O. LIESS

Uniqueness for the characteristic Cauchy problem and analytic regularity for partial differential equations with polynomial coefficients in the presence of growth type conditions

Astérisque, tome 89-90 (1981), p. 163-201
<http://www.numdam.org/item?id=AST_1981__89-90__163_0>

UNIQUENESS FOR THE CHARACTERISTIC CAUCHY PROBLEM AND ANALYTIC REGULARITY FOR PARTIAL DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS IN THE PRESENCE OF GROWTH TYPE CONDITIONS

by O. LIESS (Technische Hochschule Darmstadt)

§1. INTRODUCTION

1. In this paper we study the uniqueness of solutions for the characteristic Cauchy problem and regularity questions for solutions of a class of linear partial differential operators with polynomial coefficients when growth conditions are imposed at infinity. Let $s > 1$ and $h > 0$ be given and denote, here and in the sequel, by $\sigma = s/(s-1)$, and by $R_h^{n+1} = \{(x,t) \in R^{n+1} ; |t| < h\}$. Assume that $p(x,t,D_x,D_t)$ is an operator of form

$$(1) \qquad p(x,t,D_x,D_t) = \sum_{|\alpha|/s+|\beta|/\sigma+j \leq m} a_{\alpha\beta j}(t)\, x^\alpha D_x^\beta D_t^j$$

where the $a_{\beta j}(t)$ are real analytic functions defined for $|t| < h$ and $a_{\alpha\beta j}(t) \equiv 1$ when $j = m$. (Here α,β are multiindices, $|\alpha|,|\beta|$ is their lenghth and, as usual, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, D_x^\beta = (-i\partial/\partial x_1)^{\beta_1} \dots (-i\partial/\partial x_n)^{\beta_n}$, $D_t^j = (-i\partial/\partial t)^j$).

We can then prove the following two results:

Theorem 1.1. Let $s > 1$ and $h > 0$ be given and let p be an operator of form (1). Suppose $u \in \mathscr{C}^\infty(R_h^{n+1})$ is a solution of $p(x,t,D_x,D_t)\, u = 0$ such that

a) $\operatorname{supp} u \subset \{(x,t) \in R_h^{n+1} ; t \geq 0\}$

b) there is $B > 0$ such that for any γ, ℓ, we can find $C_{\gamma\ell} > 0$ with

$$|D_x^{\gamma} D_t^{\ell} u(x,t)| \leq C_{\gamma\ell} \exp(B|x|^s) \quad , \quad \forall \ (x,t) \in R_h^{n+1} \ .$$

Then it follows that $u \equiv 0$.

Theorem 1.2. Let $p(x,t,D_x,D_t)$ be as in Theorem 1.1 and assume that

$$\sum_{|\beta|/\sigma+j=m} a_{\alpha\beta j}(t)\ \xi^{\beta}\tau^{j} \neq 0 \quad \text{if} \quad (\xi,\tau) \in R^{n+1}\setminus\{0\} \quad \text{and} \quad |t| < h \ .$$

Let also $u \in \mathscr{C}^{\infty}(R_h^{n+1})$ be a solution of $p(x,t,D_x,D_t)\ u = 0$ which satisfies condition b) from the statement of Theorem 1.1. Then u is real analytic on R_h^{n+1} . Moreover for every $h' < h$ we can find $B' > 0$ and $c > 0$ so that u extends to an analytic function $\tilde{u}$ defined on $C_{h'}^{n+1} = \{(z,t) \in C^{n+1};\ |t| < h'\}$ for which $|\tilde{u}(z,t)| \leq c \exp(B'|z|^s)$ if $(z,t) \in C_{h'}^{n+1}$.

2. Operators of form (1) have been considered previously by J. Persson [1] when studying the Cauchy-Kowalewska theorem for analytic functions which satisfy growth conditions for $|z| \to \infty$ analogous to those used in this paper. We shall give later on examples which show that it is natural to restrict oneself to operators of form (1) in the context of this paper. When p is an operator with constant coefficients, i.e. when p has the form

$$(2) \qquad p(D_x,D_t) = \sum_{|\beta|/\sigma+j\leq m} a_{\beta j} D_x^{\beta} D_t^{j}$$

for some $a_{\beta j} \in C$, then the theorems 1.1 and 1.2 are wellknown. In fact, theorem 1.1 then reduces to a wellknown result of I.M. Gel'fand-G.E. Shilov [1], which generalizes earlier results concerning the characteristic Cauchy-problem for the heat equation (due to E. Holmgren, A.N. Tichonov, M. Nicolescu and others) and Theorem 1.2 is then essentially due to V.V. Grushin [1]. For a result related to Theorem 1.2 for the case of operators with polynomial coefficients, cf. I.A. Luckij

[1].

3. Results similar to those from the theorems 1.1 and 1.2 can be proved also when the solutions u of the equation $p(x,t,D_x,D_t)u = 0$ are distributions on which "growth" restrictions at infinity have been imposed. Since this only leads to supplimentary technical complications we restrict ourselves to the $\mathscr{C}^\infty$ case.

4. In the proof of the theorems 1.1 and 1.2 we shall study the singularities of the solutions of $p(x,t,D_x,D_t)\,u = 0$ by using a simultaneous localization in the t variable and in the variables Fourier-dual to x and t (one might note here that there is also a "localization" for the growth in the x variables). In analogy to the situation from the local analysis of singularities of solutions of partial differential equations we call this a microlocalization. To make this more precise we shall introduce (cf. definition 3.3 below) a notion of analytic wave front set adapted to the study of problems in which growth-type conditions are imposed at infinity. We shall then show that both theorems are in fact consequences of a microlocal variant of Theorem 1.2. This is in complete analogy to the local situation where Holmgren's uniqueness theorem and the analytic regularity of solutions of elliptic equations with analytic coefficients are both consequences of the regularity theorem of Hörmander-Sato (cf. Hörmander [3] and M. Kashiwara [1]). The microlocal variant of Theorem 1.2, which is stated after some preparations, as proposition 3.8, and which replaces the regularity theorem of Hörmander-Sato in this context, is therefore the main technical result from this paper.

5. Starting point for this paper has been an attempt to generalize Gel'fand-Shilov's theorem on the uniqueness of the characteristic Cauchy problem. One might then consider an operator of form

$$p(x,t,D_x,D_t) = \sum_{|\beta|/\sigma+j\leq m} a_{\beta j}(x,t)D_x^\beta D_t^j \ , \quad a_{om} \equiv 1 \tag{3}$$

for some $\sigma > 1$ and ask what conditions on p one should impose if one wants to conclude that $u \equiv 0$ for any solution $u \in \mathscr{C}^\infty(R_h^{n+1})$ of $p(x,t,D_x,D_t)\, u = 0$ concentrated in $t \geq 0$ and which satisfies $\sup_{(x,t)\in R_h^{n+1}} |D_x^\gamma D_t^\ell u(x,t)|/\exp(B|x|^s)<\infty$ $\forall \gamma, \ell$. Since no condition on the type of the operator is imposed in Gel'fand-Shilov's theorem we can only put restrictions on the regularity of the coefficients $a_{\beta j}$ and on their growth at infinity. It is then natural to assume that the coefficients $a_{\beta j}$ are at least real analytic.

Before we continue this discussion we now give some elementary examples.

Example 1.3. Denote by $\phi(x,t) = \exp(i(x^3+x)/t^3 - 1/t^4)$, by $a = -(\partial\phi/\partial t)/(\partial^2\phi/\partial x^2)$ and by

$$u = \begin{cases} \phi(x,t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} .$$

Then a is real analytic, bounded, and we have $(\partial/\partial t + a(x,t)\partial^2/\partial x^2)u=0$. This example is a variant of an example considered in L. Hörmander [4].

Example 1.4. Let $g(\theta): R \to R$ be a bounded $\mathscr{C}^\infty$ function with support in $\theta \geq 0$ but $0 \in \operatorname{supp} g$. Then

$$u = \begin{cases} g(t-1/x) & \text{for } t \geq 1/x \text{ if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

satisfies $(\partial/\partial t - x^2\partial/\partial x)\, u = 0$.

From this (standard) example one can see in particular that the problem in this paper is global.

Example 1.5. Consider the equation $p(x,D_x,D_t)u = D_t u - D_x^2(xu) + iu$ and denote by $H_+(\xi)$ the Heaviside function in $\xi \in R$. Then $u = \mathscr{F}^{-1}(H_+(\xi)\exp(-1/\xi + i\tau/\xi))$ is a solution of $p(x,D_x,D_t)\, u = 0$ and its Fourier transform $H_+(\xi)\exp(-1/\xi+i\tau/\xi)$ has an analytic extension for $\operatorname{Im} \tau \geq 0$ which is bounded. It follows that u is concentrated in $t \geq 0$. If we consider $v(t) \in \mathscr{C}_0^\infty(R)$, $\operatorname{supp} v \subset \{t \in R; t \geq 0\}$ and denote by $w = \int u(x,t-t')v(t')\,dt'$, then w is still a solution of $p(x,D_x,D_t)\,w = 0$ concentrated in $t \geq 0$ and it is easy to see that it is bounded with all derivatives. Essentially the same example is obtained if one simply performs a Fourier transform in x in example 1.4.

6. It follows from example 1.3 that one cannot, in general, prove results similar to the ones from this paper for operators of form (3) if one only assumes that the coefficients are real analytic. Another natural choice is then to assume that they are entire in x. When imposing also growth conditions of polynomial type on them (which is necessary in view of the other examples), we then arrive at coefficients which are polynomial in x. (Restriction to this case seems also natural in view of the results from J. Persson [1]). The examples 1.4, 1.5 and more sophisticated ones from S.D. Eidel'man [1] of the same type suggest that the results from this paper are optimal. I cannot prove this however.

7. The present paper is an extended version of a talk at the conference on "Analytic Solutions of Partial Differential Equations", Trento, March 1981. It does not contain the proof of proposition 3.8 below.

§2. THE SPACE $\mathscr{E}_A^s(R_h^{n+1})$ AND ITS DUAL. THE SPACE $\mathscr{A}_A^s(C_h^{n+1})$

1. In this paragraph we collect a number of facts concerning function spaces defined by growth type conditions. For more details we refer e.g. to Gel'fand-Shilov [2].

Definition 2.1. Consider $s > 1, A > 0, h > 0$. Then we denote by $\mathscr{E}_A^s(R_h^{n+1})$ the space of functions f from $\mathscr{C}^\infty(R_h^{n+1})$ such that the quantities

$$\sup_{x\in R^n, |t|\le h'} |D_x^\alpha D_t^j f(x,t)| \,/\exp(D|x|^s) \quad ,$$

are finite for all α , all j , all $D > A$ and all h' with $0 < h' < h$. $\mathscr{E}^s_A(R^{n+1}_h)$ will be endowed with the topology defined by the seminorms $f \to \sup |D_x^\alpha D_t^j f(x,t)|/\exp(D|x|^s)$, the sup being for $(x,t) \in R^{n+1}_h$. It is then a Fréchet space in which $\mathscr{C}^\infty_o(R^{n+1}_h)$ is dense. Its dual, endowed with the weak topology, will be denoted by $\mathscr{E}^s_A(R^{n+1}_h)'$.

When $f \in \mathscr{E}^s_A(R^{n+1}_h)$ and $g \in \mathscr{E}^s_{A'}(R^{n+1}_h)$, then it follows that the pointwise product $f(x,t)g(x,t)$ is in $\mathscr{E}^s_{A+A'}(R^{n+1}_h)$. If $f \in \mathscr{E}^s_A(R^{n+1}_h)$, then $f(x+x^o,t)$ is in $\mathscr{E}^s_A(R^{n+1}_h)$ for every $x^o \in R^n$. Moreover, if $\Psi \in \mathscr{C}^\infty_o(R^n)$ then $\Psi *' f = \int f(x-x',t)\Psi(x')dx'$ is in $\mathscr{E}^s_A(R^{n+1}_h)$. Further, if we fix $\zeta \in C^n$ and $\tau \in C$, then $\exp(-i\langle x,\zeta\rangle - it\tau)$ is in $\mathscr{E}^s_A(R^{n+1}_h)$ for any $s > 1$ and any $A > 0$. More precisely, when $s > 1$ and $D > 0$ are fixed, then it follows with the notations

(1) $$\sigma = s/(s-1), \quad \Delta = (sD)^{-\sigma/s}(1/\sigma) \quad ,$$

that

(2) $$|D_x^\alpha D_t^j \exp(-i\langle x,\zeta\rangle - it\tau) \le |\zeta|^{\alpha}|\tau|^j \exp(D|x|^s + \Delta|\mathrm{Im}\zeta|^\sigma + h'|\mathrm{Im}\tau|), \quad \text{if } |t| < h' .$$

In fact, (2) is a consequence of Youngs inequality $|\langle x,\xi\rangle| \le |x|^s/s + |\xi|^\sigma/\sigma$.

The notation introduced in (1), i.e. the association of σ,Δ with s,D , will be used very often in this paper, sometimes without explicit reference to (1).

