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§1. Introduction. This paper continues a line of study initiated in [8], where the atomic characterization of certain classical H^p spaces was extended to very general settings. If $1/2 < p \leq 1$ the space $H^p(\mathbb{R})$ can be characterized in terms of "atoms" that are measurable functions $a(x)$, $x \in \mathbb{R}$, having support in an interval I, $\|a\|_\infty \leq 1/|I|^{1/p}$ and are of mean value zero. The elements of $H^p(\mathbb{R})$ are distributions of the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \,, \tag{1.1}$$

where the a_j's are atoms and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ (in fact, these f's are continuous linear functionals on an appropriate space of smooth functions). The "H^p-norm" of f is equivalent to $N_p(f) = \inf(\Sigma|\lambda_j|^p)^{1/p}$, the infimum being taken over all decompositions (1.1).

These notions are very simple and have obvious extensions to measure spaces endowed with a "distance" that is sufficiently regular with respect to the measure. In [8] the fundamental properties of these "atomic" H^p spaces were developed and applied in the setting of spaces of homogeneous type.

In many situations, atoms having only mean zero suffice for the development of a useful theory. When $0 < p \leq 1/2$, however, the atomic characterization of the classical $H^p(\mathbb{R})$ spaces requires atoms having higher moments that vanish and satisfy the above properties. Specifically, we must have

$$\int_{\mathbb{R}} a(t)t^k dt = 0$$

for all non-negative integers $k \leq (1/p) - 1$ (see Coifman [4]). An analogous

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condition is required for an atomic characterization of $H^p(\mathbb{R}^n)$ (see Latter [13]). Furthermore, atomic characterizations, involving higher moments, of weighted H^p spaces on $\mathbb{R}$ have been obtained by Garcia-Cuerva [11].

One of the principal purposes of [8] was to show that many of the properties of general H^p spaces, and operators acting on them, can be obtained by focusing one's attention on individual atoms. For example, the continuity of an operator can often be proved by estimating Ta when a is an atom. While it is generally not true that atoms are mapped into atoms, it was observed in [8] that for many convolution (or multiplier) operators Ta is a function enjoying many of the properties of atoms. Such functions were called molecules and their "atom-like" properties are that their local and global size conditions are combined in a single "norm" relationship and their mean value is 0. Moreover, H^p spaces have molecular characterizations that are completely analogous to their atomic characterizations (we simply introduce molecules in the rôle played above by atoms). Each atom is a molecule and each molecule has an atomic decomposition of the form (1.1) with $\Sigma|\lambda_j|^p \leq C$, where C depends only on the "molecular norm" (which will be defined later). From this we see that a linear map T is bounded if Ta is a molecule of bounded molecular norm whenever a is an atom.

In this paper we will give appropriate definitions of molecules belonging to H^p spaces associated with $\mathbb{R}^n$ and the unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$ (taking into account the necessity of having a certain number of moments that vanish). We shall show that each such molecule has an atomic decomposition. From this, the molecular characterization of H^p will be evident. We will show how this molecular characterization can be used to obtain multiplier theorems. Moreover, we shall also consider certain "weighted" H^p spaces.

While it is not always easy to see whether a given function has an atomic representation, molecules do occur naturally and the fact that they do satisfy the conditions defining a molecule can be established by direct arguments. Let us describe an example of such a situation. Coifman and Rochberg [7] give a characterization of the functions belonging to certain H^p spaces on D that turns out to be a molecular decomposition. Perhaps the simplest example of their result concerns the "solid unweighted" Bergman space $A^1(D)$ of all holomorphic functions f on D satisfying

$$A(f) = \iint_D |f(x , y)| dx\, dy < \infty .$$

They show that there exists a (fixed) sequence of points $\{\zeta_j\}$ in D such that $f \in A^1(D)$ if and only if

$$f(z) = \lambda_0 + \sum_{j=1}^{\infty} \lambda_j \, z \frac{(1-|\zeta_j|^2)^2}{(1-\bar{\zeta}_j z)^4}$$

with $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. Moreover, the functions $z(1 - |\zeta_j|^2)^2/(1 - \zeta_j z)^4$ are molecules for the atomic Hardy space $H^1(D , dx\, dy)$ that we shall define in §3) and $\Sigma|\lambda_j|$ gives us a norm that is equivalent to $A(f)$. It follows that $A^1(D)$ is the holomorphic part of $H^1(D , dx\, dy)$.

We want to extend our special thanks to our colleague R.R. Coifman. Many ideas presented here grew out of discussions with him. We are also grateful to R. Rochberg for his many helpful suggestions.

§2. The Molecular Structure of $H^p(\mathbb{R}^n)$. Let us begin by introducing the elementary building block of $H^p(\mathbb{R}^n)$: the (p , q , s)-atoms. Suppose $0 < p \leq 1 \leq q \leq \infty$, $p < q$, and s is an integer at least $[n(\frac{1}{p} - 1)]$ (the integer part of $n(\frac{1}{p} - 1)$) . A (p , q , s)-atom centered at $x_0 \in \mathbb{R}^n$ is a function $a \in L^q(\mathbb{R}^n)$, supported on a ball $Q \subset \mathbb{R}^n$ with center x_0 and satisfying:

(2.1)

(i) $$\left[\frac{1}{|Q|} \int_Q |a(x)|^q \, dx\right]^{1/q} \leq |Q|^{-1/p}$$

(ii) $\int_{\mathbb{R}^n} a(x)\, x^\alpha \, dx = 0$, where $0 \leq |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \leq s$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$.

(We follow the usual conventions so that (2.1)(i) is interpreted as: $\sup_{x\in Q} |a(x)| \leq |Q|^{-1/p}$ if $q = \infty$ and $\alpha = (\alpha_1 , \dots , \alpha_n)$ in (2.1)(ii) is a nulti-index of non-negative integers.)

The atoms described in the introduction were $(p, \infty, 0)$-atoms on $\mathbb{R}$ for $\frac{1}{2} < p \leq 1$. The atoms studied in [8] were (p, q, o)-atoms. For each p a class of Hardy spaces was defined. It was shown that these spaces coincided and, thus, that we were dealing with a single space H^p. It is not surprising that by letting $q = 2$ the use of the Plancherel theorem becomes a powerful tool for the study of H^p. (We shall see this to be the case when we apply our results to the study of multipliers). Latter [13] has considered the spaces generated by $(p, q, [n(\frac{1}{p} - 1)])$-atoms on $\mathbb{R}^n$ and has shown that these spaces are the same as the H^p spaces defined by maximal functions (see Fefferman and Stein [9] for consequences of this fact). One of the facts that we will develop in this paper is that if p is fixed (as in the case $s = 0$), the Hardy classes based on (p, q, s)-atoms all coincide.

Let us now introduce the molecules corresponding to the atoms we have just defined. For p, q, s, satisfying the conditions for (p, q, s)-atoms and $\epsilon > \max\{\frac{s}{n}, \frac{1}{p} - 1\}$ we set $a = 1 - \frac{1}{p} + \epsilon$, $b = 1 - \frac{1}{q} + \epsilon$. A (p, q, s, ϵ)-<u>molecule centered at</u> x_0 is a function M such that $M \in L^q(\mathbb{R}^n)$ and $M(x)|x|^{nb} \in L^q(\mathbb{R}^n)$ satisfying:

(2.2)

$$\text{(i)} \quad \|M\|_q^{\frac{a}{b}} \|M|x_0 - x|^{nb}\|_q^{1-\frac{a}{b}} = \mathfrak{N}(M) < \infty$$

$$\text{(ii)} \quad \int_{\mathbb{R}^n} M(x)x^\alpha dx = 0, \quad 0 \leq |\alpha| \leq s .$$

$\mathfrak{N}(M) \equiv \mathfrak{N}(p, q, s, \epsilon; M)$ is called the molecular norm of M. Observe that our hypotheses imply the existence of the integrals in (ii) and, also, the fact that M^p is integrable (in fact, it is easy to see that $\|M\|_p \leq 2^{\frac{1}{p}}\mathfrak{N}(M)$.

<u>Proposition (2.3)</u>. <u>If</u> a <u>is a</u> (p, q, s)-<u>atom then</u> a <u>is a</u> (p, q, s, ϵ)-<u>molecule for all</u> $\epsilon > 0$ <u>and</u> $\mathfrak{N}(a) \leq C$; , <u>where</u> C <u>is independent of the atom.</u>

<u>Proof</u>. Clearly, $\|a\|_q \leq |Q|^{a-b}$ and $\|a|x - x_0|^{nb}\|_q \leq C|Q|^b \|a\|_q \leq C|Q|^a$, where C is a geometric constant. Thus, $\mathfrak{N}(a) \leq |Q|^{\frac{a}{b}(a-b)} (C|Q|^a)^{\frac{b-a}{b}} = C^{1-\frac{a}{b}}$. This

proves (2.3).

As indicated in the introduction we shall show (directly) that each molecule has an atomic decomposition. In order to do this we need to give a precise definition of the atomic Hardy space $H^p(\mathbb{R}^n)$. If s is a non-negative integer, $0 \le [n\beta] \le s$, $1 \le q' \le \infty$, we define the space $L(\beta, q', s)$ as follows:

If g is locally integrable on $\mathbb{R}^n$ and Q is a ball, let $P_Q g$ be the unique polynomial of degree at most s such that

$$\int_Q (g - P_Q g) x^\alpha \, dx = 0$$

for $0 \le |\alpha| \le s$. Suppose g satisfies

$$(2.4) \qquad \|g\|_{L(\beta,q',s)} = \sup_{Q \subset \mathbb{R}^n} |Q|^{-\beta} \left\{ \frac{1}{|Q|} \int_Q |g - P_Q g|^{q'} dx \right\}^{\frac{1}{q'}} < \infty ;$$

then, clearly, if $\tilde{g} - g$ is a polynomial of degree at most s, $\tilde{g}$ also satisfies (2.4) and $\|g\|_{L(\beta,q',s)} = \|\tilde{g}\|_{L(\beta,q',s)}$. If this is the case we say that g and $\tilde{g}$ are equivalent. The space of all such equivalence classes $[g]$ will be denoted by $L(\beta, q', s)$ and (2.4) defines its norm (similar spaces were studied by Morrey in [14] and Campaneto in [2]).

The spaces $L(0, q', 0)$, for $1 \le q' < \infty$, are known to be equivalent Banach spaces; in fact, they are various descriptions of the space BMO (see [12]). We shall see below that $L(0, q', s)$, $1 \le q' < \infty$, $s \ge 0$, is also equivalent to BMO (see [1] for a related result). When $\beta > 0$ it is not hard to show that if $[g] \in L(\beta, q', s)$ then g satisfies

$$|\Delta_h^{s+1} g(x)| \le A|h|^{n\beta} ,$$

where, in the usual notation, $\Delta_h^{m+1} g = \Delta_h(\Delta_h^m g)$, $m \ge 1$; $\Delta_h g(x) = g(x) - g(x - h) = g(x) - (\tau_h g)(x)$; $\Delta_h^\circ g = g$. It follows from this that if $\beta > 0$ and $[g] = L_{(\beta,q',[n\beta])}$ then g satisfies: i) g is continuous and ii) $g(x) = O(|x|^{n\beta})$ as $|x| \to \infty$ if $n\beta$ is not an integer and $g(x) = O(|x|^{n\beta} \log |x|)$ as $|x| \to \infty$ if $n\beta$ is an integer. These facts will be established in Appendix D.

For $0 < p \le 1$ the atomic space generated by (p, q, s)-atoms will be a subspace of the space of continuous linear functionals on

$L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$. For $p = 1$ each space will be a closed subspace of $L^1(\mathbb{R}^n)$. More precisely:

Definition. The Hardy space $H^{p,q,s}(\mathbb{R}^n)$ is the collection of all continuous linear functionals f on $L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$ of the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j , \tag{2.5}$$

where each a_j is a (p, q, s)-atom and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$.

