HENNING HAAHR ANDERSEN

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LINE BUNDLES ON FLAG MANIFOLDS

Henning Haahr Andersen

We give here a survey of some recent results on cohomology of line bundles on flag manifolds. It covers mostly the author's own contributions to the theory but we have also included a brief account of the history of the subject and (in the last section) we mention some of the other work that has been done in the area. A couple of open problems are also briefly discussed.

1. The problem

Let G denote a connected reductive algebraic group defined over an algebraically closed field k and let B be a Borel subgroup. Consider a line bundle L on G/B. Without changing G/B we may replace G first by G/RG where RG denotes the radical of G and then by a semi-simple covering group. Having done that all line bundles on G/B come from characters of B. In fact $X(B) \cong \operatorname{Pic} G/B$ where $X(B)$ denotes the character group of B [27]. The isomorphism is the one that takes a character $\lambda \in X(B)$ into the homogeneous line bundle $L(\lambda)$. The sections of $L(\lambda)$ over an open subset U are the regular functions $f: \pi^{-1}(U) \to k$ which satisfy $f(xb) = \lambda(b)^{-1}f(x)$, $x \in \pi^{-1}(U)$, $b \in B$.

π is the natural map $G \to G/B$. (In the same way any rational B-module E induces a homogeneous vector bundle $L(E)$ on G/B).

The problem is now to describe the cohomology $H^{\cdot}(G/B, L(\lambda))$ (in the following when no confusion can arise denoted $H^{\cdot}(\lambda)$) of $L(\lambda)$. This problem brakes down into two:

1) The vanishing description: For which i is $H^i(\lambda) = 0$?

2) The G-module description: What are the G-module structures of the remaining $H^i(\lambda)$'s?

It turns out that it is best to treat these two aspects of the problem simultaneously.

2. A little history

In the early fifties A.Borel, F.Hirzebruch and A.Weil proved the following theorem in the case $k = \mathbb{C}$.

Theorem 2.1 ([8], [9]). Let $\lambda \in X(B)$ such that $L(\lambda)$ has a non zero global section. Then

i) $H^o(\lambda)$ is an irreducible G-module with highest weight λ.

ii) $H^i(\lambda) = o$ for $i > o$

iii) The dimension of $H^o(\lambda)$ is given by H.Weyl's formula.

Observing that for a general $\lambda \in X(B)$ the alternating sum $\sum_i (-1)^i \dim H^i(\lambda)$ is always equal to $\pm$ the dimension of an irreducible G-module, Borel and Hirzebruch conjectured that $H^i(\lambda) = o$ for all but at most one value i_o of i and that $H^{i_o}(\lambda)$ is irreducible. This conjecture was proved by R.Bott in 1957. Before we can state the precise result - now known as the Borel-Weil-Bott theorem or just Bott's-theorem we need to introduce a little more notation.

We fix a maximal torus T contained in B and let R (resp. $R_- = -R_+$) denote the roots of G (resp. B) w.r.t. T. Set ρ equal to half the sum of the positive roots and let W be the Weyl group. When $\alpha \in R$ $\alpha^\vee$ will denote the correspon- ding coroot and $s_\alpha \in W$ the associated reflection. When $\lambda \in X(B)$ and $w \in W$ we put $w \cdot \lambda = w(\lambda+\rho) - \rho$. The set of dominant weights $X(B)_+$ is $\{\lambda \in X(B) \mid \langle\alpha^\vee,\lambda\rangle \geq o\}$. ℓ denotes the usual length function on W. Then

<u>Theorem 2.2</u>. [10]. Let $\lambda \in X(B)$ and pick $w \in W$ such that $w(\lambda+\rho) \in X(B)_+$. Then

$$H^i(\lambda) = \begin{cases} o & \text{for } i \neq \ell(w) \\ H^o(w.\lambda) & \text{for } i = \ell(w) \end{cases}$$

(Note that $H^o(x) = o$ iff $x \notin X(B)_+$ (this was observed in [8]) so that $H^{\cdot}(\lambda) = o$ precisely when there exists $\alpha \in R$ with $\langle\alpha^\vee, \lambda+\rho\rangle = o$).

