},$$

where $\dot{\underline{F}}_{(j)} = \left(\frac{\partial F_1}{\partial u_j}\Big|_{u=0},\dots,\frac{\partial F_m}{\partial u_j}\Big|_{u=0}\right)^t \in (\mathcal{E}_n^{q-1})^K \qquad (j=1,\dots,m)$

(vi) Let $\underline{f} \in (\mathcal{E}_n^{q+1})^K$ and suppose there are positive integers s and p such that $s \leqq p \leqq q$ and such that

$$(\mathcal{E}_n^p)^K \big/ T^s_{\underline{f}} \cap (\mathcal{E}_n^p)^K \text{ is a finite dimensional vectorspace over } \mathbb{R}.$$

Then the dimension of this vectorspace is called the <u>strong-$(\mathcal{K}^s,p)$-codimension of $\underline{f}$</u>

<u>Example A2</u> (cf. section I)
Let $f(x) = x^k.F(x)$ and $g(x) = x^\ell.G(x)$ $(\ell \geqq k \geqq 1)$, where F and G are C^q functions such that $F(0) \neq 0$, $G(0) \neq 0$. Set $s:=\max(k,\ell)$ and suppose $q-s \geqq 1$. Then for $s+1 \geqq p \geqq q$ we have, with $k,\ell \geqq 2$:

$$(\mathcal{E}_n^p)^2 \subset \mathcal{E}_n^{p-s-1}\left\{\begin{pmatrix} f\\0\end{pmatrix},\begin{pmatrix} 0\\g\end{pmatrix},\begin{pmatrix} \frac{df}{dx}\\ \frac{dg}{dx}\end{pmatrix}\right\} \oplus \mathbb{R}\left\{\begin{pmatrix}1\\0\end{pmatrix},..,\begin{pmatrix}x^{k-2}\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},..,\begin{pmatrix}0\\x^{\ell-2}\end{pmatrix},\begin{pmatrix}x^{k-1}\\0\end{pmatrix}\right\}\dots \quad (A1)$$

The right hand side of this inclusion is a direct sum of real vectorspaces. The projection π from $(\mathcal{E}_n^q)^2$ onto the second component is given by

$$\pi\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \varphi(0).\begin{pmatrix}1\\0\end{pmatrix} + \dots + \varphi^{(k-2)}(0).\begin{pmatrix}x^{k-2}\\0\end{pmatrix} + \psi(0).\begin{pmatrix}0\\1\end{pmatrix} + \dots + \psi^{(\ell-2)}(0).\begin{pmatrix}0\\x^{\ell-2}\end{pmatrix} +$$
$$[\varphi^{(k-1)}(0) - \alpha\cdot\psi^{(\ell-2)}(0)]\cdot\begin{pmatrix}x^{k-1}\\0\end{pmatrix} \quad (A2)$$

Here $\alpha = \frac{k.F(0)}{\ell.G(0)}$ and $\varphi^{(j)}(0)$ denotes $\frac{d^j\varphi}{dx^j}(0)$.

Note that the first component of the right hand side of (A1) is just the K^{p-s-1}-tangentspace $T_{\underline{f}}^{p-s-1}$ to the pair $\underline{f} = (f,g) \in (E_n^p)^2$. As a consequence the strong (K^{p-s-1},p)-codimension of this pair is $k+l-1$ for $s+1 \leq p \leq q$.

Remark A3

For $k=\ell=1$ the second component of the right hand side of (A1) is just $\mathbb{R}\{\begin{pmatrix}1\\0\end{pmatrix}\}$. For $k=1,\ell=2$ it is $\mathbb{R}\{\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\}$. In these cases π is given by

$$\pi\begin{pmatrix}\varphi\\\psi\end{pmatrix} = [\varphi(0) - \alpha.\psi(0)]\cdot\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and}$$

$$\pi\begin{pmatrix}\varphi\\\psi\end{pmatrix} = [\varphi(0) - \alpha\cdot\dot{\psi}(0)]\cdot\begin{pmatrix}1\\0\end{pmatrix} + \psi(0).\begin{pmatrix}0\\1\end{pmatrix} \quad \text{resp.}$$

