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Elementary proofs of analytic hypoellipticity for $\Box_b$ and the $\bar{\partial}$ - Neumann problem

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ELEMENTARY PROOFS OF ANALYTIC HYPOELLIPTICITY FOR $\Box_b$ AND THE $\bar{\partial}$ - NEUMANN PROBLEM

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I. L^2 METHODS IN PROOFS OF ANALYTICITY. A SIMPLE NON-ELLIPTIC CASE

The real analyticity of solutions to (linear) elliptic partial differential equations with analytic coefficients is well known. The operator is given by

$$P = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$$

where the $a_\alpha(x)$ are real analytic (possibly matrix-valued) functions of $x \in \Omega \subset R^n$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$, and the ellipticity is:

$$p_m(x,\xi) \equiv \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0, \quad \xi \neq 0 \quad .$$

Theorem 1. Let P be a linear, elliptic partial differential with real analytic coefficients in an open set Ω in R^n . Let $Pu = f$ in Ω with $u \in \mathscr{D}'(\Omega)$ and f real analytic in Ω . Then u is also analytic in Ω .

There are several well known proofs; the most useful today are 1) via the construction of an analytic pseudo-differential parametrix for P , i.e., an analytic pseudo-differential operator ("AΨDO") Q of degree $-m$ such that $QP = 1 - R$ on $\varepsilon'(\Omega)$ where R maps $\varepsilon'(\Omega)$ into $\mathcal{A}(\Omega)$, the real analytic functions on Ω and Q preserves local real analyticity (cf., e.g., [1]) and 2) L^2 methods based on Gårding's inequality: for all $\Omega' \subset \Omega$, $\exists C$:

$$(1.1)\qquad \|v\|_m \le C(\|Pv\|_0 + \|v\|_0) \qquad v \in C_o^\infty(\Omega') ,$$

the $\| \ \|_s$ being L^2 Sobolev norms of order s in R^n .

Both of these proofs may be microlocalized, so that not only the analyticity but also the analytic wave front set of u may be reconstructed from that of f . In particular, one says that $(x_o,\xi_o) \notin WF_a(g)$ for a distribution g provided there exists a constant C_g and a neighborhood U of x_o and an open cone Γ such that for all N there exists g_N in $\mathcal{E}'(R^n)$ equal to g in U with

$$|\hat{g}_N(\xi)| \le C_g^N(1 + \frac{|\xi|}{N})^{-N}, \ \xi \in \Gamma$$

Theorem 1' (Hörmander, Sato). Under the hypotheses of Theorem 1, if $(x_o,\xi_o) \notin WF_a(f)$ then $(x_o,\xi_o) \notin WF_a(u)$.

One can construct the g_N in the above definition explicitly:

Proposition 1.1. Given open sets $U_1 \subset\subset U_2$ with $d = \mathrm{dist}(U_1,U_2^{comp}) > 0$, there exists a constant K such that: for any N , there exists $\phi_N(x)$ in $C_o^\infty(U_2)$ identically one on U_1 such that

$$(1.2)\qquad |D^\alpha\phi_N(x)| \le (Kd^{-1})^{|\alpha|}N^{|\alpha|} \quad \text{if} \quad |\alpha| \le 2N .$$

(The construction of these functions goes back to Ehrenpreis. They are "analytic up to level N" in that when $|\alpha| = N$, the growth is $(CN)^N \le (C')^N N!$).

Proposition 1.2. One may take the g_N in the above definition of $WF_a(g)$ to be $g_N(x) = \phi_N(x)g$.

To get a flavor of the proofs below, we prove Theorem 1 using the localizing functions ϕ_N above, but assuming for simplicity that u is known to be in $C^\infty(\Omega)$ already, since we are interested mostly in the passage from C^∞ to real analyticity. Also for simplicity we take $m = 2$.

<u>Proof of Theorem 1 when $u \in C^\infty$.</u>

We fix N and will show that $\sup \|D^\alpha u\|_{L^2(U_0)} \le C^{|\alpha|+1} N^N$, the supremum over all α with $|\alpha| \le N$, whence also pointwise for $K \subset\subset U_0$, $\sup_K |D^\alpha u(x)| \le C_K^{|\alpha|+1} N^N \le \tilde{C}_K^{N+1} N!$ Using (1.1), for some β with $|\beta| = 2$,

$$\|D^\alpha u\|_{L^2(U_0)} = \|\phi_N D^\beta D^{\alpha-\beta} u\|_{L^2} \le C\left(\|P\phi_N D^{\alpha-\beta} u\|_{L^2} + \|\phi_N D^{\alpha-\beta} u\|_{L^2}\right)$$

(1.3)

$$\le C\left(\|\phi_N D^{\alpha-\beta} f\|_{L^2} + \|[P,\phi_N] D^{\alpha-\beta} u\|_{L^2} + \|\phi_N [P, D^{\alpha-\beta}] u\|_{L^2} + \|\phi_N D^{\alpha-\beta} u\|_{L^2}\right)$$

Now $$[P,\phi_N] = \Sigma\, a_{ij}(x)\, D_i(\phi_N) D_j + \Sigma\, a_{ij}(x) D_i \circ (D_j \phi_N)$$

$$= \sum_j (\text{coef.})\phi_N' D_j + \sum_j (\text{coef.})\phi_N' D_j + \Sigma(\text{coef.})\phi_N''$$

and $$[P, D^{\alpha-\beta}] = \Sigma\, [\text{coef.}, D^{\alpha-\beta}] D_i D_j = \sum_{\substack{\gamma \le \alpha-\beta \\ 1 \le |\gamma|}} \binom{\alpha-\beta}{\gamma} (D^\gamma \text{coef.}) D^{\alpha-\beta-\gamma} D_i D_j$$

so that, writing now D^d for any D^δ with $|\delta| = d$,

$$\sum_{i \le 2} \|\phi_N^{(i)} D^{N-i} u\|_{L^2} \le C\Bigg(\sum_{1 \le j \le 4} \sup_{U_1} |\text{coef}|\, \|\phi_N^{(j)} D^{N-j} u\|_{L^2}$$

(1.4)

$$+ \sum_{k=1}^{N-2} \binom{N-2}{k} \sup_{U_1} |\text{coef.}^{(k)}|\, \|\phi_N D^{N-k} u\|_{L^2} + \|\phi_N D^{N-2} f\|_{L^2}\Bigg)$$

where "coef." denote any of the coefficients of P.

The above estimate may be rewritten

$$\sum_{i \le 2} \|\phi_N^{(i)} D^{N-i} u\|_{L^2} \le C\, \|\phi_N D^{N-2} f\|$$

(1.5)

$$+ \tilde{C} \sup_{\substack{i \le j+k = N \\ i,j \le 2}} (2C)^{j+k} N^k K_{\text{coeff}}^{k+1} \sum_{i \le 2} \|\phi_N^{(j+1)} D^{N-i-j-k} u\|_{L^2}$$

where we have used the bounds $\sup_{U_i}|\mathrm{coeff}^{(k)}| \leq K_{\mathrm{coeff}}^{k+1}\, k!$ and $\binom{N-2}{k} k! \leq N^k$ for $k \leq N$. We have also replaced the sum from $j+k = 1$ to N of $C^{j+k}F(j,k)$ by $\tilde{C}\sup(2C)^{j+k}F(j,k)$, the supremum over $j+k$ between 1 and N, with $\tilde{C}$ independent of N.

If one iterates this estimate until at most two derivatives are free to fall on u, noting that every gain in such derivatives has shown up as a factor of CKN or a derivative of ϕ_N and multiplication by C, in the end we have a right hand side of the form:

$$(1.6)\qquad \tilde{C} \sup_{\substack{j+k\leq N-2\\ i\ 2}} (2CK)^{j+k}N^k\|\phi_N^{(j)}D^{N-2j-k}f\|_{L^2} + \tilde{C} \sup_{\substack{j+k\leq N\\ i\leq 2}} (2CK)^{j+k}N^k\|\phi_N^{(j+i)}D^{2-i}u\|_{L^2}$$

Finally invoking the bounds in Proposition 1 for the derivatives of ϕ_N, together with similar bounds for derivatives of f, we have

$$\sum_{i\leq 2}\|\phi_N^{(i)}D^{N-i}u\|_{L^2} \leq (\tilde{\tilde{C}}^{N+1})N^N\left(1 + \sum_{|\alpha|\leq 1}\|D^\alpha u\|_{L^2(U_1)}\right)$$

uniformly in N, and hence the analyticity (in U_o).

Remarks on the L^2 approach.