The framework in which these two theorems were established was that of complex Lie groups. Kodaira's vanishing theorem was used to conclude that $H^i(\lambda) = o$ for $i > o$ when $\lambda \in X(B)_+$. In 1968 M.Demazure [18] gave an algebraic proof of Bott's theorem (for the case of an arbitrary algebraically closed field of characteristic zero). His proof avoided any reference to Kodaira vanishing. Later (1976) Demazure [20] simplified his proof drastically so that a very simple proof of Bott's theorem now is available. It should be mentioned that B. Kostant [36] also gave a proof of Bott's theorem. His method is Lie algebra cohomology (and Kodaira vanishing).

Meanwhile one was wondering how much of this would remain true in the case of characteristic $p > o$. It was clear that the $H^i(\lambda)$'s would not be irreducible in general (They fail to be so already for $G = SL(2,k)$). The best that could be said about $H^o(\lambda)$ was that it contained a unique B-stable line and so also a unique irreducible submodule, $M(\lambda)$. Moreover any irreducible G-module is isomorphic to an $M(\lambda)$ for some $\lambda \in X(B)_+$. These results go back to C.Chevalley [13], 1957.

The vanishing behaviour of the $H^i(\lambda)$'s as described by Bott's theorem does not carry over to positive characteristic either. D.Mumford was the first to find a line bundle on a flag manifold (actually on the 3 dimensional flag manifold in characteristic 2) with two non-vanishing cohomology groups.

It was, however, still believed that the Kodaira-type vanishing in Theorem 2.1.ii would hold in all characteristics. It should take until 1975 before this was proved in general [G.Kempf,35]. Partial results had been obtained in 1971 by Kempf himself [34] and in 1974 by V.Laksmibai, C.Musili and C.S.Seshadri [37]. Kempf's proof is rather long and technical involving a detailed study of a class of Schubert varieties in G/B. Last year a short proof was found independently by W.Haboush [24] and the author [5] (see section 6). (See also the recent preprint by G.Kempf, titled: Representations of algebraic groups in finite characteristics).

For the line bundles induced by non-dominant characters the first result in the direction of a vanishing description was proved in 1975 by L.Griffith [23] who obtained the vanishing behaviour of the $H^i(\lambda)$'s for $G = SL(3,k)$ (i.e. $G/B \simeq$ the 3 dimensional flag manifold). No information about the G-module structure was found.

This brings us up to the results surveyed in the next sections.

3. H^1. [1].

We preserve the notation from the previous sections.

Theorem 3.1. [1, Theorems 3.5 and 3.6]. Let λ be a non-dominant character and let α be a simple root with $\langle\alpha^\vee,\lambda\rangle < 0$. Then

a) $H^1(\lambda) \neq 0$ iff one of the following conditions holds

i) $\langle\alpha^\vee,\lambda\rangle = -ap^n-1$ for some $n \geq 0$, $1 \leq a \leq p$ and $s_\alpha\cdot\lambda$ is dominant,

ii) $-(a+1)p^n \leq \langle\alpha^\vee,\lambda\rangle \leq -ap^n-2$ for some $n \geq 1$, $1 \leq a < p$ and $\lambda + ap^n\alpha$ is dominant.

b) $H^1(\lambda)$ contains a unique irreducible submodule whose highest weight is

$s_\alpha\cdot\lambda$ when λ satisfies condition i) in a),
$\lambda + ap^n\alpha$ when λ satisfies condition ii) in a).

Before we indicate how this result is proved let us give an example.