The proofs of (A1) and (A2) are straightforward; just use the Taylor - expansions of φ and ψ up to and including terms of order k and ℓ resp., cf. [11]

As a consequence we obtain:

Corollary A4

A strongly-(K^{q-s-1},q)-transversal deformation of the pair (f,g) in example A2 is given by the following $(k+l-1)$-parameter deformation:

$$\begin{pmatrix}F(\mu_1,\ldots,\mu_{k+l-1},x)\\G(\mu_1,\ldots,\mu_{k+l-1},x)\end{pmatrix} = \begin{pmatrix}(x^k+\mu_1x^{k-1}+\mu_2x^{k-2}+\ldots+\mu_{k-1}x+\mu_k).F(x)\\(x^\ell+\mu_{k+1}x^{\ell-2}+\ldots+\mu_{k+\ell-2}x+\mu_{k+\ell-1}).G(x)\end{pmatrix} \tag{A3}$$

So K^s-transversality is rather easy to check. One of the deep results of the Thom-Mather theory of unfoldings asserts that transversality implies versality. A similar result holds within the framework of strong contact equivalence. However, we have to be careful because of the finite degree of differentiability of the germs we consider : losses of differentiability are unavoidable.

Suppose $\alpha: \mathbb{Z} \cup \{\infty\} \to \mathbb{Z} \cup \{\infty\}$ is a non decreasing, surjective, finite-to-one function such that $\alpha(p) > 0$ for $p \geq p_1$. Typical examples of such functions are $\alpha(q) = q-s+1$ (cf. example A2) and functions like $\alpha(q) = \min\{[\frac{q-s}{k+1}], q-1\}$, where $[z]$ is the greatest integer,smaller than z. (cf. [11]).

With any such α and any triple (h,n,K) of positive integers we associate a non decreasing,finite-to-one, surjective function $d := d_{(\alpha,h,n,K)} : \mathbb{Z}\to\mathbb{Z}$.

Let $c := c_{(\alpha,h,n,K)} := \min\{ p\in\mathbb{Z} \mid d_{(\alpha,h,n,K)}(p) > 0 \}$.

For a precise definition of d and c we refer to [11] .

Theorem A5

Let $F_j : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^{k(j)}, 0)$ $(j=1,\dots,m)$ be an r-parameter C^q deformation of a C^q germ $f_j : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k(j)}, 0)$. Let $\underline{F}$ denote the m-tuple $(F_1,\dots,F_m)$.

(i) If $\underline{F}$ is strongly-(K^s,p)-versal, then $\underline{F}$ is strongly-(K^s,p)-transversal ($s \leq p \leq q$).

(ii) Suppose $q \geq c(\alpha,h,n,K)$ and the strong $(K^{\alpha(p)},p)$-codimension of $\underline{f}$ is at most h for $p_1 \leq p \leq q$.
If $\underline{F}$ is strongly-$(K^{\alpha(p)},p)$-transversal for $p_1 \leq p \leq q$, then $\underline{F}$ is strongly $(K^{d(p)},p)$-versal for $c \leq p \leq q$.

Corollary A6

The deformation (A3) of the pair (f,g) is strongly $(K^{d(p)},p)$-versal.
Note that in this case $\alpha(p) = p-s+1$, $n=1$, $K=2$ and $h=k+l-1$.
The results of [11] yield for this case: $d(p) = [\frac{p-1}{k+\ell+1}] - 1$

$c = 2k + 2\ell + 3.$

Checking genericity

From the expression (A2) we obtain a useful criterion for deciding whether a given deformation is strongly-K-versal or not. An r-parameter deformation (F,G) of the pair (f,g) in example A2 is strongly-K-transversal iff $\{\pi\begin{pmatrix}\dot F_1\\ \dot G_1\end{pmatrix},\dots,\pi\begin{pmatrix}\dot F_r\\ \dot G_r\end{pmatrix}\}$ is a set of generators for the second component in the right hand side of (A1).

