Astérisque

CORRADO DE CONCINI
CLAUDIO PROCESI

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Astérisque, tome 87-88 (1981), p. 79-83
<http://www.numdam.org/item?id=AST_1981__87-88__79_0>

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HODGE ALGEBRAS : A SURVEY

by

Corrado De Concini (Pisa)
Claudio Procesi (Roma)

This paper is a survey of some of the results of [2].

1. Standard monomials.

The idea of standard tableau has been introduced by A.Young in order to describe explicit bases for the representations of the symmetric and general linear groups. These ideas have been reintroduced by Hodge [7] in more geometric setting in order to give an explicit description of the projective coordinate ring of the Grassmann and Flag varieties.

One of the motivations of Hodge was to give an explicit postulation formula for those varieties. His computations show that the Hilbert polynomial gives the dimension of the projective coordinate ring in all degrees.

This fact has suggested in more recent times (via the consideration of Riemann-Roch's theorem) the validity of some vanishing theorems in cohomology or, more precisely, that the Grassmanniann and its Schubert varieties are arithmetically Cohen-Macaulay. This has been shown by several authors independently [6],[9],[10], always using the standard bases.

If we consider the Grassmann variety of k planes in n dimensional space, $G_{k,n}$, it has a natural projective embedding given by the Plücker coordinates. Under this embedding, one can identify the projective coordinate ring of $G_{k,n}$ with the ring generated by the maximal order minors of a k n matrix of indeterminates.

Let us denote by $[i_1,i_2,\ldots,i_k]$ the (determinant of the) minor extracted from the columns of indices i_j, a Plücker coordinate. We can partially order this set of minors, setting $[i_1,i_2,\ldots,i_k]\leq[j_1,j_2,\ldots,j_k]$ if $i_1\leq j_1$, ..., $i_k\leq j_k$ (this ordering is strictly related to the Bruhat ordering on Schubert varieties). If $p_1, p_2,\ldots, p_s$ are among such minors, we say that their product $p_1p_2\ldots p_s$ is a standard monomial if $p_1\leq p_2 \ldots\leq p_s$, and we have :

Theorem. The standard monomials are a linear basis for the projective coordinate ring of $G_{k,n}$.

Two facts are important in this analysis. The first is the nature of the equations satisfied by the p_i's. We have for each pair p,q of Plücker coordinates which are not comparable in the given ordering an explicit relation, with $p_i < p$,

$pq = \Sigma a_i p_i q_i$.

These relations clearly give an algorithm to express any monomial as a linear combination of standard ones.

The second important fact is the special nature of the partially ordered set of Plücker coordinates : if we consider the simplicial complex with vertices these coordinates and whose simplices are the totally ordered subsets, its geometric realization is a cell.

These two facts will be seen to be the basis of the proof of the arithmetic Cohen-Macaulay property for the Grassmann variety.

Let us recall for this another construction. Let Δ be a finite simplicial complex with vertex set H and let R be a coefficient ring with 1 . One defines the Reisner-Stanley algebra $R[\Delta]$ to be the polynomial ring over R in a set of variables $\{x_h\}_{h\in H}$ modulo the ideal generated by the monomials $x_{h_1}x_{h_2}\cdots x_{h_s}$ where $\{h_1, h_2,\ldots, h_s\}$ is not a simplex. If R is Cohen-Macaulay, then there is a criterion for the Cohen-Macaulay property of R :

Theorem (Reisner [11]). $R[\Delta]$ is Cohen-Macaulay if and only if considering the geometric realization X_Δ of Δ we have : $\tilde{H}_i(X_\Delta;R) = H_i(X_\Delta,X_\Delta - x;R) = 0$, for x any point in X and $0 \leq i \leq \dim\Delta - 1$, where H_i (resp. $\tilde{H}_i$) denotes the ith-cohomology (resp. reduced cohomology) group.

In the following sections we will show how one can relate this theorem with the initial discussion on Grassmanniann s.

2. Hodge Algebras.

Let Δ be a finite simplicial complex whose vertex set H has been given a partial ordering (a specially interesting case is when the simplices of Δ are the totally ordered subsets of H).

A monomial in H is a mapping $M : H \longrightarrow N$ (N the set of natural numbers). We say that a monomial is standard if his support is a simplex.

Let now A be a commutative R algebra.