2. Let us now consider $v \in \mathscr{E}^s_A(R^{n+1}_h)'$. Then $v(\exp(-i\langle x,\zeta\rangle - it\tau))$ makes sense for all $(\zeta,\tau) \in C^{n+1}$ and is an entire analytic function in (ζ,τ) which will be denoted by $\hat{v}$, or sometimes, by $\mathscr{F}v$. $\hat{v}$ will be called the Fourier-Borel transform of v (the same notations will also be used for the Fourier, or Fourier-Borel transform in more standard situations). It is easy to see that $v = 0$ if $\hat{v} \equiv 0$ on C^{n+1} . Further it follows from (2) that for any

$v \in \mathcal{E}^s_A(R^{n+1}_h)'$ we can find $c > 0$, $b \geq 0$, $D > A$ and $h' < h$ such that

(3) $$|\hat{v}(\zeta,\tau)| \leq c(1 + |(\zeta,\tau)|)^b \exp(\Delta|\mathrm{Im}\zeta|^\sigma + h'|\mathrm{Im}\tau|) .$$

Conversely, if $\phi \in \mathcal{A}(C^{n+1})$ (we denote by $\mathcal{A}(U)$ the functions $f : U \to C$ which are analytic in U) is an entire analytic function which satisfies an estimate of type (3), then it is classical (cf. e.g. I.M. Gel'fand-G.E. Shilov [2] for a similar result) that there is $v \in \mathcal{E}^s_A(R^{n+1}_h)'$ such that $\phi(\zeta,\tau) = \hat{v}(\zeta,\tau)$. Moreover, if $f \in \mathcal{E}^s_A(R^{n+1}_h)$, then $|v(f)| \leq c'$ where c' depends only on c,b,Δ,h' . If (3) is valid with c,b,Δ,h' for some given $v \in \mathcal{E}^s_A(R^{n+1}_h)'$ then we shall say that v is of order b .

<u>Lemma 2.2</u>. Consider $v \in \mathcal{E}^s_A(R^{n+1}_h)'$ for which (3) is valid and choose D' with $A < D' < D$. Then we can find a sequence $v_j \in \mathcal{E}^s_A(R^{n+1}_h)'$ such that:

a) all v_j are of order zero,

b) $v_j \to v$ in $\mathcal{E}^s_A(R^{n+1}_h)'$,

c) $|\hat{v}_j(\zeta,\tau)| \leq c'(1 + |(\zeta,\tau)|)^b \exp(\Delta'|\mathrm{Im}\zeta|^\sigma + h'|\mathrm{Im}\tau|)$,

for some c' which does not depend on j (Δ' is associated with D' via (1)).

In fact, if $g \in \mathcal{C}^\infty_0(R^{n+1})$ is positiv, with integral one, and if $g_j(x,t) = j^{n+1}g(jx,jt)$, then we can define v_j by $\hat{v}_j = \hat{g}_j\hat{v}$. a) and c) are then obvious and to prove b) it remains to observe that $g_j * f \to f$ in $\mathcal{E}^s_A(R^{n+1}_{h''})$ for $h'' < h$.

3. Consider now $f \in \mathcal{E}^s_A(R^{n+1}_h)$ and $D > A$, $h' < h$, $b \geq 0$. By the above f defines a linear functional on the space of entire analytic functions $\phi \in \mathcal{A}(C^{n+1})$ for which

$$\sup_{(\zeta,\tau)\in C^{n+1}} |\phi(\zeta,\tau)| / \exp(\Delta|\mathrm{Im}\zeta|^\sigma + h'|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|)) < \infty$$

and it is continuous if we endow this space of entire functions with an obvious topology. It follows from Hahn-Banach's theorem that we can find a Radon measure $\omega : C^{n+1} \to C$ (which depends on D,h',b) with the following two properties:

$$\int \exp(\Delta|\mathrm{Im}\zeta|^{\sigma} + h'|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|))\ d\omega(\zeta,\tau) < \infty$$

and

(4) $$v(f) = \int \hat{v}(\zeta',\tau')\ d\omega(\zeta',\tau')$$

if $v \in \mathscr{E}^{s}_{A}(R^{n+1}_{h})'$ satisfies

$$\sup_{\zeta,\tau} |\hat{v}(\zeta,\tau)| \,/\, \exp(\Delta|\mathrm{Im}\zeta|^{\sigma} + h'|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|)) < \infty .$$

In particular it follows from (4) that $f(x,t) = \int \exp(-i\langle x,\zeta'\rangle - it\tau')d\omega(\zeta',\tau')$ if $|t| < h'$. We shall therefore call ω a representation measure for f. Consider now again $f \in \mathscr{E}^{s}_{A}(R^{n+1}_{h})$ and also choose $v \in \mathscr{E}^{s}_{A'}(R^{n+1}_{h})'$ for some $A' > A$. We can then define an element $fv \in \mathscr{E}^{s}_{A'-A}(R^{n+1}_{h})'$ by $(fv)(g) = v(fg)$ if $g \in \mathscr{E}^{s}_{A'-A}(R^{n+1}_{h})$ such that it makes sense to ask for $\mathscr{F}(fv)(\zeta,\tau) =$ $= v(f\exp(-i\langle x,\zeta\rangle - it\tau)) = (\exp(-i\langle x,\zeta\rangle - it\tau)v)(f)$. Since $\mathscr{F}(\exp(-i\langle x,\zeta\rangle - it\tau)v)(\zeta',\tau') = \hat{v}(\zeta+\zeta',\tau+\tau')$ it follows after a change of variables that

(5) $$\mathscr{F}(fv)(\zeta,\tau) = \int \hat{v}(\zeta',\tau')d\omega(\zeta'-\zeta,\tau'-\tau)$$

if ω is a representation measure for f (and for suitable Δ, h', b).

4. It has been observed already in I.M. Gel'fand-G.E. Shilov [1] that the class of functions $g \in \mathscr{S}(R^{n})$ for which the Fourier transform $\hat{g}$ extends to an entire analytic function on C^{n} which satisfies

(6) $$|g(\zeta)| \leq c\ \exp(-c'|\zeta|^{\sigma} + c''|\mathrm{Im}\zeta|^{\sigma}) ,$$

for some positive constants c, c', c'' is of special interest in questions related to the uniqueness of the characteristic Cauchy problem. It is wellknown that non-trivial entire functions with (6) exist. More precisely we have

<u>Lemma 2.3</u>. Let $s > 1$ and $D > 0$ be given. Then one can find $g \in \mathscr{S}(R^n), g \not\equiv 0$ and positive constants $c_i, i=1,2,3,4,5$ such that

(7) $\quad |g(x)| \leq \exp(-D|x|^s), \quad \forall x \in R^n \quad ,$

(8) $\quad |\hat{g}(\zeta)| \leq c_1 \exp(-c_2|\zeta|^\sigma + c_3|\mathrm{Im}\zeta|^\sigma), \quad \forall\zeta \in C^n \quad ,$

(9) $\quad |\hat{g}(i\xi)| \geq c_4 \exp(-c_5|\xi|^\sigma) \quad$ if $\quad \xi \in R^n$.

Note that $\hat{g}$ is an entire analytic function in view of (7).
For a proof of Lemma 2.3 we refer e.g. to I.M. Gel'fand-G.E. Shilov̌ [2]. The last property is not explicitly stated in that book, but it is immediate if we consider a positive g with (7) and (8).

5. <u>Proposition 2.4</u>. Consider $s > 1$, $A > 0$ and let g be given by Lemma 2.3 for some $D > A$. Also choose $f \in \mathscr{E}^s_A(R^{n+1}_h)$ and assume that

(10) $\quad \int g(x') \exp(i\langle x',\xi\rangle) f(x-x',t)\, dx' = 0$

for all $(x,t) \in R^{n+1}_h$ and all $\xi \in R^n$. Then it follows that $f \equiv 0$. Note that (10) makes sense in view of (7).

<u>Proof</u> (cf. I.M. Gel'fand-G.E. Shilov [2]). If x,t are fixed, then $x' \to f(x-x',t)g(x')$ is in $\mathscr{S}(R^n)$. The condition (10) then just states that the Fourier transform of this function vanishes identically. It follows that $g(x')f(x-x',t) \equiv 0$ in x,x',t . Since $g \not\equiv 0$ this implies $f \equiv 0$.

6. <u>Proposition 2.5</u>. Let $s > 1, \Delta' > \Delta > 0, h > 0, b \geq 0, d_1 > d_2 > 0, \chi > 0$ be given. Then there are constants c,B such that

a) $\quad |D^\alpha_\zeta f(\zeta)| \leq cB^{|\alpha|}[\alpha/s]! \exp(\Delta'|\zeta|^\sigma), \forall\alpha, \forall\zeta \in C^n$ for any function $f \in \mathscr{A}(C^n)$ which satisfies $|f(\zeta)| \leq \exp(\Delta|\zeta|^\sigma)$.

b) $\quad |D^\alpha_\zeta f(\zeta)| \leq cB^{|\alpha|}[\alpha/s]! \exp(\Delta'|\mathrm{Im}\zeta|^\sigma + b \ln(1+|\zeta|)), \forall\alpha, \forall\zeta \in C^n$ for any $f \in \mathscr{A}(C^n)$ which satisfies $|f(\zeta)| \leq \exp(\Delta|\mathrm{Im}\zeta|^\sigma + b\ln(1+|\zeta|))$.

c) Fix $(\widetilde{\xi},\widetilde{\tau}) \in R^{n+1}$ and denote by Ω_i the sets $\Omega_i = \{(\zeta,\tau) \in C^{n+1};$ $|\zeta-\widetilde{\xi}|^\sigma + |\tau-\tau| \leq d_i(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$. Also choose $f \in \mathscr{A}(\Omega_1)$ such that $|f(\zeta,\tau)| \leq \exp(\Delta|\mathrm{Im}\zeta|^\sigma + h|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|))$ on Ω_1.

Then it follows that

$|D_\zeta^\alpha f(\zeta,\tau)| \leq cB^{|\alpha|}[\alpha/s]!\ \exp(\Delta'|\mathrm{Im}\zeta|^\sigma + h|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|))$ for $|\alpha| \leq \chi(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$ on Ω_2.

Here and in the sequel we will denote (for $a \in R_+$) by $[a]$ the integer part of a.

Proposition 2.5 is a wellknown consequence of Cauchy's integral formula. Cf. e.g. I.M. Gel'fand-G.E. Shilov [2] for the proof of a) and b). For the convenience of the reader we scetch the proof of the more technical c). To do so, we choose at first $d > 0$ such that $|\zeta-\widetilde{\xi}|^\sigma+|\tau-\widetilde{\tau}| < d_2(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$ and $|\theta|^\sigma \leq d(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$ implies $|\zeta-\widetilde{\xi}+\theta|^\sigma+|\tau-\widetilde{\tau}| \leq d_1(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$. With this choice of d it follows from Cauchy's integral formula that

$$|D_\zeta^\alpha f(\zeta,\tau)| \leq C'\alpha!\ \min_{d'\leq d}\ \max_{|\theta|^\sigma=d'(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)} |\theta|^{-|\alpha|}|f(\zeta+\theta,\tau)|$$

for all α and all (ζ,τ) with $|\zeta-\widetilde{\xi}|^\sigma+|\tau-\widetilde{\tau}| \leq d_2(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$. Let us also fix Δ'' such that $\Delta|\mathrm{Im}(\zeta+\theta)|^\sigma \leq \Delta'|\mathrm{Im}\zeta|^\sigma+\Delta''|\theta|^\sigma$. The proof then comes to an end if we can show that for suitable c_1, B

$$\min_{|\widetilde{\theta}|^\sigma\leq d(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)} \alpha!|\widetilde{\theta}|^{-|\alpha|-b}\exp(\Delta''|\theta|^\sigma) \leq c_1\ B^{|\alpha|}[\alpha/s]! \tag{11}$$

if $|\alpha| \leq \chi(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$.

Let us in fact choose $\widetilde{\theta}$ such that $\widetilde{\theta} = (d|\alpha|/\chi)^{1/\sigma}$. It follows in particular that $|\widetilde{\theta}|^\sigma \leq d(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$ as long as $|\alpha| \leq \chi(|\widetilde{\xi}|^\sigma+|\widetilde{\tau}|)$. It is a consequence of Stirling's formula that for such $\widetilde{\theta}$, $\alpha!|\widetilde{\theta}|^{-|\alpha|-b}\exp(\Delta''|\widetilde{\theta}|^\sigma)$ can be estimated by the right hand side of (11), whence the proposition.

7. Definition 2.6. Let $s > 1$ and $A > 0$, $h > 0$ be given. Then we denote by $C_h^{n+1} = \{(z,t) \in C^{n+1};\ |t| < h\}$ and by

$$\mathcal{A}_A^s(C_h^{n+1}) = \{f \in \mathcal{A}(C_h^{n+1});\ \sup_{z\in C^n, |t|<h'} |f(z,t)|/\exp(D|z|^s) < \infty \text{ for all } D > A \text{ and all } h' < h\} .$$

One can easily prove:

Proposition 2.7. Consider $f \in \mathcal{E}_A^s(R_h^{n+1})$. Then there are equivalent:

(i) There is $A' > 0, h' > 0$ such that f can be extended to an element in $\mathcal{A}_{A'}^s(C_{h'}^{n+1})$.