For such f, we define its "norm", $\|f\|_{H^{p,q,s}}$, to be

$$\inf\left\{\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} : \text{over all such representations (2.5)}\right\}.$$

The following remarks show that these Hardy spaces are well defined: First observe that $L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$ is continuously embedded in $L(\frac{1}{p} - 1, q', s)$; that is, if $[g] \in L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$ then $[g] \in L(\frac{1}{p} - 1, \infty, s)$ and $\|g\|_{L(\frac{1}{p} - 1, q', s)} \leq \|g\|_{L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])}$. (See Lemma (8.2) in Appendix D.) Note that $s \geq [n(\frac{1}{p} - 1)]$. Lemma (8.2) shows that the norm of $L(\beta, q', s)$ is equivalent to one using a certain infimum, and the inclusion follows. Next observe that if a is a (p, q, s)-atom then

$$\begin{aligned} \left|\int_{\mathbb{R}^n} a\, g\, dx\right| &= \left|\int_Q a(g - P_Q g)dx\right| \\ &\leq \|a\|_q \left\{\int_Q |g - P_Q g|^{q'} dx\right\}^{\frac{1}{q'}} \\ &\leq \|g\|_{L(\frac{1}{p} - 1, q', s)} \leq \|g\|_{L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])} . \end{aligned} \tag{2.6}$$

This estimate and the inequality $\Sigma|\lambda_j| \leq [\Sigma|\lambda_j|^p]^{\frac{1}{p}}$ imply that expressions (2.5) are continuous linear functionals on $L(\beta, q', s)$, $\beta = \frac{1}{p} - 1$, $s = [n\beta]$. Observe that if a is a $(1, q, s)$-atom then $\|a\|_1 \leq 1$ so that (2.5) converges in L^1 and it follows that $H^{1,q,s}$ is a closed subspace of L^1, and that if $f \in H^{1,q,s}$

then $\|f\|_{L^1} \leq \|f\|_{H^{1,q,s}}$. We also note that for $0 < p < 1$, the "norm" $\| \ \|_{H^{p,q,s}}$ induces a Fréchet metric on $H^{p,q,s}$.

The following is an extension of Theorem B in [8]:

<u>Theorem (2.7).</u> <u>The dual of</u> $H^{p,q,s}$ <u>is naturally isomorphic to</u> $L(\frac{1}{p} - 1 , q' , s)$.

Apart from obvious modifications, the proof of this theorem is the same as that of Theorem B in [8]. The estimate (2.6) is the point of departure.

From Theorem (2.7) we obtain the particular result that the dual of $H^{p,\infty,[n(\frac{1}{p}-1)]}$ is $L(\frac{1}{p} - 1 , 1 , [n(\frac{1}{p} - 1)])$. We use this in Appendix D to obtain the results following (2.4) on the local and global behaviour of representative functions in $L(\beta , \infty , [n\beta])$ if $\beta > 0$. In the proof of the following theorem these reults are used to establish that the implied atomic-decomposition of an (p , q_1 , s_1)-atom into (p , q_2 , s_2)-atoms induces the same linear functional on $L(\frac{1}{p} - 1 , \infty , [n(\frac{1}{p} - 1)]$ as that given by the original atom. This next theorem is an extension of Theorem A in [8].

<u>Theorem (2.8).</u> <u>Let</u> p , q <u>and</u> s <u>be related as they were in the definition of a</u> (p , q , s)-<u>atom</u>. <u>Then</u>

$$H^{p,q,s}(\mathbb{R}^n) = H^{p,\infty,[n(\frac{1}{p}-1)]}(\mathbb{R}^n) .$$

<u>Moreover, the "norms" associated with the two spaces are equivalent.</u>

For s fixed the proof that $H^{p,q,s} = H^{p,\infty,s}$ is almost an exact copy of the induction argument given in [8] (instead of subtracting constants one must subtract appropriate polynomials as in the proof of Theorem (2.9) below). For varying s a different argument is required; it will be presented following our discussion of (2.9).

<u>Remark.</u> It follows directly from (2.7) and (2.8) that if p , q and s are related as they were in the definition of a (p , q , s)-atom then

$L(\frac{1}{p} - 1, q', s) = L(\frac{1}{p} - 1, 1, [n(\frac{1}{p} - 1)])$. This formal identification extends to an identification of representative functions in the two spaces in the following sense: Let (p, q_1, s_1) and (p, q_2, s_2) be admissible indices for p-atoms. Let $L(\beta, q_i, s_i)$, $\beta = \frac{1}{p} - 1$, represent the collection of functions in the various equivalence classes, then

$$L(\beta, q_1, s_1) = L(\beta, q_2, s_2) \bmod (\mathcal{P}_{\max[s_1, s_2]})$$

where $\mathcal{P}_s$ is the space of polynomials of degree at most s. We also note that $L(0, q', s) = \mathrm{BMO} \bmod (\mathcal{P}_s)$ if $1 \leq q' < \infty$, $s \geq 0$. We omit the details. The necessary tools can be found in Appendix D.

Theorem (2.9). *If* M *is a* (p, q, s, ϵ)*-molecule* $(p, q, s$ *and* ϵ *related as in (2.2)) then* $M \in H^{p,q,s}$ *and*

$$\|M\|_{H^{p,q,s}} \leq C'\aleph(M),$$

where C' *is independent of the molecule* M.

There is no loss in generality if we assume that M is centered at the origin. For simplicity we shall present here the proof of this theorem when $q = 2$. At the end of this section we shall indicate what changes are needed for the general case. Briefly, our argument is as follows: Let

$$\sigma^{n\left(\frac{1}{p} - \frac{1}{2}\right)} = \|M\|_2^{-1}.$$

Then we put $E_0 = \{x \in \mathbb{R}^n : |x| \leq \sigma\}$, $E_k = \{x \in \mathbb{R}^n : 2^{k-1}\sigma < |x| \leq 2^k\sigma\}$ for $k = 1, 2, 3, \ldots$; χ_k denotes the characteristic function of E_k and $M_k = M\chi_k$. For each k there exists a unique polynomial Q_k, of degree at most s, such that if $P_k = Q_k\chi_k$ then

$$\int_{\mathbb{R}^n} (M_k - P_k)x^\alpha \, dx = 0, \qquad 0 \leq |\alpha| \leq s. \tag{2.10}$$

We then show that $M_k - P_k$ is a multiple of a $(p, 2, s)$-atom and that the coefficients sum appropriately, and (using a summation-by-parts argument analogous to that presented in [8], page 595) we also show that $\sum P_k$ can be written as a sum of (p, ∞, s)-atoms and that the coefficients sum appropriately. Since a (p, ∞, s)-atom is also a $(p, 2, s)$-atom the result will follow.

There is no loss of generality if we assume $\aleph(M) = 1$. From the definition of σ, therefore, we have

$$\| M|x|^{n(\frac{1}{2}+\varepsilon)} \|_2 = \sigma^{na} \tag{2.11}$$

For each $k = 0, 1, 2, \ldots$ let $\{\varphi_\ell^k\}_{|\ell| \leq s}$ denote the Gram-Schmidt orthonormalization of the monomials $\{x^\ell\}_{|\ell| \leq s}$ (taken in some fixed order) on the set $\mathbf{E}_k$ with respect to the weight $1/|E_k|$. We consider the functions φ_ℓ^k to be defined on $\mathbb{R}^n$, having the value 0 outside E_k. If

$$a_\ell^k = \frac{1}{|E_k|} \int M_k \, \varphi_\ell^k \, dx$$

then, clearly, the restricted polynomial satisfying (2.10) must be

$$P_k = \sum_{|\ell| \leq s} a_\ell^k \varphi_\ell^k .$$

$M_k - P_k$ is supported on $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k \sigma\}$. Moreover,

$$\left\{\frac{1}{|B_k|} \int |M_k - P_k|^2 dx\right\}^{\frac{1}{2}} = C \left\{\frac{1}{|E_k|} \int |M_k - P_k|^2 dx\right\}^{\frac{1}{2}} \leq C \left\{\frac{1}{|E_k|} \int |M_k|^2 dx\right\}^{\frac{1}{2}} . \tag{2.12}$$

(Here, C is a geometric constant. Throughout this paper the letter C will denote (possibly different) constants that are independent of the essential variables in the argument. This independence will be clear from the context.)

In particicular, using the definition of σ, we have

$$\left\{\frac{1}{|B_o|} \int |M_o - P_o|^2 \, dx\right\}^{\frac{1}{2}} \leq C \, \|M\|_2 \, \sigma^{-\frac{n}{2}} = C\sigma^{-\frac{n}{p}} .$$

If $k \geq 1$, on the other hand, from (2.12) we obtain the estimate

$$\left\{\frac{1}{|B_k|} \int |M_k - P_k|^2 \, dx\right\}^{\frac{1}{2}} \leq C \left\{\frac{1}{|E_k|} \int |M_k|^2 |x|^{n(1+2\varepsilon)} (|x|^{-n(1+2\varepsilon)} \chi_k) dx\right\}^{\frac{1}{2}}$$

$$\leq C \, (2^k\sigma)^{-\frac{n}{2}} (2^k\sigma)^{-n(\frac{1}{2}+\varepsilon)} \| M|x|^{n(\frac{1}{2}+\varepsilon)} \|_2 \leq C \, (2^k\sigma)^{-n(1+2\varepsilon)} \sigma^{na} ,$$

the last inequality being a consequence of (2.11).

Since $M_k - P_k$ is supported in B_k and $|B_k| = C(2^k\sigma)^n$, then $M_k - P_k = \lambda_k a_k$ where a_k is a $(p, 2, s)$-atom and $\lambda_k = C\, 2^{-nak}$. Thus,

$$\sum_{k=0}^{\infty} |\lambda_k|^p \le C^p \sum_{k=0}^{\infty} 2^{-nakp} = \frac{C}{1-2^{-nap}},$$

since $a = 1 - \frac{1}{p} + \epsilon > 0$.

Let $\{\psi_\ell^k\}_{|\ell|\le s}$ be the dual basis of the monomials $\{x^\ell\}_{|\ell|\le s}$ (taken in the same fixed order) on E_k with respect to the weight $1/|E_k|$. If $\varphi_\ell^k = \sum_{|\nu|\le s} \beta_{\ell\nu}^k x^\nu$, then $\psi_\ell^k = \sum_{|\nu|\le s} \beta_{\nu\ell}^k \varphi_\nu^k$. We also have

$$P_k = \sum_{|\ell|\le s} m_\ell^k \psi_\ell^k \quad \text{where} \quad m_\ell^k = \frac{1}{|E_k|}\int M_k x^\ell\, dx .$$

From considerations of homogeneity and the uniqueness of the Gram-Schmidt orthogonalization process, $|\varphi_\nu^k(x)| \le C$ for $x \in E_k$ and $|\beta_{\nu\ell}^k| \le C(2^k\sigma)^{-|\ell|}$ (here C depends only on s). Consequently, for $x \in E_k$,

$$|\psi_\ell^k(x)| \le C(2^k\sigma)^{-|\ell|} ;$$

we consider ψ_ℓ^k to be defined overywhere but supported on E_k.

