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ON CONTINUATION OF REAL ANALYTIC SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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INTRODUCTION.

This lecture aims at an introduction to a special kind of problem among the vast study on continuation of solutions of linear partial differential equations. That is, we discuss here the continuation of real analytic solutions to a relatively small exceptional set, in particular the removable isolated singularity. Probably the first example of removable singularities in such a context which appeared in the history is Riemann's theorem on isolated singularity [1]. This theorem is, however, also outside our interest because it imposes an a priori bound to the solutions (of the Cauchy-Riemann equation!) on the neighborhood of the exceptional set. Hence the really first example for us is the Hartogs extension theorem [1] which guarantees especially the continuation of holomorphic functions of several variables to an isolated singularity without the assumption of any boundedness contrary to the case of one variable. This phenomenon has been explained by Bochner [1] from the standpoint of overdetermined systems, and generalized by Ehrenpreis [1], Malgrange [1] and afterwards by Komatsu [1] as a theorem even for the continuation of distribution or hyperfunction solutions. On the other hand, Grušin [1] has given a result on removable singularities for C^∞-solutions of a class of single equations (without any a priori bounds) and thus shown that the overdeterminedness is not the ultimate explanation when we treat the continuation of sufficiently regular solutions. (Shortly speaking his result says that for an operator P with constant coefficients of which every irreducible factor contains a real simple characteristic direction, every isolated singularity of a C^∞-solution u of $Pu = 0$ is removable).

The present speaker has started from the examination of this work of Grušin and given various results on continuation of real analytic or C^∞-solutions with the aim of the ultimate explication of this phenomenon. In this lecture we will introduce the outline of this study (but restricting ourselves to the real analytic solutions for the sake of simplicity and also of the title of this meeting). We divide the talk into three parts. In part I we give a typical result in the case of constant coefficients which is based on the Fourier analysis, or rather on the Fundamental Principle of Ehrenpreis [2]-Palamodov [1]. In part II we will explain the theory of non-characteristic hyperfunction boundary value problem by Komatsu-Kawai [1], Schapira [1] which is necessitated as a substitute for the Fundamental Principle in order to treat the equations with variable coefficients. By this tool the problem of continuation of regular solutions is translated to a kind of propagation of regularity along the boundary for solutions regular outside a non-characteristic boundary. Here we give a result which we think fairly sharp on this subject. The result of part I in the case of constant coefficients can be re-explained by the method of this part except for the irreducible decomposition of the operator. Finally in part III we attack the problem of irreducibility of the operators including those with variable coefficients, aiming at the same time to accomplish the superiority of the method of part II even in the case of constant coefficients. The method proposed here is a kind of Fundamental Principle in the conormal sphere bundle of a manifold with boundary. The latter space is the key tool for Kataoka's boundary value theory [1]. Though the result is yet partial, we expect that we have chosen a right way to this last unsolved important problem in the generic theory of linear partial differential equations.

As the class of generalized functions we will employ the hyperfunctions.

A hyperfunction $u(x)$ on an open set $\Omega \subset \mathbb{R}^n$ is the sum of formal boundary value expressions of holomorphic functions on various wedges with the edge Ω :

$$(0.1) \qquad u(x) = \sum_{j:\text{finite}} F_j(x + i\Gamma_j 0), \quad F_j(z) \in \mathcal{O}(\Omega + i\Gamma_j 0) .$$

Here Γ_j denotes a convex open cone with vertex at the origin, and $\Omega + i\Gamma_j 0$ denotes an infinitesimal wedge of the width Γ_j , that is, a complex open set which approaches from the interior to the product set $\Omega + i\Gamma_j$ when the imaginary coordinates approach the vertex of Γ_j . In the above expression we can perform the following reduction of terms:

$$F_j(x + i\Gamma_j 0) + F_k(x + i\Gamma_k 0) = (F_j + F_k)(x + i\Gamma_j \cap \Gamma_k 0) \quad \text{if } \Gamma_j \cap \Gamma_k \neq \emptyset .$$

An expression (0.1) is said to be equal to zero if after a finite times of such reduction (both composition and decomposition) all the terms disappear.
For an open subset $\Omega' \subset \Omega$ we can define the restriction $u|_{\Omega'}$ by simply restricting the domain of definition of $F_j(z)$. This definition satisfies the localization principle, that is, the hyperfunctions constitute a sheaf. In particular, the notion of support is legitimately defined.
We have the obvious operation of linear combination over $\mathbb{C}$, the operation of differentiation:

$$\frac{\partial u}{\partial x_k} = \sum_j \frac{\partial F_j}{\partial z_k}(x + i\Gamma_j 0) ,$$

and multiplication by a real analytic function $\phi(x)$:

$$\phi(x)u(x) = \sum_j \phi(z)F_j(z)\Big|_{z \to x + i\Gamma_j 0} ,$$

hence combining these the operation of linear differential operators with real analytic coefficients. Obviously these operations are local, that is, define sheaf homomorphisms.We have also the integration on a bounded domain $D \subset \mathbb{R}^n$ if each $F_j(z)$ is continuous up to a neighborhood of the real set ∂D :

$$\int_D u(x)\,dx = \sum_j \int_{D_j} F_j(z)\,dz ,$$

where D_j is an n-dimensional path deformed into the infinitesimal wedge $\Omega + i\Gamma_j 0$ with the fixed boundary ∂D. Especially this last is the case when the support of u is a compact subset of D. In fact, such u admits an expression of the form (0.1) with those $F_j(z)$ which can be continued analytically up to the real axis outside supp u. The integral $\int_D \phi(x)u(x)dx$ then defines a duality between the real analytic functions ϕ and the hyperfunctions u with compact supports.

The sheaf B of hyperfunctions constitutes the most wide class of localizable generalized functions. The sheaf D' of distributions is contained in B in such a way that the ideal limits $x + i\Gamma_j 0$ in (0.1) are replaced by those of distribution sense. Among others B has the property of flabbiness, that is, a hyperfunction is always prolongeable from any open subset to the whole space. This will be found a great advantage for the problem of continuation of solutions. In the sequel, the properties of B will be explained every time when they become necessary. The audience will be able to accept them without a special knowledge to the theory of hyperfunctions.

To conclude the introduction the speaker would like to refer to [3], [9] as previous survey reports by him for a better comprehension of this talk.

I. CONTINUATION OF REAL ANALYTIC SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

For the convenience of references we first list up main results on continuation of solutions for general systems. Let P(D) be a system of linear partial differential operators corresponding to the polynomial matrix $P(\zeta) : (\mathbb{C}[\zeta])^s \to (\mathbb{C}[\zeta])^t$ of size $t \times s$. We put $D = (D_1, \ldots, D_n)$, $D_j = -i\partial/\partial x_j$.

Theorem 1. (Komatsu [1]) Let K be a convex compact set in $\mathbb{R}^n$ and U be a convex open neighborhood of K. Then every hyperfunction solution u of

$P(D)u = 0$ in $U \setminus K$ can be necessarily continued to a hyperfunction solution on U if and only if P is overdetermined, that is, $\mathrm{Ext}^1(\mathrm{Coker}\,{}^tP, \mathbb{C}[\zeta]) = 0$.

Note that in the case of a single operator P , $\mathrm{Ext}^1(\mathrm{Coker}\,{}^tP, \mathbb{C}[\zeta])$ $= \mathbb{C}[\zeta]/P(\zeta)\mathbb{C}[\zeta] \neq 0$ except when $P \equiv \text{const.} \neq 0$. Thus the above assertion becomes trivial for a single operator.

Theorem 2. (Kaneko [2]) Let $K \subset U$ be as above. Then every real analytic solution u of $P(D)u = 0$ in $U \setminus K$ can be necessarily continued to a hyperfunction solution on U if and only if $\mathrm{Ext}^1(\mathrm{Coker}\,{}^tP, \mathbb{C}[\zeta])$ has no elliptic factor, that is, every irreducible component of the algebraic varieties associated with this module contains a real point at infinity.

Theorem 2'. Under the same situation, every real analytic solution u of $P(D)u = 0$ in $U \setminus K$ can be continued to a real analytic solution on U if and only if in addition to the above condition P is determined, that is, $\mathrm{Hom}(\mathrm{Coker}\,{}^tP, \mathbb{C}[\zeta]) = 0$.

The difference between these two results is whether the propagation of regularity up to K holds or not for hyperfunction solutions on U regular in $U \setminus K$. We will consider this as a secondary question and in the sequel we will mainly concern the continuation of real analytic solutions as hyperfunction solutions. Of course we will remark the possibility of continuation as real analytic solutions every time when we can know it. Thus we will say that K is an "exceptional set" for a real analytic solution u of $P(D)u = 0$ if u cannot be continued to K even as a hyperfunction solution. (We avoid the terminology "singularity" which is usually preserved for the singularity of a solution. Note however that the singularity of a holomorphic function is used not in this sense but rather as the exceptional set in our sense).