In Appendix B this criterion is used in the case $k=1,\ell=2$. So there we have to check whether (cf. remark A3):

$$\begin{vmatrix} \frac{\partial F}{\partial \mu_1}(0) - \alpha.\frac{\partial^2 G}{\partial \mu_1 \partial x}(0) & \frac{\partial G}{\partial \mu_1}(0) \\ \frac{\partial F}{\partial \mu_2}(0) - \alpha.\frac{\partial^2 G}{\partial \mu_2 \partial x}(0) & \frac{\partial G}{\partial \mu_2}(0) \end{vmatrix} \neq 0 \qquad \text{(A4)}$$

Appendix B. Genericity of the family $\overline{X}|S^2 \times \{\lambda=0\} \times \{\nu_4=\ldots=\nu_\ell=0\}$

We shall not prove the genericity of this 2 parameter family at every point of the two-sphere. In fact it suffices to check the genericity at the points of the catastrophe-set, corresponding to the occurrence of saddle-connections. Genericity at the other points of this set follows from the theory of unfoldings of functions.

We only carry out the calculations for the point $S(z_o)=(\overline{\nu}_1,\overline{\nu}_2,\overline{\nu}_3)$ of fig.II.5 corresponding to the occurrence of a saddle-node with a generalized saddle connection (fig. B.1). We consider the case: $\overline{\nu}_3 < 0$.

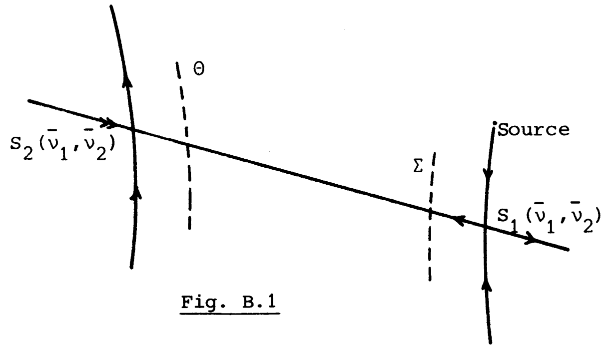


Fig. B.1

Since at $S(z_o)$ the projection from the two-sphere onto the plane $\{\nu_3=\overline{\nu}_3\}$ is a local diffeomorphism, it is sufficient to prove that the two parameter family $\{X_{(\nu_1,\nu_2,\overline{\nu}_3)}|(\nu_1,\nu_2)$ in a neighbourhood of $(\overline{\nu}_1,\overline{\nu}_2)\}$ is generic at $(\overline{\nu}_1,\overline{\nu}_2)$. The parameter $\overline{\nu}_3$ will be omitted from now on.

We have to prove that the determinant for this family, corresponding to that, appearing in (A4) of appendix A, is nonzero. However, the families F and G of (A4) are related to 'normal-form'-coordinates in a neighbourhood of S_1 and S_2. So we first introduce these local coordinates. In general it seems rather hopeless to check in a specific case whether the determinant (A4) is nonzero. However, by way of exception, Fortune is on our side this time: we have a rather detailed description of the catastrophe set in a neighbourhood of $S(z_o)$; moreover we are dealing with quadratic vectorfields, which have comfortable properties (cf. lemma II.1)

So let ξ, η be local coordinates in aneighbourhood of $S_1(\nu_1,\nu_2)$. We may and do assume that $S_1(\nu_1,\nu_2)$ corresponds to ($\xi=0$, $\eta=0$). Let $\varphi(\nu_1,\nu_2,\xi,\eta) = (\nu_1,\nu_2,\varphi_1(\nu_1,\nu_2,\xi,\eta), \varphi_2(\nu_1,\nu_2,\xi,\eta))$ be the corresponding change of coordinates; then we have: $\quad X(\varphi(\nu_1,\nu_2,0,0)) = 0$ (B1)

Moreover: $$\frac{\partial(\varphi_1,\varphi_2)}{\partial(\xi,\eta)}\Big|_{\xi=\eta=0}^{*)} = [V_1 \;\; V_2] \qquad \text{(B2)}$$

where V_1 and V_2 are the eigenvectors of $\dfrac{\partial(X_1,X_2)}{\partial(x,y)}\Big|_{(x,y)=S_1(\nu_1,\nu_2)}$, corresponding to the eigenvalues $\lambda_1(\nu_1,\nu_2) > 0$ and $\lambda_2(\nu_1,\nu_2) < 0$.