Observe that

$$\sum_{k=0}^{\infty} |E_k| m_\ell^k = \sum_{k=0}^{\infty} \int_{E_k} M\, x^\ell\, dx = \int_{\mathbb{R}^n} M(x)\, x^\ell\, dx = 0 ,$$

$0 \le |\ell| \le s$. We let $N_\ell^k = \sum_{j=k}^{\infty} |E_j| m_\ell^j$, $k = 0, 1, 2, \ldots,$ and note that $N_\ell^0 = 0$. For $k \ge 1$ we have

$$|N_\ell^k| \le \sum_{j=k}^{\infty} \int |M_j|\, |x|^{|\ell|} dx \le C \sum_{j=k}^{\infty} \left\{\frac{1}{|E_k|}\int |M_j|^2\, dx\right\}^{\frac12} (2^j\sigma)^{|\ell|+n}$$

$$\le C \sum_{j=k}^{\infty} (2^j\sigma)^{-n(1+\epsilon)} \sigma^{na} (2^j\sigma)^{|\ell|+n} = C\,\sigma^{|\ell|+n(1-\frac{1}{p})}\, 2^{k(|\ell|-n\epsilon)} ,$$

since $|\ell| \le s < \epsilon n$. Thus,

$$|N_\ell^k \psi_\ell^k |E_k|^{-1}| \le C\,\sigma^{|\ell|+n(1-\frac{1}{p})} 2^{k(|\ell|-n\epsilon)} (2^k\sigma)^{-|\ell|} (2^k\sigma)^{-n}$$

$$= C\,\sigma^{-\frac{n}{p}} 2^{-nk(1+\epsilon)} \to 0 \quad \text{as} \quad k \to \infty .$$

Hence, letting

$$f_{\ell k} = N_\ell^{k+1} \left\{ \psi_\ell^{k+1} |E_{k+1}|^{-1} - \psi_\ell^k |E_k|^{-1} \right\},$$

and summing by parts we obtain

$$\sum_{k=0}^{\infty} P_k = \sum_{|\ell|\le s} \sum_{k=0}^{\infty} (m_\ell^k |E_k|)\left(\frac{\psi_\ell^k}{|E_k|}\right) = \sum_{|\ell|\le s} \sum_{k=0}^{\infty} f_{\ell k} .$$

Since $\{\psi_\ell^k\}$ is the dual basis of $\{x^\ell\}$ on $(E_k , dx/|E_k|)$ we have that

$$\frac{1}{|E_R|} \int \psi_\ell^k x^t \, dx = \delta_{t\ell} , \quad 0 \le |t| , \ |\ell| \le s .$$

Therefore, $\int f_{\ell k} x^t \, dx = 0$ for $0 \le |t| \le s$. Moreover,

$$|f_{\ell k}| \le C\,\sigma^{-\frac{n}{p}} 2^{-nk(1+\epsilon)} = C\, 2^{-nka} |B_{k+1}|^{-\frac{1}{p}} .$$

Since $f_{\ell k}$ is supported on B_{k+1} these estimates show that

$$f_{\ell k} = \mu_{\ell k} b_{\ell k} ,$$

where $b_{\ell k}$ is a (p , ∞ , s)-atom and $|\mu_{\ell k}| = C\, 2^{-nka}$. It follows that

$$\sum_{k=0}^{\infty} |\mu_{\ell k}|^p \le \frac{C}{1-2^{-nap}} ,$$

where $C = C(p , \epsilon , s)$.

We have shown that $M = \sum_{k=0}^{\infty} \lambda_k a_k + \sum_{|\ell|\le s} \sum_{k=0}^{\infty} \mu_{\ell k} b_{\ell k}$, where a_k is a $(p , 2 , s)$-atom, $b_{\ell k}$ is a (p , ∞ , s)-atom and $\sum_{k=0}^{\infty} |\lambda_k|^p + \sum_{|\ell|\le s} \sum_{k=0}^{\infty} |\mu_{\ell k}|^p \le C = C(p , \epsilon , s)$.

We observe that the sum representing M converges pointwise (in fact, for each $x \in \mathbb{R}$ only finitely many terms are not zero). This fact, though interesting, does not imply that M and this series represent the same element of $H^{p,q,s}$ as linear functionals on $L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$. The following observations will make it clear that this is, indeed, true. The above estimates show that

$$\sum_{k=0}^{\infty} \|\lambda_k a_k\|_2 \leq C\, \sigma^{n(a-b)} \sum_{k=0}^{\infty} 2^{-nkb}, \quad \text{and}$$

$$\sum_{k=0}^{\infty} \|\mu_{\ell k}\, b_{\ell k}\|_2 \leq C\, \sigma^{n(a-b)} \sum_{k=0}^{\infty} 2^{-nkb}.$$

Thus, the series representing M converges in $L^2(\mathbb{R}^n)$. It follows from this (and the fact that $\operatorname{supp} a_k \subset B_k$) that the series

$$\sum_{k=0}^{\infty} \lambda_k a_k\, |x|^{nb'} \quad \text{and} \quad \sum_{k=0}^{\infty} \mu_{\ell k}\, b_{\ell k}\, |x|^{nb'}$$

converge in $L^2(\mathbb{R}^n)$ whenever $0 < b' < b$. Now recall that if $0 < p < 1$ and $[g] \in L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$ then g is locally bounded and $g(x) = 0(|x|^{n(\frac{1}{p} - 1)} \log |x|)$ as $|x| \to \infty$. Choose ε', $\frac{1}{p} - 1 < \varepsilon' < \varepsilon$ and let $b' = \frac{1}{2} + \varepsilon'$. Note that $g(1 + |x|^n)^{-b'} \in L^2$, $M(1 + |x|^n)^{b'} \in L^2$ and from the L^2 convergence of the series,

$$\sum_{k=0}^{\infty} \{|\lambda_k a_k|(1 + |x|^n)^{b'} + \sum_{|\ell| \leq s} |\mu_{\ell k}\, b_{\ell k}|(1 + |x|^n)^{b'}\},$$

we see that both

$$\int |Mg|\,dx \quad \text{and} \quad \int \{\sum_k \{|\lambda_k a_k g| + \sum_{|\ell| \leq s} |\mu_{\ell k}\, b_{\ell k}\, g|\}\}dx$$

are finite. Thus all we need to show is that

$$\int_{|x| \leq A} Mg\, dx = \int_{|x| \leq A} [\sum_{k=0}^{\infty} \{\lambda_k a_k + \sum_{|\ell| \leq s} \mu_{\ell k}\, b_{\ell k}\}]g\, dx$$

for all $A > 0$. But this is evident since for each $A > 0$ the sum on the right is finite and is equal to M.

For $p = 1$ the argument is much easier. Note that $L(0, \infty, 0) = L^\infty \bmod (\mathcal{P}_0)$ and that if M is a $(1, q, s, \varepsilon)$-molecule then $M \in L^1$. Note also that $\|a_k\|_1$, $\|b_{\ell k}\|_1 \leq 1$ since a_k, $b_{\ell k}$ are $(1, q, s)$-atoms and

$|\lambda_k|$, $|\mu_{\ell k}| \leq C\, 2^{-nak} = C\, 2^{-n\epsilon k}$ so the series converges in L^1 to M .

This completes the proof of (2.9) for the special case $q = 2$.

We remark that a small modification of the last argument shows that the series representing M converges in the topology induced by the $(p , 2 , s , \epsilon')$-norm when $\frac{1}{p} - 1 < \epsilon' < \epsilon$.

We now turn to the proof of (2.8). If $q_1 \leq q_2$ then

$$\left[\frac{1}{|Q|}\int_Q |a(x)|^{q_1}\, dx\right]^{\frac{1}{q_1}} \leq \left[\frac{1}{|Q|}\int_Q |a(x)|^{q_2}\, dx\right]^{\frac{1}{q_2}} .$$

Thus, any (p , q_2 , s)-atom is a (p , q_1 , s)-atom and it follows that $H^{p,q_2,s}(\mathbb{R}^n) \subset H^{p,q_1,s}(\mathbb{R}^n)$ and $\| \|_{H^{p,q_1,s}} \leq \| \|_{H^{p,q_2,s}}$. It was pointed out earlier that the proof that these two spaces are the same is a slight modification of the argument given in [8]. In order to establish theorem (2.3), therefore, it suffices to show that we can vary s . Suppose that $s_1 > s \geq [n(\frac{1}{p} - 1)]$. It is trivial that a (p , q , s_1)-atom is a (p , q , s)-atom so we need to show that a (p , q , s)-atom has a decomposition in terms of (p , q , s_1)-and (p , ∞ , s_1)-atoms. We shall use an argument that is similar to the one we used for (2.9). More precisely, we shall again restrict our attention to the special case $q = 2$ and show that if $a(x)$ is a $(p , 2 , s)$-atom then $a(x) = b_o(x) + \sum_{k=1}^{\infty} \lambda_k b_k(x)$, where b_o is a $(p , 2 , s_1)$-atom, b_k , for $k \geq 1$, is a (p , ∞ , s_1)-atom and $\sum_{k=1}^{\infty} |\lambda_k| \leq C$ (we shall indicate later the changes required for the general case)

Suppose a is centered at 0 and its support lies in a ball Q for which (2.1)(i) holds. Let Q_k be the dilation of Q by 2^k , $k = 0 , 1 , 2 , \ldots$; $\{\psi_\ell^k\}$ the dual basis of the monomials $\{x^t\}$ restricted to Q_k (taken in some fixed order), $0 \leq |\ell|$, $|t| \leq s_1$, $0 \leq k$ (with respect to the weight $|Q_k|^{-1} = 2^{-nk}|Q|^{-1}$). It is easy to check that $\psi_\ell^k(y) = 2^{-k|\ell|}\psi_\ell^0(y/2^k)$ and it follows that $|\psi_\ell^k(x)| \le C\,(2^k|Q|^{\frac{1}{n}})^{-|\ell|}$ for $x \in Q_k$. Moreover we consider ψ_ℓ^k to be defined on $\mathbb{R}^n$ but to be supported on Q_k.

Let,

$$m_\ell = \frac{1}{|Q|}\int a(x)\,x^\ell\,dx ,$$

and $P(x) = \sum_{s<|\ell|\le s_1} m_\ell\,\psi_\ell^0(x)$. We then put $b_0 = a - P$, which gives us a function supported in Q satisfying

$$\frac{1}{|Q|}\int b_o(x)x^\ell\,dx = m_\ell - m_\ell = 0 \quad \text{if} \quad 0 \le |\ell| \le s_1$$

(of course, $m_\ell = 0$ when $0 \le |\ell| \le s$). Clearly, P is the partial sum, of terms up to order s_1, of the expansion of a in terms of the Gram-Schmidt orthonormalization of the monomials restricted to Q; thus,

$$(2.13)\quad \left\{\frac{1}{|Q|}\int |b_o|^2dx\right\}^{\frac{1}{2}} = \left\{\frac{1}{|Q|}\int |a - P|^2dx\right\}^{\frac{1}{2}} \le \left\{\frac{1}{|Q|}\int |a|^2dx\right\}^{\frac{1}{2}} \le |Q|^{-\frac{1}{p}} .$$

It follows that b_o is a $(p\ ,\ 2\ ,\ s_1)$-atom.