Now a single operator $P \neq 0$ is always determined and Theorem 2' implies the following non-trivial assertion:

Corollary 3. Let $K \subset U$ be as above. The continuation of real analytic solutions of a single equation $P(D)u = 0$ from $U \setminus K$ to U is always possible if and only if each irreducible component of P is non-elliptic.

We will start our study by examining this result. We have given in [10] a fairly elementary proof without the theory of hyperfunctions. Here we prefer the original presentation based on the hyperfunctions [1] for the sake of consistency with the other parts. In the sequel we will denote by $A_P(U)$ (resp. $B_P(U)$) the space of real analytic (resp. hyperfunction) solutions of $P(D)u = 0$ on U.

Thus let $u \in B_P(U \setminus K)$. By virtue of the flabbiness of B we can choose an extension $[u] \in B(U)$. Then $P(D)[u]$ becomes a hyperfunction with support in K. Let $B[K]$ denote the totality of hyperfunctions with supports in K. As is easily seen from the definition of the integral, the Fourier transform $\hat{v}(\zeta) = \int_{\mathbb{R}^n} e^{-ix\zeta} v(x)dx$ of $v \in B[K]$ satisfies the following estimate of Paley-Wiener type:

For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$(I.1) \qquad |\hat{v}(\zeta)| \leq C_\varepsilon e^{\varepsilon|\zeta| + H_K(\operatorname{Im}\zeta)} ,$$

where $H_K(\operatorname{Im}\zeta) = \sup_{x \in K} \operatorname{Re}(-ix \cdot \zeta)$ is the supporting function of K. The ambiguity of the construction $u \to \widehat{P(D)[u]}$ obviously vanishes when restricted to the variety $P(\zeta) = 0$. If we employ the multiplicity variety $N(P) = \left\{N(P_j^{\nu_j}),\ j = 1,\ldots,s\right\}$ taking into account the multiplicity ν_j of each irreducible component P_j, then the mapping

$$(I.2) \qquad \begin{array}{ccc} B_P(U \setminus K)/B_P(U) & \xrightarrow{\ G\ } & \widehat{B[K]}(N(P)) \\ \Psi & & \Psi \\ u & \longmapsto & \left\{ \dfrac{\partial^k}{\partial \zeta_1^k} \widehat{P(D)[u]}\Big|_{N(P_j)}, 0 \leq k \leq \nu_j - 1, j = 1,\ldots,s \right\} \end{array}$$

becomes even injective, where the right-hand side denotes the space of (family of) holomorphic functions on the multiplicity variety with the growth order indicated by (I.1). This assertion is a part of the so called Fundamental Principle. (Here we are assuming that $x_1 = \text{const.}$ is non-characteristic with respect to P).

We will call the mapping G the "Grušin representation".

The real analytic solutions $A_P(U \setminus K)/A_P(U)$ constitute a subspace of the left-hand side of (I.2). (In fact we have the propagation of regularity $A_P(U \setminus K) \cap B_P(U) = A_P(U)$ for a single operator $P \neq 0$). The image $Gu = \{G_{jk}u\}$ of $u \in A_P(U \setminus K)$ has a more strict estimate of the following form: For any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $C_\varepsilon > 0$ such that

$$(I.3) \qquad |F(\zeta)| \leq C_\varepsilon e^{-\delta|\zeta| + H_K(\mathrm{Im}\zeta) + \varepsilon|\mathrm{Im}\zeta|}$$

An entire function which satisfies this estimate in the whole space would reduce to 0 as is easily seen from the Phragmén-Lindelöf principle. Because the estimate holds now only on $N(P)$, the triviality of $Gu = \{G_{jk}u\}$ depends on the situation of $N(P_j)$ relative to the real axis. If P_j is elliptic, we see easily that the function 1 satisfies (I.3) on $N(P_j)$. On the other hand, assume that P_j has a real point at infinity. Then by way of the local coordinates on a neighborhood of this point, we can apply a variant of Phragmén-Lindelöf principle (Carlson's theorem) to conclude that a holomorphic function on $N(P_j)$ satisfying (I.3) vanishes on this neighborhood. Note that the propagation of zero, or the principle of analytic continuation, holds on an irreducible algebraic variety. Thus under the assumption that <u>every</u> P_j is non-elliptic, we can conclude that $Gu = 0$, hence $u \in B_P(U)$ (and even $u \in A_P(U)$ because of the propagation of regularity).

The deduction of the estimate (I.3) can be performed by various methods according to the choice of "approximate real analytic functions with compact support". (Remark that the estimate (I.3) mentally implies that $F(\zeta)$ is the Fourier transform of a real analytic function with support in K). The method which we propose here is the use of a class of differential operators of infinite order. Consider a formal series

$$(I.4) \qquad J(D) = \sum_\alpha a_\alpha D^\alpha ,$$

with the coefficients $a_\alpha \in \mathbb{C}$ satisfying

(I.5) $$\lim_{|\alpha|\to\infty} \sqrt[\alpha]{|a_\alpha|\alpha!} = 0 \quad .$$

Then (I.4) operates to the holomorphic functions, hence to the hyperfunctions in a local manner, that is, as a sheaf homomorphism by the formula

$$J(D)u(x) = \sum_j (J(D)F_j)(x + i\Gamma_j 0) \quad , \quad J(D)F_j(z) = \sum_\alpha a_\alpha D^\alpha F_j(z) \quad .$$

(It is a good exercise of the complex function theory to show that (I.5) is the necessary and sufficient condition for the locally uniform convergence of the latter series on the domain of definition of $F_j(z)$).

A hyperfunction is differentiable of infinite order in this generalized sense. On the other hand, we have

<u>Lemma 4</u>. (Chou [1], Kaneko [5]) $u \in B(\Omega)$ is real analytic at $x_0 \in \Omega$ if and only if for every $J(D)$, $J(D)u$ is a germ of continuous function at x_0 .

An approximate real analytic function $u(x)$ in our sense is such that $J(D)u(x)$ is continuous (C^∞) for some subfamily of $J(D)$. Note also that we can choose an extension of u to K even in C^∞ functions if we accept a modification of u on the ε-neighborhood K_ε of K . Thus given $u(x) \in A_P(U \setminus K)$, we can apply this modified construction of the Grušin representation to the new element $J(D')u(x)$ of $A_P(U \setminus K)$ to obtain

(I.6) $$J(\zeta')Gu = \left\{ \frac{\partial^k}{\partial \zeta_1^k} \widehat{[P(D)(\chi(x)J(D')u(x))]_0}\Big|_{N(P_j)} , 0 \le k \le \nu_j - 1, j=1,\dots,s \right\} .$$

Here $\zeta' = (\zeta_2,\dots,\zeta_n)$ and $\chi(x)$ is an appropriate cut-off function of class C^∞ such that

$$\chi(x) \equiv 1 \quad \text{in} \quad U \setminus K_\varepsilon \quad ,$$

$$\chi(x) \equiv 0 \quad \text{in} \quad K_{\varepsilon/2} \quad ,$$

The symbol $[\quad]_0$ means that we extend the object by zero up to K . The precise deduction of the identity (I.6) follows from the fact that the difference $\widehat{[P(D)\chi(x)J(D')u]_0} - \widehat{J(D')P(D)[u]}$ produces the factor $P(\zeta)$, hence vanishes on the multiplicity variety $N(P)$. The right-hand side of (I.6) satisfies the estimate like the Fourier image of a C^∞ function with support in K_ε . Therefore we will have, given any $J(\zeta')$ and any $\varepsilon > 0$,

$$|J(\zeta')G_{jk}u| \leq C_{J,\varepsilon} e^{H_K(\mathrm{Im}\zeta) + \varepsilon|\mathrm{Im}\zeta|} .$$

The arbitrariness of J together with the non-characteristic assumption gives the estimate (I.3) for $G_{jk}u$.