Let $Y = (\varphi^{-1})_* X$, then the family F corresponds to the second component of Y. Note that:

$$\left.\begin{aligned} Y_{\nu_1,\nu_2}(\xi,\eta) &= J_{\nu_1,\nu_2}(\xi,\eta).\; X_{\nu_1,\nu_2}(\varphi(\nu_1,\nu_2,\xi,\eta)) \\ \text{with}\quad J_{\nu_1,\nu_2}(\xi,\eta) &= d\varphi^{-1}_{\nu_1,\nu_2}(\varphi(\nu_1,\nu_2,\xi,\eta)) \end{aligned}\right\} \qquad \text{(B3)}$$

Let $\tilde{\xi},\tilde{\eta}$ be normal-form coordinates in a neighbourhood of S_2, $\tilde{\varphi}$ the corresponding change of coordinates and $\tilde{Y} = (\tilde{\varphi}^{-1})_*$ of X. Then our family G of appendix A corresponds to $G = \tilde{Y}_2 \circ P$, where $P: W^s(S_1) \longrightarrow W^s(S_2)$ was introduced in section I.

In a neighbourhood of $S(z_o)$ in the plane $\{\nu_3 = \bar{\nu}_3\}$ the catastrophe set locally looks like the set of fig. B2:

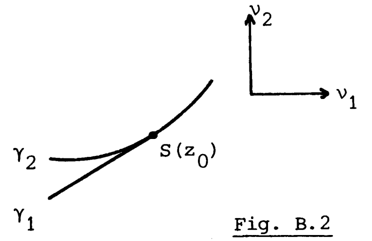


Fig. B.2

Here γ_1 is a straight halfline, tangent to γ_2 at $S(z_o)$.
Recall from section II that a parametrization for γ_2 was given by ($t \geq 1$):

$$\left.\begin{aligned} \nu_1(t) &= -\tfrac{1}{4}\bar{\nu}_3^{\,2}.\;(t^2 + \tfrac{2}{t}) \\ \nu_2(t) &= -\tfrac{1}{4}\bar{\nu}_3^{\,2}.\;(\tfrac{1}{t^2} + 2t) \end{aligned}\right\} \qquad \text{(B4)}$$

Assume that $S(z_o)$ corresponds to $(\nu_1(t_o), \nu_2(t_o))$. Along γ_2 we have a saddle node S_2. In terms of the parameter t the (x,y)-coordinates of S_2 are easily seen to be:

$$\left.\begin{aligned} x_2(t) &= \tfrac{1}{2}\bar{\nu}_3 t \\ y_2(t) &= \frac{\bar{\nu}_3}{2t} \end{aligned}\right\} \qquad \text{(B5)}$$

---

*) $$\frac{\partial(\varphi_1,\varphi_2)}{\partial(\xi,\eta)} = \begin{pmatrix} \dfrac{\partial\varphi_1}{\partial\xi} & \dfrac{\partial\varphi_1}{\partial\eta} \\[2ex] \dfrac{\partial\varphi_2}{\partial\xi} & \dfrac{\partial\varphi_2}{\partial\eta} \end{pmatrix}$$

The coordinates of the hyperbolic saddle S_1 can be solved in terms of t from the equations:

$$\begin{aligned} x^2 + \bar{\nu}_3 y + \nu_1(t) &= 0 \\ y^2 + \bar{\nu}_3 x + \nu_2(t) &= 0 \end{aligned} \qquad \text{(B6)}$$

