E. DE GIORGI

Limits of functionals and differential operators

Astérisque, tome 89-90 (1981), p. 153-161

<http://www.numdam.org/item?id=AST_1981__89-90__153_0>

LIMITS OF FUNCTIONALS AND DIFFERENTIAL OPERATORS

by E. DE GIORGI(*) (Scuola Normale Superiore di Pisa)

INTRODUCTION.

The aim of this talk is to give a survey of a broad class of problems and methods that can be reduced to the study of limits of families of functionals and differential operators. Although they mostly developed in the framework of the calculus of variations and of the differential equations of mathematical physics, possibly they will be useful in the future for the study of differential equations of more general types and in many other different fields of mathematics. May be that the choice to apply these methods to some problems rather than to others is the product of accidental and human factors more than something intrinsic to the methods. Therefore it seems useful to me to introduce the specialist of different methods to this kind of ideas, with the hope they will suggest new applications to different fields, and in turn new applications will suggest further developments of the abstract theory.

THE HOMOGENIZATION PROBLEM

As a first example, we consider a class of minimum problems:

(*) The author wishes to thank Professor Mauro Nacinovich, who friendly and carefully prepared the notes of the lecture.

$$u \in H^1_o(a,b) \quad , \quad F_h(u) = \min_{H^1_o(a,b)} F_h(v)$$

where $F_h(v) = \int_a^b \left\{ \left|\frac{dv}{dt}\right|^2 (2+\sin(ht)) + \phi(t)\,v(t) \right\} dt$
for a given $\phi \in L^2(a,b)$.

It is well known that for each integer $h \geq 1$ this problem has a unique solution $u_h \in H^1_o(a,b)$, which is a (weak) solution of the differential equation:

$$\frac{d}{dt}((2+\sin ht)\,\frac{du}{dt}) = \phi(t) \qquad \text{on} \quad (a,b) \quad .$$

If, for a fixed $\phi \in L^2(a,b)$ we let h tend to ∞ , we find out that the sequence $\{u_h\}$ of the solutions of the minimum problems above weakly converges in $H^1_o(a,b)$ to a function $u_\infty \in H^1_o(a,b)$, solution of the differential equation

$$c\,\frac{d^2 u_\infty}{dt^2} = \phi(t) \qquad \text{on} \quad (a,b)$$

where the constant c is the harmonic mean of the function $2 + \sin t$:

$$\frac{1}{c} = \frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{2+\sin t} \quad .$$

If we let

$$F_\infty(v) = \int_a^b \left\{ c\left|\frac{dv}{dt}\right|^2 + \phi(t)\,v(t) \right\} dt$$

it is easy to prove that

$$F_\infty(v) = \inf\left\{ \liminf_{h\to+\infty} F_h(v_h) : v_h \to v \text{ in } L^2(a,b) \right\} = \inf\left\{ \limsup_{h\to+\infty} F_h(v_h) : v_h \to v \text{ in } L^2(a,b) \right\}$$

for every $v \in L^2(a,b)$, that is, in the notations of Γ-convergence theory,

$$F_\infty(v) = \Gamma(\mathbf{N}, L^2(a,b)^-) \lim_{\substack{h\to+\infty \\ u\to v}} F_h(u) \quad .$$

More generally, suppose X is a metric space, K is a compact subset of X and f_h are functions defined on X such that

$$(1) \qquad \inf_{x \in X} f_h(x) = \inf_{x \in K} f_h(x) \qquad \forall\, h \in \mathbf{N} \ .$$

Suppose also that, for every $x \in X$

$$\inf\left\{\liminf_{h \to +\infty} f_h(x_h) \ : \ x_h \to x \text{ in } X\right\} = \inf\left\{\limsup_{h \to +\infty} f_h(x_h) \ : \ x_h \to x \text{ in } X\right\}$$

and denote by

$$f_\infty(x) \quad \text{or} \quad \Gamma(\mathbf{N}, X^-) \lim_{\substack{h \to +\infty \\ y \to x}} f_h(y)$$

this value. Then we have

$$\min_{x \in X} f_\infty(x) = \lim_{h \to +\infty} \left[\inf_{x \in X} f_h(x)\right] \ .$$

Moreover, if

$$\inf_{x \in X} f_h(x) = f_h(x_h)$$

for some $x_h \in X$ and

$$x_h \to x_\infty \quad \text{in} \quad X \ ,$$

then we have

$$f_\infty(x_\infty) = \min_{x \in X} f_\infty(x) \ .$$

Roughly speaking, Γ-convergence and equicoerciveness (1) imply the convergence of minima and of minimum points. This is the crucial result in order to apply the Γ-convergence theory to variational problems.