A more conventional form of this proof (see, e.g., Hörmander, [10]) does not retain the same function ϕ_N throughout but introduces a new cutoff function for each iteration of (1.5). Thus each such function need only receive a few derivatives and N such are "nested", each one identically equal to one near the support of the previous one, the distance between the region where each is equal to one and the complement of its support being proportional to $1/N$, so that in a uniform manner first derivatives of these functions may be taken proportional to N, second derivatives proportional to N^2 (and so on up to any given number (independent of N)). Later on we shall have occasion to nest $\log_2 N$ such localizing functions ϕ_{N_j} with distances d_j between the set where $\phi_{N_j} = 1$ and the complement of its support proportional to j^{-2} and N_j proportional to $N/2^j$. If the derivatives grow as in Proposition 1 with these constants and if ϕ_{N_j} receives all $N/2^j$ derivatives, the total contribution to the iterated estimates will be $C^N N!$ times a product, from $j = 1$ to $\log_2 N$, of $(j^4)^{N/2^j}$, which is bounded:

$$\prod_{j=1}^{\infty} (j^4)^{N/2^j} \leq C_o^N \tag{1.7}$$

<u>Non-Elliptic Problems</u>.

For non-elliptic operators P, the proof of Theorem 1 will not apply, since Gårding's inequality fails. One can still achieve C^∞-regularity and even some Gevrey class regularity by means of "subelliptic" estimates of the form (for $m = 2$)

$$\|v\|_\varepsilon^2 \leq C(|(Pv,v)_{L^2}| + \|v\|_{L^2}^2) \qquad \varepsilon > 0 \tag{1.8}$$

such as were used by J.J. Kohn for the $\bar{\partial}$-Neumann problem and the complex boundary Laplacian, $\Box_b$, on, for example, strongly pseudo-convex domains. Here it is possible, as we shall discuss further below, to write the operator $\Box_b$ as a determined system

$$P = \Sigma\, a_{ij}(x,y,t)X_iX_j + \Sigma\, a_i(x,y,t)X_i + a_o(x,y,t) \tag{1.9}$$

where the $a_{i(j)}$ are smooth (analytic) square matrices and

$$\begin{aligned} X_j &= \partial/\partial x_j - y_j\partial/\partial t \qquad && j = 1,\dots,n \\ X_{n+j} &= \partial/\partial y_j && j = 1,\dots,n\,. \end{aligned} \tag{1.10}$$

The substitute for Gårding's inequality here is

$$(1.11) \qquad \|v\|_1 + \sum_{i,j} \|X_i X_j v\|_0 \leq C(\|Pv\|_0 + \|v\|_0)$$

for $v \in C_o^\infty$. Here $T = \partial/\partial t$. The following is in [16], [17]:

Theorem 2. Let the operator P have the form (1.9) and assume for simplicity that the coefficients $a_{i(j)}(x,y,t)$ are constant. Let the a priori estimate (1.11) be satisfied. If $u \in D'(R^n)$ with $Pu = f$ real analytic in U , then u is real analytic in U .

Remarks. The theorem is true with no change when the coefficients are variable. See Lecture 2 for this case. The theorem is true for higher order operators with corresponding a priori estimates. The obvious microlocal versions of the theorem are true, together with microlocal real analytic regularity theorems for certain boundary value problems for elliptic operators, such as the $\bar{\partial}$-Neumann problem. We shall come back to some of these later. For the moment we wish to set forth the method of proof in as transparent a form as possible.

The Constant Coefficient Proof on the Heisenberg Group.

We denote by X^q any $X_{i_1} \dots X_{i_q}$, noting that the X_j do not commute. Using a result by Nelson, in order to establish bounds

$$\|D^\alpha u\|_{L^2(U_o)} \leq C^{|\alpha|+1}|\alpha| \quad : \qquad \text{all } \alpha$$

it suffices to show that there is a C' such that for all a,b:

$$\|X^a T^b u\|_{L^2(U_o')} \leq C'^{a+b+1}(a+b)!$$

for some $U_o' \supset\supset U_o$. Now (1.11) treats X derivatives roughly as in the elliptic case, but not the T derivatives. For in estimating high T derivatives, the naive approach would bound, say, $\|X^2\phi T^r u\|_0$ by $\|P\phi T^r u\|_0$, and the commutator of

P with ϕT^r will contain $\phi' X T^r$ and $\phi'' T^r$, which can be resubjected to (1.11) only by writing a T in terms of two X's (by commutation); but the loss is too great; two derivatives on ϕ for each T derivative gained will never lead to analyticity. In place of ϕT^r, then, we seek another expression, equal to T^r on the set U_o yet compactly supported which commutes well with the X_j.

For $r = 1$ we have

$$\left[\phi T + \sum_{j=1}^{n} (X_{n+j}\phi) X_j - \sum_{j=1}^{n} (X_j\phi) X_{n+j},\ X_k\right] = \left[(T^1)_\phi, X_k\right]$$
$$= -\sum_{j=1}^{n} (X_k X_{n+j}\phi) X_j + \sum_{j=1}^{n} (X_k X_j \phi) X_{n+j} .$$

While ϕ does receive two derivatives, no T remains, and the "corrected" ϕT is equal to T in U_o since all other terms have derivatives on ϕ.

To generalize this case, it is convenient to denote by X' the vector $(X_1,\dots,X_n)$ and by X'' the vector $(X_{n+1},\dots,X_{2n})$.
Then if we define

$$(T^2)_\phi = \phi T^2 - \Sigma (X_j''\phi) X_j' T + \Sigma (X_j'\phi) X_j'' T + \sum_{|\alpha|=2} \frac{1}{\alpha!} (X''^{\alpha}\phi) X'^{\alpha}$$
$$- \Sigma\ (X_j' X_k''\phi) X_k' X_j'' + \sum_{|\beta|=2} \frac{1}{\beta!} (X'^{\beta}\phi) X''^{\beta} ,$$

we have

$$\left[(T^2)_\phi, X_j'\right] \equiv 0 \qquad \text{and} \qquad \left[(T^2)_\phi, X_j''\right] \equiv -(T^1)_{T\phi} X_j''$$

modulo terms of the form $(X^3\phi)X^2$. This suggests the following definition, where we use multi-index notation:

$$(1.12) \qquad (T^r)_\phi = \sum_{|\alpha+\beta|\le r} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} (X'^{\alpha} X''^{\beta}\phi) X'^{\beta} X''^{\alpha}\ T^{r-|\alpha+\beta|} .$$

Proposition 1.3. With $(T^r)_\phi$ defined as above,

$$\left[(T^r)_\phi, X'_j\right] \equiv 0 \quad \text{and} \quad \left[(T^r)_\phi, X''_j\right] \equiv -\,(T^{r-1})_{T\phi}X''_j$$

modulo C^r terms of the form $(X^{r+1}\phi)X^r/r!$

The proof is an easy calculation together with a shift of index.

Now we are ready to use (1.11) to prove analyticity. As indicated above, to estimate N derivatives of u in U_o, we shall nest $\log_2 N$ open sets U_j and choose functions ϕ_{N_j}, denoted ϕ_j for short, equal to one on U_j and in $C_o^\infty(U_{j+1})$ with

$$|D^\alpha \phi_j| \leq (Kd_j^{-1})^{|\alpha|} N_j^{|\alpha|} \quad \text{for} \quad |\alpha| \leq 2N_j = q_j \tag{1.13}$$

where $d_j = \text{dist}\,(U_j, R^{2n+1}\backslash U_{j+1}) = d/j^2$ and $N_j = N/2^j$. For $a \geq 2$ and $a + b \leq N_j$, we have the estimates

$$\|X^a T^b u\|_{L^2(U_j)} \leq \|X^2 (T^b)_{\phi_j} X^{a-2} u\|_{L^2} , \tag{1.14}$$

$$\begin{aligned} \sum_{i\leq 2} \|X^i (T^b)_{\phi_j} X^{a-i} u\| &\leq \|X^2 (T^b)_{\phi_j} X^{a-2} u\|_{L^2} \\ &+ \sum_{i\leq 2} \|X^i (T^{b-1})_{T\phi_j} X^{a-i} u\|_{L^2} + C^b \sum_{i\leq 1} \|X^i (X^{b+1}\phi_j) X^{b+a-i-1} u\|_{L^2} / b\,! \end{aligned} \tag{1.15}$$

(this expresses the result of trying to move two X's to the left of $(T^b)_{\phi_j}$) and

$$\begin{aligned} \|X^2 (T^b)_{\phi_j} X^{a-2} u\|_{L^2} &\leq C \left\{ \|P (T^b)_{\phi_j} X^{a-2} u\|_{L^2} + \|(T^b)_{\phi_j} X^{a-2} u\|_{L^2} \right\} \\ &\leq C \left\{ \|\Sigma(\text{coef.}) \left[X^2, (T^b)_{\phi_j} X^{a-2}\right] u\|_{L^2} + \|(T^b)_{\phi_j} X^{a-2} u\|_{L^2} \right\} \end{aligned} \tag{1.16}$$