Eliminating x from (B6) yields a polynomial equation of degree 4 for y, possessing a double root $y_2(t)$. Using this fact we compute straightforwardly the coordinates of S_1:

$$\begin{aligned} x_1(t) &= \bar{\nu}_3 \left(-\frac{1}{2}t + \frac{1}{\sqrt{t}}\right) \\ y_1(t) &= \bar{\nu}_3 \left(-\frac{1}{2t} + \sqrt{t}\right) \end{aligned} \qquad \text{(B7)}$$

We now proceed with the computation of the expressions, entering in (A4). First observe that we may as well use ν_1 as a local coordinate on the curve γ_2 in a neighbourhood of $\nu_1(t_o)$, since $\left.\dfrac{d\nu_1}{dt}\right|_{t=t_o} \neq 0.$

So suppose that γ_2 is given by $\nu_2 = \Psi(\nu_1)$; let $\tilde{\eta}(\nu_1)$ be the $\tilde{\eta}$-coordinate of $S_2(\nu_1, \Psi(\nu_1))$. From (B5) we obtain that $\tilde{\eta}$ depends smoothly on ν_1 in a neighbourhood of $\nu_1(t_o)$. Since $P(\nu_1, \nu_2, .) : W^s_{(\nu_1,\nu_2)}(S_1) \longrightarrow W^c_{(\nu_1,\nu_2)}(S_2)$ is a local diffeomorphism, there is a smooth function $\eta(\nu_1)$ such that

$$P(\nu_1, \Psi(\nu_1), \eta(\nu_1)) = \tilde{\eta}(\nu_1) \qquad \text{(B8)}$$

Note that: $\tilde{\eta}(\bar{\nu}_1) = 0$, $\eta(\bar{\nu}_1) = 0$.

Since γ_2 corresponds to the occurrence of saddle nodes, we obtain from (B8) and the fact that $G(\nu_1, \nu_2, \eta) = \tilde{Y}_2(\nu_1, \nu_2, P(\nu_1, \nu_2, \eta))$:

$$\left.\begin{aligned} &G(\nu_1, \Psi(\nu_1), \eta(\nu_1)) = 0 \\ &\frac{\partial G}{\partial \eta}(\nu_1, \Psi(\nu_1), \eta(\nu_1)) = 0 \end{aligned}\right\} \qquad \text{(B9)}$$

Differentiation of (B9) with respect to ν_1 yields:

$$\left.\begin{aligned} &\left.\frac{\partial G}{\partial \nu_1} + \frac{d\Psi}{d\nu_1} \cdot \frac{\partial G}{\partial \nu_2}\right|_{(\nu_1, \Psi(\nu_1), \eta(\nu_1))} = 0 \\ &\left.\frac{\partial^2 G}{\partial \nu_1 \partial \eta} + \frac{d\Psi}{d\nu_1} \cdot \frac{\partial^2 G}{\partial \eta \partial \nu_2} + \frac{d\eta}{d\nu_1} \cdot \frac{\partial^2 G}{\partial \eta^2}\right|_{(\nu_1, \Psi(\nu_1), \eta(\nu_1))} = 0 \end{aligned}\right\} \qquad \text{(B10)}$$

From (B1) we obtain: $\frac{\partial F}{\partial \nu_1}\Big|_{(\bar\nu_1,\bar\nu_2,o)} = \frac{\partial F}{\partial \nu_2}\Big|_{(\bar\nu_1,\bar\nu_2,o)} = 0$ (B11)

Finally, the constant α appearing in (A4) is easily seen to be:

$$\alpha = \left\{\frac{\partial F}{\partial \eta}\cdot\left(\frac{\partial^2 G}{\partial \eta^2}\right)^{-1}\right\}_{(\bar\nu_1,\bar\nu_2,o)} \tag{B12}$$