Now we turn to another situation where K is now the part in $\{x_n < 0\}$ of a convex compact set $L = \overline{K} \subset \{x_n \leq 0\}$, and U is a neighborhood of K in $\mathbb{R}^n$ (in the sense that $K \subset U$ is a relatively closed subset). As remarked by Malgrange [1], we have an affirmative result on continuation of general solutions analogous to Theorem 1 under the assumption of overdeterminedness, which, however, is no more necessary. The precise condition can be expressed by a kind of hyperbolicity of the module $\mathrm{Ext}^1(\mathrm{Coker}\,{}^tP, C[\zeta])$ (see Kaneko [4]). Now the Grušin representation takes the following form of relative nature:

$$\begin{array}{ccc} B_P(U \setminus K)/B_P(U) & \xrightarrow{\ G\ } & \widehat{B[L]}(N(P))/\widehat{B[L \setminus K]}(N(P)) \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ u & \longmapsto & \left\{ \dfrac{\partial^k}{\partial \zeta_1{}^k} \widehat{[[P(D)[u]]]}\Big|_{N(P_j)}, 0 \leq k \leq \nu_j - 1, j=1,\dots,s \right\} \end{array} .$$

Here $[u] \in B(U)$ denotes an extension of $u \in B_P(U \setminus K)$ and $[[P(D)[u]]] \in B[L]$ an extension of $P(D)[u] \in B(U)$. (We are always assuming that $x_1 = 0$ is non-characteristic with respect to P). Thus this time the ambiguity of the extension on the "lid" $L \setminus K$ remains up to the final stage. By way of $J(D)$'s a real analytic solution $u \in A_P(U \setminus K)$ can be characterized by the following estimate of relative type for every component $F(\zeta)$ of Gu :

(I.7) For any $J(\zeta')$ and $\varepsilon > 0$, there exists a decomposition

$J(\zeta')F(\zeta) = V(\zeta) + W(\zeta)$ such that

$$|V(\zeta)| \leq Ce^{H_L(\mathrm{Im}\zeta) + \varepsilon|\mathrm{Im}\zeta|} ,$$

$$|W(\zeta)| \leq C_\gamma e^{\gamma|\zeta| + H_{L\setminus K}(\mathrm{Im}\zeta) + \varepsilon|\mathrm{Im}\zeta|} , \text{ for every } \gamma > 0 .$$

This estimate can be obtained by the same way as above on decomposing the support of $P(D)(\chi(x)J(D')u)$ by another C^{∞} function ϕ on a neighborhood of the lid $L \setminus K$. Thus we obtain the following abstract theorem:

Theorem 5. We have $A_P(U \setminus K)/A_P(U) = 0$ if and only if every global holomorphic function $F(\zeta)$ on $N(P)$ satisfying (I.7) satisfies simply

$$\text{(I.8)} \qquad |F(\zeta)| \leq C_{\gamma} e^{\gamma|\zeta| + H_{L\setminus K}(\mathrm{Im}\zeta)}, \quad \text{for every } \gamma > 0 ,$$

that is, the estimate corresponding to $\widehat{B[L \setminus K]}(N(P))$.

Note that the propagation of regularity $B_P(U) \cap A(U \setminus K) = A_P(U)$ is known by Kawai [1] even for our present situation.

The assumption of Theorem 5 is far from practical. We will give here two practical sufficient conditions for that. First note that if we have the following estimate on $N(P)$:

$$\text{(I.9)} \qquad H_L(\mathrm{Im}\zeta) \leq H_{L \setminus K}(\mathrm{Im}\zeta) + \varepsilon|\zeta| + C_{\varepsilon} , \quad \text{for every } \varepsilon > 0 ,$$

then the image of every $u \in B_P(U \setminus K)$ will satisfy (I.8). This corresponds to the continuation of all the hyperfunction solutions and (I.9) implies the weak hyperbolicity of P to the direction dx_n . Considering the unique continuation property for real analytic solutions we can directly obtain the first half of the following theorem:

Theorem 6. (Kaneko [4]) Assume that every irreducible component P_j of P satisfies either of the following conditions:

1) There exists a sequence of conormal directions θ_k converging to dx_n such that P_j is weakly hyperbolic to θ_k .
2) $K \subset \{x_1 = 0\}$ and the roots $\tau_k(\zeta'), k=1,\dots,m_j$ of the equation $P_j(\zeta_1,\zeta') = 0$ for ζ_1 satisfy the estimate

$$\text{(I.10)} \qquad |\mathrm{Im}\tau_k(\zeta')| \leq \varepsilon|\zeta_n| + b_{\zeta''}|\mathrm{Im}\zeta_n| + C_{\zeta'',\varepsilon} , \quad \text{for every } \varepsilon > 0 \text{ and } \zeta'' \in \mathbf{C}^{n-2}.$$

(Here m_j is the order of P_j and $\zeta'' = (\zeta_2,\dots,\zeta_{n-1})$).

Then we have $A_p(U \setminus K)/A_p(U) = 0$.

The second half of this theorem constitutes the main subject of further discussion. Note that (I.10) implies the hyperbolicity of the "frozen" operator $P_j(D_1,0,D_n)$ in two variables x_1, x_n . This is far from the hyperbolicity in all the variables and the second part of this theorem is much deeper than the first part. As in the case of compact K , we reduce the problem to the estimation on one variable ζ_n . Then we can apply the following Phragmén-Lindelöf type theorem of relative nature:

Lemma 7. (Kaneko [4]) Let $F(z)$ be a holomorphic function of one variable z on $\operatorname{Im} z \geq 0$. Assume that for any $J(z)$ and any $\varepsilon > 0$, there exists a decomposition $J(z)F(z) = V(z) + W(z)$ such that

$$|V(z)| \leq Ce^{\varepsilon|z|} ,$$
$$|W(z)| \leq Ce^{b|\operatorname{Im} z|} .$$

Then $F(z)$ simply satisfies, for any $\varepsilon > 0$,

$$|F(z)| \leq C_\varepsilon e^{\varepsilon|z|} .$$

After we improve the estimate with respect to ζ_n by way of this lemma, we consider the symmetric functions of $F(\tau_k(\zeta'),\zeta'), k = 1,\dots,m_j$ and employ a lemma by Martineau for the growth order of entire functions to obtain a global uniform estimate in ζ' , hence (I.8). Note that the zero propagates along an irreducible variety and correspondingly the assumption of non-ellipticity requires the existence of only one approximately real roots. On the contrary, we cannot expect the propagation of estimate, and consequently we must impose here the condition (I.10) to all the roots concerned.

Remark. The article Kaneko [4] contains other examples of operators for which the continuation is possible. For example, for a heat equation of any space dimension

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} - \frac{\partial}{\partial x_n}\right) u(x) = 0$$

the conclusion of Theorem 6 is true even for K not in $\{x_1 = 0\}$. Since the calculus there essentially concerns the principal part only, the same conclusion is true for the Schrödinger equation with constant coefficients. In order to explain this result from the local theory, we would have to replace the non-characteristic boundary value theory of the next part by one for a family of boundaries possessing characteristic points, or by a somewhat global substitute relying on the analytic functionals.

II. HYPERFUNCTION BOUNDARY VALUE THEORY AND PROPAGATION OF MICRO-ANALYTICITY ALONG THE BOUNDARY

Now we intend to interpret our results from the local theory with the attempt of generalization to the case of variable coefficients. Which tool can replace the Fourier transform? For the function $F(\zeta) \in Gu$ on $N(P)$ constructed in part I, there corresponds always an entire function of polynomial form in ζ_1 :

$$F(\zeta) = \zeta_1^m + f_1(\zeta')\zeta_1^{m-1} + \dots + f_m(\zeta')\big|_{N(P)} , \quad f_j(\zeta') \in \mathcal{O}(\mathbb{C}^{n-1}) .$$

Such an entire function can easily be obtained via the global interpolation formula. Since the exceptional set K is contained in the non-characteristic hyperplane $x_1 = 0$, it satisfies also the same estimate as $F(\zeta)$. Therefore we can consider it (or rather its coefficients $f_j(\zeta')$) as a representative for the Grušin transform of u . In fact these coefficients are essentially the Fourier image of the difference of the boundary values of $u(x)$ to $x_1 = 0$ from both sides. Thus we are led to the idea of employing the boundary value theory.

Let $P(x, D)$ be a linear partial differential operator of order m with real analytic coefficients defined on a neighborhood of $0 \in \mathbb{R}^n$. Let K be a locally closed set contained in the non-characteristic hypersurface $x_1 = 0$. Assume that $0 \in K$ and let U be a neighborhood of 0 in $\mathbb{R}^n$ on which P is defined and which contains K as a closed subset. Put $U_{\pm} = U \cap \{\pm x_1 > 0\}$. We will discuss the continuation of real analytic solution $u \in A_P(U \setminus K)$. Let $[u]_{\pm} \in B(U)$ be the canonical extension of the solution $u|_{U_\pm}$. By definition it has support in $\overline{U_\pm}$ respectively and satisfies the identity of the form

$$\pm P(x,D)[u]_{\pm} = \sum_{j=0}^{m-1} u_j^{\pm}(x')\delta^{(m-j-1)}(x_1) \quad \text{in } U . \tag{II.1}$$

The coefficients $u_j^{\pm}(x')$ are uniquely determined by this identity and called the boundary values of u from $\pm x_1 > 0$ (with respect to some normal boundary system which we need not specify here). The identity (II.1) can be obtained by the following way: By the flabbiness of B we can obtain anyway an extension $\tilde{u} \in B(U)$ of $u|_{U_+} \in B(U_+)$ such that $\operatorname{supp} \tilde{u} \subset \overline{U_+}$. Then $\operatorname{supp} P(x,D)\tilde{u} \subset \{x_1 = 0\}$, hence we will have the expansion

$$P(x,D)\tilde{u} = \sum_{j=0}^{\infty} u_j(x')\delta^{(j)}(x_1) \quad . \tag{II.2}$$

In case where the left-hand side has compact support, such expansion holds in the sense of topology. In the contrary case the expansion is formal and even not faithful. The hyperfunction boundary value theory (cf. Komatsu-Kawai [1]) shows however that we can "divide" the right-hand side by $P(x,D)$ so that the residue gives the right-hand side of (II.1) and the quotient serves the modification of $\tilde{u}$ to the canonical extension. (The division holds in the sense of topology in the case of compact support, and it can be localized in the general case. The calculus is just the dual of the Cauchy-Kowalevsky theorem). Note that we are employing here the flabbiness of B in an essential way. (In the category of distributions such a construction is only possible when we restrict ourselves to the solutions a priori prolongeable as distributions beyond the boundary. But then the calculus becomes purely algebraic and works even for the characteristic boundary of parabolic type).