Example 3.2. Let $G = GL(n,k)$ and let B be the set of upper triangular matrices. A character λ of B may then be thought of as an n-tuple of integers $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$. Then $\lambda' = (\lambda_1',\lambda_2',\ldots,\lambda_n')$ defines the same line bundle on G/B as λ iff $\lambda_1' - \lambda_1 = \lambda_2' - \lambda_2 = \ldots = \lambda_n' - \lambda_n$. λ is dominant iff $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. In characteristic zero we have

$H^1(\lambda) \neq 0$ iff there exists i such that

$$(*)\quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{i-1} \geq \lambda_{i+1} + 1 \geq \lambda_i - 1 \geq \lambda_{i+2} \geq \ldots \geq \lambda_n$$

Now Theorem 3.1 says that in characteristic p $H^1(\lambda) \neq o$ iff one of the following 2 conditions holds

i) there exists i s.t. $\lambda_i - \lambda_{i+1} = -ap^n - 1$ for some $n \geq o$, $1 \leq a \leq p$ and (*) holds

ii) there exists i s.t. $-(a+1)p^n \leq \lambda_i - \lambda_{i+1} \leq -ap^n - 2$ for some $n \geq 1$, $1 \leq a \leq p$ and

(**) $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{i-1} \geq \lambda_i + ap^n \geq \lambda_{i+1} - ap^n \geq \lambda_{i+2} \geq \ldots \geq \lambda_n$.

Also the highest weight of the irreducible submodule of $H^1(\lambda)$ is given in i) by the n-tuple from (*) and in ii) by the n-tuple from (**).

Proof of Theorem 3.1. (sketch): Let P_α denote the minimal parabolic subgroup corresponding to α. Then $P_\alpha/B \simeq \mathbb{P}^1$ and the restriction of $L(\lambda)$ to P_α/B is by this isomophism identified with $\mathcal{O}(\langle\alpha^\vee, \lambda\rangle)$. Therefore only $H^1(P_\alpha/B, L(\lambda)|) \neq o$ and by Serre duality this P_α-module is the dual of $H^o(P_\alpha/B, L(-\lambda-\alpha)|)$. The P_α-module structure is thus explicitly known (this is an $SL(2,k)$ - problem).

Now if P is an arbitrary parabolic subgroup containing B and E is a P-module then we get an induced vector bundle $L_P(E)$ on G/P. If π: $G/B \to G/P$ is the natural map then it is easy to see that $\pi_* L(\lambda) \simeq L_P(H^o(P/B, L(\lambda)|)$. Furthermore one proves that in fact $R^q\pi_* L(\lambda) \simeq L_P(H^q(P/B, L(\lambda)|)$.

In the case $P = P_\alpha$ this identification together with the Leray spectral sequence give

(3.3) $H^i(\lambda) \simeq H^{i-1}(G/P_\alpha, L_{P_\alpha}(H^1(P_\alpha/B, L(\lambda)|)))$. The advantage of (3.3) is that we have replaced an i-th cohomology group with an (i-1)-th cohomology group. The disadvantage is that at the same time the line bundle $L(\lambda)$ has been replaced by a vector bundle. Only in the case $i = 1$ is the advantage bigger than the disadvantage. In this case the explicit knowledge of the structure of $H^1(P_\alpha/B, L(\lambda)|)$ together with a careful analysis of the B-stable lines in $H^0(G/P_\alpha, L_{P_\alpha}(H^1(P_\alpha/B, L(\lambda)|))$ enable us to derive Theorem 3.1.

Remark 3.4 Part b) of Theorem 3.1 was the first result that gave information about the G-module structure of a cohomology group of a non-dominant line bundle. Note that it implies in particular that $H^1(\lambda)$ is indecomposable, a property that - strangely enough - is not shared by all higher cohomology groups [3, p.58].

4. The linkage principle [3]

In characteristic zero the composition factors of a Verma module for a semi-simple Lie algebra over k satisfy a linkage principle (or Harish-Chandra principle). D.N. Verma conjectured that a similar principle would hold for the composition factors of an $H^0(\lambda)$, $\lambda \in X(B)_+$ [45]. Partial results in that direction were obtained by J.E.Humphreys [25], R.Carter and G.Lusztig [11], V.Kac and B.Weisfeiler [32] and by J.C.Jantzen [28],[29].