(assuming that $Pu = 0$, as we may locally in the analytic case; the estimates merely get a bit more complicated looking if we keep f at each stage). Now

$$\left[X^2,(T^b)_{\phi_j}X^{a-2}\right] = \left[X^2,(T^b)_{\phi_j}\right]X^{a-2} + (T^b)_{\phi_j}\left[X^2,X^{a-2}\right]$$

$$= X(T^{b-1})_{T\phi_j}X^{a-1} + (T^{b-1})_{T\phi_j}X^a + \underline{Ca}\,(T^b)_{\phi_j}X^{a-2}T$$

$$+ \underline{C}^b(\phi_j^{(b+1)})X^{a-1+b}/b! + \underline{C}^b X \circ (\phi_j^{(b+1)})X^{a-2+b}/b!$$

where underlining a coefficient indicates the number of terms of the form which follows. Thus we obtain:

$$\begin{aligned}(1.17)\qquad &\sum_{i\le 2}\|X^i(T^b)_{\phi_j}X^{a-i}u\|_{L^2} \le C\sum_{i\le 2}\|X^i(T^{b-1})_{T\phi_j}X^{a-i}u\|_{L^2}\\ &+ \underline{Ca}\sum_{i\le 2}\|X^i(T^b)_{\phi_j}X^{a-2-i}Tu\|_{L^2} + C\underline{C}^b\sum_{i\le 2}\|X^i(\phi_j^{(b+1)})X^{a-1-i+b}u\|_{L^2}/b!\end{aligned}$$

Upon iteration, the terms still containing $(T^{b'})_{T^{b-b'}\phi_j}$ look like $CC^{b-b'}\sum_{i<2}(Ca)^k\|X^i(T^{b'})_{T^{b-b'}\phi_j}X^{q-2k-i}T^ku\|_{L^2}$, $b'\le b$ for some $k\le (a-2)/2$. If we subject such a term with three X's to (1.11) we may wind up with a term only containing one X. No matter; (1.11) also bounds terms with one X, though not quite so well, and this will occur only once in the lifetime of any given term. From this point on two X's will survive. The end result of these iterations will be to halve the total order, and one additional derivative will harmlessly be absorbed at this stage. We shall return to this point below.

When the fundamental estimate applied to an expression such as that in the previous paragraph yields a term containing only X's, i.e., a term such as the last in (1.15), it will be

$$CC^{b-b'}(Ca)^k\sum_{i\le 2}\tilde{C}^{b'}\|X^i(\phi_j^{(b+1)})X^{a-2k-i-1+b'}T^k u\|_{L^2}/(b')!$$

for some $b'\le b$ and $k\le(a-2)/2$. Thus we have at last

$$(1.18)\quad \|X^aT^bu\|_{L^2(U_j)} \leq C^{a+b} \sup_{b'\leq b,\, k\leq (a-2)/2} \sum_{i\leq 2} a^k \, \|X^i(\phi_j^{(b+1)})X^{a-2k-i-1+b'}\, T^k u\|_{L^2} / (b')!$$

Of course we also may use (1.11) to yield, as in the proof of Theorem 1 but with $Pu = 0$,

$$(1.19)\quad \sum_{i\leq 2} \|X^i\Psi X^{a'}T^k u\|_{L^2} \leq C \sum_{i\leq 2}\sum_{\ell\leq 2} \|X^i\Psi^{(\ell)}X^{a'-\ell}T^k u\|_{L^2} + C\sum_{i\leq 2} \underline{Ca'}\,\|X^i\Psi X^{a'-2}T^{k+1}u\|_{L^2}$$

(iterating)

$$\leq C^{a+1} \sup_{\substack{2k'+\ell\leq a' \\ \tilde{a}=2 \text{ or } 3 \\ i\leq 2}} (a')^{k'} \, \|X^i\Psi^{(\ell)}X^{\tilde{a}-i}T^{k+k'}u\|_{L^2}$$

(i.e., keep applying (1.11) until all but 2 or 3 X's either give T's or land on Ψ). Together with (1.17), this is

$$\|X^aT^bu\|_{L^2(U_j)} \leq C'^{\,a+b+1} \sup \|X^i(\phi_j^{(b+1+\ell)}X^{\tilde{a}-i}T^{k-k'}u\|_{L^2}/(b')!$$

where the supremum is taken over all k,k',ℓ,b', i and $\tilde{a}$ with $i \leq 2$, $b' \leq b$, $2k \leq a-2$, $\tilde{a} = 2$ or 3, and $2k' + \ell \leq a-2k + b' - 1$. Simplifying,

$$\|X^aT^bu\|_{L^2(U_j)} \leq C^{a+b+1} \sup a^k \|X^i\phi_j^{(b+1+\ell)})X^{\tilde{a}-i}T^{k-k'}u\|_{L^2}/(b')!$$

the supremum now over $\tilde{a} = 2$ or 3, $2k + \ell \leq a+b' -1$, $b' \leq b$.

Notice: the number of remaining derivatives is less than $(a+b)/2 + 3$.

In view of the estimate (recall $q_j \geq a+b$, $q_{j+1} = q_j/2$)

$j^{2(b+\ell+1)} k^{b+\ell+1} q_j^{k+b+\ell+1-b'} (q_j^{b'}/b') \leq C^{q_j} j^{4q_j} q_j^{q_j} q_{j+1}^{-k}$ under the above

constraints on $\tilde{a}, k, \ell,$ and b', we have finally

$$(1.20) \qquad \underset{a+b\leq q_j}{\text{supremum}} \frac{\|X^a T^b u\|_{L^2(U_j)}}{q_j!} \leq C^{q_j} j^{4q_j} \underset{\tilde{a}+\tilde{b}\leq q_{j+1}+4}{\text{supremum}} \frac{\|X^{\tilde{a}} T^{\tilde{b}} u\|_{L^2(U_j+1)}}{(q_{j+1})!} .$$

Iterating this result at most $\log_2 N$ times yields analyticity in view of (1.7), since the additional 4 derivatives on the right are harmless:

$$\underbrace{((\ldots(((N+4)/2)+4)/2\ldots/2+4)/2\ldots/2+4)}_{\log_2 N \text{ times}} \leq 8$$

II. NON-HYPOELLIPTIC OPERATORS; VARIABLE COEFFICIENTS

§1. Introduction.

The methods and results of the first lecture extend with no significant change to higher order systems satisfying analogous estimates. Using this, the author and L.P. Rothschild are currently studying questions of analytic regularity of solutions to operators which have only "relative" fundamental solutions and whose kernels and cokernels are infinite dimensional.

In particular, we consider homogeneous, left-invariant differential operators on the Heisenberg group H^n

$$L = \sum_{|I|=m} a_I X_{i_1} \ldots X_{i_I}$$

which are "elliptic in the generating directions", i.e., for which

$$\sum_{|I|=m} a_I (\partial/\partial x)_I \quad \text{is elliptic in} \quad R^{2n}$$

(and the a_I are constants). Thus here the X_j are (may be taken)* as in the first lecture, i.e., $X' = (X_1,\ldots,X_n)$, $X'' = (X_{n+1},\ldots,X_{2n})$ with

$$(2.1) \qquad X_i' = \partial/\partial x_i - y_i\partial/\partial t, \quad X_i'' = \partial/\partial y_i, \quad \text{and} \quad T = \partial/\partial t$$

Then Geller ([9]) proves the existence of a "regular homogeneous distribution" K (plus log terms if $\deg L \geq 2n+2$) and a principal value distribution P with $LK = \delta - P$. Here $f \to f*P$ is the L^2 projection onto the orthogonal complement of L acting on Schwartz functions. P is analytic except at 0, and $Lu = f \in \mathscr{D}'$ for some u in $\mathscr{D}'$ if and only if $f*P$ is real analytic near the point under consideration. When L is $\Box_b$ on functions, i.e.,

$$L = L_n = -\frac{1}{2}\Sigma\,(Z_j\overline{Z}_j + \overline{Z}_jZ_j) + inT, \quad Z_j = X_j' + iX_j''$$

in the conventional notation, then P is the Cauchy-Szegö kernel. In this case, the above results were obtained by Greiner-Kohn-Stein ([7]). When $P = 0$, the methods of the first lecture show that u is real analytic whereever Lu is.