Now because of the uniqueness of the expression (II.1), the coefficients give the boundary values in the usual sense at every point of the boundary where the solution $u(x)$ is prolongeable as a usual function. Especially $u_j^{\pm}(x')$ are real analytic outside K, and hence the differences

(II.3) $\quad u_j(x') = u_j^+(x') - u_j^-(x')\,, \quad j = 0,\ldots,m-1\,,$

have support in K. We propose that $u \to \{u_j\}$ replaces the role of the Grušin representation, because in the case of constant coefficients this one has been nearly equivalent to the Fourier image of $\{u_j\}$. Then the decay property for Gu coming from the analyticity of u will correspond to a kind of micro-analyticity of $\{u_j\}$. The Phragmén-Lindelöf principle will correspond to a kind of micro-local unique continuation property. For the latter we have the following famous result, called the Holmgren type theorem:

<u>Lemma 8</u>. (Kashiwara-Kawai) Let $u(x)$ be a hyperfunction defined on a neighborhood of $0 \in \mathbb{R}^n$ such that $\operatorname{supp} u \subset \{x_1 \geq 0\}$. Assume that u is micro-analytic at either of the conormal points $(0, \pm i dx_1 \infty)$. Then u vanishes on a neighborhood of 0.

<u>Remark</u>. We say by definition that $u(x)$ is micro-analytic at $(0, i\xi dx\infty)$ if u admits the analytic continuation into the half space $\{z \in \mathbb{C}^n;\ \mathrm{Re}\langle i\xi, z\rangle < 0\}$ near the point 0, that is, if u has, near the point 0, an expression of the form

$$u(x) = \sum F_j(x + i\Gamma_j 0)\quad ,$$

with $\Gamma_j \cap \{\langle \xi, y\rangle > 0\} \neq \phi$. This is equivalent to say that on a neighborhood of 0, u is the sum of a real analytic function and the inverse Fourier image of a (Fourier) hyperfunction exponentially decreasing on a conical neighborhood of ξ. The set of points $(x, i\xi dx\infty) \in iS_\infty^* \mathbb{R}^n \cong \mathbb{R}^n \times iS_\infty^{*\,n-1}$ where $u(x)$ is not micro-analytic, is called the singular spectrum (S.S. for short) of u and is written S.S.u. This is a closed subset of $iS_\infty^* \mathbb{R}^n$ and agrees with $WF_A(u)$ when $u(x)$ is a distribution (cf. Bony [1]).

Lemma 8 means that we have the unique continuation property to the micro-analytic direction. The fact that only one choice of the sign suffices is a little delicate. Kashiwara's proof can be found in Kaneko [11]. Since we have no other reference in European languages, we will give a simplified proof in the appendix of this part.

Thus we can obtain the following abstract theorem indicating the direction of our study:

Theorem 9. Assume that K is contained in one side of a hypersurface $\phi(x') = 0$ of class C^1 in $\mathbb{R}^{n-1} \cong \{x_1=0\}$ passing through 0. Assume that every real analytic solution u of $P(x,D)u = 0$ on $U_\pm$ has the boundary values $u_j^\pm(x')$ which are micro-analytic at either of the points $(0,\pm id\phi(x')\infty) \in iS^*_\infty \mathbb{R}^{n-1}$. Then $u \in A_P(U \setminus K)$ can be continued as a hyperfunction solution to a neighborhood of 0.

The proof will be clear: Since then (II.3) all vanish by virtue of Lemma 8, the continuation of u will be given by $[u]_+ + [u]_-$ in view of (II.1). Note that the continuation is unique. This follows from the uniqueness of the expression (II.1), where those $[u]_\pm$ with support in $x_1 = 0$ are not allowed except for the trivial one: $[u]_\pm \equiv 0$.

Thus the problem is now to seek such a class of operators as the above propagation of micro-analyticity up to the boundary holds for their real analytic solutions. In this direction we have given the following result.

Theorem 10. (Kaneko [8]) Let $P(x,D)$ be an operator with real analytic coefficients having $x_1 = 0$ as non-characteristic hypersurface. Assume that $P(x,D)$ is semi-hyperbolic to $x_1 < 0$ (resp. to $x_1 > 0$) on a neighborhood I of $(0,i\nu'dx'\infty)$ in $\mathbb{R}^{n-1} \times iS^{*\,n-2}_\infty$ in the following sense:

(II.4) There exists $\varepsilon > 0$ such that all the roots of the characteristic equation $P_m(x,\zeta_1,\xi') = 0$ for ζ_1 have non-positive (resp. non-negative) imaginary parts when $(x',i\xi'dx'\infty) \in I$ and $0 \leq x_1 \leq \varepsilon$ (resp. $-\varepsilon \leq x_1 \leq 0$).

Then the S.S. of the boundary values of real analytic solutions of $P(x,D)u = 0$ in U_+ (resp. in U_-) does not contain the point $(0,i\nu'dx'\infty)$.

Corollary 11. (Kaneko [8]) Let K and $\phi(x')$ be as in Theorem 9. Assume that $P(x,D)$ is semihyperbolic to $x_1 < 0$ and to $x_1 > 0$ on a neighborhood of either of the points $(0,\pm id\phi(x')\infty)$. Then $u \in A_p(U \setminus K)$ can be continued as a hyperfunction solution to a neighborhood of 0 .

This time the propagation of real analyticity does not necessarily take place, hence we cannot assert the continuation as a real analytic solution in general.

For an operator with constant coefficients the assumption (II.4) of Theorem 10, when we choose $i\nu' dx'\infty = idx_n\infty$, means the following inequality for the roots $\zeta_1 = \tau_k(\zeta')$ of $P(\zeta_1,\zeta') = 0$:

$$\text{(II.5)} \qquad \pm \operatorname{Im}\tau_k(\zeta') \leq a|\operatorname{Re}\zeta_n|^q + b|\operatorname{Im}\zeta_n| + C \quad \text{on} \quad \operatorname{Re}\zeta_n \geq c|\operatorname{Re}\zeta''| \quad ,$$

where $q < 1$ and a, b, c, C are some constants. This inequality is far stronger than (I.10). Therefore Corollary 11 is much weaker than the second part of Theorem 6 when we compare them for a convex K . In fact, we cannot deduce Theorem 6 from Theorem 9, as we have the following example of non-propagation of micro-analyticity up to the boundary under the condition (I.10): Let $u(x)$ be a solution of the wave equation

$$\text{(II.6)} \qquad P(D)u = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} - \frac{\partial^2}{\partial x_n^2}\right) u(x) = 0 \quad ,$$

whose S.S. agrees with the bicharacteristic strip

$$\left\{(x, i\xi dx\infty)\,;\; x_1 = \ldots = x_{n-2} = 0,\; x_{n-1} = -x_n,\; \xi_1 = \ldots = \xi_{n-2} = 0,\; \xi_{n-1} = \xi_n > 0\right\} .$$

(For the construction of such a solution see e.g. Kawai [2]). Then u is a real analytic solution on $x_1 > 0$. Choosing the function $x_{n-1} + x_n$ as the new x_n-coordinate, we see that the characteristic roots of this operator satisfy (I.10). But either of the boundary values $u|_{x_1 \to +0}, \frac{\partial u}{\partial x_1}|_{x_1 \to +0}$ must contain in S.S. the direction $idx_n\infty$ along the bicharacteristic curve, because otherwise u should be real analytic there.

To utilize the convexity of K we must generalize Theorem 9 in the following way:

Theorem 9'. Let K and $\phi(x')$ be as in Theorem 9. (To fix the situation we now assume that $K \subset \{\phi(x') \geq 0\}$). Assume that for every real analytic solution u of $P(x,D)u = 0$ on $U_{\pm}$, the following propagation of micro-analyticity along the boundary holds: If the boundary values $u_j^{\pm}(x')$ are real analytic in $\phi(x') < 0$, then they become all automatically micro-analytic at either of the points $(0, \pm id\phi(x')\infty) \in iS^*_\infty \mathbb{R}^{n-1}$. Then $u \in A_P(U \setminus K)$ can be continued as a hyperfunction solution to a neighborhood of 0.