The previous one-dimensional homogenization example has been generalized to the following case: let a_{jk} (for $1 \leq j, k \leq n$) be bounded measurable functions on $\mathbb{R}^n$, periodic with respect to each variable. We make the ellipticity assumption:

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_k\xi_j \geq \nu|\xi|^2 \quad \text{a.e. on } \mathbb{R}^n, \ \forall\xi \in \mathbb{R}^n$$

with a constant $\nu > 0$.

Let Ω denote an open bounded set in $\mathbb{R}^n$ and let us consider, for a fixed $\phi \in L^2(\Omega)$, the family of Dirichlet problems:

$$(*)_h \qquad \begin{cases} u \in H^1_o(\Omega) \\ \sum\limits_{j,k} \dfrac{\partial}{\partial x_j}(a_{jk}(hx)\dfrac{\partial u}{\partial x^k}) = \phi(x) \end{cases}$$

(h an integer ≥ 1).

For each h, this problem has a unique solution $u_h \in H^1_o(\Omega)$ and, as in the one dimensional case, it has been proved that the u_h's weakly approximate in $H^1_o(\Omega)$ a (weak) solution $u_\infty \in H^1_o(\Omega)$ of an elliptic partial differential equation with constant coefficients:

$$\sum_{j,k} b_{jk} \frac{\partial^2 u_\infty}{\partial x_j \partial x_k} = \phi$$

where the constants b_{jk} can be determined from the functions a_{jk}, however by methods more complicated than in the one-dimensional case, involving solutions of related partial differential equations and therefore not reducing to quadratures. We note that

$$\sum_{j,k} b_{jk}\, \xi_k\xi_j \geq \nu|\xi|^2 \quad \forall\xi \in \mathbb{R}^n$$

with the same ellipticity constant ν.

The problem arises of what happens when considering different types of differential operators with periodic coefficients. So far, results have been obtained for parabolic and hyperbolic equations generalizing the heat and the wave equation: the problem is still open for many other types of equations.

AN EXAMPLE WITH FIRST ORDER DIFFERENTIAL OPERATORS

Let $a_1,\dots,a_n$ be periodic functions of all the coordinates in $\mathbb{R}^n$ and let us consider the Cauchy problems:

$$(*)_h \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{j=1}^{n} a_j(hx)\,\dfrac{\partial u}{\partial x_j} \\ u(x,o) = \phi(x) \ . \end{cases}$$

(h is an integer ≥ 1).

For $n = 1$ and $a_1(x) = a(x) = 2 + \sin x$, it has been shown that the solutions u_h of problem $(*)_h$ approximate a solution u_∞ of a Cauchy problem for a differential operator with constant coefficients:

$$\begin{cases} \dfrac{\partial u_\infty}{\partial t} = c \cdot \dfrac{\partial u_\infty}{\partial x} \\ u_\infty(x,o) = \phi(x) \ . \end{cases}$$

But for $n = 2$, and $a_1(x) = a_1(x_1,x_2) = (\sin x_2) + 2$, $a_2(x) = 0$, although the solutions u_h of the Cauchy problems $(*)_h$ converge to a function u_∞ , it is easy to prove that the limit u_∞ is not, for general ϕ , a solution of any first order partial differential equation with constant coefficients.

It will be interesting to find out general sufficient conditions for the convergence of the solutions of $(*)_h$ and for the limit to satisfy a partial differential equation of the same type. Analogous problems should be investigated for differential equations of any order.

HYPERBOLIC EQUATIONS

For limits of hyperbolic problems, Colombini, De Giorgi and Spagnolo made the following remark:

consider the family of Cauchy problems

$$(*)_h \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = (2+\sin ht)\,\dfrac{\partial^2 u}{\partial x^2} \\ u(x,o) = \phi(x) \\ \dfrac{\partial u}{\partial t}(x,o) = \psi(x) \end{cases} .$$

If one requires that the solutions u_h's of these problems converge in some Sobolev type space, then he must assume that ϕ and ψ are real analytic. Dropping this drastic assumption, one has to content himself with convergence as analytic functionals, as Sobolev norms of the solutions cannot be uniformly bounded in terms of Sobolev norms of the initial data.

The study of the homogenization problem above had a great importance to suggest methods to investigate the Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \alpha(t)\,\dfrac{\partial^2 u}{\partial x^2} \\ u(x,o) = \phi(x) \\ \dfrac{\partial u}{\partial t}(x,o) = \psi(x) \end{cases}$$

under assumptions on the regularity of α weaker than the usual requirements of α being Lipschitz continuous or at least of bounded variation. One finds out that the weaker the assumptions of α the stronger should be the assumptions upon the initial data: for instance, if α is assumed Hölder continuous with an exponent < 1, then ϕ and ψ must belong to suitable Gevray classes to obtain classical solutions.