The conjecture was finally proved in complete generality by including in the consideration all cohomology groups of line bundles on G/B, see Theorem 4.2 below.

<u>Definition 4.1</u> Let $\lambda, x \in X(B)$. We write $\lambda \uparrow x$ if there exists $\alpha \in R_+$, $n \geq o$ such that $n \leq \langle\alpha^\vee, x+\rho\rangle$ and $\lambda = s_\alpha . x + np\alpha$. λ is said to be strongly linked to x if there exists a sequence $\mu_1, \mu_2, \ldots, \mu_n \in X(B)$ such that $\lambda = \mu_1 \uparrow \mu_2 \uparrow \ldots \uparrow \mu_n = x$.

The following theorem (Theorem 1 of [3]) is called the strong linkage principle.

<u>Theorem 4.2</u> Let $x, \lambda \in X(B)$ such that $x+\rho$ and λ are dominant. If $M(\lambda)$ is a composition factor of $H^i(w.x)$ for some $w \in W$, $i \geq o$ then λ is strongly linked to x.

The idea to the proof of this result came from M.Demazure's proof of Bott's theorem [20]. He noticed that if α is a simple root with $\langle\alpha^\vee, x\rangle \geq o$ and $V_\alpha(x) = H^o(P_\alpha/B, L(x)|)$ then we have the following two exact sequences of B-modules

$$0 \to K_\alpha(x) \to V_\alpha(x) \to x \to 0$$

$$0 \to s_\alpha(x) \to K_\alpha(x) \to \bar{V}_\alpha(x) \to 0$$

where the quotient $\bar{V}_\alpha(x)$ in characteristic zero is isomorphic to $V_\alpha(x-\alpha)$. This observation together with the easely proved fact that $H^o(V\otimes -\rho) = o$ for all P_α-modules V imply that $H^i(x) \simeq H^{i+1}(s_\alpha . x)$ for all i.

Now in positive characteristic there is still a B-equivariant map $\bar{V}_\alpha(x) \to V_\alpha(x-\alpha)$ but it is not always an isomorphism. The kernel $C_\alpha(x)$ and cokernel $Q_\alpha(x)$ both have the property that their weights (w.r.t.T) have the form $s_\alpha(x) + kp\alpha$ with $o < kp < \langle\alpha^\vee, x\rangle$. Instead of having an isomorphism $H^{i+1}(s_\alpha . x) \not\simeq H^i(x)$ we get therefore the two long exact sequences

$$\ldots \to H^{i+1}(s_\alpha . x) \to H^i(x) \to H^{i+1}(\bar{V}_\alpha(x+\rho)\otimes-\rho) \to \ldots$$

and

$$\ldots \to H^{i+1}(C_\alpha(x+\rho)\otimes -\rho) \to H^{i+1}(\bar{V}_\alpha(x+\rho)\otimes-\rho) \to H^i(Q_\alpha(x+\rho)\otimes-\rho) \to \ldots$$

Now the theorem is proved by induction first after $\ell(w)$ and then w.r.t. the ordering in $X(B)_+$ induced by the positive roots. In addition to the above two long exact sequences a key observation used in the proof is that any non-zero homomorphism $H^{\dim G/B}(w_o . x) \to H^o(x)$, where w_o is the longest element in W, has image equal to $M(x)$. This follows from Serre-duality which gives that $M(x)$ is the only irreducible quotient of $H^{\dim G/B}(w_o . x) \simeq H^o(-w_o(x))^*$.