Our main subject of this part is to introduce the following new result on the propagation of micro-analyticity along the boundary:

Theorem 12. (Kaneko [13]) Assume that the roots $\zeta_1 = \tau_k(\zeta')$ of $P(\zeta_1, \zeta') = 0$ satisfy:

$$\text{(II.7)} \qquad \pm \mathrm{Im}\,\tau_k(\zeta') \leq \varepsilon|\zeta_n| + b|\mathrm{Im}\,\zeta_n| + C_{\zeta'',\varepsilon} \quad \text{on } \mathrm{Re}\,\zeta_n \geq 0 .$$

Let u be a hyperfunction solution of $P(D)u = 0$ in $\pm x_1 > 0$, micro-analytic to the directions $\rho^{-1}(idx_n\infty)$, where $\rho : S^{*\,n-1}_\infty \setminus \{\pm idx_1\infty\} \to S^{*\,n-2}_\infty$ is the projection from the poles to the equator. Then for the S.S. of the boundary values $u_j^{\pm}(x')$ of u we have the following assertion:

$$\text{(II.8)} \qquad \bigcup_{j=0}^{m-1} \mathrm{S.S.}\,u_j^{\pm}(x') \cap \{(x', idx_n\infty);\ x_n = \text{const.}\} \quad \text{must not be compact.}$$

When we apply this result combining with Theorem 9' and the method of sweeping out, we can recapture the second part of Theorem 6 completely from the viewpoint of local theory of hyperfunction boundary value problems.

In the rest of this part we will sketch the proof of Theorem 12. First we employ a reduction which will be found a very powerful tool for operators with constant coefficients. Recall the "curved wave decomposition" of the delta function:

$$\text{(II.9)} \qquad \delta(x) = \int_{S^{n-1}} W(x,\omega)\,d\omega .$$

The most classical is the "plane wave decomposition" where the component satisfies

$$W_0(x,\omega) = \frac{(n-1)!}{(-2\pi i)^n} \frac{1}{(x\omega+i0)^n} ,$$

$$S.S.W_0(x,\omega) = \{(x,\omega;i(\omega dx+xd\omega)\infty);\ x\omega = 0\} .$$

As is discovered by Kashiwara, we can improve this estimate by twisting the phase into the complex domain. The general form of $W(x,\omega)$ is

$$\text{(II.10)} \qquad W(x,\omega) = \frac{(n-1)!}{(-2\pi i)^n} \frac{\det(\text{grad}_\omega \psi(x,\omega))}{(\langle x,\psi(x,\omega)\rangle+i0)^n} ,$$

where $\psi(x,\omega)$ is a real analytic vector-function of $(x,\omega) \in \mathbb{R}^{2n}$, positively homogeneous of degree 1 in ω, $\psi(0,\omega) = \omega$, and such that $\langle x,\psi(x,\omega)\rangle$ is of positive type, i.e. $\text{Im}\langle x,\psi(x,\omega)\rangle \geq 0$ when $\text{Re}\langle x,\psi(x,\omega)\rangle = 0$ (see S-K-K [1]). The case where $\langle x,\psi(x,\omega)\rangle \neq 0$ outside the origin is of particular importance, because then as a hyperfunction of x the component $W(x,\omega)$ contains in S.S. the only one point $(0,i\omega dx\infty)$. For our purpose is also useful such a component as allows the flow-out of S.S. along a certain linear subvariety. Multiplying the component by e^{-x^2} we can obtain a decomposition by rapidly decreasing components thus accepting the Fourier transformation in x. If we further choose the phase such that $\langle x,\psi(x,\omega)\rangle$ is asymptotically linear in x as $|x| \to \infty$, then the Fourier image of $W(x,\omega)$ admits a good estimate, i.e. exponentially decreasing outside any conic neighborhood of ω, and slowly increasing everywhere.

Employing this decomposition we can paraphrase the micro-analyticity of hyperfunctions as follows:

<u>Lemma 13</u>. Let $u(x)$ be a hyperfunction with compact support. Then $u(x)$ is micro-analytic at $(0,i\nu dx)$ if (and only if) $u(x) \underset{x}{*} W(x,\omega)$ is real analytic when (x,ω) runs in a neighborhood Ω of $(0,\nu)$.

This follows from the decomposition

$$u(x) = \int_\Omega u \underset{x}{*} W(x,\omega)d\omega + \int_{S^{n-1}\setminus\Omega} u \underset{x}{*} W(x,\omega)d\omega ,$$

where the last term is always micro-analytic to the direction $i\nu dx\infty$. (The only if part holds if $S.S.W(x,\omega) = \{(0,i\omega dx\infty)\}$ or if the S.S. of u outside the origin is such that it does not propagate to 0 by convolution with $W(x,\omega)$).

Employing a refinement of Lemma 4, the above assertion can be improved into the following ultimate form:

Lemma 13'. (Kaneko [6]) Assume that for every $J(D_\omega)$, having constant coefficients with respect to a fixed system of local coordinates ω on a neighborhood of ν , $u(x) \underset{x}{*} J(D_\omega)W(x,\omega)\big|_{\omega=\nu}$ is real analytic at $x = 0$. Then $u(x)$ is micro-analytic at $(0,i\nu dx\infty)$.

In view of this lemma the following proposition micro-localizes the problem of analyticity of the boundary value problem to exactly one direction in the case of constant coefficients.

Proposition 14. (Kaneko [12]) Let u be a hyperfunction solution of $P(D)u = 0$ in U_+ such that $S.S.u \cap \rho^{-1}(idx_n\infty) = \emptyset$. Put $U' = U \cap \{x_1=0\}$ which we consider as an open subset of $\mathbb{R}^{n-1}_{x'}$. Let $u_j(x') \in B(U')$ be the boundary values of u and let $f_j(x') \in B[\overline{U'}]$ be their extensions with compact supports. Then for every $V' \subset\subset U'$ we can find $V \subset\subset U$ such that $V' = V \cap \{x_1=0\}$ and a real analytic solution v of $P(D)v = 0$ in V_+ whose boundary values agree with $f_j(x') \underset{x'}{*} J(D_{\omega'})W(x',\omega')\big|_{\omega'=\nu'}$ on V' . Here $\nu' = (0,\dots,0,1)$ and $W(x',\omega')$ is the component of a decomposition of $\delta(x')$ such that $W(x',\omega')$ is regular outside $x'=0$.

In case where we know a priori that the boundary values $u_j(x')$ are micro-analytic to the direction $idx_n\infty$ on some part of $\partial U'$, then we can also employ those $W(x',\omega')$ which cause by convolution the flow-out of S.S. from the interior of U' to such part of $\partial U'$. We can prove this proposition by convoluting $J(D_{\omega'})W(x',\omega')\big|_{\omega'=\nu'}$ to both sides of the identity

$$[[P(D)[u]_+]] = \sum_{j=0}^{m-1} f_j(x')\delta^{(m-j-1)}(x_1) + w \quad ,$$

which is a modification of (II.1), and thereafter adjusting superfluous terms. For the latter purpose is very useful the following lemma which can be considered as an extension of the Cauchy-Kowalevsky theorem.

Lemma 15. (Kaneko [10]) Let $P(x,D)$ be an operator with real analytic coefficients with respect to which $x_1=0$ is non-characteristic. Let f be a hyperfunction locally defined on a neighborhood of $x_1=0$ such that

$$\text{(II.11)} \qquad \text{supp } f \subset \left\{x_1 \geq 0\right\}, \quad \text{S.S.} f \subset \left\{x_1 \geq 0\right\} \times \left\{\pm i dx_1 \infty\right\} .$$

Then on a smaller neighborhood of $x_1=0$ we can find a unique solution u of $P(x,D)u = f$ which satisfy the same condition (II.11).

Note that the Cauchy-Kowalevsky theorem corresponds to the data of particular form $f = Y(x_1)g(x) + \sum_{j=0}^{m-1} v_j(x')\delta^{(m-j-1)}(x_1)$, where g, v_j are real analytic.