(ii) There are $D > 0, \eta > 0, c > 0$ such that $|v(f)| \leq c$ for all $v \in \mathcal{E}_D^s(R_h^{n+1})'$ which satisfy

$$(12) \qquad |\hat{v}(\zeta,\tau)| \leq \exp(\Delta|\zeta|^{\sigma}+\eta|\tau|) .$$

8. The estimates (3) respectively (12) correspond to the Paley-Wiener respectively Martineau-Ehrenpreis theorem. When multiplying v with an element from $\mathcal{A}_{D'}^s(C_{\eta'}^{n+1})$ then we obtain of course estimates of the same type. The following result shows that this is essentially also true locally. More precisely we have:

Lemma 2.8. Let $\varepsilon > 0, \tilde{D} > 0, \tilde{\eta} > 0, d', 0 < d' \leq 1$ be given and consider a holomorphic function $a(t)$ defined for $|t| < \eta$ such that $|a(t)| \leq 1$. Then we can find $c > 0, A > 0, \varepsilon' > 0, D' > 0, \eta' > 0$, which do not depend on $a(t)$, with the following property:

if $(\tilde{\xi},\tilde{\tau}) \in R^{n+1}$ is fixed and if $v \in \mathcal{E}_{D'}^s(R_{\eta'}^{n+1})'$ satisfies

$$(13) \qquad |\hat{v}(\zeta,\tau)| \leq \exp((\varepsilon'/2)(|\mathrm{Re}\zeta|+|\mathrm{Re}\tau|)+\Delta'|\mathrm{Im}\zeta|^{\sigma}+\eta'|\mathrm{Im}\tau|+b\ \ln(1+|(\zeta,\tau)|) \quad \forall(\zeta,\tau) \in C^{n+1}$$

respectively

(14) $\quad |\hat{v}(\zeta,\tau)| \le \exp(\Delta'|\mathrm{Im}\zeta|^{\sigma}+\eta'|\mathrm{Im}\tau|+b\ \ln(1+|(\zeta,\tau)|))$ for

$$|\zeta-\tilde{\xi}|^{\sigma}+|\tau-\tilde{\tau}| \ge d'(|\xi|^{\sigma}+|\tau|) ,$$

then the functions $f_{\nu}(\zeta,\tau)$ defined for $|\nu| \le (|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)$ by

$f_{\nu}(\zeta,\tau) = v(\exp(-i<x,\zeta>-it\tau)\ a(t)\cdot x^{\nu})$,

satisfy the estimates

(15) $\quad |f_{\nu}(\zeta,\tau)| \le cA^{|\nu|}[\nu/s]!\exp(\tilde{\Delta}|\mathrm{Im}\zeta|^{\sigma}+\tilde{\eta}|\mathrm{Im}\tau|+b\ \ln(1+|(\zeta,\tau)|))$ for

$$|\zeta-\tilde{\xi}|^{\sigma}+|\tau-\tilde{\tau}| \ge 3d'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|) ,$$

respectively

(16) $\quad |f_{\nu}(\zeta,\tau)| \le cA^{|\nu|}[\nu/s]!\exp(\varepsilon(|\mathrm{Re}\zeta|^{\sigma}+|\mathrm{Re}\tau|)+\tilde{\Delta}|\mathrm{Im}\zeta|^{\sigma}+\tilde{\eta}|\mathrm{Im}\tau|$

$$+b\ \ln(1+|(\zeta,\tau)|)) \quad \text{for} \quad |\zeta-\tilde{\xi}|^{\sigma}+|\tau-\tilde{\tau}| \le 3d'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|) .$$

Proof. We only prove (15). (16) may be proved with similar arguments but is easier.

Let us then choose $\chi', 0 < \chi' \le 1$ and consider a function $\rho(t) \in \mathscr{C}^{\infty}(R)$ (which depends on $\tilde{\xi},\tilde{\tau}$) such that for some $A_1 > 0$ and some $\eta', 0 < \eta' < \tilde{\eta}$:

a) $\operatorname{supp} \rho \subset \{t \in R;\ |t| \le \tilde{\eta}\}$,

b) $\rho(t) = 1$ for $|t| < \eta'$,

c) $|D_t^{\ell+b+2}\ \rho(t)| \le A_1^{\ell+1}\ \chi'^{\ell}(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)^{\ell}$ for $\ell \le \chi'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)$.

Here and in the sequel we shall denote by A_i and also by c_i positive constants which do not depend on $\tilde{\xi},\tilde{\tau}$ and v . The "b+2" has been inserted in c) for later convenience.

Later on we shall put restrictions on χ' , but A_1 must not depend on $\tilde{\xi},\tilde{\tau},\chi'$. It is wellknown that such functions exist, provided A_1 is great enough (cf. e.g. L. Hörmander [3]).

To simplify notations we denote $\chi'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)$ in the sequel by k' . We will also denote temporarily by $\mathscr{F}$ the Fourier-Borel transform in t . Since the same notation shall be used (still further) also for the Fourier-Borel transform in

(x,t) , the precise meaning of $\mathcal{F}$ must be clear from the context.

Now we observe that

$$(17) \qquad f_\nu(\zeta,\tau) = (1/2\pi)\int_{-\infty}^{\infty} (-D_\zeta)^\nu \hat{v}(\zeta,\tau')\mathcal{F}(\rho(t)a(t))(\tau-\tau')d\tau' \quad .$$

To estimate the integrand in (17) we apply Proposition 2.5. Since $\tau' \in R$ it follows that

$$|D_\zeta^\nu \hat{v}(\zeta,\tau')| \leq A_2^{|\nu|+1}[\nu/s]!\ \exp(\Delta|\mathrm{Im}\zeta|^\sigma + b\ \ln(1+|(\zeta,\tau')|))$$

$$\text{if} \quad |\zeta-\tilde{\xi}|^\sigma + |\tau'-\tilde{\tau}| \geq 2d'(|\tilde{\xi}|^\sigma + |\tilde{\tau}|) \quad \text{and} \quad |\nu| \leq (|\tilde{\xi}|^\sigma + |\tilde{\tau}|)$$

respectively

$|D_\zeta^\nu \hat{v}(\zeta,\tau')| \leq A_2^{|\nu|+1}[\nu/s]!\ \exp(\varepsilon'(|\mathrm{Re}\zeta|^\sigma + |\tau'|) + \tilde{\Delta}|\mathrm{Im}\zeta|^\sigma + b\ \ln(1+|(\zeta,\tau')|))$ in all other cases, provided that $\Delta' < \tilde{\Delta}$.

$\varepsilon' > 0$ is here chosen with $\varepsilon' \leq \varepsilon$, but later on we will impose another restriction on ε' .

At this moment we use the hypothesis on $a(t)$ and $\rho(t)$. This gives in view of Cauchy's inequalities

$$|D_t^{\ell+b+2}\ \rho(t)a(t)| \leq A_3^{\ell+1}\ k'^{\ell} \quad \text{if} \quad \ell \leq k'$$

such that

$$|\mathcal{F}(\rho(t)a(t))(\tau-\tau')| \leq c_1\ A_3^\ell\ k'^\ell (1+|\tau-\tau'|)^{-\ell-b-2}\ \exp(\tilde{\eta}|\mathrm{Im}\tau|) \quad \text{for} \quad \ell \leq k' \quad .$$

We conclude that

$$(18) \qquad |D_\zeta^\nu \hat{v}(\zeta,\tau')\mathcal{F}(\rho(t)a(t))(\tau-\tau')| \leq c_2\ A_4^{|\nu|+\ell}\ k'^\ell (1+|\tau-\tau'|)^{-\ell-b-2}[\nu/s]!\ \exp G$$

$$\text{for} \quad \ell \leq k'$$

where $G = \tilde{\Delta}|\mathrm{Im}\zeta|^\sigma + \tilde{\eta}|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau')|)$ when $|\zeta-\tilde{\xi}|^\sigma + |\tau-\tilde{\tau}| \geq 2d'(|\tilde{\xi}|^\sigma + |\tilde{\tau}|)$, respectively $G = \varepsilon'(|\mathrm{Re}\zeta|^\sigma + |\tau'|) + \tilde{\Delta}|\mathrm{Im}\zeta|^\sigma + \tilde{\eta}|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau')|)$ for

$|\zeta-\tilde{\xi}|^{\sigma}+|\tau-\tilde{\tau}| \le 2d'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)$.

When estimating the integrand from (17) we now distinguish between two cases. The first is when $|\zeta-\tilde{\xi}|^{\sigma}+|\tau'-\tilde{\tau}| \ge 2d'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)$. In this case we use (18) for $\ell = 0$ such that we can estimate this integrand by

$$(19) \qquad c_2 A_4^{|\nu|} \exp(\tilde{\Delta}|\mathrm{Im}\zeta|^{\sigma}+\tilde{\eta}|\mathrm{Im}\tau|+b \ln(1+|(\zeta,\tau)|))/(1+|\tau-\tau'|)^2 \quad .$$

The second situation is when $|\zeta-\tilde{\xi}|^{\sigma}+|\tau-\tilde{\tau}| \le 2d'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)$, such that $|\zeta|^{\sigma}+|\tau'| \le c_3(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)$ for some c_3 . Since in (15) we are interested only in points (ζ,τ) for which $|\zeta-\tilde{\xi}|^{\sigma}+|\tau-\tilde{\tau}| \ge 3d'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)$ we then have $|\tau'-\tau| \ge d'(|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|) \ge c_4(|\zeta|^{\sigma}+|\tau'|)$. We now apply (18) for $\ell = k'$ and choose χ' and ε' (in the order: first χ' and then ε') so small that

$$A_4^{k'}\, k'^{k'} c_4^{k'} (|\tilde{\xi}|^{\sigma}+|\tilde{\tau}|)^{-k'} \exp(\varepsilon'(|\mathrm{Re}\zeta|^{\sigma}+|\tau'|)) \le c_5 \quad .$$

The integrand in (17) can therefore also in the second case be estimated by (19), perhaps with c_2 replaced by some greater constant. The estimate (15) now follows.

Remark 2.9. It is clear that we now can also estimate a finite number of derivatives of f by (15) or (16).

Remark 2.10. In the proof of Lemma 2.8 we have introduced the constant χ' , which must be small. No other restriction is put on χ' however and the choice of χ' only affects the ε' .

§3. THE ANALYTIC WAVE FRONT SET WF_A^s. DEFINITIONS AND STATEMANT OF THE RESULTS

1. In this paragraph we introduce a notion of analytic wave front set adapted to the study of problems when growth type conditions in part of the variables

appear. To justify our definition we recall the following result:

Proposition 3.1. Let $f \in \mathscr{C}^{\infty}(R^n), x^o \in R^n$ and $\xi^o \in R^n \setminus \{0\}$ be given. Then there are equivalent:

a) $(x^o, \xi^o) \notin WF_A f$.

b) There are $\varepsilon > 0, c > 0$ and an open cone $\Gamma \subset R^n \setminus \{0\}$ which contains ξ^o such that $|v(f)| \leq c$ for any $v \in \mathscr{E}'(R^n)$ which satisfies the estimates

$$|\hat{v}(\zeta)| \leq \exp(\varepsilon|\mathrm{Im}\zeta| + \langle x^o, \mathrm{Im}\zeta\rangle) , \quad \text{for } \mathrm{Re}\zeta \notin -\Gamma ,$$

and

$$|\hat{v}(\zeta)| \leq \exp(\varepsilon|\zeta| + \langle x^o, \mathrm{Im}\zeta\rangle) \qquad \text{for } \mathrm{Re}\zeta \in -\Gamma .$$

Here $WF_A f$ denotes the (standard) analytic wave front set of f , introduced by M. Sato [1] and L. Hörmander [2]. Proposition 3.1 is a consequence of results from O. Liess [1] cf. also J.E. Björk [1], pp. 310-313, for arguments which immediately give this result. Note that $\varepsilon|\xi|+\langle x^o,\xi\rangle$ is just the support-function of an ε-neighborhood of x^o .

2. When trying to define a notion similar to WF_A in the case when growth type conditions appear, we have to replace conic neighborhoods of points by quasi-conic neighborhoods of a certain kind. At first we therefore explain what we shall call a quasi-cone in this paper (as well as other notions in this paper this will depend on s , but, since we may keep s fixed everywhere we will not make this dependance explicit in the terminology or notation).

Definition 3.2. $\Gamma \subset R^{n+1} \setminus \{0\}$ (or $\Gamma \subset C^{n+1} \setminus \{0\}$) will be called a quasi-cone if $(\xi,\tau) \in \Gamma$ implies $(t^{1/\sigma}\xi, t\,\tau) \in \Gamma$ for all $t > 0$.

Similar or also more general objects have been considered in connection with (standard) wave front sets e.g. by L. Hörmander [2] and B. Lascar [1].

Definition 3.3. Consider $f \in \mathscr{E}^s_A(R^{n+1}_h)$, $t^o \in R$, $|t^o| < h$ and

$(\xi^o,\tau^o) \in R^{n+1} \setminus \{0\}$. Then we shall say that $(t^o,\xi^o,\tau^o) \notin WF_A^s f$ if we can find $c > 0, \varepsilon > 0, D > 0, \eta > 0$ and an open quasi-cone $\Gamma \subset R^{n+1} \setminus \{0\}$ which contains (ξ^o,τ^o) such that $|v(f)| \leq c$ for any $v \in \mathscr{E}_D^s(R_h^{n+1})'$ which satisfies

(1) $$|\hat{v}(\zeta,\tau)| \leq \exp(\Delta|\mathrm{Im}\zeta|^\sigma + t^o\,\mathrm{Im}\tau + \eta|\mathrm{Im}\tau|) \text{ if } \mathrm{Re}(\zeta,\tau) \notin -\Gamma ,$$

respectively

(2) $$|\hat{v}(\zeta,\tau)| \leq \exp(\varepsilon(|\zeta|^\sigma + |\tau|)+\Delta|\mathrm{Im}\zeta|^\sigma+t^o\mathrm{Im}\tau + \eta|\mathrm{Im}\tau|) , \text{ if } \mathrm{Re}(\zeta,\tau) \in -\Gamma .$$

(Δ is here associated with D by (1), §2).

For technical reasons it is very useful to have the following equivalent characterization of WF_A :

<u>Proposition 3.4</u>. Consider $f \in \mathscr{E}_A^s(R_h^{n+1})$. Then there are equivalent:

(i) $(t^o,\xi^o,\tau^o) \notin WF_A^s f$.