Remark 4.3.

i) In [4] the arguments in the above proof are used to obtain a bound on the set of weights in a cohomology group $H^i(\lambda)$ and then strengthened so as to give the existence of certain intertwining homomorphisms between Weyl modules.

ii) The strong linkage principle implies that if two irreducible modules $M(\lambda)$ and $M(\mu)$ are in the same block then λ and μ are linked. Recently, S.Donkin [21] has determined all blocks, showing (roughly) that the converse also holds.

5. Further vanishing results [6, sections 4 and 5]. Jantzen's translation principle [29] is extended to all cohomology modules in section 2 of [6]. We shall not dicuss this here but only mention that the above strong linkage principle is a key ingredient. What we want to point out now is that the following general vanishing theorem can be derived from it. Recall first that a character x is called regular if p does not divide $\langle\alpha^\vee, x+\rho\rangle$ for any $\alpha \in R$.

Theorem 5.1. Let $x \in X(B)$ be regular and suppose i_o is a non-negative integer such that $H^i(x) = o$ for $i \geq i_o$. Then $H^i(x+\lambda) = o$ for $i \geq i_o, \lambda \in X(B)_+$.

Remark 5.2. It can be seen from the strong linkage theorem that $H^i(o) = o$ for $i > o$. If therefore o is regular (i.e. if p is bigger than or equal to the Coxeter number) then it follows from Theorem 5.1 that $H^i(\lambda) = o$ for all dominant weights λ, i.e. the vanishing theorem for dominant line bundles, is in this case contained in our result. On the other hand the theorem is empty for p small.

6. An isomorphism theorem [5]

Consider the category M_B (resp. M_G) of rational B-modules (resp. G-modules). The induction functor $|^G : M_B \to M_G$ takes a B-module E into the G-module $E|^G = \{f: G \to E \mid f(gb) = b^{-1}f(g), g \in G, b \in B\}$, i.e $E|^G = H^o(E)$. M_B and M_G are abelian categories, have enough injectives etc., and we can thus ask for the higher derived functors of $|^G$. Now the minimal injective objects in M_B have the form $I_\lambda = \varinjlim_n M((p^n-1)\rho) \otimes (\lambda+(p^n-1)\rho), \lambda \in X(B)$. This and the wellknown fact that $L(\rho)$ is an ample line bundle show that $L(I_\lambda)$ is acyclic for $H^o(-)$. Hence the i-th derived functor of $|^G$ coincide with $H^i(-)$. So to compute $H^i(\lambda)$ we may use this characterization of H^i.

Let $F_n: G \to G$ be the n-th Frobenius homomorphism (given via raising to the p^n-th power in k). If V is a G-module the composite $G \xrightarrow{F_n} G \to GL(V)$ gives a new G-module structure on V which we denote $V^{(p^n)}$. We can then state the isomorphism theorem.

<u>Theorem 6.1</u> Let E be a rational B-module and let $n \geq o$. Then

$$H^i(E)^{(p^n)} \otimes M((p^n-1)\rho) \simeq H^i(E^{(p^n)} \otimes (p^n-1)\rho)$$

(as G-modules).

The isomorphism in the theorem is the composite of the Frobenius $H^i(E)^{(p^n)} \to H^i(E^{(p^n)})$ tensored by $M((p^n-1)\rho)$ and the cup product $H^i(E^{(p^n)}) \otimes H^o((p^n-1)\rho) \to H^i(E^{(p^n)} \otimes (p^n-1)\rho)$.

The proof of Theorem 6.1 uses the above explicit description of the injective objects in M_B and is almost purely representation theoretic. The only fact from algebraic geometry involved is the vanishing of $H^i(X \otimes n\rho)$ for $i > o$ and n large, $X \in M_B$. Actually the key to the proof lies in the following simple (and wellknown) lemma.

<u>Lemma 6.2</u> For n large $M((p^n-1)\rho) \simeq H^o((p^n-1)\rho)$

In fact this lemma holds for all n (as it will follow from the theorem).