Now let

$$m_\ell^k = \frac{1}{|Q_k|}\int_{Q_k} a(x)x^\ell\,dx = 2^{-nk}m_\ell .$$

Thus,

$$|m_\ell^k| \le 2^{-nk}\frac{1}{|Q|}\int |a(x)|\,|x|^{|\ell|}dx \le C\,2^{-nk}|Q|^{\frac{|\ell|}{n}}\left\{\frac{1}{|Q|}\int |a|^2dx\right\}^{\frac{1}{2}} \le C2^{-nk}|Q|^{\frac{|\ell|}{n}-\frac{1}{p}} .$$

Consequently, $|m_\ell^k\,\psi_\ell^k| \le C\,2^{-nk}|Q|^{\frac{|\ell|}{n}-\frac{1}{p}}(2^k|Q|^{\frac{1}{n}})^{-|\ell|} = C\,2^{-k(|\ell|+n)}|Q|^{-\frac{1}{p}} = o(1)$ as $k \to \infty$. It is also convenient to write this in the form

$$(2.14)\qquad |m_\ell^k\,\psi_\ell^k| \le C2^{-nk(\frac{|\ell|}{n}+1-\frac{1}{p})}\,|Q_k|^{-\frac{1}{p}} .$$

If we now write

$$P(x) = \sum_{s<|\ell|\le s_1} m_\ell\psi_\ell^0(x)$$

$$= \sum_{s<|\ell|\le s_1} \sum_{k=1}^{\infty} (m_\ell^{k-1} \psi_\ell^{k-1}(x) - m_\ell^k \psi_\ell^k(x))$$

$$= \sum_{s<|\ell|\le s_1} \sum_{k=1}^{\infty} f_{k\ell}(x)$$

we see that $f_{k\ell}$'s are supported in Q_k and by (2.14)

$$|f_{k\ell}(x)| \le C\, 2^{-nk(\frac{|\ell|}{n}+1-\frac{1}{p})} |Q_k|^{-\frac{1}{p}} .$$

Finally, if $0 \le |t| \le s_1$, we have

$$\int f_{k\ell}(x)x^t\, dx = m_\ell^{k-1} \int \psi_\ell^{k-1}(x)x^t\, dx - m_\ell^k \int \psi_\ell^k(x)x^t\, dx$$

$$= \begin{cases} 2^{-n(k-1)} m_\ell |Q_{k-1}| - 2^{-nk} m_\ell |Q_k| = 0 , & \text{if } t = \ell . \\ 0 - 0 = 0 , & \text{if } t \ne \ell \end{cases}$$

This shows that $f_{k\ell} = \lambda_{k\ell} b_{k\ell}$, where $b_{k\ell}$ is a (p , ∞ , s_1)-atom and $|\lambda_{k\ell}| \le C\, 2^{-nk\left(\frac{|\ell|}{n}+1-\frac{1}{p}\right)}$. From $|\ell| > s \ge [n(\frac{1}{p} - 1)]$ we have $\frac{|\ell|}{n} > \frac{1}{p} - 1$; consequently,

$$\sum_{s<|\ell|\le s_1} \sum_{k=1}^{\infty} |\lambda_{k\ell}|^p \le C ,$$

where C depends only on n , p and s_1 .

We have, therefore, the desired atomic decomposition:

$$a = (a - P) + P = b_0 + \sum_{s<|\ell|\le s_1} \sum_{k=1}^{\infty} \lambda_{k\ell}\, b_{k\ell} . \tag{2.15}$$

The estimate for $m_\ell^k \psi_\ell^k$ immediately preceding (2.14) shows that the last series converges pointwise for all x . (The atomic decomposition of a molecule involved a series with only a finite number of non-zero terms for each x . This is not the case here.) Moreover, $\|b_{k\ell}\|_2 \le |Q_k|^{\frac{1}{2}-\frac{1}{p}} = 2^{-nk(\frac{1}{p}-\frac{1}{2})} |Q|^{\frac{1}{p}-\frac{1}{2}}$ since $b_{k\ell}$ is a $(p , 2 , s_1)$-atom. We can then argue as we did for molecules that the series (2.15) represents the same element as $a(x)$, as a linear functional on

$L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$. For $0 < p < 1$ we use the estimate on $\lambda_{k\ell}$ to show that $b_o(x)(1 + |x|^n)^\beta + \sum_{s<|\ell|\leq s_1} \sum_{k=1}^{\infty} \lambda_{k\ell}\, b_{k\ell}(x)(1 + |x|^n)^\beta$ converges in $L^2(\mathbb{R}^n)$ if $\beta < \frac{|\ell|}{n} + \frac{1}{2}$. We find that if $[g] \in L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$ that $g(x)(1 + |x|^n)^{-\beta} \in L^2(\mathbb{R}^n)$ if $\beta > \frac{1}{p} - \frac{1}{2}$. Since $|\ell| > s \geq [n(\frac{1}{p} - 1)]$, $\frac{|\ell|}{n} > (\frac{1}{p} - 1)$ and so there is such a β and the proof proceeds as before. For $p = 1$ we use the fact that $\Sigma|\lambda_{k\ell}| < \infty$ to see that the series converges in L^1. Thus the series and $a(x)$ represent the same linear functional.

These arguments complete the proof of (2.8) and (2.9) in the case $q = 2$. If $q \neq 2$, apart from obvious changes (such as setting $\sigma^{n(\frac{1}{p} - \frac{1}{q})} = \|M\|_q^{-1}$ and the use of Hölder's inequality instead of Schwartz's inequality) one needs to obtain the analog of the inequality in (2.12) and (2.13). (The inequality in (2.13) is the case $k = 0$ of (2.12) with $s = s_1$. We recall that the Gram-Schmidt polynomials $\{\varphi_\nu^k\}_{|\nu|\leq s}$ satisfy the inequality $|\varphi_\nu^k(x)| \leq C(s, n)$ and that the polynomials P_k have the form

$$P_k(x) = \sum_{|\nu|\leq s} a_\nu^k\, \varphi_\nu^k(x),$$

where

$$a_\nu^k = \frac{1}{|E_k|} \int M_k\, \varphi_\nu^k\, dx .$$

Thus,

$$\sup_{x\in E_R} |P_k(x)| \leq \frac{C}{|E_k|} \int |M_k|\, dx .$$

From this we obtain the desired result:

$$\left[\frac{1}{|E_k|} \int |M_k - P_k|^q dx\right]^{\frac{1}{q}} \leq \left[\frac{1}{|E_k|} \int |M_k|^q dx\right]^{\frac{1}{q}} + \sup_{x\in E_k} |P_k(x)|$$

$$\leq \left(\frac{1}{|E_k|}\int |M_k|^q dx\right)^{\frac{1}{q}} + \frac{C}{|E_k|}\int |M_k|\, dx$$

$$\leq C\left[\frac{1}{|E_k|}\int |M_k|^q\, dx\right]^{\frac{1}{q}} .$$

This establishes theorems (2.8) and (2.9). We see, therefore, that the spaces $H^{p,q,s}$ are, for p fixed, all the same as long as p and q are related as indicated at the beginning of this section. Moreover, any two (p , q , s)-"norms" (p fixed) are equivalent. In the same way, spaces determined by (p , q , s , ϵ)-molecules (p fixed) are all the same and any two norms are equivalent.

We shall often use the symbol H^p and $\| \ \|_{H^p}$ to denote any of these admissible atomic $H^{p,q,s}$ spaces (or molecular $H^{p,q,s,\epsilon}$ spaces) and associated norms. Similarly a "p-atom" or a "p-molecule" will be names for (p , q , s)-atoms and (p , q , s , ϵ)-molecules when we are not necessarily interested in their dependence on the parameters q , s and ϵ .

§3. A Family of Hardy Spaces Associated with the Disk. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane. For each $\alpha > 0$ put $\omega(z) = \omega_\alpha(z)$ $= \frac{\alpha}{\pi}(1 - |z|^2)^{\alpha-1}$ for $z \in D$. The "weight" function ω gives rise to measure on D , which we also denote by ω , defined for each Borel set $E \subset D$ by

$$\omega(E) = \int_E \omega(z)d\mu(z) ,$$

where μ is two-dimensional Lebesgue measure (the choice of $\frac{\alpha}{\pi}$ is made so that $\omega(D) = 1$.)

Proposition (3.1). D endowed with the measure ω and Euclidean distance as a metric is a space of homogeneous type (as defined in [8]).

In order to show this we must prove that there exists a constant $C = C_\alpha$

such that

$$\omega(B_{z_0}(2\varepsilon)) \leq C\, \omega(B_{z_0}(\varepsilon)) \tag{3.2}$$

whenever $z_0 \in \bar{D}$ and $\varepsilon > 0$, where $B_{z_0}(\delta) = \{z \in D : |z - z_0| < \delta\}$ $= \{z \in \mathbf{C} : |z - z_0| < \delta\} \cap D$ is the ball centered at z_0 of radius δ. This inequality is an easy consequence of the estimate

$$\omega(B_{z_0}(\varepsilon)) \sim \begin{cases} (1 - |z_0|)^{\alpha-1}\varepsilon^2 , & 0 \leq \varepsilon \leq 1 - |z_0| \\ \varepsilon^{\alpha+1} , & 1 - |z_0| \leq \varepsilon \leq 1 + |z_0| \end{cases} \tag{3.3}$$

whenever $z_0 \in \bar{D}$ and $0 \leq \varepsilon \leq 1 + |z_0|$ (the symbol "$\sim$" denotes the fact that the ratios of the quantities on the left to the quantities on the right are bounded below and above by positive constants). The estimate (3.3) will be proved in Appendix A.

Since (D, ω) is a space of homogeneous type one can develop an atomic H^p space theory as is done in [8]. As in the case in $\mathbb{R}^n$, however, there are "natural reasons for considering atoms and molecules having vanishing higher order moments, the number of vanishing moments increasing with $1/p$. In many ways the theory of these atomic spaces on D is similar to the one we considered on $\mathbb{R}^n$. For example, the fact that molecules have an atomic decomposition can be proved by using the same ideas that were exploited for the proof of Theorem (2.9). There are, however, differences creating some technical difficulties, due to the fact that the underlying domain is compact: the "balls" we introduced are either disks or intersections of disks with D. This fact creates some difficulties in the Gram-Schmidt estimates on the analogs of the "rings" E_k. In addition to the regular atoms having an appropriate number, s, of moments vanishing (s depends on both p and α), we must consider atoms that are polynomials of degree not exceeding s. Moreover, some difficulties arise from the fact that certain integral estimates involve the measure ω. In fact, care must be taken since the "moment condition" involves only Lebesgue measure, while the "size condition" is given in terms of ω . The reason why these two different measures occur naturally, in this manner, will be made clearer at the end of this section.

It is useful to keep in mind that the atomic spaces we shall study now are related to the Bergman spaces A_α^p of those holomorphic functions $F(z)$, $z \in D$, satisfying

$$(3.4) \qquad \left\{\frac{\alpha}{\pi}\int_D |F(z)|^p(1-|z|^2)^{\alpha-1}\,d\mu(z)\right\}^{\frac{1}{p}} = \|F\|_{p,\alpha} < \infty .$$

When $\alpha = 1$ we are dealing with the space A_1^p which is sometimes referred to as "solid" $H^p(D\ ,\ dx\,dy)$; in this case $\omega = (1/\pi)\mu$. Letting $\alpha \to 0$ (3.4) reduces to the finiteness of

$$\int_0^{2\pi} |F(e^{i\theta})|^p\, d\theta .$$

Thus the family of spaces A_α^p , $\alpha \geq 0$, can be considered to be a parametrized family of spaces containing the classical Hardy spaces and the solid spaces $H^p(D\ ,\ dx\,dy)$.

Let us now pass to the definition of the atoms associated with the domain $(D\ ,\ \omega)$. Suppose $0 < p \leq 1$, $q > \max\{1\ ,\ \alpha\}$ (if $\alpha = 1$, $0 < p < 1$ we allow $q = 1$) and $s > \max\{[2(\frac{1}{p}-1)]\ ,\ \frac{1+\alpha}{p}-2]\}$ then a function $a(z)$, $z \in D$ is a (<u>regular</u>) $(p\ ,\ q\ ,\ s)$-<u>atom centered at</u> $z_0 \in \bar{D}$ if it is supported in a ball $B_{z_0} \subset D$ and satisfies:

$$(3.5) \qquad \text{(i)} \quad \left\{\frac{1}{\omega(B_{z_0})}\int_{B_{z_0}} |a(z)|^q\, \omega(z)d\mu(z)\right\}^{\frac{1}{q}} \leq [\omega(B_{z_0})]^{-\frac{1}{p}}$$

$$\text{(ii)} \quad \int_{B_{z_0}} a(z)z^\nu d\mu(z) = 0 ,$$

where ν is any ordered pair of non-negative integers $(\nu_1\ ,\ \nu_2)$ such that $0 \leq \nu_1 + \nu_2 = |\nu| \leq s$ and $z^\nu = (x+iy)^\nu \equiv x^{\nu_1} y^{\nu_2}$. Any polynomial of degree not exceeding s (in x and y) that is bounded by 1 will be called an exceptional (p, q, s)-atom. These exceptional atoms obviously span a finite dimensional space and are needed to represent the entire space $H^p(D, \omega)$.