Now fix $U' = \left\{|x''| < A\right\} \times \left\{|x_n| < r\right\}$. By virtue of this proposition and Lemma 13', we can assume without loss of generality that u is a real analytic solution and the boundary values may be non-micro-analytic only to the direction $idx_n\infty$, and it suffices to prove that when $u_j(x')$ are further real analytic on a neighborhood of $|x''| = A$, then they are real analytic everywhere. Remark that the estimate (II.7) implies the following more handy one: There exists $N > 0$ such that

$$\pm \mathrm{Im}\tau_k(\zeta') \leq b|\mathrm{Im}\zeta'| + c|\mathrm{Re}\zeta''|^{1/N}|\mathrm{Re}\zeta_n|^{1-1/N} + C \quad .$$

Then we introduce the function

$$\text{(II.12)} \qquad E(x',\varepsilon) = \mathscr{F}_{x'}^{-1}(\exp(-2c\varepsilon(\sqrt{1+|\xi''|^2})^{1/N}(\sqrt{1+|\xi'|^2})^{1-1/N}) \quad .$$

Our next task is to show that $f_j(x') \underset{x'}{*} E(x',\varepsilon)$ becomes real analytic in $U' \times \{\mathrm{Re}\,\varepsilon > 0\}$, and complex holomorphic in ε there. This can be proved by Green's formula coupled with the solution F_k of the boundary value problem

$$\text{(II.13)}\qquad \begin{cases} {}^tP(D)F_k = 0 \\ \dfrac{\partial^j}{\partial x_1^j} F_k\Big|_{x_1\to+0} = \delta_{jk}\Big\{J(D_{\omega'})W_0(x',\omega')\Big|_{\omega'=\nu'} * E(x',\varepsilon)\Big\}(y'-x',\varepsilon), j=0,\dots,m-1. \end{cases}$$

Here ${}^tP(D)$ is the transposed operator of $P(D), \nu'=(0,\dots,0,1)$ and $W_0(x',\omega')$ denotes the component of the plane wave decomposition of $\delta(x')$. Remark that (II.11) just assures the solvability of (II.13) (Kaneko [12]). (Or we can employ the method of part I based on the Fourier analysis (Kaneko [13]): We choose a cut-off function χ such that $\chi \equiv 1$ on a neighborhood of $U \cap \{x_1=0\}$ in U. Then we obtain an identity of the form

$$[[P(D)(1-\chi)J(D')u]] = \sum_{j=0}^{m-1} J(D')f_j(x')\delta^{(m-j-1)}(x_1) + w$$

modulo $P(D)B_*(\mathbb{R}^n)$ which vanishes on $N(P)$ after the Fourier transformation. $[[P(D)(1-\chi)J(D')u]]$ is a suitable extension with support in the closure of $\{0 < \chi(x) < 1\}$. The residue term w has support in $\partial U'$ and along $|x''| = A$, $|x_n| < r$ it has the form $\sum_{j=0}^{m-1} w_j(x')\delta^{(m-j-1)}(x_1)$ with coefficients micro-analytic outside the direction $dx_n = 0$. Thus the real analyticity of u is reflected in the decay property of $[[P(D)(1-\chi)J(D')u]]$, and hence to the continuity of $J(D')f_j(x') \underset{x'}{*} E(x',\varepsilon)$).

Then we study the analytic continuation of $f_j(x') \underset{x'}{*} E(x',\varepsilon)$ beyond $\mathrm{Re}\,\varepsilon = 0$ through the part $\mathrm{Im}\, z_n > 0$. This can be done directly examining the expression (II.12). As a result $f_j(x') \underset{x'}{*} E(x',\varepsilon)$ becomes the boundary value of a function $V(z',\varepsilon)$ holomorphic on a domain of the following form:

$$\Big\{(z',\varepsilon)\in\mathbb{C}^n;\ |x''| < A-\delta,\ |x_n| < r-\delta,\ |\mathrm{Re}\,\varepsilon| < B,$$
$$B > \mathrm{Im}\, z_n > \phi(|\mathrm{Im}\, z''| + \max\{0,-\mathrm{Re}\,\varepsilon\}) - \psi(\max\{0,\mathrm{Re}\,\varepsilon\})\Big\}\ .$$

Here $\phi(t) \geq 0$, $\psi(t) \geq 0$ are convex functions satisfying $\phi(0) = \psi(0) = 0$, $\phi(t)/t \to 0$, $\psi(t)/t \to 0$ as $t \to 0$.

Finally we recall the local Bochner theorem which, after Kashiwara's remark, has become a very useful tool in the analytic theory of linear partial differential equations.

Lemma 16. (see e.g. Komatsu [2]) Let $F(z)$ be a function holomorphic on a neighborhood of the thin wedge

$$\left\{z=x+iy\in\mathbb{C}^n;\ |x|<A,\ 0<y_1<B,\ y'=0\right\} .$$

Then $F(z)$ can be continued to a wedge with positive width:

$$\left\{z=x+iy\in\mathbb{C}^n;\ |x|<A-\delta,\ c_\delta|y'|<y_1<B-\delta\right\} .$$

We apply this lemma in two steps changing the role of $\mathrm{Re}\,\varepsilon$ and $\mathrm{Im}\,\varepsilon$. Thus we conclude that $V(z',\varepsilon)$ can be continued on $|x''|<A-2\delta$, $|x_n|<r-2\delta$ up to the boundary $\varepsilon=0$. Since on the other hand $f_j(x')\underset{x'}{*}E(x',\varepsilon)\to f_j(x')$ is the process of boundary value $\varepsilon\to 0$ with respect to the differential equation

$$\text{(II.14)}\qquad \left\{\frac{\partial^{2N}}{\partial\varepsilon^{2N}}+(2c)^{2N}(\Delta_{x''}-1)(1-\Delta_{x'})^{N-1}\right\}V(x',\varepsilon)=0 ,$$

we thus conclude that $f_j(x')$ is real analytic on $|x''|<A-2\delta$, $|x_n|<r-2\delta$, thereby proving Theorem 12.

Remark 1. There are operators beyond the range of Theorem 10 which admit the same conclusion. The partial Laplacian $\Delta_{x''}$ is a distinguished example among them (Schapira [3], Kaneko [12]). Thus for the operators to which we can choose $N=1$ in (II.11), we can simplify the last part of the above proof by applying this result to (II.14). We infer, however, that under such assumption we can expect the conclusion of Theorem 10 rather than that of Theorem 12.

Remark 2. Theorem 12 is best possible in the sense that if a root does not satisfy (II.7) then we can construct a real analytic solution of $Pu=0$ whose boundary values possess a compact, but in total non-void S.S. In fact, such a root defines an elliptic pseudo-differential factor for which the corresponding boundary value problem is micro-locally solvable on a neighborhood of the direction $+idx_n\infty$, e.g. with the boundary data $W(x',\nu')$, $\nu'=(0,\ldots,0,1)$. This gives rise to a required solution of the original equation. There are, however, results of various types concerning a more precise propagation of regularity along the boundary. For example, we can strengthen the observation up to the bicharacteristic level for the wave equation (Kaneko [12]). Note also that there are studies of similar kind in relation to the propagation of waves. For example, Sjöstrand [1] seems to consider similar problem for the wave equation. He assumes, however, a boundary condition while we do not, naturally according to the different origin of problems. Therefore the conclusion must be a little different.

APPENDIX TO PART II.

Here we give a simplified proof of the Kashiwara-Kawai theorem of Holmgren type (Lemma 8). Let $u(x)$ be such a hyperfunction. By the so called Holmgren transformation

$$\begin{cases} \tilde{x}_1 = x_1 + \varepsilon(x_2^2+\dots+x_n^2) \\ \tilde{x}' = x' , \end{cases}$$

we can assume that $\operatorname{supp} u \subset \left\{x_1 \geq \varepsilon(x_2^2+\dots+x_n^2)\right\}$. Then we can calculate the convolution

$$u_j(x) = u(x) \underset{x'}{*} W(x', \Delta_j) ,$$

where $\bigcup \Delta_j = S^{n-2}$ is a decomposition by piramids and $W(x', \Delta_j) = \int_{\Delta_j} W(x', \omega')d\omega'$. Hence we have $\sum u_j = u$. We have obviously

$$\operatorname{supp} u_j \subset \left\{x_1 \geq 0\right\}, \quad \text{S.S.}u_j \subset \rho^{-1}(\Delta_j) \cup \left\{\pm i dx_1\infty\right\} .$$

A more precise estimate of S.S. shows that $\text{S.S.}u_j$ is free from either of $(0, \pm i dx_1\infty)$ which has been absent in $\text{S.S.}u$. Thus $\text{S.S.}u_j$ is contained in a convex proper cone Γ_j° and therefore u_j admits a boundary value expression by unique term: $u_j(x) = F_j(x+i\Gamma_j 0)$. Then the fact that $u_j(x) \equiv 0$ in $x_1 < 0$ implies $F_j(z) \equiv 0$ there, hence $u_j(x) \equiv 0$ everywhere as long as this expression is valid. q.e.d.