When $P \neq 0$, we have studied the analyticity properties of K (and P) by using an idea of Beals and Greiner together with the estimates of the first lecture. Namely, for small, non-zero complex λ, $L_\lambda = L + \lambda T^{m/2}$ is again homogeneous and has no cokernel, i.e., the corresponding P_λ is zero, and hence the corresponding K_λ preserves local analyticity. Then, writing

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* The left invariant vector fields on the Heisenberg group are usually given as $X_j' = \partial/\partial x_j + 2y_j\partial/\partial t$, $X_j'' = \partial/\partial y_j - 2x_j\partial/\partial t$; since the group law is of no interest to us, a simple coordinate change brings them into the above form. The coefficients of L remain constant, of course.

$$Q_o = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \lambda^{-1} L_\lambda^{-1} d\lambda \quad \text{and} \quad B_o = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} L_\lambda^{-1} \delta\lambda$$

where Γ_ε is a small contour containing 0, we have

$$LQ_o = I - T^{m/2} B_o \quad .$$

The estimates of the first lecture, (uniform in λ!) then show that Q_o (and B_o) preserve (micro-)local real analyticity. We hope to study variable coefficient operators of this form soon.

§2. $\square_b$ and the $\bar\partial$-Neumann problem.

The (real analytic) boundary, Γ, of an open set Ω in $\mathbb{C}^n$ inherits a natural splitting of its complexified tangent space:

$$\mathbb{C}T\Gamma = T' \oplus T'' \oplus F \quad \text{where} \quad T' = \overline{T''}$$

and sections of T' are the restrictions to Γ of holomorphic vector fields (from $\mathbb{C}^n$) that are tangent to Γ. F has complex dimension 1 and is spanned by $J\partial/\partial\nu$, where ν is the (unit) interior normal to Γ. Letting $L_1,\ldots,L_{n-1}$ be certain local real analytic sections of T', and defining $\square_b$ in terms of the L_j, the operator

$$\square_b = \bar\partial_b \bar\partial_b^* + \bar\partial_b^* \bar\partial_b$$

may be viewed as a determined system mapping smooth $(p,q)_b$ forms to smooth $(p,q)_b$ forms, and, in a local frame, has the form

$$P = \sum_{i,j} a_{ij} \overset{(-)}{L_i} \overset{(-)}{L_j} + \sum_i a_i \overset{(-)}{L_i} + a_o$$

with real analytic matrix coefficients. When the Levi form, c_{ij} :

(2.2) $$[L_j, \overline{L_k}] \equiv c_{jk}T \mod (L_1,\dots,\overline{L_{n-1}})$$

satisfies Kohn's eigenvalue condition $Y(q)$, we have the a priori estimate

$$\sum_{i,j} \|\overset{(-)}{L_i}\,\overset{(-)}{L_j}v\|_{L^2} \leq C(\|Pv\|_{L^2} + \|v\|_{L^2})$$

for v of compact support. Writing $X_1,\dots,X_{2n-2}$ for the real and imaginary parts of the L_i, this estimate is (1.11).

§3. Proof of analytic hypoellipticity.

Now the operator has variable coefficients and the vector fields $\{X_j\}$ are not those given by (1.10). But a simple consequence (due to A. Dynin) of the Darboux theorem then asserts that when $\det(c_{jk}) \neq 0$, (i.e., the Levi form is non-degenerate) then we may change coordinates analytically and obtain the form (1.10) without changing the linear span of the X_j. And clearly we still have the "maximal" estimate:

(2.3) $$\sum_{i,j=1}^{2n-2} \|X_iX_jv\|_{L^2} \leq C(\|Pv\|_{L^2} + \|v\|_{L^2}), \quad v \in C_o^\infty(\omega)$$

The proof of analytic hypoellipticity here is the same as before except that the coefficients are variable. Thus in estimating, for $a \geq 2 \geq i, \|X^i(T^b)_\phi T^cX^{a-i}u\|_{L^2}$ we encounter, in $[P,(T^b)_\phi T^cX^{a-2}]$, obvious terms such as those in (1.17) where $(T^b)_\phi[\text{coef}.X^2,T^cX^{a-2}]$ enters, which are handled as before, and terms of this form

$$[g(x,y,t),(T^b)_\phi]T^cX^a$$

where g is one of the coefficients of P. This has been treated in [1], but we have a simpler version of this essential Lemma which we give here:

Lemma 2.1. Let $g(x,y,t) \in C^\infty(R^{2n+1})$. Then for any $\phi,v \in C_o^\infty$ and any $s \geq 0$,

$$|[(T^s)_\phi, g]v| \le \sum_{0<i+2j+k<s} c_{ijks} \underline{C^{i+j}} \, |(D^{i+j+k}g)|$$

$$(2.4) \qquad \cdot \operatorname*{supremum}_{|\mu'|\le i, |\nu''|+|\nu'|\le j=|\nu|} |Y^{\mu'+\nu'+\nu''} \circ X'^{\nu-\nu'-\nu''} \circ$$

$$\circ (T^{s-i-2j-k})_{D^{i+j+|\nu''|}\phi} \circ X''^{\nu'} v| \; .$$

Here $c_{jiks} = (s-i-j)!/i!j!k!(s-i-j-k)!$ All X's act on everything and D stands for a $\frac{\partial}{\partial z}$, $z = x_\ell, y_\ell, t$, or T or an X . The underlined coefficient indicates that for each i,j there are C^{i+j} terms of the form which follow.

Admitting the Lemma, which we shall prove in "easy" stages, note that $c_{ijks} \le \frac{s^k}{i!j!k!}$; the $(i!j!k!)^{-1}|(D^{i+j+k}g)|$ will be bounded by C_g^{i+j+k} ; derivatives still able to hit ϕ or g are reduced by k , hence the s^k . The rest of the decrease in s merely turns up as free X's or passes to ϕ .

<u>Proof</u>. First consider functions g independent of t . Then

$$\left[(T^p)_\phi, g\right] = \sum_{|a+b|\le p} \left[\frac{(-1)^b}{a!b!} (X'^a X''^b \phi) X'^b X''^a, g\right] T^{p-a-b}$$

$$(2.5) \qquad = \sum_{|a+b|\le p} \sum_{\substack{1\le|a'+b'|\le|a+b| \\ a'\le a \\ b'\le b}} \frac{(-1)^b}{a!b!} \binom{a}{a'}\binom{b}{b'} \cdot$$

$$\cdot (X'^{a'} X'^{a-a'} X''^{b-b'} X''^{b'} \phi) X'^{b-b'} X''^{a-a'} T^{p-|a+b|} \; .$$

Thinking of $a-a' = a''$ and $b-b' = b''$ as new indices depending on a',b' , associating $X''^{b'}$ directly with ϕ and, thanks to the special form of the $X'_k = \frac{\partial}{\partial x_k} - y_k \frac{\partial}{\partial t}$, so that $\left[\frac{\partial}{\partial x_k}, X''\right] = \left[\frac{\partial}{\partial t}, X''\right] = 0$, the extra $X'^{a'}$ to the left have their $\frac{\partial}{\partial x}$'s and $\frac{\partial}{\partial t}$'s pass directly to ϕ and the coefficients y_k remain with g . In all, then $X'^{a'}$ gives rise to $2^{|a'|}$ terms each of the form

$$\sum\sum \frac{g^{(a'+b')}}{a!b!} (-y)^{a'-\tilde{a}'} X'^{a''} X''^{b''} \left(\frac{\partial}{\partial t}\right)^{a'-\tilde{a}'} X''^{b'} \phi \circ$$

$$\circ X'^{b''} X''^{a''} T^{(p-|a'+b'|)-|a''+b''|} \Big/ a''!b''!$$

for some $\tilde{a}' \le a'$. That is, for g independent of t,

$$\left[(T^P)_\phi, g\right] = \sum_{1\le|a'+b'|\le p} (-1)^{b'} \sum_{\tilde{a}'\le a'} \binom{a'}{\tilde{a}'} (-y)^{a'-\tilde{a}'} \frac{(X'^{b'} X''^{a'} g)}{a'!b'!} \circ$$

$$\circ (T^{p-(a'+b')})\left(\frac{\partial}{\partial t}\right)^{a'-\tilde{a}'} \left(\frac{\partial}{\partial x}\right)^{\tilde{a}'} X''^{b'} \phi \quad .$$

When g depends on t, we must also consider the effect of $[T^{p-|a+b|}, g] = \sum_{1\le k\le p-|a+b|} \binom{p-|a+b|}{k} g^{(k)} T^{p-|a+b|-k}$. The difficulty here is that the coefficient $\binom{p-|a+b|}{k}$ should not involve $|a+b|$ explicitly or else it distorts the sum

(2.6) $$\Sigma \frac{(-1)^b}{a!b!} (X'^a X''^b \phi) X'^b X''^a T^{(p-k)-|a+b|}$$

with different weights for each $|a+b|$. To resolve this we use an elementary expansion for the binomial coefficient:

$$\binom{p-|a+b|}{k} = \sum_{\substack{t\le k\\ t\le|\alpha+\beta|}} \binom{p-t}{k-t} (-1)^t \binom{|a+b|}{t} .$$

Together with a change of indices this shows, after a tedious calculation, that

(2.7) $$\pi_s(A+A', B+B') = \sum_{i+2j+k\le s} C_{ijks} B'^{k+j} A'^{i} (-A)^j \pi_{s-2j-i-k}(A,B)$$

where $C_{ijks} = \dfrac{(s-i-j)!}{i!j!k!(s-i-j-k)!}$. Here $\pi_s(\underline{A},\underline{B}) = \sum_{\ell\le s} \underline{A}^{\ell} \underline{B}^{s-\ell}/\ell!$.