The point is that the ampleness of $L(\rho)$ immediately gives that <u>for n large</u> the dimension of $H^o((p^n-1)\rho)$ equals the Euler character of $L((p^n-1)\rho)$ which is in turn seen to be equal to the dimension of $M((p^n-1)\rho)$ (usually called the n-th Steinberg module).

Important corollaries of Theorem 6.1 are

<u>Corollary 6.3</u> Let E be a rational B-module and let $j \geq o$. Then

i) The Frobenius homomorphism $H^j(E)^{(p^n)} \to H^j(E^{(p^n)})$ is injective (as conjectured by E.Cline, B.Parshall and L.Scott).

ii) The cup-product $H^j(E^{(p^n)}) \otimes M((p^n-1)\rho) \to H^j(E^{(p^n)} \otimes (p^n-1)\rho)$ is surjective.

<u>Corollary 6.4</u> (The vanishing theorem for dominant line bundles). Let $\lambda \in X(B)_+$. Then $H^i(\lambda) = o$ for $i > o$.

<u>Corollary 6.5</u> Let $\lambda \in X(B)$ and assume that $i \geq o$ such that $H^i(\lambda) \neq o$. Then for all $\nu \in X(B)_+$ with $\langle\alpha^\vee,\nu\rangle < p^n$ for all simple roots α we have $H^i(p^n\lambda + \nu) \neq o$.

The last corollary shows that in general there are lots of line bundles on G/B with more than one non-vanishing cohomology group. In [6] it is illustrated how Corollary 6.5 together with Theorem 5.1 can be used to give a complete description of the vanishing behaviour for line bundles on G/B when G has semi-simple rank 2.

7. Related results and some open problems

We mention here briefly a few results that are closely related to the above theory (and to the author's field of interest).

In an attempt to prove the vanishing theorem for dominant line bundles the geometry of the Schubert varieties in G/B have been studied intensely. In addition to the papers already mentioned in section 2 we should like here first of all to mention M.Demazure's work [19]. In this paper it is proved that if L is a dominant line bundle on G/B and if X is any Schubert variety then $H^i(X, L|_X) = o$ for $i > o$ provided that either $\mathrm{char}(k) = o$ or that a certain conjecture on integral lattices holds. Under the same assumptions a character formula for $H^o(X, L|_X)$ is given. The paper inspired the author's partial generalization to characteristic p [2] as well as the recent work of Seshadri et al, see [40]. The interest in studying L's restrictions to Schubert varieties comes not only from the fact that it gives a way of proving the vanishing theorem. Along the same line it is also possible to prove that the Schubert varieties have rational singularities and so are normal and Cohen-Macaulay.

Demazure derived this in characteristic zero. In positive characteristic it has not yet been proved in general but the work of Kempf [35] shows that it holds for a class of "special" Schubert varieties and V.Laksmibai, C.Musili and C.S.Seshadri ([38]-[41]) have further results. Actually the latter work is concerned with giving explicit nice bases for $H^o(G/B,L)$ (generalized Hodge-Young theory). This is in turn also related to the work of C.de Concini and C.Procesi on classical invariant theory [17].

Among other places where the vanishing theorem plays an important rôle we should like just to mention the following articles of E.Cline, B.Parshall and L.Scott [14] (With W.van der Kallen), [15] and [16].

The importance of the vanishing theorem for representation theory comes from the fact that it allows us to compute the dimension of the $H^o(\lambda)$'s. It shows that these modules are dual to certain modules constructed via reduction mod p from irredu cible characteristic zero representations. These modules were called Weyl modules by Carter and Lusztig [11] and they have been extensively studied in particular by J.C.Jantzen, [28] and [29].

It is imposible to end this brief discussion on related results without mentioning the recent conjecture of G.Lusztig. The focal point for much work on representation theory is the (still unsolved) problem of computing the character of the irreducible modules.