III. PROBLEM OF IRREDUCIBILITY OF THE OPERATOR WITH REAL ANALYTIC COEFFICIENTS

In the preceding part the role of real characteristic roots has been clarified from the standpoint of local theory. Now we try to cover the second feature of Corollary 3: Which operators can we consider as irreducible from the standpoint of local theory? This is a very delicate question because the two operators $D_1\Delta$ and $D_1\Delta+1$ behave differently concerning Corollary 3, although the latter can be decomposed via pseudo-differential operators in a very good sense. Recall here the result in the case of variable coefficients which we have formerly presented as the correspondent to Corollary 3 (or rather to the result of Grušin).

Theorem 17. (Kaneko [7]) Let $P(x,D)$ be a linear partial differential operator of order m with real analytic coefficients defined on a neighborhood U of 0. Assume that $P_m(x,D)$ is real, simply characteristic, has $x_1 = 0$ as a non-characteristic boundary at $x = 0$, and that the roots of $P_m(x,\zeta_1,\xi') = 0$ for ζ_1 are all real and simple when (x,ξ') runs in a neighborhood of some point $(0,\nu')\in \mathbb{R}^n \times S^{n-2}$. Then $A_P(U\setminus\{0\})/A_P(U) = 0$ (and also $C_P^\infty(U\setminus\{0\})/C_P^\infty(U) = 0$).

In this part, as the first trial to this problem we will content ourselves by weakening the condition of this theorem to require only one real simple root instead of all, with the introduction of an assumption of "irreducibility" of $P_m(x,D)$ which would guarantee a phenomenon corresponding to "the propagation of zero" on the irreducible variety in $\mathbb{C}^n$.

The space on which we undertake the work corresponding to the Fundamental Principle in the case of variable coefficients is the one introduced by Kashiwara-Kawai [1] or Kataoka [1]. Recall that the cotangential sphere bundle iS_∞^{*n-1} on which we discuss the micro-analyticity of hyperfunctions is naturally introduced as the conormal sphere bundle S_M^*X of $M = \mathbb{R}^n$ in $X = \mathbb{C}^n$. Kashiwara-Kawai [1] has introduced the conormal sphere bundle S_N^*X of $N = \{x_1=0\} \subset X$ in order to analyse the hyperfunctions with supports in a hypersurface in relation to the boundary value problem. Kataoka [1] has worked out their idea by introducing the conormal sphere bundle $S^*_{M_+}X$ of the real half space $M_+ = \{x_1 \geq 0\} \subset X$. These spaces are written in local coordinates as follows:

$$S^*_N X = \{(x', (\zeta_1 dx_1 + i\eta' dx')\infty);\ x' \in \mathbb{R}^{n-1},\ \zeta_1 \in \mathbb{C},\ \eta' \in \mathbb{R}^{n-1},\ (\zeta_1, \eta') \neq 0\}$$

$$S^*_{M_+} X = \{(x, i\eta dx\infty) \in S^*_M X,\ x_1 > 0\} \cup \{(0, x', (\zeta_1 dx_1 + i\eta' dx')\infty) \in S^*_N X,\ \mathrm{Re}\zeta_1 \geq 0\}.$$

Let π denote the projection from $S^*_M X$, $S^*_N X$, $S^*_{M_+} X$ to the corresponding base spaces. Then there exist objects (sheaves) C_M, $C_{N|X}$, $C_{M_+|X}$ on these spaces respectively, such that

(III.1) $\pi_* C_M = B/A$ (this means simply that the classes of hyperfunctions by the micro-analyticity can be expressed as an object on $S^*_M X$, called the microfunctions),

(III.2) $0 \to \Gamma_N(M,B) \to \pi_* C_{N|X} \to A_M|_N \to 0$ is exact (that is, the hyperfunctions with supports in N are completely decomposed without ambiguous regular part, but with the excess of a kind of "period"),

(III.3) $C_{M_+|X}\big|_{\mathrm{Int}\,(M_+)} = C_M\big|_{\mathrm{Int}\,(M_+)}$,

(III.4) $\pi_* C_{M_+|X}\big|_N = \Gamma_{M_+}(M,B)\big|_N$,

(III.5) $C_{M_+|X}\big|_{\{\mathrm{Re}\zeta_1 > 0\}} = C_{N|X}\big|_{\{\mathrm{Re}\zeta_1 > 0\}}$ (that is, $C_{M_+|X}$ is an object which is the usual microfunctions on the interior $x_1 > 0$, decomposes along $x_1 = 0$ just the germs of hyperfunctions with supports in M_+ , and behaves like $C_{N|X}$ on the hemisphere $\mathrm{Re}\zeta_1 > 0$).

Among various properties of these sheaves most important is the unique continuation property with respect to the complex variable ζ_1 , which holds also along $\mathrm{Re}\zeta_1 = 0$ with respect to the "real" parameter $\mathrm{Im}\zeta_1$. As a consequence, we have the beautiful theorem on watermelon slicing describing the shape of S.S. of the hyperfunctions with support limited by a half space (see the appendix to this part). Recall also that the pseudo-differential operators operate to these sheaves where they are defined.

The complex parameter ζ_1 accepts the operation of restriction or the boundary value to the "real" which induces the relation

(III.6) $\quad C_{N|X}\Big|_{S_M^*X \underset{X}{\times} N} \hookrightarrow C_{M_+|X}\Big|_{S_M^*X \underset{X}{\times} N} \hookrightarrow C_M\Big|_{S_M^*X \underset{X}{\times} N}$,

where $S_M^*X \underset{X}{\times} N$ denotes the restriction of S_M^*X on N , that is, the "real part" (more precisely the pure imaginary part) of the fibres of S_N^*X or $S_{M_+}^*X$.

These constructions can be unified to the following general situation: Let $K \subset \left\{x_1=0\right\}$ be a linear subvariety with boundary, say

$$K = \left\{x_1=0,\ x_2=\ldots=x_k=0,\ x_{k+1} \geq 0,\ldots,x_{k+\ell} \geq 0\right\} .$$

Then its conormal sphere bundle S_K^*X in X is defined in the same way. The corresponding sheaf on this space is $C_{K|X}$ whose precise definition is

$$C_{K|X} = \mathscr{H}^n_{S_K^*X}(\pi^{-1}\mathcal{O}_X) \otimes \omega_{K|X} ,$$

where $\pi : \widetilde{K}_{X^*} = (X\setminus K) \sqcup S_K^*X \to X$ denotes the comonoidal transformation with center K and $\omega_{K|X}$ denotes the corresponding orientation sheaf. The sections of $C_{K|X}$ have the unique continuation property with respect to the complex parameters $\zeta_1,\ldots,\zeta_{k+\ell}$ for $\zeta_1,\ldots,\zeta_k \in \mathbb{C}, \mathrm{Re}\zeta_{k+1} \geq 0,\ldots,\mathrm{Re}\zeta_{k+\ell} \geq 0$. Put $\partial K = \left\{x_1=\ldots=x_{k+\ell}=0\right\}$. We have

(III.7) $\quad \Gamma_K(M,B)\Big|_{\partial K} \overset{\pi^*}{\hookrightarrow} C_{K|X}\Big|_{\partial K}$,

(III.8) $\quad C_{K|X}\Big|_{S_{M_+}^*X} \hookrightarrow C_{M_+|X}$.

<u>Definition 18</u>. Put $\zeta^{I} = (\zeta_1,\ldots,\zeta_{k+\ell})$, $\zeta^{II} = (\zeta_{k+\ell+1},\ldots,\zeta_n)$. We will say that $P_m(x,D)$ is irreducible at $(0,\nu')$ along S_K^*X if the m points $(\tau_1(i\nu'),i\nu'),\ldots,(\tau_n(i\nu'),i\nu')$ of $P_m(0,\zeta_1,i\nu') = 0$ lying over $i\nu'$ can be joined by a non-singular complex analytic submanifold of dimension $k+\ell-1$ in the conic algebraic variety

$$\left\{\zeta^{I} \in C^{k+\ell};\ (0,(\zeta^{I}dx^{I}+i\nu'^{II}dx^{II})\infty) \in S_K^*X,\ P_m(0,\zeta^{I},i\nu'^{II})=0\right\} .$$

(Here ν'^{II} denotes $(\nu'_{k+\ell+1},\dots,\nu'_n)$).