Before introducing the notion of a molecule, let us make a general observation about spaces of homogeneous type. When working with such spaces it is convenient to introduce a "quasi-distance" which produces the same spheres that were obtained from the original distance and, furthermore, satisfies the homogeneity property that a ball of radius $\gamma > 0$ has measure on the order of γ. (Recall that in the definition of a molecule for $H^p(\mathbb{R}^n)$ we used the quasi-distance $\delta(x_0, x) = |x_0 - x|^n$). By letting

$$\delta(z_0, z) = \begin{cases} |z - z_0|^2(1 - |z_0|)^{\alpha-1}, & |z - z_0| \le 1 - |z_0| \\ |z - z_0|^{\alpha+1}, & 1 - |z_0| \le |z - z_0| \le 1 + |z_0| \end{cases}$$

we obtain a function that is equivalent to such a quasi-distance. More precisely,

(3.6) $\qquad \omega(\{z : \delta(z_0, z) \le r\} \cap D) \sim r \quad \text{for} \quad 0 \le r \le (1 + |z_0|)^{\alpha+1}$.

The proof of (3.6) is given in Appendix A; in fact, the "homogeneous type" properties of (D, ω) that we shall need are presented in this appendix. An advantage of using δ is that the ball $\{z : \delta(z_0, z) \le r\} \cap D$ has a boundary that is made up of Euclidean circular arcs. This fact facilitates certain computations.

We can now give the definition of a (regular) molecule: A function $M \in L^q(D, \omega)$ is a (p, q, s, ε)-molecule centered at $z_0 \in \bar{D}$ provided p, q, s satisfy the conditions given immediately preceeding (3.5), $\varepsilon > \max\{\frac{1}{p} - 1, \frac{s}{2}, \frac{s+1-\alpha}{1+\alpha}\}$ and

(3.7)
$$\text{(i)} \quad \|M\|_q^{\frac{a}{b}} \|M(\delta(z_0, z))^b\|^{1-\frac{a}{b}} = \aleph(M) < \infty \ ;$$

$$\text{(ii)} \quad \int_D M(z) z^\nu d\mu(z) = 0 \ ,$$

where ν is as in (3.5)(ii), and, as was the case for (2.2), $a = 1 - \frac{1}{p} + \varepsilon$, $b = 1 - \frac{1}{q} + \varepsilon$.

It follows immediately from the definition of δ and (3.3) that if $B = B_{z_0}(\epsilon)$ then

$$\sup_{z \in B} \delta(z_0, z) \sim \psi(B) . \tag{3.8}$$

We use (3.8) and an argument completely analogous to the one given in the proof of (2.3) to obtain the fact that any (regular) (p, q, s)-atom $a(z)$ is such a (p, q, s, ϵ)-molecule with $\aleph(a) \le C$, where C is independent of the atom.

We shall show that each molecule has an atomic decomposition. As we observed in the $\mathbb{R}^n$-case, in order to do this we must define the atomic Hardy spaces $H^p(D, \psi)$. Again this forces us to introduce spaces $L_{(\beta, q', s)}$ associated with (D, ψ_α). We assume $1 \le q' \le \frac{\alpha_0}{\alpha_0 - 1}$, where $\alpha_0 = \max\{1, \alpha\}$, and $0 \le [2\beta] \le s$. (If $\alpha = 1$, $1 \le q' \le \infty$ is permitted when $0 < p < 1$.) A function g is said to belong to $L(\beta, q', s)$ is and only if

$$\sup_{B \subset D} \psi_\alpha(B)^{-\beta} \left[\frac{1}{\psi_\alpha(B)} \int_B \left| \frac{g(z) - P_B(z)}{\psi_\alpha(z)} \right|^{q'} \psi_\alpha(z) d\mu(z) \right]^{\frac{1}{q'}} < \infty , \tag{3.9}$$

where P_B is the unique polynomial (in x and y) of degree at most x such that $g - P_B$ is orthogonal to z^ν on B, when $|\nu| \le s$. The norm $\|g\|_{L(\beta, q', s)}$ is the sum of the expression (3.9) and

$$\sup_{|\nu| \le s} \left| \int g(z) z^\nu d\mu(z) \right| .$$

Observe that on (D, ψ) the space $L(\beta, q', s)$ is an actual function space (not a space of equivalence classes, as was the situation for the unbounded case on $(\mathbb{R}^n, dx)$.)

We are not in a position to define the Hardy spaces associated with (D, ψ), characterize their duals and study their molecular structure.

The atomic space generated by $(1, q, s)$-atoms is the subspace of $L^1(D, \psi)$ of those functions having the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j , \tag{3.10}$$

where each a_j is a $(1, q, s)$-atom and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. The atomic space generaged by (p, q, s)-atoms is the collection of all continuous linear functionals of the form (3.10), acting on $L(\frac{1}{p} - 1, \infty, [2(\frac{1}{p} - 1)])$, where each a_j is a (p, q, s)-atom and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. If a is a (p, q, s)-atom its action as a linear functional on $L(\frac{1}{p} - 1, \infty, [2(\frac{1}{p} - 1)])$ is given by

$$\int_D g(z)a(z)d\mu(z) .$$

An argument similar to the one used in establishing (2.6) shows that the linear functionals of the form (3.10) are well defined. Two ingredients are missing: 1) The proof that the integral is well defined is more technical (see the argument before (3.21)). 2) The embedding of $L(\frac{1}{p} - 1, \infty, [2(\frac{1}{p} - 1)])$ into $L(\frac{1}{p} - 1, q', s)$ is not as easy as it was without weights. The analog of (8.2) of Appendix D can be found in Cuerva [11]. The "norm" of f is the $\inf(\sum_{j=1}^{\infty} |\lambda_j|^p)^{\frac{1}{p}}$ over all representations (3.10); this definition applies to all cases, $0 < p \leq 1$ (of course, even in the case $p = 1$ we could have defined the Hardy space as a space of linear functionals on $L(0, \infty, 0)$). As we did in §2 we denote the spaces $H^{p,q,s} \equiv H^{p,q,s}(D, \omega)$ and the norms, $\| \ \|_{H^{p,q,s}}$.

The results corresponding to theorems (2.7) and (2.8) are valid for (D, ω):

<u>Theorem (3.11).</u> <u>The dual of</u> $H^{p,q,s}(D, \omega)$ <u>is naturally isomorphic to</u> $L(\frac{1}{p} - 1, q', s)$.

<u>Theorem (3.12).</u> <u>If</u> p, q <u>and</u> s <u>are admissible indices for a</u> (p, q, s)-<u>atom then</u>

$$H^{p,q,s}(D, \omega) = H^{p,\infty,[2(\frac{1}{p}-1)]}(D, \omega) .$$

<u>Moreover, the norms associated with these two spaces are equivalent.</u>

The proof of (3.11) follows the line of the argument given for Theorem B in

[8]; the technical changes forced on us by the weight ω are the same as those encountered by Cuerva in [11]. Again, for s fixed, the equivalence of $H^{p,q,s}$ and $H^{p,\infty,s}$ can be established by reasoning that is similar to that in the proof of Theorem A in [8] (see the comments following (2.8)). We shall discuss the situation occurring when s varies toward the end of this section. It will be apparent there that the exceptional atoms are, indeed, necessary.

Corollary (3.13). *If* $1 \le q' < \frac{\alpha_0}{\alpha_0 - 1}$ *and* $s \ge [2(\frac{1}{p} - 1)]$ *then the spaces* $L(\frac{1}{p} - 1 , q' , s)$ *and* $L(\frac{1}{p} - 1 , 1 , [2(\frac{1}{p} - 1)])$ *are equivalent.*

Theorem (3.14). *If* M *is a* (p , q , s , ϵ)-*molecule then* $M \in H^{p,q,s}$ *and*

$$\|M\|_{H^{p,q,s}} \le C \ \aleph(M) ,$$

where C *is independent of the molecule* M.

We shall now prove (3.14). The basic ideas used in the proof of (2.9) to obtain the atomic decomposition of the molecule M on $\mathbb{R}^n$ are applicable to (D , ω). The boundedness of D and the weight ω, however, create certain differences and, for this reason, we shall present some of the details of the argument.

Let us fix $\alpha > 0$ and suppose that M is a (p , q , s , ϵ)-molecule centered at $z_0 \in \bar{D}$ with $\aleph(M) = 1$. Let $\sigma = \|M\|_q^{1/(a-b)}$ where $a = 1 - \frac{1}{p} + \epsilon$, $b = 1 - \frac{1}{q} + \epsilon$. Thus, by (3.7)(i) we have

$$\|M(z)[\delta(z_0 , z)]^b\|_q = \sigma^a . \tag{3.15}$$

If $\sigma \ge \frac{1}{2}$ then, clearly, $M(z) = 2^{\frac{1}{p} - \frac{1}{q}} a(z)$, where $a(z)$ is a (p , q , s)-atom. Thus, we can assume $0 < \sigma < \frac{1}{2}$. We construct a dequence of balls $\{B_k\}_{k=0}^n$, $n = [\log_2 \frac{1}{\sigma}] \ge 1$, such that $B_k = B_{z_0}(\rho_k)$ satisfies $\omega(B_k) = 2^k \sigma$ for $0 \le k < n$ and $\omega(B_n) = 1$. (Observe that this implies that $B_n = D$ and, also, $\frac{1}{2} < 2^n \sigma \le 1$.

The fact that $\omega(B_{n-1}) \leq \frac{1}{2}$ follows from this.)

It follows easuly from (3.3) that $\rho_k \sim \{2^k\sigma(1 - |z_0|^{1-\alpha}\}^{\frac{1}{2}}$ if $\rho_k \leq 1 - |z_0|$, while $\rho_k \sim (2^k\sigma)^{\frac{1}{1+\alpha}}$ if $1 - |z_0| \leq \rho_k \leq 1 + |z_0|$. We thus have $\rho_{k-1}/\rho_k \sim 1$. Observe that $\rho_n = 1 + |z_0|$.

We now put $E_0 = B_0$ and $E_k = B_k - B_{k-1}$ for $1 \leq k \leq n$. Let $M_k = M\chi_{E_k}$ and denote by Q_k the unique polynomial in x and y (of degree at most s) satisfying

$$\int_{E_k} (M_k - P_k)z^\nu \, d\mu(z) = 0$$

for $|\nu| \leq s$, where $P_k = Q_k\chi_{E_k}$.

Let $\{\varphi_\nu^k\}$ be the Gram-Schmidt orthonormalization, with respect to the measure $d\mu(z)/\mu(E_k)$, of the functions $\{((z - z_o)e^{-i\theta})^\nu\}$, $|\nu| \leq s$, where the ordered pairs ν are taken in some fixed order, $\theta = \arg z_0$ and all these functions are defined on D but are supported on E_k.