If we let $\underline{A} = \sigma(X_v, X_\phi) = \langle X''_v \cdot X'_\phi\rangle - \langle X'_v \cdot X''_\phi\rangle$ (X_v is the vector field X which "sees" only v and its derivatives, not ϕ, similarly with X_ϕ), then it is easy to see that

$$\pi_s(\sigma(X_v, X_\phi), T_v) = (T^s)_\phi v$$

provided X' always is taken to act to the left of X''. Now we have

$$(T^s)_\phi gv = \pi_s(A+A', B+B')gv$$

with $A = \sigma(X_v, X_\phi)$, $A' = \sigma(X_g, X_\phi)$, $B = T_v$, $B' = T_g$ so that (2.7) expresses our commutator, if we take $1 \leq i+2j+k \leq s$ in the sum, except that $A'^i(-A)^j \pi_{s-i-2j-k}(A,B)$ will contain the order restriction (X' to the left of X''). That is, $(-A)^j$ will contain some X'_v's just to the right of ϕ and A'^i will contain some "extra", X'_ϕ 's as the left-most derivatives of ϕ.

Written out in our previous notation, (2.7) looks like

$$(T^s)_\phi gv = \sum_{1 \leq i+2j+k \leq s} c_{ijks}\, T_g^{k+j} \sum_{i' \leq i} \sum_{j' \leq j} \binom{i}{i'} \binom{j}{j'} \cdot$$

$$\cdot\ (X''_g \cdot X'_\phi)^{i'} (-X'_g \cdot X''_\phi)^{i-i'} (X''_v \cdot X'_\phi)^{j'} (-X'_v \cdot X''_\phi)^{j-j'} g\ \cdot$$

$$\cdot \sum_{|a+b| \leq s-i-2j-k} \frac{(-1)^b}{a!b!} (X'^b X''^a \phi) X'^a X''^b T^{s-i-2j-k-|a+b|}\ v$$

but rember that $\underline{\text{all}}$ X'_ϕ act to the left of all X''_ϕ, all X'_v to the left of X''_v (and likewise for X'_g, X''_g). The X''_ϕ just sit against ϕ, and the X''_v just sit against v. As above, the extra X'_ϕ (within dot products) are written out:

$$X'_\phi = \left(\frac{\partial}{\partial x}\right)_\phi - y\left(\frac{\partial}{\partial t}\right)_\phi \quad ,$$

the derivatives passed onto ϕ, the powers of y left to the (extreme) left.

Extra X'_v (also within dot product) are written as $(X'_v + X'_\phi) - X'_\phi$ and the X'_ϕ treated as above.

Finally, expanding the dot products:

$$(A\cdot B)^\ell = \sum_{|\lambda|=\ell} \binom{\ell}{\lambda} \pi (A_j B_j)^{\lambda_j} = \sum_{|\lambda|=\ell} \binom{\ell}{\lambda} A^\lambda B^\lambda$$

where $\binom{\ell}{\lambda} = \frac{\ell!}{\lambda_1!\lambda_2!\ldots\lambda_n!}$, we have the statement of the Lemma. (Though this may seem complicated and combinatorial, the proof in [17] is more so).

To complete the proof of the theorem we have only to observe that the iterations of (1.17) of the first lecture proceed very much as in that case. We have seen how the terms in the Lemma contribute; losing at least $i+j+k$ free derivatives we allow $C_{(g)}^{i+j+k}$ since g belongs to the (finite) set of (analytic) coefficients of P ; $|y| < 1$ in supp ϕ so powers of y to the left of all derivatives are harmless; in the Lemma, of course, new X' derivatives appear to the left of $(T^{s'})_\phi$, - but these may be commuted to the right quite harmlessly.

The actual proof, as in [17], defines a formal norm for finite expressions of the form

$$G = \Sigma\, C_A \|G_{A,\phi}\|_{L^2} \quad \text{with} \quad G_{A,\phi} = X^a T^c (T^b)_{D^h \phi} T^d X^e u$$

where $A = (a,c,b,d,e,h)$, namely

$$\|G\|'_N = \Sigma |C_A| K^b N^{|A|+h+b}/b!$$

where $|A| = a+c+b+d+e$. Also, we set $\|A\| = |A| + c + d$, i.e. doubly weighting pure T derivatives. One then shows that applying (2.3) to an expression $X^2 G_{A,\phi}$ of the form above leads to $G = \Sigma\, C_{A',\phi} \|G_{A',\phi}\|$ whose $\|\ \|'_N$ norm is bounded by C times $\|X^2 G_{A,\phi}\|'_N$ but all A' in G have $|A'| < |A| + 2$ and $\|A'\| \leq \|A\| + 2$ (provided K is well chosen and we assume $|D^\alpha g| \leq \varepsilon^{|\alpha|}|\alpha|!$ in the (small) support of ϕ with ε small, as we may by dilation). Calling $G_{A',\phi}$ <u>simple</u> if $b = 0$ and $a+c \leq 1$, we may iterate the above for $G_{A',\phi}$ not (yet) simple. One apparent hitch is that if $b \neq 0$ and $a + c = 3$, two X's may commute to give a T ($[P,X]$ will contain coef.XT in general):

$$G_{A,\phi} = X^3 G'_{A',\phi} \longrightarrow XTG'_{A',\phi} = XG''_{A'',\phi}$$

So far $|A|$ has dropped by 1, $\|A\|$ remained the same. How to treat one X? A temporary excursion into forbidden territory is the solution:

$$\|XTG'_{A',\phi}\|^2_{L^2} \leq \frac{1}{2}\left(\|X^2TG'_{A',\phi}\|^2_{L^2} + \|TG'_{A',\phi}\|^2_{L^2}\right)$$

and since $TG'_{A',\phi}$ contains a $T = [X'_k, X''_k]$, in both terms on the right we have <u>two</u> X's, $|A|$ is back up, $\|A\|$ possibly up by 1. But on the next iteration, $\|X^2T\phi'_{A',\phi}\|_{L^2} \longrightarrow$ terms with at least two X's, and to get into this situation again with 3 X's $[\text{coef}, (T^b)_\phi]$ must generate an X; it does so (cf. (2.4)) in the form

$$Y^{\nu''}X'^{\nu-\nu'-\nu''} \circ (T^{b-2j})_{D^{i+|\nu''|}\phi} \circ X''^{\nu'} \quad ,$$

$|\nu| = j$, i.e. $|A|$ is now reduced below its original level, $\|A\|$ no greater. The $\| \ \|'_N$ norm thus rose briefly but is back down (after <u>two</u> iterations).

Thus the iterations may continue until each $G_{A,\phi}$ is simple (at most one X and $b = 0$). At this point the total order has dropped by half and the rest of the argument is just as in the constant coefficient case (cf. (1.20)).

III. THE ANALYTIC PSEUDODIFFERENTIAL CASE

In this lecture we want to show how these L^2 methods are able to appreciably simplify recent proofs of Métivier covering hypoelliptic pseudo-differential operators.

We consider a pseudo-differential operator (or system) with symbol

$$p(x,\xi) \sim p_m(x,\xi) + p_{m-1}(x,\xi) + \ldots$$

where $p_{m-j}(x,\xi)$ is homogeneous of degree $m-j$ in ξ. The characteristic variety $\Sigma = p_m^{-1}(0)$ is assumed to be a submanifold on which p_m vanishes to exactly order k and p_{m-j} vanishes to order $k-2j$. As in [4] and [13], we assume that P is subelliptic with loss of $\frac{k}{2}$ derivatives.

Theorem (Métivier [11]). If P is an analytic pseudo-differential operator as above and if, in addition, Σ is symplectic then P is analytic hypoelliptic.