(ii) There are $\varepsilon > 0, D > 0, \eta > 0$ an open quasi-cone $\Gamma \subset R^{n+1} \setminus \{0\}$ which contains (ξ^o,τ^o) and for every $b \geq 0$ some $c > 0$ such that $|v(f)| \leq c$ for any $v \in \mathscr{E}_D^s(R_\eta^{n+1})'$ which satisfies

(3) $$|\hat{v}(\zeta,\tau)| \leq \exp(\Delta|\mathrm{Im}\zeta|^\sigma + t^o\mathrm{Im}\tau+\eta|\mathrm{Im}\tau|+b\ln(1+|(\zeta,\tau)|)) , \text{ if } \mathrm{Re}(\zeta,\tau) \notin -\Gamma ,$$

respectively

(4) $$|\hat{v}(\zeta,\tau)| \leq \exp(\varepsilon(|\zeta|^\sigma+|\tau|)+\Delta|\mathrm{Im}\zeta|^\sigma+t^o\mathrm{Im}\tau+\eta|\mathrm{Im}\tau|+b\ln(1+|(\zeta,\tau)|)), \quad \text{if } \mathrm{Re}(\zeta,\tau) \in -\Gamma .$$

(iii) There $\varepsilon > 0, D > 0, \eta > 0$ and an open quasi-cone $\Gamma \subset R^{n+1} \setminus \{0\}$ which contains (ξ^o,τ^o) and for every $b \geq 0$ some $c > 0$ such that $|v(f)| \leq c$ for any $v \in \mathscr{E}_D^s(R_\eta^{n+1})'$ of order zero which satisfies (3) and (4).

In particular it follows from Proposition 3.4 that $(t^o,\xi^o,\tau^o) \notin WF_A^s D_s D_t^j f$ if $(t^o,\xi^o,\tau^o) \notin WF_A^s f$.

Proposition 3.4 will be proved in §6.

3. The notation WF_A^s from definition 3.3 is also justified by the fact that WF_A^s has a number of properties which one expects an analytic wave front set to have. Before we mention some of them we introduce a notation. Let then $s > 1$ be given and denote by $T : R^{n+1} \setminus \{0\} \to R^{n+1}$ the map

$$T(\xi,\tau) = (\xi/(|\xi|^\sigma + |\tau|)^{1/\sigma}, \tau/(|\xi|^\sigma+|\tau|)) \quad \text{for} \quad \xi \in R^n, \tau \in R \quad .$$

Thus the image of T is $\{(\xi,\tau) \in R^{n+1}; |\xi|^\sigma+|\tau| = 1\}$ and $T^{-1}(\xi,\tau) = \{(t^{1/\sigma}\xi, t\tau);\ t > 0$ if $|\xi|^\sigma + |\tau| = 1\}$. Further $\Gamma \subset R^{n+1} \setminus \{0\}$ is a quasi-cone precisely when $T^{-1}T\,\Gamma = \Gamma$.

<u>Proposition 3.5</u>. Let U be a closed set in $E = \{(\xi,\tau) \in R^{n+1}; |\xi|^\sigma+|\tau| = 1\}$ and consider $f \in \mathscr{E}_A^s(R_h^{n+1})$. Suppose that $(0,\xi^o,\tau^o) \notin WF_A^s f$ for all $(\xi^o,\tau^o) \in U$. Then we can find $\varepsilon > 0, D > 0, \eta > 0$ and for every $b \geq 0$ some $c > 0$ such that $|v(f)| \leq c$ for any $v \in \mathscr{E}_D^s(R_h^{n+1})'$ which satisfies

$$(5) \qquad |\hat{v}(\zeta,\tau)| \leq \exp(\Delta|\mathrm{Im}\zeta|^\sigma+\eta|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|)) \quad \text{if} \quad T(\mathrm{Re}\zeta,\mathrm{Re}\tau) \notin -U \quad ,$$

respectively,

$$(6) \qquad |\hat{v}(\zeta,\tau)| \leq \exp(\varepsilon(|\zeta|^\sigma+|\tau|)+\Delta|\mathrm{Im}\zeta|^\sigma+\eta|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|))$$
$$\text{if} \quad T(\mathrm{Re}\zeta,\mathrm{Re}\tau) \in -U \quad .$$

In particular it follows from this result that $f \in \mathscr{A}_B^s(C_\eta^{n+1})$ for some B and η if $(0,\xi^o,\tau^o) \notin WF_A^s f$ for all $(\xi^o,\tau^o) \in R^{n+1} \setminus \{0\}$. The converse is of course also true (cf. Proposition 2.7). Proposition 3.5 corresponds to H. Epstein's version of the edge-of-the-wedge theorem. It will be proved in §6.

<u>Proposition 3.6</u>. Consider $f_2, f_2 \in \mathscr{E}_A^s(R_h^{n+1})$ and assume that $WF_A^s f_1 + WF_A^s f_2 \overset{\text{def}}{=} \{(t,\xi^1 + \xi^2, \tau^1 + \tau^2); (t,\xi^i,\tau^i) \in WF_A^s f_i\} \subset \{(t,\xi,\tau) \in R^{n+2};\ |t| < h, (\xi,\tau) \neq 0\}$ Then we have

$$WF_A^s(f_1 \cdot f_2) \subset WF_A^s f_1 \cup WF_A^s f_2 \cup (WF_A^s(f_1) + WF_A^s(f_2))$$

(by definition we put $WF_A^s\, f_1 + WF_A^s\, f_2 = \emptyset$ if $WF_A^s\, f_1$ or $WF_A^s\, f_2$ is void). The hypothesis from Proposition 3.6 is in particular satisfied when $WF_A^s\, f_1 = \emptyset$ and it follows that $WF_A^s(f_1 \cdot f_2) \subset WF_A^s\, f_2$ then. Proposition 3.6 will be proved in §6.

4. The following two results are of particular interest in this paper.

<u>Proposition 3.7</u>. Consider $f \in \mathscr{E}_A^s(R_h^{n+1})$ such that $f(x,t) \equiv 0$ for $t < 0$ and assume that $(0,0,a) \notin WF_A^s\, f$ for $a = 1$ and $a = -1$. Then there is $\eta > 0$ such that $f(x,t) \equiv 0$ for $t \leq \eta$.

Proposition 3.7 will be proved in §6.

<u>Proposition 3.8</u>. Let $s > 1$ be a rational number and consider a linear partial differential operator of form

$$p(x,t,D_x,D_t) = \sum_{|\alpha|/s+|\beta|/\sigma+j\leq m} a_{\alpha\beta j}(t)\, x^\alpha D_x^\beta\, D_t^j \tag{7}$$

where the $a_{\alpha\beta j}$ are holomorphic functions defined for $|t| < \eta$, $\eta > 0$ and $a_{0,0,m} \equiv 1$. Let also $(t^o,\xi^o,\tau^o) \in R\times(R^{n+1} \setminus \{0\})$ be such that

$$\sum_{|\beta|/\sigma+j=m} a_{0\beta j}(t^o)\,(\xi^o)\,(\tau^o)^j \neq 0 \tag{8}$$

and consider $f \in \mathscr{E}_A^s(R_h^{n+1})$ such that $p(x,t,D_x,D_t)f = 0$.
Then it follows that $(t^o,\xi^o,\tau^o) \notin WF_A^s\, f$.

<u>Definition 3.9</u>. If $(t^o,\xi^o,\tau^o) \in R\times(R^{n+1} \setminus \{0\})$ satisfies (8) then we shall say that it is noncharacteristic for p.
Thus Proposition 3.8 replaces the regularity theorem of L. Hörmander-M. Sato for operators of type (7). It will be proved in a forthcoming paper.

It is now clear that Theorem 1.1 is a consequence of the Propositions 3.7 and 3.8. Similarly Theorem 1.2 is a consequence of the Propositions 3.5 and 3.8.

5. We conclude this paragraph with one more definition:

Definition 3.10. Consider $t^o \in R$ and $(\xi^o,\tau^o) \in R^{n+1} \setminus \{0\}$. Then we say that $U \subset R^{2n+2}$ is a microlocal neighborhood of $(t^o,\xi^o \tau^o)$ if it contains a set of form $\{(x,t,\xi,\tau) \in R^{2n+2}; x \in R^n, |t-t^o| < \varepsilon, |T(\xi,\tau)-T(\xi^o,\tau^o)| < \varepsilon\}$ for some $\varepsilon > 0$. Note that microlocal neighborhoods are global in x. This is justified by the fact that the conditions which define elements from $\mathscr{E}^s_A(R^{n+1}_h)$ are global in x.

§4. PREPARATIONS FOR THE STUDY OF WF^s_A

1. Let $s > 1$ be given. The notation σ has been introduced in (1), §2, and the notations T and E in §3.

Lemma 4.1. For every $c' > 0$ we can find $c > 0$ such that $|\xi-\xi^o|^\sigma + |\tau-\tau^o| \le c(|\xi^o|^\sigma + |\tau^\sigma)$ implies

$$(1) \qquad |T(\xi,\tau) - T(\xi^o,\tau^o)| \le c' .$$

This is obvious if we observe that it suffices to prove (1) under the additional assumption that $(\xi,\tau) \in E$.

Lemma 4.2. Consider U, V closed sets in E such that $0 \notin T^{-1}(U) + T^{-1}(V)$. Then $T(T^{-1}(U) + T^{-1}(V))$ is a closed set in E.

Proof. The set $A = T^{-1}(U) + T^{-1}(V)$ is quasiconic, so it suffices to show that $A \cap E$ is closed. This follows if we show that there is M such that $\lambda' \in T^{-1}(U)$, $\lambda'' \in T^{-1}(V)$, $\lambda' + \lambda'' \in E$ implies $|\lambda'| \le M, |\lambda''| \le M$. Assume then by contradiction that there is no such M. It follows that we can find sequences (ξ'^j,τ'^j), (ξ''^j,τ''^j) in $T^{-1}(U)$, respectively in $T^{-1}(V)$ such that $(\xi'^j,\tau'^j)+(\xi''^j,\tau''^j) \in E$ and such that $(\xi'^j,\tau'^j) \to \infty$. Denote $|\xi'^j|^\sigma + |\tau'^j|$ by t_j and consider $\lambda^j = ((\xi'^j+\xi''^j)/t_j^{1/\sigma}, (\tau'^j+\tau''^j)/t_j)$, $\lambda'^j = T(\xi'^j,\tau'^j)$,

$\lambda''^j = (\xi''^j/t_j^{1/\sigma}, \tau''^j/t_j)$.

Then $\lambda^j \to 0$ and $|\lambda'^j| + |\lambda''^j|$ is bounded. If λ'^o, λ''^o are limit points for λ'^j, λ''^j , we get $0 = \lambda'^o + \lambda''^o$, which contradicts our assumption.

Notation 4.3. Let $s > 1$ be given and consider $U \subset R^{n+1} \setminus \{0\}$. We denote by $\mathscr{R}(U)$ the set of functions $r : R^{n+1} \to R_+$ with the following properties:

a) there are constants $c_1, c_2, c_1 > 0$ such that $r(\xi,\tau) \geq c_1(|\xi|^\sigma + |\tau|) - c_2$ if $T(\xi,\tau) \in T(U)$,

b) there is a constant c such that

$|r(\xi,\tau)-r(\xi',\tau')| \leq c(1 + |\xi-\xi'|^\sigma + |\tau-\tau'|), \forall\xi,\xi' \in R^n, \forall\tau,\tau' \in R$.

In particular $r(\xi,\tau) \leq c(|\xi|^\sigma + |\tau|) + c_3$.

If U consists of just one point (ξ^o,τ^o) , then we will also use the notation $\mathscr{R}(\xi^o,\tau^o)$ instead of $\mathscr{R}(\{(\xi^o,\tau^o)\})$.

Note that the two inequalities (1) and (2) from §3 can now be replaced by

$$|\hat{v}(\zeta,\tau)| \leq \exp(r(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)+\Delta|\mathrm{Im}\zeta|^\sigma + t^o\mathrm{Im}\tau + \eta|\mathrm{Im}\zeta|) \quad \text{for some } r \in \mathscr{R}(\xi^o,\tau^o).$$

Lemma 4.4. Consider $U \subset E$ and $r \in \mathscr{R}(U)$. Then there is $\varepsilon > 0$ such that $r \in \mathscr{R}(U')$ for $U' = \{(\xi,\tau) \in E; |\xi - \xi'| + |\tau-\tau'| < \varepsilon\}$ for some $(\xi',\tau') \in U'\}$.

This is immediate.

Lemma 4.5. Let $U \subset E$ and for every $(\xi,\tau) \in U$ some $r_{(\xi,\tau)} \in \mathscr{R}(\xi,\tau)$ be given. Then we can find a finite set of points $(\xi^i,\tau^i), i=1,\dots,k$ and constants c_1, c_2 such that

$$\max_i r_{(\xi^i,\tau^i)}(\xi,\tau) \geq c_1(|\xi|^\sigma + |\tau|) - c_2 \quad \text{if } T(\xi,\tau) \in U .$$

This is an obvious consequence of Lemma 4.4.