RELATIONS BETWEEN STATIONARY AND EVOLUTION PROBLEMS

To study the convergence of solutions of a sequence of equations $A_h u_h = \phi$ to the solution of an equation $A_\infty u_\infty = \phi$, Spagnolo introduced the method of investigating the convergence of the corresponding evolution equations:

$$\frac{\partial u_h}{\partial t} - A_h u_h = \phi \quad \text{to} \quad \frac{u_\infty}{\partial t} - A_\infty u_\infty = \phi \quad .$$

The problem arises to find out general conditions to pass from the stationary to a corresponding evolution problem (even with $\frac{\partial^s}{\partial t^s}$ substituting $\frac{\partial}{\partial t}$) and vice-versa.

A result of this kind has been obtained, for instance, by Trotter and Kato; they proved that the convergence of the problems

$$A_h u_h + \lambda u_h = \phi \quad \text{to} \quad A_\infty u_\infty + \lambda u_\infty = \phi$$

is necessary and sufficient for the convergence of the corresponding evolution equations if the operators A_h are monotone.

It would be interesting to find more larger classes of operators A for which solutions of

$$(*) \qquad \frac{\partial u}{\partial t} - Au = f$$

for suitable initial data converge (is some function space) for $t \to \infty$ to a solution u_∞ of $Au_\infty = f$.

When $(*)$ is not solvable, then one should consider solvable near-by problems:

$$\frac{\partial u}{\partial t} = \varepsilon\, Eu + Au - f$$

(with for instance an elliptic or monotone E) and investigate

$$\lim_{\substack{\varepsilon\to o\\ t\to\infty}} u(t,\varepsilon)$$

(with all possible combinations of these limits, for instance choosing sequences $\{t_h\}$ and $\{\varepsilon_h\}$ and letting t_h tend to ∞ and ε_h go to o with various rules).

Vice-versa one can also study methods to reduce evolution problems to corresponding stationary problems.

For instance one can investigate conditions for the existence of solutions of the recursive problems:

$$Au_{h+1} - f = \lambda(u_{h+1} - u_h)$$
$$u_o = \phi \quad .$$

If $u_h(\lambda)$ denotes the solution to this problem, then one looks for the limits:

$$\lim_{h\to\infty} u_h(th) = w(t)$$

that would be comparable with the solutions of

$$\frac{dw}{dt} = Aw - f \quad .$$

Analogously solutions of the recursive system

$$Au_{h+1} - f = \lambda^2(u_{h+1} - 2u_h + u_{h-1})$$

would lead to solutions of the equations

$$\frac{d^2w}{dt^2} = Aw - f \quad .$$

To end with, I will mention the fact that, as in the theory of Γ-convergence often it is easier to solve variational rather than non variational problems, it could be convenient in some instances to substitute to the study of the equations

$A_h u = f$ that of the minima of the functionals

$$F_h(u,\Omega) = \int_\Omega |A_h u - f|^2 dx$$

(with Ω a variable bounded open set). For problems on an unbounded domain, one could also consider functionals of the type

$$\int_\Omega (|A_h u - f|^2 + \varepsilon|u|^2 + \eta|\Delta^k u|^2) e^{-x^2/\tau^2} dx$$

and take limits for $\varepsilon, \eta \to o$ and $\tau \to \infty$.

REFERENCES

[1] E. DE GIORGI: Convergence problems for functionals and operators. Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis, edited by E. De Giorgi, E. Magenes, V. Mosco, Pitagora Bologna 1979.

[2] F. COLOMBINI, E. DE GIORGI, S. SPAGNOLO: Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps. Annali Scuola Normale Superiore Pisa, Volume VI (1979), 511-559.

[3] E. DE GIORGI, F. FRANZONI: Su un tipo di convergenza variazionale. Rendiconti Classe Scienze Fisiche Matematiche Naturali, Accademia Nazionale Lincei, Volume 58 (1975), 842-850.

[4] E. DE GIORGI, F. FRANZONI: Su un tipo di convergenza variazionale. Rendiconti Seminario Matematico Brescia, Volume III (1979), 63-101.

[5] S. SPAGNOLO: Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. Annali di Scuola Normale Superiore Pisa, Volume XXII (1968), 571-597.

[6] E. DE GIORGI, S. SPAGNOLO: Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine. Bollettino Unione Matematica Italiana, Volume VIII (1973), 391-411.

Ennio DE GIORGI
Scuola Normale Superiore
Piazza Cavalieri, 7
I-56100 Pisa
(Italy)