Let $\{\psi_\nu^k\}_{|\nu|\leq s}$ denote the unique set of polynomials in z^ν, $|\nu| \leq s$, restricted to E_k satisfying

$$\frac{1}{\mu(E_k)} \int_{E_k} \psi_\lambda^k(z)((z - z_o)e^{-i\theta})^\nu \, d\mu(z) = \delta_{\lambda,\mu} .$$

In Appendix B we shall obtain the following estimates:

Lemma (3.16). $\sup_{z\in E_k} |\varphi_\nu^k(z)| \leq C$, $\sup_{z\in E_k} |\psi_\nu^k(z)| \leq C\,\rho_k^{-|\nu|}$, *where* C *depends only on* α *and* s.

Corollary (3.17). *If* $q > \max\{1, \alpha\}$ *then*

$$\left[\frac{1}{\omega_\alpha(B_k)} \int_{B_k} |P_k(z)|^q \, \omega_\alpha(z)d\mu(z)\right]^{\frac{1}{q}} \leq C \left[\frac{1}{\omega_\alpha(B_k)} \int_{B_k} |M_k(z)|^q \, \omega_\alpha(z)d\mu(z)\right]^{\frac{1}{q}} ,$$

where C *depends only on* α, q *and* s (*the inequality also holds for* $q = 1$ *when* $\alpha = 1$).

Proof. Clearly

$$\left[\frac{1}{w_\alpha(B_k)}\int_{B_k}|P_k(z)|^q\, w_\alpha(z)d\mu(z)\right]^{\frac{1}{q}} \le \sup_{z\in E_k}|P_k(z)| .$$

It is easy to see that $P_k(z) = \sum_{|\nu|\le s} m_\nu^k\, \psi_\nu^k(z)$ where

$$m_\nu^k = \frac{1}{\mu(E_k)}\int_{E_k} M(z)((z - z_0)e^{-i\theta})^\nu\, d\mu(z) .$$

Thus, by Lemma (3.16) and the fact that $|((z - z_0)e^{-i\theta})^\nu| \le C\,\rho_k^{|\nu|}$ on E_k we have

$$\sup_{z\in E_k}|P_k(z)| \le C\left[\frac{1}{\mu(E_k)}\int_{B_k}|M_k(z)|d\mu(z)\right] .$$

From (5.9) in Appendix A we obtain that there is a $B_\alpha < 1$, independent of σ, such that $\rho_{k-1}/\rho_k \le B_\alpha$ for $k = 0, 1, \ldots, n$. It follows that the last expression is dominated by

$$\frac{C}{\mu(B_k)}\int_{B_k}|M_k(z)|d\mu(z) .$$

Finally, it follows from (5.7) and (5.8), in Appendix A that this last quantity does not exceed

$$\left[\frac{1}{w_\alpha(B_k)}\int_{B_k}|M_k(z)|^q\, w_\alpha(z)d\mu(z)\right]^{\frac{1}{q}}$$

and Corollary (3.17) is proved.

We now proceed, as in the proof of Theorem (2.9), to show that $M = \sum_{k=0}^{n} M_k = \sum_{k=0}^{n}(M_k - P_k) + \sum_{k=0}^{n} P_k$ has an appropriate atomic decomposition. First we shall show that

$$(3.18) \qquad M_k - P_k = C\, 2^{-ka}\, a_k$$

where a_k is a (p, q, s)-atom. In order to do this (in view of (3.17)) we prove

$$(3.19)\ \left[\frac{1}{w(B_k)}\int_{B_k}|M_k(z)|^q\, w(z)d\mu(z)\right]^{\frac{1}{q}} \le C\, 2^{-k(1+\varepsilon)}\sigma^{-\frac{1}{p}} \sim 2^{-ka}[w(B_k)]^{-\frac{1}{p}} .$$

If $k = 0$ then

$$\left[\frac{1}{\omega(B_0)} \int_{B_0} |M_0(z)|^q \, \omega(z)d\mu(z)\right]^{\frac{1}{q}} \leq \|M\|_q [\omega(B_0)]^{-\frac{1}{q}}$$

$$= [\omega(B_0)]^{\frac{1}{q}-\frac{1}{p}} [\omega(B_0)]^{-\frac{1}{q}} = [\omega(B_0)]^{-\frac{1}{p}} .$$

If $k \geq 1$ we make use of (3.8) and (3.15),

$$\left[\frac{1}{\omega(B_k)} \int_{B_k} |M_k(z)|^q \, \omega(z)d\mu(z)\right]^{\frac{1}{q}}$$

$$\leq \left[\frac{1}{\omega(E_k)} \int_{E_k} |M_k(z)|^q [\delta(z_0 , z)]^{bq} (\chi_{E_k}(z)[\delta(z_0 , z)]^{-bq})\omega(z)d\mu(z)\right]^{\frac{1}{q}}$$

$$\leq \frac{C}{(2^k\sigma)^{\frac{1}{q}}} \frac{\|M\delta(z_0,\cdot)^b\|_q}{(\min_{z\in E_k} \delta(z_0,z))^b} \leq \frac{C\sigma^\alpha}{(2^k\sigma)^{\frac{1}{q}}(2^{k-1}\sigma)^b}$$

$$\sim \begin{cases} \sigma^{-\frac{1}{p}} 2^{-k(1+\varepsilon)} \\ [\omega(B_k)]^{-\frac{1}{p}} 2^{-ka} \end{cases} .$$

This proves (3.19) and (3.18) follows immediately. Thus,

$$M = \sum_{k=0}^{n} \lambda_k a_k + \sum_{k=0}^{n} P_k ,$$

where the a_k 's are (p , q , s)-atoms and $|\lambda_k| \leq C\, 2^{-ka}$.

As we did in §2 we shall show that $\sum_{k=0}^{n} P_k$ can be represented as a sum of (p , ∞ , s)-atoms. We have $\sum_{k=0}^{n} P_k = \sum_{|\nu|\leq s} \sum_{k=0}^{n} m_\nu^k \psi_\nu^k$, where

$$m_\nu^k = \int_{E_k} M(z)((z - z_0)e^{-i\theta})^\nu \frac{d\mu(z)}{\mu(E_k)} .$$

From (3.7)(ii) we know that

$$\sum_{k=0}^{n} m_\nu^k \mu(E_k) = 0 , \quad |\nu| \leq s .$$

Let $N_\nu^k = \sum_{j=k}^{n} m_\nu^j \mu(E_j)$ (Note that $N_\nu^0 = 0$, $|\nu| \leq s$) . Thus,

$$\sum_{k=0}^{n} P_k = \sum_{|\nu|\leq s} \sum_{k=1}^{n} N_\nu^k \left\{ \frac{\psi_\nu^k}{\mu(E_k)} - \frac{\psi_\nu^{k-1}}{\mu(E_{k-1})} \right\}$$

$$\equiv \sum_{|\nu|\leq s} \sum_{k=1}^{n} f_{\nu k} \,.$$

Using Lemma (3.16), (3.19), (5.7), (5.8) and the defining properties of B_k , E_k and σ we obtain the estimates

$$\sup_{z\in E_k} \frac{|\psi_\nu^k(z)|}{\mu(E_k)} \leq \frac{C}{\rho_k^{(|\nu|+2)}} \,, \text{ and}$$

$$|m_\nu^k| \leq \rho_k^{|\nu|} \left\{ \frac{1}{\mu(E_k)} \int_{B_k} |M_k(z) d\mu(z) \right\}$$

$$\leq C \quad \rho_k^{|\nu|} \left\{ \frac{1}{\omega(B_k)} \int_{B_k} |M_k(z)|^q \, \omega(z) d\mu(z) \right\}^{\frac{1}{q}}$$

$$\leq C \quad \rho_k^{|\nu|} \, 2^{-k(1+\epsilon)} \, \sigma^{-\frac{1}{p}} \,.$$

It follows from the second of these inequalities and Proposition (7.1) of Appendix C that

$$|N_\nu^k| \leq \frac{C}{\sigma^{1/p}} \sum_{j=k}^{n} \frac{\rho^{|\nu|+2}}{2^{j(1+\epsilon)}} \leq \frac{C}{\sigma^{1/p}} \frac{\rho_k^{(|\nu|+2)}}{2^{k(1+\epsilon)}} \,.$$

Thus,

$$|f_{\nu k}(z)| \leq \frac{C}{\sigma^{1/p}} \frac{\rho_k^{(|\nu|+2)}}{2^{k(1+\epsilon)}} \quad \frac{1}{\rho_k^{(|\nu|+2)}} = \frac{C}{(2^k\sigma)^{1/p}} \, 2^{-ka}$$

$$\leq C \quad 2^{-ka} [\omega(B_k)]^{-\frac{1}{p}} \,.$$

Observe that $f_{\nu k}$ is supported on B_k and using the defining property of the functions ψ_ν^k , we have

$$\int_D f_{\nu k}(z) \, z^\ell \, d\mu(z) = 0 \,, \; |\ell| \,, \; |\nu| \leq s \,.$$

This shows that $f_{\nu k} = \mu_{\nu k} \, b_{\nu k}$ where $b_{\nu k}$ is a $(p\,,\,\infty\,,\,s)$-atom and $|\mu_{\nu k}| \leq C2^{-ka}$.

Thus,

$$M = \sum_{k=0}^{n} \lambda_k a_k + \sum_{|\nu|\le s} \sum_{k=1}^{n} \mu_{\nu k} b_{\nu k} . \tag{3.20}$$

We have shown that M is a finite linear combination of (p , q , s)-atoms (since (p , ∞ , s)-atoms are, clearly (p , q , s)-atoms) and since $a = 1 + \varepsilon - \frac{1}{p} > 0$ we see that

$$\sum_{k=0} |\lambda_k|^p + \sum_{|\nu|\le s} \sum_{k=1}^{n} |\mu_{\nu k}|^p \le C ,$$

where C depends only on p, ε and s. It only remains to show that both sides of (3.20) generate the same linear functionals on $g \in L(\frac{1}{p} - 1 , \infty , [2(\frac{1}{p} - 1)])$. Since $M , a_k , b_{\nu k} \in L^q(D , \omega)$ and the sum in (3.20) is finite it suffices to show that $(g/\omega) \in L^{q'}(D , \omega)$. Thus, if $f \in L^q(D , \omega)$ we obtain

$$\int_D |f(z)g(z)|d\mu(z) = \int_D |f(z)(g(z)/\omega(z))(\omega(z)d\mu(z) \le \|M\|_q \|(g/\mu)\|_{q'} .$$

From the definition of $L(\frac{1}{p} - 1 , \infty , [2(\frac{1}{p} - 1)])$ we have that there is a constant $A \ge 0$ and a polynomial P_D such that

$$\sup_{z\in D} \left|\frac{g(z)}{\omega(z)} - \frac{P_D(z)}{\omega(z)}\right| \le A .$$

Thus, $|g(z)/\omega(z)| \le A + |P_D(z)/\omega(z)|$. There are two cases to consider. If $\alpha = 1$ then $q' = \infty$, $\omega(z) \equiv 1/\pi$ and we see, easily, that g is bounded since P_D is bounded. If $\alpha \ne 1$ then from (5.8) we can use (5.6) if $q > \max\{1 , \alpha\}$ and (5.6) applied to $B = D$ asserts that

$$\int_D \omega(z)^{1-q'} d\mu(z) \le C \tag{3.21}$$

since $(1 - q)(1 - q') = 1$. Since P_D is bounded and $\omega \in L^1(D , \mu)$ we see that $g/\omega \in L^{q'}(D , \omega)$. This completes the proof of (3.14)

Let us now turn to a sketch of the proof of Theorem (3.12): we must show that a (p , q_2 , s_2)-atom decomposes into (p , q_1 , s_1)-atoms. If $s_1 = s_2$, but $q_1 \ne q_2$ we can use the fact that $g/\omega \in L^{q_j'}(D , \omega)$ (as we just did) in order to make the obvious adaptations (subtract polynomials, not constants) of the argument found in [8]. Certain special cases are obvious if $q_1 = q_2$, but $s_1 \neq s_2$: if $s_1 < s_2$ and $q_1 = q_2 = q$ then an exceptional s_1-atom is an exceptional s_2-atom and a regular s_2-atom is a regular s_1-atom.