3. The following lemma is helpful when we study the wave front set of a product of two functions.

Lemma 4.6. Consider $V_1, V_2 \subset E$ closed sets such that $0 \notin T^{-1}(V_1) + T^{-1}(V_2)$ and choose (ξ^o, τ^o) in E, $(\xi^o, \tau^o) \notin V_1 \cup V_2 \cup (T^{-1}(V_1) + T^{-1}(V_2))$. Also choose $r_i \in \mathscr{R}(CV_i), i=1,2$. Then there is $r \in \mathscr{R}(\xi^o, \tau^o)$ such that

$$r(\xi', \tau') \leq \min_{(\xi,\tau) \in R^{n+1}} (r_1(\xi'-\xi, \tau'-\tau) + r_2(\xi, \tau)) .$$

Proof. It suffices to show that for some $\varepsilon > 0, c_1 > 0, c_2$ we have

$$(2) \qquad r_1(\xi'-\xi, \tau'-\tau) + r_2(\xi,\tau) \geq c_1(|\xi'|^\sigma + |\tau'|) - c_2 \quad \text{if} \quad |T(\xi',\tau') - T(\xi^o, \tau^o)| \leq \varepsilon .$$

If we choose ε small enough, then we may assume that $(\xi', \tau') \in CV_1 \cap CV_2$. In view of Lemma 4.2 we may further assume that $(\xi', \tau') \notin T^{-1}(V_1) + T^{-1}(V_2)$ then. For such (ξ', τ') and for some fixed (ξ, τ) we can now have one of the following three situations:

a) $(\xi, \tau) \in V_2$

b) $(\xi, \tau) \notin V_2$, $T(\xi'-\xi, \tau'-\tau) \in V_1$

c) $(\xi, \tau) \notin V_2$, $T(\xi'-\xi, \tau'-\tau) \notin V_1$.

We shall study the three cases one after the other.

Case a). If we choose ε small then it follows from $(\xi, \tau) \in V_2$ and Lemma 4.1 that $|\xi'-\xi|^\sigma + |\tau'-\tau| \geq c(|\xi'|^\sigma + |\tau'|)$ for some constant $c > 0$ which does not depend on ξ, τ, ξ', τ'. Further we must have $T(\xi'-\xi, \tau'-\tau) \notin V_1$ then such that $r_1(\xi'-\xi, \tau'-\tau) \geq c_3(|\xi'-\xi|^\sigma + |\tau'-\tau|) - c_4$ in view of $r_1 \in \mathscr{R}(CV_1)$. This gives (2).

Case b). $|T(\xi', \tau') - T(\xi^o, \tau^o)| \leq \varepsilon$ and $T(\xi'-\xi, \tau'-\tau) \in V_1$ implies $(|\xi|^\sigma + |\tau|) \geq c_5(|\xi'-\xi|^\sigma + |\tau'-\tau|)$. Thus $|\xi'|^\sigma + |\tau'| \leq 2^\sigma |\xi'-\xi|^\sigma + 2^\sigma |\xi|^\sigma + |\tau'-\tau| + |\tau| \leq c_6(|\xi|^\sigma + |\tau|)$. We then obtain (2) from $r_2 \in \mathscr{R}(CV_2)$.

Case c). Again we use $|\xi'|^\sigma + |\tau'| \leq 2^\sigma |\xi'-\xi|^\sigma + 2^\sigma |\xi|^\sigma + |\tau'-\tau| + |\tau|$, and then we apply $r_i \in \mathscr{R}(CV_i)$.

4. Lemma 4.7. Let $r : R^{n+1} \to R_+$ be a function such that

$$(3)\qquad |r(\xi,\tau) - r(\xi',\tau')| \leq 1+|\xi-\xi'|^{\sigma}+|\tau-\tau'| \quad \text{for all} \quad \xi,\xi' \in R^n \ , \ \tau,\tau' \in R \ .$$

Then there are constants c,γ and a plurisubharmonic function $\rho : C^{n+1} \to R$ such that

$$r(\mathrm{Re}\zeta,\ \mathrm{Re}\tau) \leq \rho(\zeta,\tau)+\gamma(|\mathrm{Im}\zeta|^{\sigma}+|\mathrm{Im}\tau|) \quad ,$$

respectively

$$\rho(\zeta,\tau) \leq 2r(\mathrm{Re}\zeta,\ \mathrm{Re}\tau)+\gamma(|\mathrm{Im}\zeta|^{\sigma}+|\mathrm{Im}\tau|) + c \quad .$$

Moreover, γ and c do not depend on r here.

Proof (for a related result, cf. O. Liess [1]). We start by choosing a continuous function $\psi(\tau,t) : C \times R_+ \to R$, which is plurisubharmonic in τ for each fixed t and which satisfies the following inequalities:

a) $\psi(\tau,t) \leq C + B|\mathrm{Im}\tau|$,

b) $\psi(\tau,t) \leq -|\tau| + B|\mathrm{Im}\tau|$, for $(1/4)t \leq |\tau| \leq t, \tau \in C$,

c) $\psi(\tau,t) \leq -t + B|\mathrm{Im}\tau|$, for $|\tau| \geq t,\ \tau \in C$,

d) $\psi(i\tau,t) \geq -B|\tau|$, if $\tau \in R$,

for suitable constants C,B (cf. O. Liess [1]).

Further we choose $\phi \in \mathscr{A}(C^n)$, $\phi(0)=1$, such that $|\phi(\zeta)| \leq c_1\exp(-|\zeta|^{\sigma}+c_2|\mathrm{Im}\zeta|^{\sigma})$, respectively $|\phi(i\xi)| \geq c_3\exp(-c_4|\xi|^{\sigma})$ if $\xi \in R^n$ for some positive constants $c_i, i=1,2,3,4$ (cf. Lemma 2.3).

For every $(\xi^o,\tau^o) \in R^{n+1}$ we now define a plurisubharmonic function $\rho_{(\xi^o,\tau^o)} : C^{n+1} \to R$ by $\rho_{(\xi^o,\tau^o)}(\zeta,\tau) = r(\xi^o,\tau^o)+4\ \ln|\phi(\zeta-\xi^o)| + 4\psi(\tau-\tau^o,r(\xi^o,\tau^o))$.

We claim that the following inequality is then valid:

$$(4)\qquad \rho_{(\xi^o,\tau^o)}(\zeta,\tau) \leq 2r(\mathrm{Re}\zeta,\mathrm{Re}\tau) + 4c_2|\mathrm{Im}\zeta|^{\sigma} + 4B|\mathrm{Im}\tau| + c_5 \quad ,$$

where c_5 does not depend on (ξ^o,τ^o) .

Proof of (4). Three cases are to be considered:

I. When $|\zeta-\xi^o|^\sigma + |\tau-\tau^o| \leq r(\xi^o,\tau^o)/2$, then it follows from (3) that $|r(\mathrm{Re}\zeta,\mathrm{Re}\tau)-r(\xi^o,\tau^o)| \leq 1+r(\xi^o,\tau^o)/2$, such that $r(\xi^o,\tau^o) \leq 2r(\mathrm{Re}\zeta,\mathrm{Re}\tau) + 2$, whence also (4).

II. When $|\zeta-\xi^o|^\sigma+|\tau-\tau^o| > r(\xi^o,\tau^o)/2$, but $|\tau-\tau^o| < r(\xi^o,\tau^o)/4$, then $|\zeta-\xi^o|^\sigma > r(\xi^o,\tau^o)/4$ such that $r(\xi^o,\tau^o)+4\ \ln|\phi(\zeta-\xi^o)| \leq 4c_2|\mathrm{Im}\zeta|^\sigma + 4\ \ln c_1$, which again gives (4).

III. Finally, when $|\tau-\tau^o| > r(\xi^o,\tau^o)/4$, then we can apply b) or c) and conclude that $r(\xi^o,\tau^o) + 4\psi(\tau-\tau^o,r(\xi^o,\tau^o)) \leq 4\ B|\mathrm{Im}\tau| + 4C$ such that (4) is valid also in this case.

We have now proved (4) and return to the proof of Lemma 4.7. In fact let us define $\rho'(\zeta,\tau) = \sup_{(\xi^o,\tau^o)\in R^{n+1}} \rho_{(\xi^o,\tau^o)}(\zeta,\tau)$. Then $\rho'(\zeta,\tau) \leq 2r(\mathrm{Re}\zeta,\mathrm{Re}\tau) + \gamma(|\mathrm{Im}\zeta|^\sigma + |\mathrm{Im}\tau|) + c_5$ for $\gamma = \max(4c_2,4B)$. In particular ρ' is finite at every point. It is easy to see that ρ' is plurisubharmonic and from d) and the corresponding property for ϕ it follows that $r(\mathrm{Re}\zeta,\mathrm{Re}\tau) \leq \rho'(\zeta,\tau) + \gamma(|\mathrm{Im}\zeta|^\sigma+|\mathrm{Im}\tau|) + c_6$. We now set $\rho = \rho'+c_6$.

5. With the aid of Lemma 4.7 one can now prove:

Proposition 4.8. Consider $r(\xi,\tau) : R^{n+1} \to R_+$ a function which satisfies $|r(\xi,\tau) - r(\xi',\tau')| \leq c(1+|\xi-\xi'|^\sigma+|\tau-\tau'|)$, and let $k \geq 0$ be given. Then there are constants c',k' and b' such that the following is true:
if $u \in \mathscr{C}^\infty(C^{n+1})$ satisfies

$$|\bar\partial(\partial/\partial\zeta)^\alpha(\partial/\partial\bar\zeta)^{\alpha'}\ u(\zeta,\tau)| \leq \exp(r(\mathrm{Re}\zeta,\mathrm{Re}\tau)/2+\Delta|\mathrm{Im}\zeta|^\sigma+h|\mathrm{Im}\tau|+b\ \ln(1+|(\zeta,\tau)|))$$
$$\text{for } |\alpha| + |\alpha'| \leq k'$$

then there is $v \in \mathscr{C}^\infty(C^{n+1})$ such that $\bar\partial v = \bar\partial u$ and such that

$$|(\partial/\partial\zeta)^{\beta}(\partial/\partial\bar{\zeta})^{\beta'} v(\zeta,\tau)| \leq c' \exp(r(\mathrm{Re}\zeta,\mathrm{Re}\tau) + (\Delta+c\gamma)|\mathrm{Im}\zeta|^{\sigma} + (h+c\gamma)|\mathrm{Im}\tau| + \\ + b \ln(1+|(\zeta,\tau)|)) \quad \text{for} \quad |\beta| + |\beta'| \leq k .$$

γ is here the constant from Lemma 4.7 and $\bar{\partial}$ stands for the Cauchy-Riemann operator in ζ,τ .

In view of Lemma 4.7, Proposition 4.8 is a consequence of classical results from L. Hörmander [1]. For details of a proof in a similar situation, cf. O. Liess [1], and also O. Liess [2].

§5. PREPARATIONS FOR THE STUDY OF WF_A^s. SPLITTING OF ENTIRE FUNCTIONS

1. Proposition 5.1. Consider $\phi_i : C^{n+1} \to R, i=1,2,3,4,$ and $\psi : C^{n+1} \to R$ continuous functions with the following properties:

a) $\phi_1(\zeta,\tau) \leq \phi_3(\zeta,\tau)$, $\phi_2(\zeta,\tau) \leq \phi_4(\zeta,\tau)$,

b) ψ is plurisubharmonic,

c) $\min(\phi_1(\zeta,\tau), \phi_2(\zeta,\tau)) \leq \psi(\zeta,\tau) \leq \min(\phi_3(\zeta,\tau), \phi_4(\zeta,\tau))$,

d) there is C such that $|\phi_i(\zeta,\tau) - \phi_i(\zeta',\tau')| \leq C(1 + |\zeta-\zeta'|^{\sigma}+|\tau-\tau'|)$, $i=1,2,3,4$.

Then there are constants c and μ such that every $h \in \mathscr{A}(C^{n+1})$ with $|h(\zeta,\tau)| \leq \exp\max(\phi_1(\zeta,\tau), \phi_2(\zeta,\tau))$ can be decomposed in the form $h = h_1 + h_2$, $h_i \in \mathscr{A}(C^{n+1})$ with h_i satisfying

$|h_1(\zeta,\tau)| \leq c \exp(\phi_3(\zeta,\tau) + \mu \ln(1+|(\zeta,\tau)|))$,

$|h_2(\zeta,\tau)| \leq c \exp(\Phi_4(\zeta,\tau) + \mu \ln(1+|(\zeta,\tau)|))$.

A similar result has been proved in O. Liess [2]. In fact (by means of a partition of unity) it is quite easy to decompose h in the form $h = f_1 + f_2$ where the f_i are in $\mathscr{C}^{\infty}(C^{n+1})$ and satisfy $|f_i(\zeta,\tau)| \leq \exp\phi_i(\zeta,\tau)$, respectively $|\bar{\partial} f_i(\zeta,\tau)| \leq c' \exp(\min(\phi_1(\zeta,\tau),\phi_2(\zeta,\tau)) + \mu' \ln(1+|(\zeta,\tau)|))$ for some

c' and μ'. Using b), c), d) and results from L. Hörmander [1] one can then show that there is $g \in \mathscr{C}^\infty(C^{n+1})$ such that $\bar\partial f_1 = \bar\partial g$ and such that $|g(\zeta,\tau)| \le c'' \exp(\min(\Phi_3(\zeta,\tau),\Phi_4(\zeta,\tau)) + \mu \ln(1+|(\zeta,\tau)|))$ for some c'' and μ. We then define h_1 by $h_1 = f_1 - g$ and h_2 by $h_2 = f_2 + g$. For more details, cf. O. Liess [2].