Combination of (B10), (B11) and (B12) yields:

$$\begin{vmatrix} \frac{\partial F}{\partial \nu_1} - \alpha.\frac{\partial^2 G}{\partial \nu_1 \partial \eta} & \frac{\partial G}{\partial \nu_1} \\ \frac{\partial F}{\partial \nu_2} - \alpha.\frac{\partial^2 G}{\partial \nu_2 \partial \eta} & \frac{\partial G}{\partial \nu_2} \end{vmatrix}_{(\bar\nu_1,\bar\nu_2,o)} = -\frac{\partial F}{\partial \eta}\cdot\left(\frac{\partial G}{\partial \eta^2}\right)^{-1}\cdot\begin{vmatrix} -\frac{d\eta}{d\nu_1}\cdot\frac{\partial^2 G}{\partial \eta^2} & 0 \\ * & \frac{\partial G}{\partial \nu_2}\end{vmatrix}_{(\bar\nu_1,\bar\nu_2,o)}$$

$$= \left(\frac{\partial F}{\partial \eta}\cdot\frac{\partial G}{\partial \nu_2}\cdot\frac{d\eta}{d\nu_1}\right)_{(\bar\nu_1,\bar\nu_2,o)}$$

Since $\eta = 0$ corresponds to the hyperbolic saddle S_1 for $X_{\bar\nu_1,\bar\nu_2}$, we already know $\frac{\partial F}{\partial \eta}\Big|_{(\bar\nu_1,\bar\nu_2,o)} \neq 0$.

So we only have to prove: $\frac{\partial G}{\partial \nu_2}\Big|_{(\bar\nu_1,\bar\nu_2,o)} \neq 0$ and $\frac{\partial \eta}{\partial \nu_1}\Big|_{\bar\nu} \neq 0$.

I. $\frac{\partial G}{\partial \nu_2}\Big|_{(\bar\nu_1,\bar\nu_2,o)} \neq 0$.

Proof: Since $\tilde Y = (\tilde\varphi^{-1})_* X$, we have for $\tilde Y$ an expression $(\widetilde{B3})$ similar to (B3). Observe that $\frac{\partial \tilde Y_2}{\partial \nu_2} = \frac{\partial G}{\partial \nu_2}$ at $(\bar\nu_1,\bar\nu_2,o)$, since $\frac{\partial \tilde Y_2}{\partial \tilde\eta}\Big|_{(\bar\nu_1,\bar\nu_2,\tilde\eta=0)} = 0$.

Differentiating $(\widetilde{B3})$ and using $X(\tilde\varphi(\bar\nu_1,\bar\nu_2,\tilde\xi=0,\ \tilde\eta=0)) = 0$ we obtain at $(\nu_1=\bar\nu_1,\nu_2=\bar\nu_2,\tilde\xi=0,\tilde\eta=0)$:

$$\begin{pmatrix} \frac{\partial \tilde Y_1}{\partial \nu_1} & \frac{\partial \tilde Y_1}{\partial \nu_2} \\ \frac{\partial \tilde Y_2}{\partial \nu_1} & \frac{\partial \tilde Y_2}{\partial \nu_2} \end{pmatrix} = \tilde J\cdot\left\{\frac{\partial(X_1,X_2)}{\partial(\nu_1,\nu_2)} + \frac{\partial(X_1,X_2)}{\partial(x,y)}\cdot\frac{\partial(\tilde\varphi_1,\tilde\varphi_2)}{\partial(\nu_1,\nu_2)}\right\} \tag{B13}$$

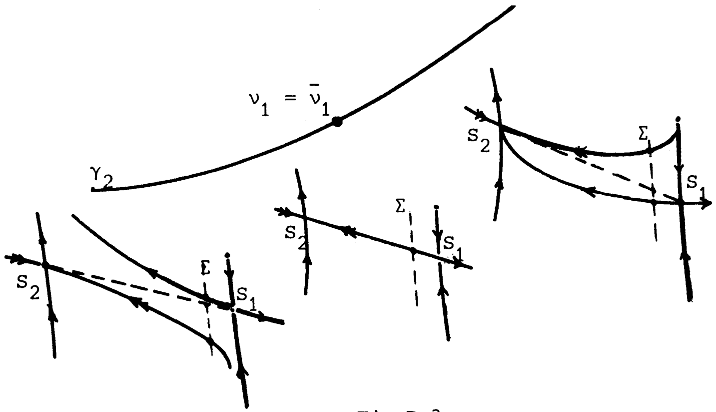
Observe that $\tilde{J} \cdot \frac{\partial(X_1,X_2)}{\partial(x,y)} \Big|_{(\bar{\nu}_1,\bar{\nu}_2,S_2)} = \begin{pmatrix} \tilde{\lambda}_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \tilde{J}$, where $\tilde{\lambda}$ is the negative eigenvalue of the linear part of $X_{(\bar{\nu}_1,\bar{\nu}_2)}$ at S_2. Hence the second term between accolades in (B13) does not add to the second row of the matrix in the left hand side. Moreover, $\tilde{J} = [\tilde{V}_1\ \tilde{V}_2]^{-1}$, where $\tilde{V}_1$ and $\tilde{V}_2$ are in the direction of the strong stable separatrix and the centermanifold of $S_2(\bar{\nu}_1,\bar{\nu}_2)$ resp.
Suppose $\tilde{V}_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then we obtain from (B13):