Suppose now that a is an exceptional s_2-atom. We write $a = b_1 + b_2$, where b_1 consists of the terms of a of order not exceeding s_1. Clearly $b_1 = P_0 a$, so, as before,

$$\sup_{z \in D} |b_1(z)| \leq C \int_D |a(z)|\, d\mu(z) \leq C \quad ;$$

thus, $\sup_{z \in D} |b_2(z)| \leq 1 + C$. If follows that $b_1 = C\, a_1$ where a_1 is an exceptional s_1-atom and $b_2 = a - P_D a = (1 + C) a_2$ where a_2 is a regular (p, ∞, s_1)-atom.

Finally, suppose a is a regular (p, q, s_1)-atom supported in the ball B centered at z_0. Let $\sigma = \omega(B)$. If $\sigma \geq 1/2$, then $a = (a - P_D a) + P_D a$ is the required decomposition (here, $P_D a$ is the unique polynomial of degree at most s_2 such that

$$\int_D (a - P_D a) z^\nu \, d\mu(z) = 0 \ ,$$

for $|\nu| \leq s_2$). If $\sigma < 1/2$ we construct, as in the proof of (3.14), a sequence of balls $\{B_k\}_{k=0}^n$ such that $\omega(B_k) = 2^k \sigma$, $0 \leq k < n$, $\omega(B_n) = 1$, $1/4 < \omega(B_{n-1}) \leq 1/2$, $B_k = B_{z_0}(\rho_k)$. Observe that $B_0 = B$ and $B_n = D$. Now let $\{\psi_\nu^k\}$, $|\nu| \leq s_2$, be the "dual basis" of $\{((z - z_0)e^{-i\vartheta})^\nu\}$, $|\nu| \leq s_2$, on B_k with respect to the measure $\mu/\mu(B_k)$. With this setup one can reproduce the argument that gave us the decomposition (2.15). The only differences are that the exceptional atoms appear naturally in the decomposition of a on D as ultimate terms in the finite sum and that the estimates on the coefficients involve the radii ρ_k. More precisely, we need to check that if $s_1 < |\nu| \leq s_2$

$$\sum_{k=1}^{n} (\rho_0/\rho_k)^{p(|\nu|+2)}\, 2^k \leq C \ , \tag{3.22}$$

where C depends on α, p and s_2 but does not depend on z_0 nor σ. This inequality is established in Appendix C .

We mentioned in the introduction that there is a connection between the Bergman spaces associated with D and the atomic spaces we have just studied. We will discuss this connection in the light of a similar connection for the Bergman spaces and the atomic Hardy spaces on the upper half-plane $\mathbb{R}^2_+ = \{z = x + iy \in \mathbb{C} : y > 0\}$. In many ways the Hardy spaces with $\mathbb{R}^2_+$ are easier to study than the ones we have just studied and certain notions are more natural in this unbounded case.

The unit disk D and the upper half-plane $\mathbb{R}^2_+$ are particular examples of Siegel domains of type II. For such spaces there is a Bergman kernel $B_0(z , \zeta)$ that is analytic in z, anti-analytic in ζ, conjugate symmetric in the arguments (z , ζ) and is a reproducing kernel for holomorphic functions in $L^2(D , d\mu)$, or respectively $L^2(\mathbb{R}^2_+ , d\mu)$. For D, $B_0(z , \zeta) = C(1 - z\bar{\zeta})^{-2}$; for $\mathbb{R}^2_+$, $B_0(z , \zeta) = C(z - \bar{\zeta})^{-2}$.

Let us note that, if we let $\omega_\alpha(z) = [B_0(z , z)]^{\frac{1-\alpha}{2}}$, $\alpha > 0$, we have $\omega_\alpha(z) = C(1 - |z|^2)^{\alpha-1}$ for D and $\omega_\alpha(z) = -4c\, y^{\alpha-1}$ for $\mathbb{R}^2_+$. The corresponding Bergman kernel for the weighted space $L^1(D , \omega_\alpha d\mu)$ (respectively $L^2(\mathbb{R}^2_+ , \omega_\alpha d\mu)$ is $B_r(z , \zeta) = [B_0(z , \zeta)]^{r+1}$ where $r = (\alpha - 1)/2$.

We will now describe in sone detail the (technically) easiest case: $\mathbb{R}^2_+$ with $\alpha = 1$ (this is the unweighted case). Suppose $p \le 1 \le q$, $p < q$, $\varepsilon > \frac{1}{p} - 1$, $r > \varepsilon > s/2$, where s is a non-negative integer such that $s \ge [2(\frac{1}{p} - 1)]$. If $\zeta \in \mathbb{R}^2_+$ then, as a function of z,

$$M_\zeta(z) = \frac{[B_0(\zeta,\zeta)]^{\frac{1}{p}}}{B_r(\zeta,\bar{\zeta})} B_r(z , \zeta)$$

is a $(p , q , s , \varepsilon)$-molecule on $\mathbb{R}^2_+$ centered at ζ with a molecular norm that is uniformly bounded in $\zeta \in \mathbb{R}^2_+$. What is meant by a "molecule on $\mathbb{R}^2_+$" is of course, that it meets the size and moment condition (2.2) with $n = 2$, except that M_ζ is supported on $\mathbb{R}^2_+$. What is meant by "uniformly bounded molecular norm" is that $\aleph(M_\zeta)$ is uniformly bounded in $\zeta \in \mathbb{R}^2_+$.

Coifman and Rochberg [7] consider the space $\mathfrak{a}_p$, $0 < p$, of holomorphic functions F on $\mathbb{R}^2_+$ such that

$$\mathfrak{A}_p(F) = \left[\int_{\mathbb{R}^2_+} |F(z)|^p \, d\mu(z) \right]^{\frac{1}{p}} < \infty .$$

They show that there exists a fixed sequence of $\{\zeta_k\} \subset \mathbb{R}^2_+$ such that $F \in \mathfrak{A}_p$ if and only if there is a sequence $\{c_k\}$ of complex numbers for which

$$F(z) = \sum_{k=1}^{\infty} c_k M_{\zeta_k}(z)$$

(the convergence can be taken in the space of continuous linear functionals on an appropriate space of smooth functions - or, equivalently, uniform convergence on compact sets) and $\mathfrak{A}_p(F)$ is equivalent to the infimum of all expressions $\left[\sum_{k=1}^{\infty} |c_k|^p\right]^{\frac{1}{p}}$ corresponding to the representations (2.23) of F.

This shows that $\mathfrak{A}_p$ consists of holomorphic functions contained in atomic $H^p(\mathbb{R}^2_+)$ where the atomic-H^p space is defined in a manner that is completely analogous to the one we gave for $H^p(D, d\mu)$ in this section $(0 < p \leq 1)$. Consequently, for $F \in \mathfrak{A}_p$, F has an atomic decomposition in terms of p-atoms, supported on $\mathbb{R}^2_+$, which, a fortiori are p-atoms on $\mathbb{R}^2$. An interesting consequence of this fact is that the function $\tilde{F}$ defined by

$$\tilde{F}(z) = \begin{cases} F(z) & \text{for } z \in \mathbb{R}^2_+ \\ 0 & \text{for } z \in \mathbb{C} - \mathbb{R}^2_+ \end{cases}$$

belongs to the atomic space $H^p(\mathbb{R}^2)$ which (as follows from Latter [13]) is also the maximal H^p space.

An interesting consequence of this last observation is that a locally integrable function, f, in atomic H^p is in $L^p(\mathbb{R}^2_+)$ (i.e., $\mathfrak{A}_p(f) < \infty$). It follows that $\mathfrak{A}_p$ consists, precisely, of the holomorphic functions in H^p.

The Bergman theory for D is quite similar. The main difference is that the functions $M_\zeta(z)$ no longer satisfy the moment condition and we need to introduce an exceptional term which is a polynomial in z. For p, q, s, ε and γ related as above, Coifman and Rochberg show that there is a fixed sequence $\{\zeta_k\}$ of complex numbers in D such that F is a holomorphic function with

$$\mathfrak{A}_p(F) = \left[\int_D |F(z)|^p \, d\mu(z)\right]^{\frac{1}{p}} < \infty$$

(i.e., $F \in \mathfrak{A}_p(D)$) iff

$$F(z) = c_0\, P(z) + \sum_{k=1}^{\infty} c_k\, z^s\, M_{\zeta_k}(z) ,$$

where $P(z)$ is a polynomial in z (of degree at most s) that is bounded by 1 on D and $\left[\sum_{k=0}^{\infty} |c_k|^p\right]^{\frac{1}{p}} < \infty$. It turns out that the M_ζ are (p , q , s , ϵ)-molecules for $H^p(D , d\mu)$ and it can be concluded that $\mathcal{A}_p$ is, exactly, the holomorphic part of H^p.

When $\alpha \neq 1$ $(\alpha > 0)$ then there is an entirely analogous theory of weighted Bergman spaces on $\mathbb{R}^2_+$ and D, and corresponding atomic Hardy spaces $H^p(\mathbb{R}^2_+ , \omega_\alpha\, d\mu)$ and $H^p(D , \omega_\alpha\, d\mu)$, together with molecular characterizations of the atomic spaces. In this section we developed the theory for the atomic spaces on D. In generaly, the situation for $\mathbb{R}^2_+$ is much simpler than it is for D. One never has to deal with exceptional atoms and the "balls" can be taken to be rectangles (squares if the center is far enough from the boundary, $y = 0$) with their sides parallel to the coordinate axes, so the geometry is almost trivial and the weighted measure, $\omega_\alpha(z)d\mu(z) = \alpha y^{\alpha-1}\, dx\, dy$, is easily computed on such "balls." Thus, the results analogous to those in appendices A, B and C are, relatively speaking, obtained with ease.

This connection between the Bergman spaces and the Hardy spaces explains the moment condition that we imposed, where the moments were taken with respect to Lebesgue measure $d\mu$. Thus we require that if $X = D$ or $\mathbb{R}^2_+$ that

(3.24) $$\int_X M(z)\, x^{\nu_1}\, y^{\nu_2}\, d\mu(z) = 0 , \quad \nu_1 + \nu_2 \leq s .$$

The molecules $M_\zeta(z)$ that occur in the Bergman theory satisfy this condition. But one could just as well have required that, alternatively,

(3.25) $$\int_X M(z)\, x^{\nu_1}\, y^{\nu_2}\, \omega_\alpha(z)d\mu(z) = 0 , \quad \nu_1 + \nu_2 \leq s ,$$

for if M is holomorphic (3.24) and (3.25) are equivalent. With this moment condition for atoms and molecules we could develop another collection of atomic and molecular Hardy spaces and the corresponding weighted Bergman space is in the intersection of both.

In terms of technical details of the proofs: in this second version, with the weight "inside" (as in (3.25)) there is no need to establish the weighted norm inequalities and many of the technical calculations are simplified. For example, we can dispense with (5.9). On the other hand, the Gram-Schmidt estimates in Appendix B are now more delicate since we must account for a changing measure associated with each domain; one that changes continuously with the domain. Given the choices: D or $\mathbb{R}^2_+$, weight "inside" the moment integral or no weight ((3.24) vs. (3.25)) the example we develop in this section is the technically most difficult. Details for the other cases can be left to the reader.