2. Proposition 5.2. Consider a finite set of functions $r_i : R^{n+1} \to R_+, i=1,\dots,j$, such that

$$(1) \qquad |r_i(\xi,\tau) - r_i(\xi',\tau')| \le C(1+|\xi-\xi'|^\sigma+|\tau-\tau'|), i=1,\dots,j$$

for some C. Let $D' > D > 0, \eta > \eta' > 0$ and $b \ge 0$ be given. Then there are $c > 0, d, 0 < d < 1, \mu' \ge 0$ such that every $h \in \mathscr{A}(C^{n+1})$ which satisfies

$$(2) \qquad |h(\zeta,\tau)| \le \exp(d \max_i r_i(\operatorname{Re}\zeta,\operatorname{Re}\tau)+\Delta'|\operatorname{Im}\zeta|^\sigma+\eta'|\operatorname{Im}\tau| + b \ln(1+|(\zeta,\tau)|))$$

can be decomposed in the form

$$h = \sum h_i$$

for some $h_i \in \mathscr{A}(C^{n+1})$ such that

$$|h_i(\zeta,\tau)| \le c \exp(r_i(\operatorname{Re}\zeta,\operatorname{Re}\tau)+\Delta|\operatorname{Im}\zeta|^\sigma + \eta|\operatorname{Im}\tau| + (b+\mu')\ln(1+|(\zeta,\tau)|)) \quad .$$

Here d depends only on j,C,D,D',η,η' (and not explicitly on the r_i) and μ', which is related to μ from Proposition 5.1, depends only on j (and n).

Moreover, if h is of form $h = \hat v$ for some $v \in \mathscr{E}^s_{D'}(R^{n+1}_{,})'$, then we can choose the h_i to be of form $h_i = \hat v_i$ for some $v_i \in \mathscr{E}^s_D(R^{n+1})'$.
(D and D' are here related to Δ,Δ' via relation (1) from §2).

Proof. Arguing by induction we may assume that $j = 2$. The first part of the proposition is then an immediate consequence of Proposition 5.1. To see this, we introduce $\phi_i = d\, r_i(\operatorname{Re}\zeta,\operatorname{Re}\tau)+\Delta'|\operatorname{Im}\zeta|^\sigma + \eta'|\operatorname{Im}\tau| + b \ln(1+|(\zeta,\tau)|)$ for $i = 1,2$

and $\phi_i = r_{i-2}(\mathrm{Re}\zeta,\mathrm{Re}\tau)+\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|) + c_1$ for some c_1 if $i = 3,4$. The only thing which remains then to be shown is that for suitably choosen d and c_1 we can find a plurisubharmonic function ψ such that $\min(\phi_1,\phi_2) \le \psi \le \min(\phi_3,\phi_4)$. The existence of such a function ψ is a consequence of Lemma 4.7. In fact in view of that lemma and for d sufficiently small we can find a plurisubharmonic function $\phi : C^{n+1} \to R$ such that $d\min(r_1(\mathrm{Re}\zeta,\mathrm{Re}\tau),r_2(\mathrm{Re}\zeta,\mathrm{Re}\tau)) \le \phi(\zeta,\tau) \le \min(r_1(\mathrm{Re}\zeta,\mathrm{Re}\tau),r_2(\mathrm{Re}\zeta,\mathrm{Re}\tau)) + (\Delta-\Delta')$ $|\mathrm{Im}\zeta|^\sigma + (\eta-\eta')|\mathrm{Im}\tau| + c_2$. Since $|\mathrm{Im}\zeta|^\sigma$, $|\mathrm{Im}\tau|$ and $b\ \ln(1+|(\zeta,\tau)|)$ are plurisubharmonic it then remains to define ψ by
$\psi = \phi + \Delta'|\mathrm{Im}\zeta|^\sigma + \eta'|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|)$.

Let us now turn our attention to the second statement from the proposition. Thus assume $h = \hat{v}$ for some $v \in \mathscr{E}^s_{D'}(R^{n+1}_{\eta'})'$. We want to show that the second assertion is a consequence of the first applied for some modified r_i.

The first thing to observe here is that we can find $C_1 > 0, b_1 \ge 0, \Delta_1 < \Delta'$ and $\eta_1 < \eta'$ (which all may depend on v) such that

(3) $$|h(\zeta,\tau)| \le C_1 \exp(\Delta_1|\mathrm{Im}\zeta|^\sigma + \eta_1|\mathrm{Im}\tau| + b_1\ln(1+|(\zeta,\tau)|)) .$$

In particular it follows that

(4) $$|h(\zeta,\tau)| \le \exp(\Delta_1|\mathrm{Im}\zeta|^\sigma + \eta_1|\mathrm{Im}\tau| + (b_1+1)\ln(1+|(\zeta,\tau)|)) \quad \text{for} \quad |(\zeta,\tau)| \ge C_1 .$$

Let us also fix Δ_2,η_2 with $\Delta' < \Delta_2 < \Delta, \eta' < \eta_2 < \eta$ and let d be the one given by the first part of the proposition for $2C, D', D_2$ (associated with Δ_2), η',η_2.

We now choose $r'_i : R^{n+1} \to R_+$ such that

a) $|r'_i(\xi,\tau) - r'_i(\xi',\tau')| \le 2C(1+|\xi-\xi'|^\sigma + |\tau-\tau'|)$,

b) $r'_i(\xi,\tau) = r_i(\xi,\tau)$ if $|(\xi,\tau)| \le C_2$ for some $C_2 \ge C_1$ which will be chosen later on,

c) $r_i(\xi,\tau) \geq r_i'(\xi,\tau) \geq \min(r_i(\xi,\tau), 2((b_1+1)/d)\ln(1+|(\xi,\tau)|))$, if $|(\xi,\tau)| \geq C_2$,

d) $r_i'(\xi,\tau) = \min(r_i(\xi,\tau), 2((b_1+1)/d)\ln(1+|(\xi,\tau)|))$ if $|(\xi,\tau)| \geq C_3$ for some $C_3 > C_2$.

Let us then assume that h satisfies (2) for the d introduced just before and assume that it satisfies (3). We claim that h then also satisfies

$$(5) \qquad |h(\zeta,\tau)| \leq \exp(d \max_i r_i'(\mathrm{Re}\zeta,\mathrm{Re}\tau)+\Delta'|\mathrm{Im}\zeta|^\sigma + \eta'|\mathrm{Im}\tau| + b\ln(1+|(\zeta,\tau)|))$$

if C_2 is chosen great enough.

In fact when $|\mathrm{Re}(\zeta,\tau)| \leq C_2$ then (5) follows from b) and (2) and when $|\mathrm{Re}(\zeta,\tau)| \geq C_2$ but $|\mathrm{Im}(\zeta,\tau)| \leq |\mathrm{Re}(\zeta,\tau)|$, then (5) follows from c) and (4). It therefore remains to consider the case $|\mathrm{Im}(\zeta,\tau)| \geq |\mathrm{Re}(\zeta,\tau)| \geq C_2$ when (5) is a consequence of (3) if we choose C_2 great enough in order to have $(\Delta'-\Delta_1)|\mathrm{Im}\zeta|^\sigma + (\eta'-\eta_1)|\mathrm{Im}\tau| \geq (b_1+1)\ln(1+|(\zeta,\tau)|)$ for the (ζ,τ) under consideration.

We have now proved that we may assume that h satisfies (5). It follows from the first part of the proposition that we can find $h_i \in \mathcal{A}(C^{n+1})$ such that $h = \Sigma h_i$ and such that $|h_i(\zeta,\tau)| \leq c' \exp(r_i'(\mathrm{Re}\zeta,\mathrm{Re}\tau) + \eta_2|\mathrm{Im}\tau| + (b+\mu')\ln(1+|(\zeta,\tau)|))$. In view of d) from the definition of r_i' it is then clear that h_i is of form $h_i = \hat{v}_i$ for some $v_i \in \mathcal{E}_D^s(R_\eta^{n+1})'$.

3. <u>Remark 5.3</u>. It is possible to apply Proposition 5.2 also if instead of (2) we only have the weaker estimate

$$|h(\zeta,\tau)| \leq \exp(d \sum_i r_i(\mathrm{Re}\zeta,\mathrm{Re}\tau) + \Delta'|\mathrm{Im}\zeta|^\sigma + \eta'|\mathrm{Im}\tau| + b\ln(1+|(\zeta,\tau)|)) .$$

In fact then $|h(\zeta,\tau)| \leq \exp((jd) \max_i r_i(\mathrm{Re}\zeta,\mathrm{Re}\tau) + \ldots)$ and we must only change the notation for d.

4. <u>Proposition 5.4</u>. Consider $r : R^{n+1} \to R_+$ such that

$|r(\xi,\tau) - r(\xi',\tau')| \leq 1+|\xi-\xi'|^\sigma+|\tau-\tau'|$ and let $D' > D > 0, \eta > \eta' > 0, b \geq 0$ and $\mu \geq 0$ be given. If μ is sufficiently great then we can find $d > 0$ and $c > 0$ such that every $h \in \mathscr{A}(C^{n+1})$ which satisfies

$$(6) \qquad |h(\zeta,\tau)| \leq \exp(d\, r(\mathrm{Re}\zeta,\mathrm{Re}\tau)+\Delta'|\mathrm{Im}\zeta|^\sigma + \eta'|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|))$$

can be written in the form $h = h_1 + h_2$, $h_{1,2} \in \mathscr{A}(C^{n+1})$, with $h_{1,2}$ satisfying

$$|h_1(\zeta,\tau)| \leq c\ \exp(r(\mathrm{Re}\zeta,\mathrm{Re}\tau)+\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + \mu\ \ln(1+|(\zeta,\tau)|)) \quad ,$$

respectively

$$|h_2(\zeta,\tau)| \leq c\ \exp(\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + (b+\mu)\ \ln(1+|(\zeta,\tau)|)) \quad .$$

Moreover, μ does not depend on $r,\Delta',\Delta,\eta',\eta,b$ here.
Further if $h = \hat{v}$ for some $v \in \mathscr{E}^s_{D'}(R^{n+1}_{\eta'})'$ then we can choose the h_i such that $h_i = \hat{v}_i$ for some $v_i \in \mathscr{E}^s_D(R^{n+1})'$.

Proof. (cf. O. Liess [2] for a similar result). Let us choose $\Delta' < \Delta'' < \Delta$, $\eta' < \eta'' < \eta$. Then we can find c_1 such that

$$|h(\zeta,\tau)| \leq c_1\ \exp(d\ r(\mathrm{Re}\zeta,\mathrm{Re}\tau)+b\ \ln(1+|\mathrm{Re}(\zeta,\tau)|)+\Delta''|\mathrm{Im}\zeta|^\sigma + \eta''|\mathrm{Im}\tau|)$$

if h satisfies (6). We would like to reduce ourselves to remark 5.3 by essentially taking $j = 2, r_1 = r, r_2 = (1/d)\ b\ \ln(1+|(\zeta,\tau)|)$, where $d > 0$ is the constant from the conclusion of Proposition 5.2 (when combined with Remark 5.3) for $C = 1,\Delta'',\eta'',\Delta,\eta$. Remark 5.3 is however not directly applicable since for small ξ,τ,ξ',τ' we do not necessarily have $|r_2(\xi,\tau)-r_2(\xi',\tau')| \leq 1 + |\xi-\xi'|^\sigma + |\tau-\tau'|$. We therefore introduce an auxiliary function r_2' and a constant c' such that $r_2'(\xi,\tau) = r_2(\xi,\tau)$ if $|(\xi,\tau)| \geq c'$ and $|r_2'(\xi,\tau) - r_2'(\xi,\tau)| \leq 1+|\xi-\xi'|^\sigma + |\tau-\tau'|$ everywhere. For some c_2 we will then have

$$|h(\zeta,\tau)| \leq c_2\ \exp(d(r_1(\mathrm{Re}\zeta,\mathrm{Re}\tau) + r_2'(\mathrm{Re}\zeta,\mathrm{Re}\tau))+\Delta''|\mathrm{Im}\zeta|^\sigma + \eta''|\mathrm{Im}\tau| \quad .$$

The proposition is now a consequence of Remark 5.3.

5. Proposition 5.5. Let $r' \in \mathscr{R}(\xi^o,\tau^o), D' > 0, \eta' > 0, d' > 0$ be given. Then we can find $c > 0, r \in \mathscr{R}(\xi^o,\tau^o), D > 0, \eta > 0, b \geq 0$ and a sequence of points $(\xi^k,\tau^k) \in R^{n+1}$, $= 1,2,3,\ldots$ with the following property: if $v \in \mathscr{E}^s_D(R^{n+1})'$ satisfies

$$(7) \qquad |\hat{v}(\zeta,\tau)| \leq \exp(r(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)+\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau|) \quad ,$$

then there is a sequence of entire analytic functions $h_k : C^{n+1} \to C$, $k = 1,2,\ldots,$ such that only a finite number of them are $\not\equiv 0$ (how many functions are $\not\equiv 0$ may depend on v) and such that

$$(8) \qquad \hat{v}(\zeta,\tau) = \sum_k h_h(\zeta,\tau) \quad ,$$

$$(9) \qquad |h_k(\zeta,\tau)| \leq (c/(1+k^2))\exp(r'(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)+\Delta'|\mathrm{Im}\zeta|^\sigma + \eta'|\mathrm{Im}\tau| + \\ + b\ \ln(1+|(\zeta,\tau)|)) \quad \text{for all} \quad \zeta,\tau \quad ,$$

$$(10) \qquad |h_k(\zeta,\tau)| \leq (c/(1+k^2))\exp(\Delta'|\mathrm{Im}\zeta|^\sigma + \eta'|\mathrm{Im}\tau| + \\ + b\ \ln(1+|(\zeta,\tau)|)) \quad \text{if} \quad |\zeta-\xi^k|^\sigma + |\tau-\tau^k| \geq d'(|\xi^k|^\sigma + |\tau^k|) \quad .$$

Proposition 5.5 is closely related to Proposition 5.2. On the other hand the main difference is here that the constants must not depend on the number of terms which effectively appear in (8). Induction in this number is not allowed therefore and we must obtain all terms from the decomposition at the same time. We prepare the proof by a lemma in which we show that it suffices to obtain (10) on a smaller set (smaller appart from a renotation).