It is wellknown that the classes $\{[H^o(\lambda)] \mid \lambda \in X(B)_+\}$ constitute a basis for the representation ring $R(G)$ of G. In particular we can write in $R(G)$

$$[M(\lambda)] = \Sigma_\mu a_{\mu,\lambda} \quad [H^o(\mu)], \; \lambda \in X(B)_+$$

with $a_{\mu,\lambda} \in \mathbb{Z}$ (The linkage principle tells us that $a_{\mu,\lambda} = o$ unless μ is linked to λ) Lusztig's conjecture [42] gives for λ regular an expression for the $a_{\mu,\lambda}$ in terms of certain polynomials indexed by the representatives corresponding to μ and λ in the affine Weyl group.

$R(G)$ is identified with the ring of W-invariants in $X(B)$. It is completely formal to check that if A is a G-module then

$$(7.1) \quad [A] = \sum_\lambda \Big(\sum_{i \geq o} (-1)^i \dim \operatorname{Ext}^i_G (A, H^o(\lambda))\Big)[H^o(\lambda)].$$

Lusztig's conjecture can therefore (and has been, see [33]) also be formulated as a conjecture about $\operatorname{Ext}^{\cdot}_G(M(\lambda), H^o(\mu))$, $\lambda, \mu \in X(B)_+$. In this context we should like to state a conjecture for which we have good evidence in low rank cases.

<u>Conjecture 7.2</u> Let $\mu, \lambda, \nu \in X(B)_+$ with $\langle\alpha^\vee, \nu\rangle < p$ for all simple roots α. Suppose that μ is strongly linked to the regular weight $p\lambda+\nu$ and maximal with that property. Then $\operatorname{Ext}^1_G (M(\mu), H^o(\lambda)^{(p)} \otimes M(\nu)) \neq o$.

It can be shown that this conjecture implies Lusztig's conjecture.

Returning to the question of finding the vanishing behaviour of the $H^i(\lambda)$'s and comparing with (7.1) we would like to draw attention to the following problem.

<u>Problem 7.3</u> Find $\mathrm{Ext}_G^i\,(H^j(\lambda),\ H^o(\mu))$ for all $\lambda \in X(B)$, $\mu \in X(B)_+$, $i,j \geq o$

Of course one might more generally ask for all $\mathrm{Ext}_G^i\,(H^j(\lambda),\ H^k(\nu)\ )$, $\lambda,\nu \in X(B)$, $i,j,k \geq o$. (7.1) shows, however, that if we could just compute $\sum_{i\geq o}(-1)^i \dim \mathrm{Ext}_G^i\,(H^j(\lambda),\ H^o(\mu))$ then not only would the vanishing/non-vanishing of $H^j(\lambda)$ follow but we would also get information about the G-module structure of the non-zero $H^j(\lambda)$'s (in terms of certain $H^o(\mu)$'s).

The strong linkage principle gives that if $\mathrm{Ext}_G^i\,(H^j(\lambda),\ H^o(\mu)) \neq o$ then λ and μ are linked. We should also point out that $\mathrm{Ext}_G^i\,(H^j(\lambda),\ H^o(\mu)) \simeq \mathrm{Ext}_B^i\,(H^j(\lambda)|_B,\mu)$ and when $w.\lambda$ is not bigger than μ for any $w \in W$ then $\mathrm{Ext}_G^i\,(H^j(\lambda),\ H^o(\mu)) = o$ for $i > o$ [14].

Problem 7.3 is open even for $i = j = o$ (for partial results in this case see [11], [12], [4] and [6]). In particular the problem described in section 1 is still wide open.

We should like to finish up by mentioning that the representation theory for G is closely related to the representation theory for certain infinitesimal subgroups G_n, $n \geq o$. G_n is defined as the kernel of the n-th Frobenius homomorphism on G.

For this aspect of the theory see [26], [30] and [31]. Without giving any details let us just mention that the PIM's for G_n, i.e. the injective hulls of the irreducible modules for G_n, play a crucial role. The point is that (at least for p big) they sit nicely inside the corresponding injective G-modules, [44] and [22].

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