Definition. We say that A is a Hodge algebra on Δ over R if :

i) There is an injection $j : h \longrightarrow x_h$ from H to A.

ii) Extending j to the monomials, the standard monomials are a basis of A over R .

iii) If $K = \{k_1,k_2,\ldots,k_t\} \subseteq H$ is a minimal non simplex of Δ , write the relation

$$(\star) \qquad x_{k_1} x_{k_2} \cdots x_{k_t} = \Sigma\, r_i\, j(M_i)$$

with $r_i \neq 0$ and M_i standard monomials. Then for each $h \in K$ and each i we have that there is a $h_i \in \mathrm{Supp}(M_i)$ with $h_i < h$.

It is clear that both the projective coordinate ring of the Grassmann and Schubert varieties and the Reisner-Stanley algebras fall into this more general setting.

For simplicity we will assume that A is graded with the x_h 's of positive degree.

One of the most interesting features of a Hodge algebra on Δ lies in the fact that one may introduce in suitable way parameters in the defining relations $(\star)$ in order to perform a faithfully flat deformation from the given Hodge algebra to the corresponding Reisner-Stanley algebra $R[\Delta]$. As a corollary one obtains :

Corollary. If A is a graded Hodge algebra and $R[\Delta]$ is Cohen-Macaulay, so is A.

Let us give a sketch of the proof of how one obtains the faithfully flat deformation from A to $R[\Delta]$. Let $\mathrm{Ind}(A)$ be the union of the supports of all standard monomials M_i appearing on the right handside of the relations $(\star)$. Clearly $\mathrm{Ind}(A) = \emptyset$ if and only if $A = R[\Delta]$. The idea is to choose a minimal element $h \in \mathrm{Ind}(A)$ and perform a faithfully flat deformation over $R[t]$ (t is a parameter) whose general fiber is A and whose special fiber is a Hodge algebra A' on Δ with $\mathrm{Ind}(A') \subseteq \mathrm{Ind}(A) - h$.

In order to do this, one considers the ideal $I = A\, x_h$ and the corresponding Rees algebra :

$$\Sigma_1^{\infty}\, I^n\, t^{-n} + A[t] = \mathcal{R}(I,A)\ .$$

One shows that $\mathcal{R}(I,A)$ is a Hodge algebra on Δ over $R[t]$ with generators $\{\bar{x}_k\}_{k\in H}$ with $\bar{x}_k = x_k$ if $k \neq h$ and $\bar{x}_h = x_h\, t^{-1}$. Furthermore analysing the relations for the minimal non simplices, one verifies that at the special fiber ($t=0$) the monomials containing $\bar{x}_h$ disappear. We remark that $A' = \mathcal{R}(I,A)\,/\,(t)$ is the graded algebra associated to I.

3. Examples.

There are many interesting examples that can be treated by these methods. We give here a list referring to the original papers for the details.

First of all one may treat by the previous method the determinantal varieties, since they are suitable open sets of special Schubert varieties.

Similarly one can treat the Pfaffian varieties associated to skew symmetric matrices, in fact also these varieties are related to special Schubert varieties of the orthogonal groups. This and the case of the Grassmann variety are both special cases of the theory of G/P where G is a reductive group and P a parabolic subgroup associated to a minuscule weight (cf.[12]).

The flag variety and its Schubert subvarieties can also be treated and their multihomogeneous coordinate ring shown to be Cohen-Macaulay ; the fact that these algebras are Hodge can be found in [8] , the geometric realizations of the associated simplicial complexes turn out to be again cells, although the proof of this fact can be given for the moment only by a lengthy case by case analysis.

A further interesting set of examples is given by the Buchsbaum-Eisenbud varieties of complexes [5].

If one wants to treat the determinantal varieties of symmetric matrices one sees that this method cannot be directly applied, although a certain theory of standard basis can be developed also in this case [4] .

This in fact turns out to be part of a general theory studied in [8] for special representations (classical weights) of reductive algebraïc groups. A suitable axiomatization of this situation has been developed in [3] ; there it is shown that, even in this more general case the Cohen-Macaulay property of the algebras involved can be deduced by the study of certain posets. Let us remark that in this theory the basic posets occuring are the following :

Let G be a reductive group, P a parabolic subgroup, W and W_P the respective Weyl groups and be the Bruhat ordering on W/W_P . If $x \leqslant y$ are two elements in W/W_P one may consider the segment $[x,y] = \{z \in W/W_P \mid x \leqslant z \leqslant y\}$; if $X_{[x,y]}$ is the geometric realization of the simplicial complex associated to the poset x,y one has the following remarkable fact :

Theorem (Björner-Wachs [1]). $X_{[x,y]}$ is a cell.

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