Lemma 5.6. There is c' such that every h_k which satisfies (9) and

$$(11) \qquad |h_k(\zeta,\tau)| \leq (c/(1+k^2))\exp((\Delta'/2)|\mathrm{Im}\zeta|^\sigma + (\eta'/2)|\mathrm{Im}\tau| + \\ + b\ \ln(1+|(\zeta,\tau)|)) \quad \text{for} \quad |\mathrm{Re}\zeta-\xi^k|^\sigma + |\mathrm{Re}\tau-\tau^k| \geq (d'/2)(|\xi^k| + |\tau^k|) \quad ,$$

also satisfies (10) with c replaced by cc', if $r' \in \mathscr{R}(\xi^o,\tau^o)$ is small enough.

Proof of Lemma 5.6. When $|\mathrm{Re}\zeta-\xi^k|^\sigma + |\mathrm{Re}\tau-\tau^k| \geq (d'/2)(|\xi^k|^\sigma + |\tau^k|)$, then (10) follows from (11). In the remaining case we have for $|\zeta-\xi^k|^\sigma + |\tau-\tau^k| \geq d'(|\xi^k|^\sigma + |\tau^k|)$ that $(|\mathrm{Re}\zeta|^\sigma + |\mathrm{Re}\tau|) \leq c_1(|\mathrm{Im}\zeta|^\sigma + |\mathrm{Im}\tau|)$ for some c_1. If we choose r' small enough in order to have $r'(-\mathrm{Re}\zeta,-\mathrm{Re}\tau) \leq (\Delta'/2)|\mathrm{Im}\zeta|^\sigma + (\eta'/2)|\mathrm{Im}\tau|+c_2$ for such ζ,τ then it follows from (9) that (10) is satisfied with c replaced by $c \exp c_2$.

We now return to the proof of Proposition 5.5. The situation is here similar to the one in Lemma 3.6 from O. Liess [1] or (even more so) to the one from Proposition 5.18, chapt. VII in J.E. Björk [1].

Proof of Proposition 5.5. We choose a $\mathscr{C}^\infty$ partition of unity g_k on R^{n+1}, $g_k \geq 0, k=1,2,\ldots,$ a sequence of functions $r_k : R^{n+1} \to R_+, k=1,2,\ldots,$ and a sequence of points $(\xi^k,\tau^k) \in R^{n+1}$ with the following properties:

a) there are $c_1 > 0, c_2 \geq 0$ such that $(\xi,\tau) \in \mathrm{supp}\, g_k$ implies $|(\xi,\tau)| \geq c_1 k^{1/\sigma} - c_2$,

b) $\mathrm{supp}\, g_k \cap \mathrm{supp}\, g_j = \emptyset$ if $|k-j| \geq 2$

c) there are c_3 and ν such that $|\mathrm{grad}_{\xi,\tau} g_k(\xi,\tau)| \leq c_3(1+|(\xi,\tau)|)^\nu$ for all k,

d) for $k \geq 2$ we can decompose $\mathrm{grad}_{\xi,\tau} g_k$ into the form $\mathrm{grad}_{\xi,\tau} g_k = g_k' + g_k''$ for some g_k', g_k'' such that

 d_1) $\mathrm{supp}\, g_k' \cap \mathrm{supp}\, g_k'' = \emptyset$

 d_2) $g_k'' = -g_{k-1}'$ if $k \geq 2$, $g_1' = \mathrm{grad}_{\xi,\tau} g_1$ (by definition),

e) there is a sequence of positive numbers $t_k \to \infty$ such that $(\xi^k,\tau^k) = (t_k^{1/\sigma}\xi^o, t_k\tau^o)$,

f) $r_k(\xi,\tau) \leq r'(\xi,\tau)$, $\forall (\xi,\tau) \in R^{n+1}$,

g) $r_k(\xi,\tau) = 0$ if $|\xi-\xi^k|^\sigma + |\tau-\tau^k| \geq (d'/2)(|\xi^k|^\sigma + |\tau^k|)$,

h) $|r_k(\xi,\tau) - r_k(\xi',\tau')| \leq 1+|\xi-\xi'|^\sigma + |\tau-\tau'|$, for all ξ,ξ',τ,τ',

i) $\max\limits_k r_k(\xi,\tau) \stackrel{\text{def}}{=} \tilde{r}(\xi,\tau) \in \mathscr{R}(\xi^o,\tau^o)$,

j) $(\xi,\tau) \in \operatorname{supp} g_k \cap \operatorname{supp} r_j \Rightarrow r_j(\xi,\tau) \leq r_k(\xi,\tau)$, $\forall k,j$

k) $(\xi,\tau) \in \operatorname{supp} g_k' \cap \operatorname{supp} r_j \Rightarrow r_j(\xi,\tau) \leq \min(r_k(\xi,\tau), r_{k+1}(\xi,\tau))$, $\forall k,j$.

Such r_k, g_k and (ξ^k,τ^k) are easy to construct. (Note that e) will not be used later on -it only serves to make the construction more transparent- and that k) is in fact a consequence of b), d) and j)). In fact, we may, e.g., consider the sets $U_k \subset R^{n+1}$ defined by

$$U_1 = \{(\xi,\tau) \in R^{n+1};\ |\xi|^\sigma+|\tau| < 2\}, U_k = \{(\xi,\tau) \in R^{n+1}; 2k-3 \leq |\xi|^\sigma+|\tau| \leq 2k\} ,$$

and choose the partition g_k to be subordinate to this covering of R^{n+1} . Further we choose some "small" $r^\circ \in \mathscr{R}(\xi^o,\tau^o)$ and define r_k on $U_k \cup U_{k+1}$ to be the restriction of r° to this set. We then extend r_k to R^{n+1} by letting it die out as rapidly as this is allowed by h). This immediately gives the desired properties, if r° has been choosen suitably (in particular r° must vanish in a neighborhood of the origin in order to avoid the difficulties which might come from the fact that for small d' and small k the sets $|\xi-\xi^k|^\sigma + |\tau-\tau^k| \leq (d'/2)(|\xi^k|^\sigma + |\tau^k|)$ can be small)). A similar construction is implicit in J.E. Björk [1], loc. cit. We omit further details.

Once we have constructed r_k, g_k and (ξ^k,τ^k) with the properties a),...,k), the proof of Proposition 5.5 is straightforward. Let us in fact assume that $v \in \mathscr{E}^s_D(R^{n+1})'$ satisfies (7) for $r = d\,\tilde{r}$, where $\tilde{r}$ is from i) and $d > 0$ shall be specified later on. In view of Proposition 5.4 we may assume that v is of order μ such that we can find c_4 for which

$$|\hat{v}(\zeta,\tau)| \leq \exp(\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + (\mu+1)\ln(1+|(\zeta,\tau)|)) \quad \text{for} \quad |(\zeta,\tau)| \geq c_4 .$$

It follows in particular that for some k^o

$$(12) \qquad |\hat{v}(\zeta,\tau)| \leq \exp(d \max_{k\leq k^o} r_k(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)+\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + (\mu+1)\ln(1+|(\zeta,\tau)|))$$

As a first attempt to find h_k we introduce functions f_k by

$$f_k(\zeta,\tau) = g_k(\mathrm{Re}\zeta,\mathrm{Re}\tau)\ h(\zeta,\tau) \quad .$$

In view of a) and j) it follows that

$$|f_k(\zeta,\tau)| \leq c_5(1+k^3)^{-1} \exp(d\ r_k(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)+\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + \\ + (3\sigma+\mu+1)\ \ln(1+|(\zeta,\tau)|)) \quad .$$

Further $\bar\partial f_k = (1/2)\ \mathrm{grad}_{\xi,\tau} g_k(\mathrm{Re}\zeta,\mathrm{Re}\tau)h = (1/2)(g_k'(\mathrm{Re}\zeta,\mathrm{Re}\tau)h + g_k''(\mathrm{Re}\zeta,\mathrm{Re}\tau)h)$.
In view of d_2) it is now clear that $(1/2)(g_k'(\mathrm{Re}\zeta,\mathrm{Re}\tau)h) = \bar\partial((\sum_{j<k} g_j)h)$ (one might alternatively observe that in view of d_1), $\bar\partial g_k'(\mathrm{Re}\zeta,\mathrm{Re}\tau)h = \bar\partial g_k''(\mathrm{Re}\zeta,\mathrm{Re}\tau)h = 0$, where we have now interpreted $g_k'h$ and $g_k''h$ as (0,1)-forms, and use a variant of Proposition 4.8 later on). We now use a), j) and k) and conclude that

$$|g_k'(\mathrm{Re}\zeta,\mathrm{Re}\tau)h(\zeta,\tau)| \leq c_6(1+k^3)^{-1} \exp(d\ \min(r_k(-\mathrm{Re}\zeta,-\mathrm{Re}\tau), \\ r_{k+1}(-\mathrm{Re}\zeta,-\mathrm{Re}\tau))+\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + (3\sigma+\mu+\nu+1)\ \ln(1+|(\zeta,\tau)|))$$

if $k \leq k^o$, respectively

$$|g_k'(\mathrm{Re}\zeta,\mathrm{Re}\tau)h(\zeta,\tau)| \leq c_6(1+k^3)^{-1} \exp(\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + (3\sigma+\mu+\nu+1)\ \ln(1+|(\zeta,\tau)|))$$

for $k > k^o$.
Now fix Δ'',η'' such that $\Delta < \Delta'' < \Delta',\eta < \eta'' < \eta'$. If d is sufficiently small then we can find $\tilde f_k \in \mathscr{C}^\infty(\mathbb{C}^{n+1})$ such that $\bar\partial\tilde f_k = (1/2)g_k'(\mathrm{Re}\zeta,\mathrm{Re}\tau)h$ and such that

$$|\tilde f_k(\zeta,\tau)| \leq c_7(1+k^3)^{-1} \exp(\min(r_k(-\mathrm{Re}\zeta,-\mathrm{Re}\tau),r_{k+1}(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)) + \\ + \Delta''|\mathrm{Im}\zeta|^\sigma + \eta''|\mathrm{Im}\tau| + b''\ln(1+|(\zeta,\tau)|)) \quad \text{if } k \leq k^o$$

respectively

$$|\tilde f_k(\zeta,\tau)| \leq c_7(1+k^3)^{-1} \exp(\Delta''|\mathrm{Im}\zeta|^\sigma + \eta''|\mathrm{Im}\tau| + b''\ln(1+|(\zeta,\tau)|))$$

for $k > k^o$ and suitable c_7, b''. This follows from Proposition 4.8. The proof of Proposition 5.5 comes now, in view of Lemma 5.6, to an end if we set

$h_1 = f_1 - \tilde{f}_1, h_k = f_k - \tilde{f}_k + \tilde{f}_{k-1}$ for $2 \le k \le k^o$ and

$h_{k^o+1} = \sum_{k>k^o} (f_k - \tilde{f}_k + \tilde{f}_{k-1})$, and observe that $\sum_{k>k^o} (1+k^3)^{-1} \le c_8(1+(k^o+1)^2)^{-1}$.

§6. PROOF OF THE PROPOSITIONS 3.4, 3.5, 3.6 and 3.7.

1. Proof of Proposition 3.4 (first part).

(ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are trivial. If (iii) is valid then (ii) follows from Lemma 2.2. It remains to prove that (i) $\Rightarrow$ (ii). In doing so, we may of course assume that $t^o = 0$.

Proposition 6.1. Suppose $(0,\xi^o,\tau^o) \notin WF_A^s f$ and let $b \ge 0$ be given. Then we can find $r \in \mathscr{R}(\xi^o,\tau^o), D > 0, \eta > 0$, which do not depend on b and c such that

(1) $\quad |v(f*g)| \le c\int |g(x,t)|\, dx\, dt$ for any $v \in \mathscr{E}_D^s(R_\eta^{n+1})'$ which satisfies

(2) $\quad |\hat{v}(\zeta,\tau)| \le \exp(r(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)+\Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + b\ln(1+|(\zeta,\tau)|))$, and for any $g \in \mathscr{E}\mathscr{C}_o^\infty(R^{n+1})$ such that $\mathrm{supp}\, g \subset \{(x,t) \in R^{n+1}; |x| \le \Delta, |t| \le \eta\}$.

In particular we obtain from (1) that $|v(f)| \le c$ for any v satisfying (2), by just inserting in (1) for g a sequence of $\mathscr{C}_o^\infty$ functions g_j such that $\int |g_j(x,t)|\, dx\, dt = 1$ which approximates the δ-distribution in $\mathscr{E}'(R^{n+1})$ and have support in $|x| \le \Delta, |t| \le \eta$. Thus Proposition 6.1 shows that (i) $\Rightarrow$ (ii). We have taken the idea to estimate $v(f)$ via $v(f*g)$ from W. Rudin [1]. For a similar argument, cf. O. Liess [2].

Before we prove Proposition 6.1, we mention a simple lemma which follows essentially from the fact that the Laplace operator in $n+1$ variables is elliptic.