The condition that Σ be symplectic, used in [17], is the same as the non-degeneracy on the Levi form in [16] and was actually introduced in [15] two years earlier to "break the Gevrey class 2 barrier"; unfortunately only quasi-analytic and $\bigcap_{s>1} G^s$ hypoellipticity was achieved in [15]. Also it was clear from [9] that some additional assumption on Σ was necessary for better than G^2 regularity, see also [2].

Métivier's proof, like Trèves' for the case $k = 2$ ([18]), constructs a micro-local pseudo-differential parametrix of type $\left(\frac{1}{2},\frac{1}{2}\right)$, and the estimates required are very delicate. After a microlocalization, which seems unavoidable here even for partial differential operators, he uses an analytic canonical transformation to bring Σ near $(x_o,\xi_o) \in \Sigma$ into a standard form

$$\Sigma = \{x_1 = \ldots = x_\nu = \xi_1 = \ldots = \xi_\nu = 0\}$$

and an associated elliptic Fourier integral operator with analytic real phase and classical analytic amplitude. The problem is reduced to one concerning

$$P = \sum_{|I|=m} C_I(x,D)A_I \quad .$$

The C_I are analytic pseudo-differential operators of order 0 in R^n, $2\nu + 1 \leq n$ and the A_i are given by $A_j = \partial/\partial x_j$, $1 \leq j \leq \nu$, and $A_{\nu+j} = x_j\, \partial/\partial x_n$, $1 \leq j \leq \nu$. The addition of variables brings P into the form (with the X_j of lecture I, $t = x_n$ and $y_j = x_{\nu+j}$)

$$P = \sum_{|I|\leq k} C_I(x,D)X_I$$

(after taking a power of P and multiplying by an elliptic factor so that $m = k > \nu$). From [4] and [5], we know that the hypotheses on P are preserved under these operations, and that what we must show is that $(0;0,\dots,0,1) \notin WF_a(u)$. The condition that P be microlocally hypoelliptic with loss of $k/2$ derivatives is that: the kernel in $\mathscr{S}$ of $\sigma^k_{(x_o,\xi_o)}(P)$ is $\{0\}$ for each $(x_o,\xi_o) \in \Sigma$; where

$$\sigma^k_{(x_o,\xi_o)}(P)(y,D_y) = \sum_{2j+|\alpha|+|\beta|=k} \frac{1}{\alpha!\beta!} \partial^\alpha_x \partial^\beta_\xi P_{m-j}(x_o,\xi_o) y^\alpha D^\beta_y$$

(cf [4], [13]), and this condition is equivalent to the "maximal hypoellipticity" of

$$P_{(x_o,\xi_o)} = \sum_{|I|=k} C_I(x_o,\xi_o)X_I$$

(see Beals [3], Helffer-Nourrigat [8]) that is, $\exists C_\omega : \forall v \in C^\infty_o(\omega)$, ω fixed,

$$(3.1) \qquad \sum_{|I|=k} \|X_I v\|_{L^2} \leq C_\omega\Big(\|P_{(x_o,\xi_o)} v\|_{L^2} + \|v\|_{L^2}\Big) \quad .$$

We claim that a similar estimate holds for P , provided the functions v are consider have suitably restricted conic support.

Since u belongs to ε' , it belongs to some $H^s(R^n)$. We shall assume that $s = 0$ for simplicity and in fact later, also for simplicity, that $u \in C_o^\infty$ (actually the C^∞ regularity is well understood ([4],[13])). The functions v we shall insert into our estimates will be of the form $v = \phi(x)\Psi(D)D^\alpha u$ where ϕ and Ψ have small (small conic) support, $\Psi(\xi)$ being homogeneous of degree zero outside a large sphere. Thus v has small x-support and the difference

$$\left(P_{(x_o,\xi_o)} - \Sigma\, c_I(x,\xi_o)X_I\right)v$$

has L^2 norm bounded by ε times the left hand side of (3.1). To replace $c_I(x,\xi_o)$ by $c_I(x,D)$, we introduce $\tilde{\Psi}(\xi) \equiv 1$ near $\operatorname{supp}\Psi(\xi)$ and obtain $(P_{(x,\xi_o)} = \Sigma\, c_I(x,\xi_o)X_I)$:

$$\|X_I v\| \le C\|P_{(x,\xi_o)}\phi\tilde{\Psi}\Psi D^\alpha u\| + \ldots$$

$$(3.2)\qquad \le C\|\Sigma c_I(x,\xi_o)\tilde{\Psi}X_I\phi\Psi D^\alpha u\|_{L^2} + \Sigma\|c_I(x,\xi_o)[X_I\phi,\tilde{\Psi}]\Psi D^\alpha u\|_{L^2} + \ldots$$

$$\le C\|\Sigma c_I(x,D)\tilde{\Psi}X_I\phi\Psi D^\alpha u\|_{L^2} + \varepsilon\Sigma\|X_I v\|_{L^2} + C\|v\|_{L^2} + \Sigma\|R_1^I\Psi u\|_{L^2} + \ldots$$

where $R_1^I = [X_I\phi,\tilde{\Psi}]D^\alpha$ (since, modulo terms of order -1, $c(x,\xi_o)$ differs from $c(x,D)$ on functions $\tilde{\Psi}\,w$ in norm by $\varepsilon\|\tilde{\Psi}w\|_{L^2}$ if the cone supporting Γ is narrow enough). Here the ...'s refer to the last term in (3.1), which will reappear below. Commuting $\tilde{\Psi}$ back past $X_I D^\alpha\phi$ only adds to the R^I , term; thus

$$(3.3)\qquad \Sigma\|X_I v\|_{L^2} \le C(\|Pv\|_{L^2} + \|v\|_{L^2} + \sum_I\|R_1^I\Psi u\|)$$

To control R^I (which has zero symbol) we present a lemma next which will be used again below.

<u>Lemma 3.1</u>. Let $\phi(x) \in C_o^\infty$ and $\Psi(\xi) \in C^\infty$ satisfy the estimates:

$$(3.4)\qquad \begin{aligned} |D_x^\alpha \phi| &\le K_\phi^{|\alpha|+1} N_\phi^{|\alpha|} && |\alpha| \le 2N_\phi \\ |D_\xi^\beta \Psi(\xi)| &\le K_\Psi^{|\beta|+1}\left(1 + \frac{|\xi|}{N_\Psi}\right)^{-|\beta|} && |\beta| \le 2N_\Psi \end{aligned}$$

Then for $k + s \le \min(N_\phi, N_\Psi), M$

$$[\phi, \Psi(D)] = \sum_{|\alpha| \le M} (D_x^\alpha \phi)((D_\xi^\alpha \Psi)(D))/\alpha! + R_M$$

where the L^2 norm of $D^k R_M D^s$ in a compact set K is bounded by

$$C_K C^s K_\Psi^{M+1} K_\phi^{M+k+n+1} N_\phi^{M+k+n+1} (N_\phi^s + N_\Psi^s)/M!$$

<u>Proof</u>. The symbol of R_M, $r_M(x,\xi)$, is $(2\pi)^{-n} \int r_M(x,\eta,\xi)\,d\eta$ where (cf. [1])

$$r_M(x,\eta,\xi) = \sum_{|\gamma|=M+1} i^{|\gamma|} e^{ix\cdot\eta} \eta^\gamma \hat{\phi}(\eta) \int_0^1 D_\xi^\gamma \Psi(\xi + t\eta)(1-t)^{|\gamma|-1} dt/\gamma!$$

For $x \in K$ and $2|\eta| \le |\xi|$, (3.4) implies

$$|\xi|^s |D_\xi^\gamma \Psi(\xi + t\eta)| \le N_\Psi^s K_\Psi^{|\gamma|+1} C^s ,$$

and

$$|\eta^{\tilde{\alpha}+\gamma} \hat{\phi}(\eta)| \le C K_\phi^{|\tilde{\alpha}+\gamma|+n+1} N_\phi^{|\tilde{\alpha}+\gamma|+n+1}/(1+|\eta|)^{n+1}$$

so that for $2|\eta| \le |\xi|$

$$|\xi|^s |D_x^{\tilde{\alpha}} r_M(x,\eta,\xi)| \le C^{s+1} K_\phi^{|\tilde{\alpha}+\gamma|+n+1} K_\Psi^{|\gamma|+1} N_\Psi^s N_\phi^{|\tilde{\alpha}+\gamma|+n+1}/(1+|\eta|)^{n+1}$$

When $|\xi| \leq 2|\eta|$,

$$|D_\xi^\gamma \Psi(\xi + t\eta)| < K_\Psi^{|\gamma|+1} \quad \text{and} \quad |\xi|^s |\eta^{\gamma+\tilde{\alpha}} \hat{\phi}(\eta)| \leq C^s K_\phi^{|\gamma+\tilde{\alpha}|+s} N_\phi^{|\gamma+\alpha|+s} .$$

Crudely estimating the L^2 norm in K of $D^k R_M D^s$ by a constant C_K times $\sup_{\substack{|\alpha|\leq n+1+k \\ x\in K,\xi}} |D_x^\alpha r_M(x,\xi)|\xi|^s|$, these estimates prove the Lemma.