Now consider $u \in A_P(U \setminus K)$. By the construction of part II, we have

$$\text{(II.1)bis} \qquad \pm P(x,D)[u]_{\pm} = \sum_{j=0}^{m-1} u_j^{\pm}(x')\delta^{(m-j-1)}(x_1) \quad .$$

Hence

$$\text{(III.9)} \qquad P(x,D)([u]_+ + [u]_-) = \sum_{j=0}^{m-1} u_j(x')\delta^{(m-j-1)}(x_1) \quad ,$$

and this hyperfunction becomes a section of $\pi_* C_{N\,X}$. Now choose <u>any</u> extension $[u] \in B(U)$ of u . Then $P(x,D)[u]$ has support in $x_1=0$, hence gives also a section of $\pi_* C_{N|X}$. How it differs from (III.9)? Obviously by a term of the form $P(x,D)\pi_* C_{N|X}$. Since the space $S_N^* X$ directly concerns only the principal part, we cannot annihilate this ambiguity by simply restricting to some variety. Note however that $P(x,D)$ is invertible as a pseudo-differential operator outside the characteristic variety $P_m(x,\zeta) = 0$. Thus the ambiguity has the meaning still only along $P_m(x,\zeta) = 0$. To continue u to a hyperfunction solution to U is, therefore, to show that (III.9) goes into the ambiguity term even along $P_m(x,\zeta) = 0$. Assume that on a neighborhood of $(x',\zeta) \in S_N^* X$, $P_m(x,\zeta) = 0$ has simple component $\zeta_1 - \tau_k(\zeta') = 0$. Then by virtue of the Weierstrass preparation theorem (S-K-K [1]), we can decompose $P(x,D)$ in such a way that

$$(D_1 - \tau_k(D'))P'(x,D)([u]_+ + [u]_-) = \sum_{j=0}^{m-1} u_j(x')\delta^{(m-j-1)}(x_1) \quad .$$

In view of the division theorem in $S_N^* X$ (see Schapira [2], Kataoka [3]), we obtain

$$\text{(III.10)} \qquad (D_1 - \tau_k(D'))\Big(P'(x,D)([u]_+ + [u]_-) - \sum_{j=0}^{m-2} v_j(x')\delta^{(m-j-2)}(x_1)\Big) = u_0^{(k)}(x')\delta(x_1) .$$

We claim that this coefficient $u_0^{(k)}(x')$ is, near (x',ζ) , the substitute for the object $\widehat{P(D)[u]}\Big|_{N(P)}$ considered in the Fundamental Principle. In fact we have

<u>Lemma 19</u>. Assume that $P_m(x,\zeta_1,\xi')$ has a simple real root $\zeta_1 = \tau_k(\xi')$ near $(x,\xi') = (0,\nu')$. Then the coefficient $u_0^{(k)}(x')$ vanishes there (as an element

of C_N , that is, becomes micro-analytic near $(x',\xi') = (0,\nu')$).

This is a special case of the propagation of micro-analyticity up to the boundary for micro-hyperbolic operators (Schapira [2], Kataoka [3]). Our main lemma where the irreducibility concerns is the following:

Lemma 20. Assume that $P_m(x,D)$ is irreducible at $(0,\nu')$ along $S^*_K X$. Then the propagation of zero holds for the coefficient in (III.10).

In fact the restriction of support $\mathrm{supp}\, u_j(x') \subset K$ allows us to consider $u_j(x')$ as sections of $\pi_* C_{K|X}$. Thus we can let ζ^I run in the complex with respect to which $u_j(x')$ enjoys the unique continuation property. This property is inherited to the coefficient in (III.10), because the division process conserves the complex holomorphic parameter.

Summing up we have obtained

Theorem 21. Let $P(x,D)$ be a linear partial differential operator of order m with real analytic coefficients defined on a neighborhood U of 0 . Let K be as above and let ν' be a conormal of K in $x_1 = 0$. Assume that $x_1 = 0$ is non-characteristic, $P_m(x,D)$ is irreducible at $(0,\nu')$ along $S^*_K X$ in the sense of Definition 18 and that the roots of $P_m(x,\zeta_1,\xi') = 0$ for ζ_1 are simple and one of them is real for (x,ξ') near $(0,\nu')$. Then every $u \in A_P(U \setminus K)$ can be continued as a hyperfunction solution to a neighborhood of 0 .

Corollary 22. Let $K = \{0\}$. Then under the same assumption we have $A_P(U \setminus \{0\})/A_P(U) = 0$.

In fact this time the singularity of the extended solution, if it existed at 0 , would flow out along the bicharacteristic strip corresponding to the real simple root (Kawai [2]).

Let us examine an example to Theorem 21. Consider the following operator with constant coefficients on a neighborhood U of 0 :

$$P(D) = D_1^3 + D_2^3 + D_3^3 \quad \text{on } \mathbb{R}^3 \quad .$$

First choose $K = \{x_1=0,\ x_2 \geq 0\}$. Then $\nu' = (1,0) \in \mathbb{R}^2$ is the unique conormal of K in $x_1 = 0$. The operator $P(D)$ is not irreducible at ν' along S_K^*X . In fact, at $i\nu' = (i,0)$, as a function of ζ_1 and ζ_2 , $P(\zeta)$ can be decomposed as follows:

$$\zeta_1^3 + \zeta_2^3 = (\zeta_1 + \zeta_2)(\zeta_1^2 - \zeta_1\zeta_2 + \zeta_2^2) \quad .$$

We can even construct an element $u \in A_P(U \setminus K)$ which cannot be continued as a hyperfunction solution to a neighborhood of 0 : Choose $f(x) = Y(x_2)\delta(x_1)$. Since the operator $D_1^2 - D_1D_2 + D_2^2$ is elliptic, we can find a hyperfunction solution u of $(D_1^2 - D_1D_2 + D_2^2)u = f$ (hence of $P(D)u = (D_1 + D_2)f$) which is real analytic outside K . By the uniqueness of the boundary values supp u contains certainly the points outside K .

Next choose $K = \left\{x_1=0,\ x_2 \geq 0,\ x_3 \geq 0\right\}$. Choose $\nu' \in \mathbb{R}^2$, $\nu_2 \geq 0$, $\nu_3 \geq 0$, $\nu_2^2 + \nu_3^2 = 1$ as a conormal of K in $x_1 = 0$. This time an elementary consideration shows that the non-singular variety

$$\left\{\zeta \in \mathbb{C}^3 \setminus \{0\};\ \zeta_1^3 + \zeta_2^3 + \zeta_3^3 = 0,\ \mathrm{Re}\zeta_2 \geq 0,\ \mathrm{Re}\zeta_3 \geq 0\right\}$$

is connected, that is, $P(D)$ is irreducible at ν' along S_K^*X in the sense of Definition 18. Therefore combined with the existence of a real simple zero, we can conclude by Theorem 21 that $u \in A_P(U \setminus K)$ can be continued locally to a neighborhood of 0 as a hyperfunction solution (hence as a real analytic solution in view of the propagation of regularity by virtue of the convexity of K). Note however that if we consider $\nu'x'$ as the new x_n coordinate and take for U a neighborhood of $K \cap \{\nu'x' < \text{const.}\}$, we cannot assert $A_P(U \setminus K)/A_P(U) = 0$ globally as in Theorem 6.

APPENDIX TO PART III.

We introduce here the statement of the <u>theorem on watermelon slicing</u> (or chopping) which exhibits a distinguished feature of the analytic singular spectrum as compared with the C^∞ .

Theorem. Let $u(x)$ be a hyperfunction defined on a neighborhood of $0 \in \mathbb{R}^n$ such that $0 \in \operatorname{supp} u \subset \left\{x_1 \geq 0\right\}$. Then the fiber at $x = 0$ of S.S.u has the following remarkable structure: There exists a closed subset F in the equator S_∞^{*n-2} such that

$$\text{(III.11)} \qquad \text{S.S.}u\Big|_{x=0} = \rho^{-1}(F) \cup \left\{\pm i dx_1 \infty\right\} ,$$

where $\rho : S_\infty^{*n-1} \setminus \left\{\pm i dx_1 \infty\right\} \to S_\infty^{*n-2}$ denotes the projection from the poles along the meridians.

This theorem has its origin in the works of V.S. Vladimirov, M. Morimoto etc. on the interdependence of support and S.S. translating the law of causality in quantum mechanics (see Morimoto [1]). It was conjectured by M. Sato in the conversation with Morimoto on this subject. As remarked in the text, it is based on a kind of unique continuation property with respect to ζ_1, which had been conjectured by Morimoto. The latter was afterwards proved by Kashiwara by virtue of the contact transformation and the unique continuation property for micro-functions with holomorphic parameter. The proof has been reconstructed by Kataoka [1] which seems the first written one. An elementary version is given in Kaneko [11] where only the hyperfunctions with holomorphic parameter is used. Remark that $\left\{\pm i dx_1 \infty\right\} \subset$ S.S.u is the conclusion of the Holmgren type theorem (Lemma 8). By virtue of this theorem the treatment of boundary wave front set becomes very simple in the analytic case: It suffices to consider the S.S. of the traces (Schapira [2], Kaneko [12]). C.f. the talk of P. Lousberg in this meeting for the C^∞ case.

By a figure of the form (III.11) (for n=3), we Japanese imagine at once a slice of the watermelon which is fully round in Japan, whence the denomination due to Morimoto. In Kaneko [11], however, the melon is referred to instead of the watermelon, because the latter in France seems of long, ellipsoidal form. The speaker does not yet see the Italian watermelon. But the sketch of "cocomero" in the dictionary seems also long.

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