Lemma 6.2. Denote by ∇ the Laplacean in $n+1$ variables and let $\Delta > 0, \eta > 0, b' \geq \mu$ be given. μ is here the constant which appears in the conclusion of Proposition 5.4. Then we can find $c_1 > 0, \ell > 0$, ℓ an integer, and maps

$$S_{1,2} : \mathscr{C}_o^\infty(\{(x,t) \in R^{n+1}; |x| \leq \Delta, |t| \leq \eta\}) \to \mathscr{C}_o^\infty(\{(x,t) \in R^{n+1}; |x| \leq 2\Delta, |t| \leq 2\eta)$$

such that

$$g = \nabla^\ell S_1 g + S_2 g$$

and such that

$$\sup_{(x,t)\in R^{n+1}} \sum_{|\alpha|+j\leq b'+1} |D_x^\alpha D_t^j S_{1,2} g(x,t)| \leq c_1 \int |g(x,t)| dx\, dt \quad .$$

It follows in particular that

$$(3) \qquad (1+|(\zeta,\tau)|)^{b'} |\mathscr{F}(S_{1,2}g)(\zeta,\tau)| \leq c_2 \int |g(x,t)| dx\, dt \cdot \\ \cdot \exp(2\Delta|\mathrm{Im}\zeta| + 2\eta|\mathrm{Im}\tau|) \quad ,$$

for some c_2.

2. Proof of Proposition 6.1. In view of the hypothesis on f we can find $r' \in \mathscr{R}(\xi^o,\tau^o)$, $D' > 0, \eta' > 0$ and c_3 such that $|v(f)| \leq c_3$ for any $v \in \mathscr{E}^s_{D'}(R^{n+1}_{\eta'})'$ which satisfies $|\hat{v}(\zeta,\tau)| \leq \exp(r'(-\mathrm{Re}\zeta,-\mathrm{Re}\tau) + \Delta'|\mathrm{Im}\zeta|^\sigma + \eta'|\mathrm{Im}\tau|)$. We now apply Lemma 6.2 with $\Delta = \Delta'/4$ and $\eta = \eta'/4$ such that $v(f*g) = (v*S_1g)(f) + (\nabla^\ell v*S_2g)(f)$. Let also $v \in \mathscr{E}^s_D(R^{n+1}_\eta)'$ satisfy (2) for these Δ,η and consider $r \in \mathscr{R}(\xi^o,\tau^o)$. If r (e.g. of form dr') is small enough, then we can apply Proposition 5.4 in order to split $\nabla^\ell v$ in the form

$$\nabla^\ell v = v_1 + v_2$$

for some $v_i \in \mathscr{E}^s_{D''}(R^{n+1}_{\eta'/2})'$ which satisfy

$$|\hat{v}_1(\zeta,\tau)| \leq c_4 \exp(r'(-\mathrm{Re}\zeta,-\mathrm{Re}\tau) + (\Delta'/2)|\mathrm{Im}\zeta|^{\sigma} + (\eta'/2)|\mathrm{Im}\tau| + \mu \ln(1+|(\zeta,\tau)|)) ,$$

respectively

$$|\hat{v}_2(\zeta,\tau)| \leq c_4 \exp((\Delta'/2)|\mathrm{Im}\zeta|^{\sigma} + (\eta'/2)|\mathrm{Im}\tau| + b_1 \ln(1+|(\zeta,\tau)|)$$

and where $\Delta'/2 = (sD'')^{-\sigma/s}(1/\sigma)$.

We now estimate $v(f*g)$ using $v(f*g) = (v*S_1g)(f) + (v_1*S_2g)(f) + (v_2*S_2g)(f)$

It is then clear that the proposition follows as soon as we can show that the inequalities (4), (5), (6) below are valid

(4) $\quad |(v*S_1g)(f)| \leq c_5\int|g(x,t)|\, dx\, dt$,

(5) $\quad |(v_1*S_2g)(f)| \leq c_6\int|g(x,t)|\, dx\, dt$,

(6) $\quad |(v_2*S_2g)(f)| \leq c_7\int|g(x,t)|\, dx\, dt$.

a) Proof of (4) and (5). It follows from (3) that

$$|\mathscr{F}(v*S_1g)(\zeta,\tau)| \leq c_2\exp(r'(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)+\Delta'|\mathrm{Im}\zeta|^{\sigma} + \eta'|\mathrm{Im}\tau|) \cdot \int|g(x,t)\, dx\, dt .$$

Thus (4) follows with $c_5 = c_2 \cdot c_3$ by the choice of r',Δ',η' . A similar argument gives (5).

b) Proof of (6). This follows from the fact that $v_2*S_2g/\int|g(x,t)|dx\, dt$ is a bounded set in $\mathscr{E}^s_A(R^{n+1}_h)'$ if Δ',η' are small, as we may assume. This concludes the proof of Proposition 6.1 and thus also that of Proposition 3.4.

3. <u>Proof of Proposition 3.5</u>. Using the assumption on f , Proposition 3.4 and Lemma 4.5 we conclude that we can find a finite set of functions $r_1,\ldots,r_j : R^{n+1} \to R_+$, some $r_o \in \mathscr{R}(U)$ and $c > 0, D > 0, \eta > 0$ which have the following properties:

a) for every $b' \geq 0$ we can find c' such that $|v(f)| \leq c'$ for any

$v \in \mathscr{E}^s_D(R^{n+1}_\eta)'$ for which $|\hat{v}(\zeta,\tau)| \le \exp(r_k(-\mathrm{Re}\zeta,-\mathrm{Re}\tau)+\Delta|\mathrm{Im}\zeta|^\sigma +$

$+ \eta|\mathrm{Im}\tau| + b'\ln(1+|(\zeta,\tau)|))$ for some $k, 1 \le k \le j$,

b) $|r_i(\xi,\tau) - r_i(\xi',\tau')| \le 1 + |\xi-\xi'|^\sigma + |\tau-\tau'|$, $i=1,\ldots,j$,

c) $\max\limits_{i\ge 1} r_i(\xi,\tau) \ge r_o(\xi,\tau)$.

Let us also fix $b \ge 0$, and apply Proposition 5.2. It follows that we can find $d > 0, \Delta' > 0, \eta' > 0, b' \ge 0$ and c'' such that every $v \in \mathscr{E}^s_D(R^{n+1}_\eta)'$ which satisfies $|\hat{v}(\zeta,\tau)| \le \exp(d\ r_o(-\mathrm{Re}\zeta,-\mathrm{Re}\tau) + \Delta'|\mathrm{Im}\zeta|^\sigma + \eta'|\mathrm{Im}\tau| + b\ \ln(1+|(\zeta,\tau)|))$ can be decomposed in the form $v = \sum\limits_{i\ge 1} v_i$, for some $v_i \in \mathscr{E}^s_{D'}(R^{n+1}_{\eta'})'$ which satisfy

$$|\hat{v}_i(\zeta,\tau)| \le c''\ \exp(r_i(-\mathrm{Re}\zeta,-\mathrm{Re}\tau) + \Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau| + b'\ln(1+|(\zeta,\tau)|)) .$$

The proposition now follows since we obtain that $|v(f)| = |\Sigma v_i(f)| \le j\ c'c''$.

4. <u>Proof of Proposition 3.6</u>. Consider $f_1,f_2 \in \mathscr{E}^s_A(R^{n+1}_h)$ as in the proposition and consider quasi-cones $U_1,U_2 \subset R^{n+1} \setminus \{0\}$ such that $\{0\} \times U_i \cap WF^s_A f_i = \emptyset$ for $i = 1,2$ and such that the sets $T(U_i)$ are closed. We may assume that $CU_1 + CU_2 \subset R^{n+1} \setminus \{0\}$. To prove the proposition it then suffices to show that $(0,\xi^o,\tau^o) \notin WF^s_A(f_1 \cdot f_2)$ if $(\xi^o,\tau^o) \notin (CU_1 \cup CU_2 \cup (CU_1 + CU_2))$. In order to prove this last assertion, let us choose $r_i \in \mathscr{R}(U_i), i=1,2$, $D > 0, \eta > 0$ and $c > 0$ such that $|v(f_i)| \le c$ if $v \in \mathscr{E}^s_D(R^{n+1}_\eta)'$ satisfies

$$(7) \qquad |\hat{v}(\zeta,\tau)| \le \exp(r_i(-\mathrm{Re}\zeta,-\mathrm{Re}\tau) + \Delta|\mathrm{Im}\zeta|^\sigma + \eta|\mathrm{Im}\tau|) .$$

Such v_i,D,η exist in view of Proposition 3.5.

Let us now apply Lemma 4.6 for r_1,r_2 and denote by r the element from $\mathscr{R}(\xi^o,\tau^o)$ given by that lemma. We want to show that $|v(f_1 \cdot f_2)| \le c'$ if $v \in \mathscr{E}^s_{D'}(R^{n+1}_\eta)'$ satisfies

$$|\hat{v}(\zeta,\tau)| \le \exp(r(-\mathrm{Re}\zeta,-\mathrm{Re}\tau) + (\Delta/2^{\sigma})|\mathrm{Im}\zeta|^{\sigma} + \eta|\mathrm{Im}\tau|) \tag{8}$$

where D' is associated with $(\Delta/2^{\sigma})$ via (1), §2. In order to show this we shall show that

$$|\mathscr{F}(f_1 v)(\zeta,\tau)| \le c''\exp(r_2(-\mathrm{Re}\zeta,-\mathrm{Re}\tau) + \Delta|\mathrm{Im}\zeta|^{\sigma} + \eta|\mathrm{Im}\tau|) \tag{9}$$

in this case.

To prove (9) we choose a suitable representation measure ω for f_1. In fact from the definition of r_1,Δ,η and from Hahn-Banach's theorem it follows that we can find a Radon measure ω on C^{n+1} and c_1 such that

a) $\int\exp(r_1(-\mathrm{Re}\zeta',-\mathrm{Re}\tau')+\Delta|\mathrm{Im}\zeta'|^{\sigma} + \eta|\mathrm{Im}\tau'|)d\,\omega(\zeta',\tau') < c_1$,

b) $v(f_1) = \int\hat{v}(\zeta',\tau')\,d\omega(\zeta',\tau')$ for all v which satisfy (7) for $i=1$.

In particular we get from (5), §2, that

$$\mathscr{F}(f_1 v)(\zeta,\tau) = \int v(\zeta',\tau')\,d\omega'(\zeta'-\zeta,\tau'-\tau) \tag{10}$$

for all such v. It now remains to estimate the integral from (10) when v satisfies (8) with the aid of Lemma 4.6.

5. Proof of Proposition 3.7. We consider $f \in \mathscr{E}^s_A(R^{n+1}_h)$ as in the proposition. Further we choose $g \in \mathscr{S}(R^n)$ such that $|g(x)| \le \exp(-A'|x|^s)$ and $|\hat{g}(\zeta)| \le c_1\exp(-c_2|\zeta|^{\sigma} + c_3|\mathrm{Im}\zeta|^{\sigma})$ for some $A' > A, c_1 > 0, c_2 > 0, c_3$ and $g \not\equiv 0$. It is easy to see that $H(x,t) = \int f(x-x',t)\,g(x')\,dx'$ is in $\mathscr{E}^s_{A''}(R^{n+1}_h)$ and that $\{0\} \times (R^{n+1}\setminus\{0\}) \cap WF^s_A\, h = \emptyset$. In particular H is therefore an analytic function for $|t|$ small. Since it vanishes for $t \le 0$ we conclude that $H \equiv 0$. We can now apply Proposition 2.4.

REFERENCES

J.E. BJÖRK: [1] Rings of differential operators, North-Holland Publishing Comp. Amsterdam-Oxford-New York.

S.D. EIDEL'MAN: [1] Parabolic systems, North-Holland Publ. Comp. Amsterdam-London Wolters-Noordhoff Publ. Groningen, (translated from the Russian).

I.M. GEL'FAND and G.E. SHILOV: [1] Fourier transforms of rapidly increasing functions and problems of uniqueness for solutions of the Cauchy problem (in Russian), U.M.N. 8:6 (1953), 3-54.

I.M. GEL'FAND: [2] Generalized functions vol. 2 and 3, Fizmatgiz. Moscow (1958).

V.V. GRUSHIN: [1] On the relation between the local and global properties of solutions of hypoelliptic equations with constant coefficients Mat. Sb. 66 (108) :4 (1965), 525-550.

L. HÖRMANDER: [1] An introduction to complex analysis in several complex variables, D. Van Nostrand Comp., New-York 1966.

[2] On the existence and regularity of solutions of linear pseudodiff. operators, L'enseignement math. XVII: 2, 99-163.

[3] Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, C.P.A.M. XXIV (1971), 671-704.

[4] Non uniqueness for the Cauchy problem, Colloque Int. Nice 1974 Fourier integral operators and partial differential equations, Springer Verlag, Lecture Notes 459 (1975), 36-73.

M. KASHIWARA: [1] Thesis, Univ. of Tokyo, 1971 (in Japanese).

R. LASCAR: [1] Propagation des singularites des solutions d'equations pseudodifferentielles quasi-homogenes, Ann. Inst. Fourier t XXVII: 2 (1977), 79-125.

O. LIESS: [1] Intersection properties of weakly analytic uniform classes of functions, Ark. Mat. 14: 1 (1976), 93-111.

[2] Necessary and suffcient conditions for propagation of singularities for const. coeff. lin. partial differential operators, Preprint 526 (January 1980), Darmstadt.

I.A. LUCKIJ: [1] Double pseudodifferential operators in spaces of generalized functions that increase exponentially, D.A.N. 235: 3 (1977), 527-530.

J. PERSSON: [1] Linear characteristic Cauchy problems for partial differential equations with variable not only time dependent coefficients, Boll. Un. Mat. Ital. IV Ser 4 (1971), 91-102.

W. RUDIN: [1] Lectures on the edge-of-the-wedge-theorem, Conf. board math. sc., Regional conf. series in math. nr. 6 A.M.S. Providence Rhode Island.

M. SATO: [1] Hyperfunctions and partial differential equations, Conf. on functional analysis and related topics, Tokyo 1969, 91-94.

Otto LIESS,
Fachbereiche Mathematik,
Technische Hochschule Darmstadt,
Schlossgartenstr. 7,
D-6100 Darmstadt (W. Germany).