The localizing functions $\phi_N(x)$ we use will be as in lecture I:

$$|D^\alpha \phi_{N_j}(x)| \leq (Cd_j^{-1})N_j^{|\alpha|} \qquad |\alpha| \leq 2N_j \tag{3.5}$$

with $d_j = d/j^2$, $N_j = N/2^{j-1}$. For the functions $\Psi_{N_j}(\xi)$ we have

<u>Proposition 3.1</u>. Given two cones, $\Gamma_1 \subset\subset \Gamma_2$ with the infimum on the sphere of $\left\{|\xi-\eta| \text{ for } y \in \Gamma_1, \xi \notin \Gamma_2\right\} = e$ there exist $\Psi_N(\xi) \in C_o^\infty(\Gamma_2)$ and identically one in $\Gamma_1 \cap \{|\xi| \geq 2N\}$ with

$$|D^\alpha \Psi_N(\xi)| \leq (\underline{K}e^{-1})^{|\alpha|+1} \left(\frac{N}{|\xi|}\right)^{|\alpha|} , \qquad |\alpha| \leq 2N$$

$\underline{K}$ is independent of N and the Ψ_N are zero for $|\xi| \leq N$.

Such functions are constructed in [1]. The restriction that the Ψ_N are zero for $|\xi| \leq N$ is minimal since for $|\xi| < N$, $|\alpha| \leq N$, $|\widehat{D^\alpha v}(\xi)|^2 \leq N^{2N}|\hat{v}(\xi)|^2$, and thus

$$||\widehat{D^\alpha v}(\xi)||_{L^2(|\xi|\leq N)} \leq C^N N! \; ||v||_{L^2} \tag{3.6}$$

To prove the theorem, we show that $(x_o,\xi_o) \notin WF_a(u)$, $x_o = 0$, $\xi_o = (0,\ldots,0,1)$. That is, by Proposition 1.2 of lecture I, that for some cone $\Gamma \supset (0,\ldots,0,1)$,

$$\|(1+|\xi|^2)^{(N+n_o)/2}\Psi_N(\xi)\widehat{\phi_N(x)u}(\xi)\|_{L^2} \leq C^N N^N$$

where n_o depends only on n. In the support of Ψ_N, $|\xi| \leq C|\xi_n|$ so it suffices to show that

$$\|T^N\phi_N(x)\Psi_N(D)u\|_{L^2} \leq C^N N^N$$

(using the lemma above to interchange ϕ_N and Ψ_N) and since $[X_j', X_j''] = T$ and k is even, we shall estimate

$$\|X^k T^N \phi_N(x)\Psi_N(D)\|_{L^2} \leq C^N N^N \quad . \tag{3.7}$$

As we saw in the previous lectures, merely localizing T^N in this fashion will not do. The suitable analogue of $(T^p)_\phi$ here is

$$(T^p)_{\phi\Psi} \sum_{|a+b|\leq p} \frac{(-1)^{|a|}}{a!b!} \ ad^a_{X'} \ ad^b_{X''}(\phi\Psi) \circ X'^b X''^a T^{p-|a+b|}$$

where $ad^{(i_1,i_2,\ldots,i_{|a|})}(g)$ stands for

$$[X'_{i_1},[X'_{i_2},[\ldots[X'_{i_{|\alpha|}},g]]\ldots]$$

etc. We have, as one may verifies at once,

$$\begin{aligned} &[X_j',(T^p)_{\phi(x)\Psi(D)}] \equiv 0 \quad \text{and} \\ (3.8)\qquad &[X_j'',(T^p)_{\phi(x)\Psi(D)}] \equiv (T^{p-1})_{[T,\phi(x)\Psi(D)]}X_j'' \end{aligned}$$

modulo C^p terms of the form

$$\mathrm{ad}_X^{p+1}(\phi\Psi) \circ X^p/p!$$

Sketch of the Proof.

This series of lectures is not the place to write out all details of the proof; they will appear in a forthcoming article. But aside from handling the localizing functions, via the above Lemma 3.1, along the same lines as in the case of function coefficients, the main difficulty arises in connecting the pseudo-differential coefficient with $(T^s)_{\phi\Psi}$. We outline below the result of this commutation.

We next open sets $U_o \subset\subset U_1 \subset\subset \ldots \subset\subset U_{\log_2 N} \subset\subset U$ and choose the separations $d_j = \mathrm{dist}(U_j, U_{j+1}^{\mathrm{comp.}}) = d/2^j$ as before, $(d_o = d)$ and let $N_j = N/2^{j-1}$, $\phi_j(x) \equiv 1$ near U_i, $\in C_o^\infty(U_{j+1})$ with

$$|D^\alpha \phi_j| \le C(kd_j^{-1})^{|\alpha|} N_j^{|\alpha|} \qquad |\alpha| \le 2N_j \quad .$$

We also nest cones $\Gamma_o \subset\subset \Gamma_1 \subset\subset \ldots \subset\subset \Gamma_{\log_2 N} \subset\subset \Gamma$ with separations

$$e_j = \mathrm{dist}(\Gamma_j \cap \{|\xi|=1\}, \Gamma_{j+1}^{\mathrm{comp}} \cap \{|\xi| = 1\})$$

with $e_j = e/2^j$, $e_o = e$, and $\Psi_j(\xi) \equiv 1$ on $\Gamma_j \cap \{|\xi| \ge N_j\}$, zero outside $\Gamma_{j+1} \cap \{|\xi| \ge N_{j+1}\}$, C^∞ , and homogeneous of degree zero in ξ for $|\xi| \ge N_j$ and with

$$|D^\alpha \Psi_j(\xi)| \le C(Ke_j^{-1})^{|\alpha|+1} \left(\frac{N_j}{|\xi|}\right)^{|\alpha|}$$

for $|\alpha| \le 2N_j$ $(= N/2^j)$.

The first step is to replace (3.7) by an expression whose main T-dependence is in the form of $(T^N)_{\phi\Psi}$. Taking the Ψ_N , ϕ_N in (3.7) to be ϕ_o, Ψ_o , and assuming $f = 0$ for simplicity and $u \in C_o^\infty$, we write

$$X^k T^N \phi_o(x)\Psi_o(D)u = X^k T^N \phi_o \Psi_o u = \phi_o \Psi_o X^k (T^N)_{\Psi_1 \phi_1} u + R_1 u$$

where

$$R_1 = [X^k T^N, \phi_o \Psi_o] + \phi_o \Psi_o\ X^k (T^N - (T^N)_{\Psi_1 \phi_1})$$

$$= \sum_{k'+N'=1}^{k+N} \binom{N}{N'} \binom{k}{k'} \mathrm{ad}_X^{k'} \mathrm{ad}_T^{N'} (\phi_o \Psi_o) X^{k-k'} T^{N-N'} + R_2$$

with

$$R_2 = \phi_o \Psi_o X^k (T^N - (T^N)_{\phi_1 \Psi_1})$$

(which has symbol $\equiv 0$).

Thus

$$X^k T^N \phi_o \Psi_o u = \sum_{k'+N'=0}^{k+N} \left\{ \binom{N}{N'} \binom{k}{k'} \mathrm{ad}_X^{k'} \mathrm{ad}_T^{N'} (\phi_o \Psi_o) X^{k-k'} (T^{N-N'})_{\phi_1 \Psi_1} \right\} + R_3^{k',N'} \tag{3.9}$$

with

$$R_3^{k',N'} = \binom{N}{N'} \binom{k}{k'} \mathrm{ad}_X^{k'} \mathrm{ad}_T^{N} (\phi_o \Psi_o) X^{k-k'} \left(T^{N-N'} - (T^{N-N'})_{\phi_1 \Psi_1} \right)$$

whose symbols are all zero.

These remainders are estimated via Lemma 3.1. Here we pursue the main argument.