$$\begin{pmatrix} * & * \\ \frac{\partial\tilde{Y}_2}{\partial\nu_1} & \frac{\partial\tilde{Y}_2}{\partial\nu_2} \end{pmatrix}_{(\bar{\nu}_1,\bar{\nu}_2,0,0)} = c. \begin{pmatrix} * & * \\ v_2 & -v_1 \end{pmatrix} \begin{pmatrix} 1 & z \\ z & \beta \end{pmatrix}$$

where c is a nonzero constant.
Hence $\frac{\partial\tilde{Y}_2}{\partial\nu_2}(\bar{\nu}_1,\bar{\nu}_2,o) = c.\ (-\beta v_1 + z v_2) \neq 0$ for $|z|$ small, since $\frac{v_2}{v_1} = O(z)$ is the slope of the saddle connection occurring at $(\bar{\nu}_1,\bar{\nu}_2)$.

II. $\underline{\frac{d\tilde{\eta}}{d\nu_1}\Big|_{\bar{\nu}_1} \neq 0.}$

<u>Proof</u>: Along γ_2 we have the following situation (fig. B3)



<u>Fig.B.3</u>

In figure (B3) the point $\eta = \eta(\nu_1)$ corresponds to the intersection of Σ and the strong stable separatrix of S_2; the point $\eta = 0$ corresponds to the intersection of Σ and the unstable separatrix of S_1. This is obvious in view of (B8). Note that we may use η as a coordinate on the transversal section Σ. Geometrically the condition $\frac{d\eta}{d\nu_1}(\bar{\nu}_1) \neq$ means that the separatrices of S_1 and S_2 should cross at nonzero velocity at $\nu_1 = \bar{\nu}_1$.
In general this condition is not easy to check. However, here we succeed in the following way.
First observe that the straight line joining S_1 and S_2 has slope $-\frac{1}{\sqrt{t}}$. This is an easy consequence of (B5) and (B7) In the sequel we consider η as a function of the parameter t. We shall prove: $\frac{d\eta}{dt}\big|_{t_o} \neq 0$.
Next observe that for $t \neq t_o$ the vectorfield is transversal to the straight-line S_1S_2. If not we should have a saddle connection in view of the proof of lemma II.1. Hence the intersection $\eta = \rho(t)$ of this straight line and Σ lies between $\eta = 0$ and $\eta = \eta(t)$.
Since η and ρ depend smoothly on t we have: $\frac{d}{dt}(\eta - \rho)\big|_{t=t_o} > 0$. Hence it suffices to prove $\frac{d\rho}{dt}(t_o) \neq 0$, since it is obvious that $\frac{d\rho}{dt}(t_o) \geq 0$.

First we compute the slope m(t) of the expanding eigenvector of S_1. The condition $\frac{d\rho}{dt}(t_0) \neq 0$ is equivalent to $\frac{dm}{dt}(t_o) \neq \frac{d}{dt}\left(-\frac{1}{\sqrt{t}}\right)\big|_{t=t_o}$. Since (B7) provides all the ingredients for checking this condition, a straightforward, though tedious, computation shows that the latter condition is satisfied indeed. We omit further details.

Q.E.D.

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