In the first terms on the right in (3.9), we recognize that $\mathrm{ad}_X^{k'} \mathrm{ad}_T^{N'} (\phi_o \Psi_o)$ is a bounded operator in L^2. It consists of at most $2^{k'}$ terms of the form

$$\left(X^{k'-k''-2\ell} T^{N'+\ell} \phi_o \right) \circ \left(\mathrm{ad}_y^{k''} (\Psi_o) T^{k''} \right)$$

for some ℓ, k'' with $2\ell + k'' \leq k$. Now $\mathrm{ad}_y^{k''} (\Psi_o) T^{k''}$ is bounded in L^2 by construction of Ψ_o by $C(K e^{-1})^{k''} N_o^{k''}$ while

$$|X^{k'-k''-2\ell} T^{N'+\ell} \phi_o| \leq C(kd^{-1}N_o)^{k'-k''+N'-\ell}$$

so that the L^2 norm of $ad^{k'} ad_T^{N'}(\phi_o\Psi_o)$ is no greater than $C^2(KN_o(d^{-1}+e^{-1}))^{k'+N'}$

To estimate $\|X^{k-k'}(T^{N-N'})_{\phi_1\Psi_1} u\|$ we use (3.3), with errors $R_1^I u$ as indicated there which have symbol zero, and must bound the commutator

$$\left[P,(T^{N-N'})_{\phi_1\Psi_1}\right]u = \sum_{|I|=k}\left[C_I(x,D),(T^{N-N'})_{\phi_1\Psi_1}\right]X^k u \tag{3.10}$$
$$+ \sum_{|I|=k} C_I\left[X^k,(T^{N-N'})_{\phi_1\Psi_1}\right]u \;.$$

The C_I being bounded, the L^2 norm of the second term on the right is bounded by a constant times

$$\sum_{i\leq k} \|X^i(T^{N-N'-1})_{ad_T(\phi_1\Psi_1)} X^{k-i}u\|_{L^2} + \tag{3.11}$$
$$+ \sum_{i\leq k-1} \underline{C^{N-N'}}\, \|X^i \circ ad_X^{N-N'+1}(\phi_1\Psi_1) \circ X^{k-i-1+N-N'}u\|_{L^2}\Big/(N-N')!$$

just as in the earlier lectures, in view of (3.8). In fact, the further fate of the terms in (3.11) is just as before, with $\phi_j^{(\ell)}$ replaced here by $ad_X^\ell(\phi_1\Psi_1)$.

Writing $(T^s)_{\phi\Psi} = \sum_{|a+b|\leq s} \frac{(-1)^{|a|}}{a!b!} A_{ab}(\phi\Psi) \circ B_{ab}$ with $B_{ab} = X'^b X''^a T^{s-|a+b|}$,

we write

$$\left[C_I,(T^s)_{\phi\Psi}\right] = \sum\frac{(-1)^{|a|}}{a!b!}\left\{\left[C_I,A_{ab}\right]B_{ab} + A_{ab}\left[C_I,B_{ab}\right]\right\} \;. \tag{3.12}$$

Now the analysis of the second term is simple. For $[X_j,C_I]$ is the pseudo-differential operator given completely by the symbol $H_{\sigma(X_j)}\sigma(C_I)$, which we shall write $H_{X_j} C_I$ abusively. The analysis of the second lecture applies completely, and gives us a sum of the form

$$\sum_{\substack{1\le\ell\le s\\ i+2j+k=\ell\\ i'\le i,j'\le j,j''\le j-j'}} \sum_{|a+b|\le s-\ell} \frac{(-1)^{|a|}}{a!b!} X'^{j''} \circ H_{X'}^{j-j''+i'}(H_{X'}^{a} H_{X''}^{b}(H_{X''}^{i-i'+j-j'}\phi\Psi)) \circ$$

$$\circ\ ((-H_{X'})^{i-i'} H_{X''}^{i'} H_T^{k+j} C_I) \circ X'^{b} X''^{a} T^{s- -a-b} \circ X''^{j'} \Big/ (s-\ell)!$$

which have a good form except form except for the presence of $(H_{X'}),(H_{X''})C_I$ in the middle. One might think that $H_{X'}^{a''}$ on the left was a problem; but $H_{X'_j}$ consists of three terms: $\frac{\partial}{\partial x_j}$, $x_{\nu+j}\frac{\partial}{\partial x_n}$, and $\xi_n \frac{\partial}{\partial \xi_{\nu+j}}$. The first and third of these commute with $H_{X'}^{a}$, $H_{X''}^{b}$, as does $\frac{\partial}{\partial x_n}$. The "coefficients" of these $\frac{\partial}{\partial x_n}$, $x_{\nu+j}$, remain where they are or are even commuted with $X'^{j''}$ and brought out of the norm, just as in the second lecture. It remains to treat the problem of commuting c_I (or its derivatives) with operators such as $\tilde{\tilde{A}}_{ab}(\phi\Psi)$.

Here we need a special form of Leibnitz formula in terms of operators, not just symbols.

Proposition 3.2. Let G and H be classical pseudo-differential operators and let $G_{(\gamma)}^{(\delta)} = \mathrm{Op}(\partial_x^\delta D_\xi^\gamma \sigma(G))$ (full symbols). Let $\{G\}_s = G - \mathrm{Op}(\sigma_o(G)+ \dots + \sigma_s(G))$ where $\sigma(G) \sim \sigma_o(G) + \sigma_1(G) + \dots$. Then

$$[G,H] = \sum_{|\beta|=1}^{M} (-1)^{|\beta|+1}(G_{(\beta)} \circ H^{(\beta)} - H_{(\beta)} \circ G^{(\beta)})/\beta! + \tilde{R}_M \tag{3.12}$$

for any M , where

$$\tilde{R}_M = \sum_{|\beta|=0}^{M} (-1)^{|\beta|+1}\Big(G_{(\beta)}\circ H^{(\beta)} - \Big\{G_{(\beta)}\circ H^{(\beta)}\Big\}_{M-|\beta|} - (H_{(\beta)}\circ G^{(\beta)} - \Big\{H_{(\beta)}\circ G^{(\beta)}\Big\}_{M-|\beta|})\Big)/\beta! \tag{3.13}$$

In practice, $H = C_I$ (or some $H_X^i H_T^j C_I$) and $G = H_{X'}^a H_{X''}^b (\widetilde{\widetilde{\phi\Psi}})$ for some $\tilde{\phi}, \tilde{\Psi}$. That is,

$$(3.14) \qquad H_{X'}^a H_{X''}^b (\widetilde{\widetilde{\phi\Psi}}) \circ C_I \equiv \sum_{|\beta|=1}^{M} (-1)^{|\beta|+1} (H_{X'}^a H_{X''}^b (\widetilde{\widetilde{\phi\Psi}}))_{(\beta)} \circ C_I^{(\beta)}/\beta! + \sum_{|\beta|=0}^{M} (-1)^{|\beta|} C_{I_{(\beta)}} \circ (H_{X'}^a H_{X''}^b (\widetilde{\widetilde{\phi\Psi}}))^{(\beta)}/\beta!$$

modulo the remainder $\tilde{R}_M$ which consists entirely of terms of order $-M$. The operators $C_{I_{(\beta)}}(x,D)$ and $C_I^{(\beta)}$ are treated like C_I , while we rewrite $(H_{X'}^a H_{X''}^b (\widetilde{\widetilde{\phi\Psi}}))^{(\beta)}$ and $(H_{X'}^a H_{X''}^b (\widetilde{\widetilde{\phi\Psi}}))_{(\beta)}$ somewhat. Since $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial \xi_j}$ do not in general commute with $H_{X'}$ and $H_{X''}$, we cannot simply write $H_{X'}^a H_{X''}^b ((\widetilde{\widetilde{\phi\Psi}})^{(\beta)})$ and $H_{X'}^a H_{X''}^b ((\widetilde{\widetilde{\phi\Psi}})_{(\beta)})$; but we may write $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial \xi_j}$ as linear combinations of vector fields Y_k on T^*R^n with coefficients which are linear functions of the variables x_j , such that $[Y_k, H_{X'_j}] = [Y_k, H_{X''_j}] = 0$, pass these vector fields out $\widetilde{\widetilde{\phi\Psi}}$, and, as before, leave the powers of x to the left.

Using the Proposition, and the discussion before and after it, we can write $[C_I, (T^s)_{\phi\Psi}]$ as a sum of terms of the form $\tilde{C}_I \circ X^{\nu'} \circ (T^{s'})_{\phi'\Psi'} \circ X^{\nu''}$ plus $\tilde{R}_M$ with $\tilde{C}_I$ bounded in L^2 and some gain in $X^{\nu'}(T^{s'})_{\phi'\Psi'}X^{\nu''}$ as in Lemma 2.1 in lecture II.

All remainders encountered lead directly (i.e., without further iterations) to bounds of the form $C^N N^N$. We shall not work out the bounds in more detail here as they will appear in a forthcoming paper.

With these ingredients one may follow the iterations presented in the second lecture.

Concluding note. We do not claim that the above proof is "simple". But it is essentially elementary and accessible and, we feel, demonstrates the power of purely L^2 methods plus purely classical results on pseudo-differential operators.

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