If the reader does carry out the details he will note that the condition on s (for atoms and molecules) is: $s > \max\{[2(\frac{1}{p} - 1)]\ ,\ [(1 + \alpha)(\frac{1}{p} - 1)]\}$; and on ε (for molecules) is: $\varepsilon > \max\{\frac{1}{p} - 1\ ,\ \frac{s}{2}\ ,\ \frac{s}{1+\alpha}\}$.

Perhaps even more interesting than this theory for holomorphic functions is the fact that Coifman and Rochberg have developed an analogous theory for harmonic functions on the $(n + 1)$-dimensional space $\mathbb{R}^{n+1}_+ = \{(x\ ,\ y) \in \mathbb{R}^{n+1}\ :\ x \in \mathbb{R}^n\ ,\ y > 0\}$. This theory is based on two facts: the first is the reproducing property of the derivatives $P^{(k)}(x\ ,\ y) = \frac{\partial^k}{\partial y^k} P(x\ ,\ y)$ of the Poisson kernel, $P(x\ ,\ y) = (1/C_n)(y/(|x|^2 + y^2)^{\frac{n+1}{2}})$. That is,

$$h(x\ ,\ y) = \frac{(-2)^k}{\Gamma(k)} \int_{\mathbb{R}^n} \int_0^\infty h(\xi\ ,\ \eta) P^{(k)}(x - \xi\ ,\ y + \eta)\eta^{k-1}\, d\eta\, d\xi$$

for appropriate harmonic functions on $\mathbb{R}^{n+1}_+$ (this is just the usual Poisson integral representation modified by integration -by-parts). The second fact is that a multiple of $P^{(k)}$ is a molecule; namely,

$$M^{(k)}_{(x,y)}(\xi\ ,\ \eta) = C_k \frac{y^{k-1}}{y^{(n+1)(\frac{1}{p}-1)}} \frac{\partial}{\partial y^k} P(x - \xi\ ,\ y + \eta) \quad \text{is a} \tag{3.26}$$

$(p\ ,\ s\ ,\ q\ ,\ \varepsilon)$-molecule on $\mathbb{R}^{n+1}_+$ centered at $(x\ ,\ y)$ (or $(x\ ,\ 0)$) of uniformly bounded molecular norm, where $(p\ ,\ q\ ,\ s\ ,\ \varepsilon)$ are related as before (for $\mathbb{R}^{n+1}$!) provided $k - 1 > (n + 1)\varepsilon$. Coifman and Rochberg show that if a harmonic function satisfies

$$\int_{\mathbb{R}_+^{n+1}} |h(x , y)|^p \, dx \, dy > \infty$$

it then has a molecular representation similar to (3.23) in terms of a fixed sequence of such molecules (3.26).

§4. Convolution and Multiplier Transforms. It is a consequence of the molecular characterization of H^p that if T is a linear map, then to show that T is bounded from H^{p_0} to H^{p_1} it is sufficient to show that whenever a is a p_0-atom then Ta is a p_1-molecule and $\aleph(Ta) \leq C$ for some constant C. Even in those cases where Ta is always a multiple of an atom (say, $Ta = k * a$, k bounded with compact support) one cannot expect to gain much information from this fact since the support of a is "smeared about". It was such observations that led Coifman and Weiss [8] to consider molecules. (See their theorems (1.29) and (1.30).)

We will illustrate this approach to the estimation of operators on H^p spaces with several results. On the one hand we will exploit the smoothness of the kernels of the fractional integration operators and obtain a rather elementary proof that these operators "act the way they should" on the Hardy spaces. (See [8] Theorem (1.35) for a model of this argument in the "atomic theory".) On the other hand we will exploit the Plancherel relations to show that if a is a $(p , 2 , s)$-atom for s large enough and m satisfies the expected Hörmander condition then $m\,\hat{a}$ is the Fourier transform of a $(p , 2 , [n(\frac{1}{p} - 1)] , \epsilon)$-molecule for a suitable $\epsilon > 0$, with a bound on $\aleph((m\,\hat{a})^{\vee})$. An elementary version of this argument is found in Theorem 1.29 if [8].

In both of these situations the results are not new in any essential way (although there are certain technical improvements in the formulations); but are meant as a vehicle to introduce an approach to the study of operators on H^p spaces. (See Stein and Weiss [17] for the first result and A.P. Calderòn and Torchinsky [3] for the second.) It is an approach that is conceptually quite simple and straightforward and will be applicable whenever there is a corresponding atomic and molecular structure available. To keep the exposition simple, all results in this section will be for $\mathbb{R}^n$.

Sobolev Theorems. Let us define the Riesz potential operators in the usual way. Thus, $(I^\alpha f)^\wedge = |x|^{-\alpha} \hat{f}$ and equivalently, for $0 < \alpha < n$ and f "nice enough";

$$I^\alpha f(x) = \gamma_\alpha \int \frac{f(y)}{|x-y|^{n-\alpha}} dy ,$$

where γ_α is an appropriate constant. For $1 < p < \infty$ let $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Theorem (4.1). *Suppose* $0 < p_1 < \infty$, $0 \le \alpha < n/p_1$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, *then* I^α *maps* $H^{p_1}(\mathbb{R}^n)$ *continuously into* $H^{p_2}(\mathbb{R}^n)$. *If we replace* $H^\infty(\mathbb{R}^n)$ *with* BMO *the result holds for* $\alpha = \frac{n}{p_2}$.

Proof. The result will follow from the repeated application four cases below:

I. $1 < p_1 < p_2 < \infty$. Well known result of Sobolev.

II. $p_1 \le 1 < p_2$, $0 < \alpha < n$. Choose $s + 1 > n(\frac{1}{p_1} - 1)$ and $1 < q_1 < q_2 < \infty$ so that $\frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{q_1} - \frac{1}{q_2} = \frac{\alpha}{n}$. If a is a (p_1, q_1, s)-atom we show that $\|I^\alpha a\|_{p_2} \le C_{\alpha, p_1, p_2}$.

III. $p_1 < p_2 \le 1$, $0 < \alpha < 1$. Choose s, q_1 and q_2 as in II. If a is a (p_1, q_1, s)-atom we show that I^α_a is a $(p_2, q_2, [n(\frac{1}{p_2} - 1)], \epsilon)$-molecule for $0 < \epsilon - \frac{s}{n} < (1 - \alpha)/n$ and $\aleph(I^\alpha_a) \le C_{\alpha, p_1, p_2}$.

IV. $p > 1$, $\alpha = \frac{n}{p}$. We need to show that $I^{\frac{n}{p}} : L^p \to$ BMO continuously. But $\text{BMO} = (H^1)^*$ and from I and II we have that $I^{\frac{n}{p}} : H^1 \to L^{p'}$ $(\frac{1}{p'} + \frac{1}{p} = 1)$, so the result follows by duality.

We give the details for II and III.

Case II. Let Q be the support of a (Q is a ball), Q^* its double and we assume that Q is centered at the origin.

$$\|I^{\alpha}a\|_{p_2} \le \left[\int_{k\in Q^*} |I^{\alpha}a(x)|^{p_2}\,dx\right]^{\frac{1}{p_2}} + \left[\int_{x\notin Q^*} |I^{\alpha}a(x)|^{p_2}\right]^{\frac{1}{p_2}} = I_1 + I_2 .$$

From $\frac{1}{p_2} = \frac{1}{q_2} + (\frac{1}{p_1} - \frac{1}{q_1})$ we have $I_1 \le C\|I^{\alpha}a\|_{q_2}|Q|^{\frac{1}{p_1}-\frac{1}{q_1}} \le C_{\alpha}\|a\|_{q_1}|Q|^{\frac{1}{p_1}-\frac{1}{q_1}} \le C_{\alpha}$.

$$I_2 = |\gamma_{\alpha}|\left[\int_{x\notin Q^*}\left|\int_{y\in Q}\frac{a(y)}{|x-y|^{n-\alpha}}\,dy\right|^{p_2}dx\right]^{\frac{1}{p_2}}$$

$$\le C_{\alpha}\left[\int_{x\notin Q^*}\left[\int_{y\in Q}\frac{|a(y)|\,|y|^{s+1}}{|x|^{n-\alpha+s+1}}\,dy\right]^{p_2}dx\right]^{\frac{1}{p_2}}$$

$$\le C_{\alpha}\|a\|_{q_1}|Q|^{\frac{s+1}{n}+1-\frac{1}{q_1}}\left[\int_{x\notin Q^*}\frac{dx}{|x|^{(n-\alpha+s+1)p_2}}\right]^{\frac{1}{p_2}}$$

$$\le C_{\alpha}\|a\|_{q_1}|Q|^{\frac{s+1}{n}+1-\frac{1}{q_1}}|Q|^{-1+\frac{\alpha}{n}-\frac{s+1}{n}+\frac{1}{p_2}}$$

$$= C_{\alpha}\|a\|_{q_1}|Q|^{\frac{1}{p_2}-\frac{1}{q_1}+\frac{\alpha}{n}} = C_{\alpha}\|a\|_{q_1}|Q|^{\frac{1}{p_1}-\frac{1}{q_1}} \le C_{\alpha}$$

Case III. This is where the molecular theory is used. We need to check the size and the moments of $I^{\alpha}a$. $\|I^{\alpha}a\|_{q_2} \le C_{\alpha,q_1,q_2}\|a\|_{q_1}$ from Case I. Now let $b = 1 - \frac{1}{q_2} + \epsilon$, for $0 < \epsilon < (s+1-\alpha)/n$.

$$\left[\int |I^{\alpha}a(x)|x|^{nb}|^{q_2}\,dx\right]^{\frac{1}{q_2}} \le \left[\int_{x\in Q^*}|I^{\alpha}a(x)|x|^{nb}|^{q_2}\,dx\right]^{\frac{1}{q_2}}$$

$$+C_{\alpha}\left[\int_{x\notin Q^*}\left[\int_{y\in Q}|a(y)|\,|y|^{s+1}\,dx\right]^{q_2}\frac{|x|^{nbq_2}}{|x|^{(n-\alpha+s+1)q_2}}\,dx\right]^{\frac{1}{q_2}}$$

$$= I_1 + I_2 .$$

$$I_1 \le \|I^{\alpha}a\|_{q_2}|Q|^{b} \le C_{\alpha,p_1,p_2}\|a\|_{q_1}|Q|^{b} .$$

$$I_2 \le C_{\alpha}\|a\|_{q_1}|Q|^{\frac{s+1}{n}+1-\frac{1}{q_1}}\left[\int_{x\notin Q^*}\frac{dx}{|x|^{(n-\alpha+s+1-nb)q_2}}\right]^{\frac{1}{q_2}}$$

$$\leq C_\alpha \|a\|_{q_1} |Q|^{\frac{s+1}{n}+1-\frac{1}{q_1}} |Q|^{-1+\frac{\alpha}{n}-\frac{s+1}{n}+b+\frac{1}{q_2}} = C_\alpha \|a\|_{q_1} |Q|^b .$$

Thus $\| I^\alpha a |x|^{nb} \|_{q_2} \leq C_{\alpha, p_1, p_2} \|a\|_{q_1} |Q|^b$. Let $a = 1 - \frac{1}{p_2} + \epsilon$, and we see that

$$(\|a\|_{q_1})^{\frac{a}{b}} (\|a\|_{q_1} |Q|^b)^{1-\frac{a}{b}} = \|a\|_{q_1} |Q|^{b-a} = \|a\|_{q_1} |Q|^{\frac{1}{p_2}-\frac{1}{q_2}} = \|a\|_{q_1} |Q|^{\frac{1}{p_1}-\frac{1}{q_1}} \leq 1 .$$

Thus, the size condition is met.

From the fact that $I^\alpha a \in L^{q_2}$ and $I^\alpha a |x|^{nb} \in L^{q_2}$ we get that $I^\alpha a \, |x|^\nu \in L^1$ if $|\nu| \leq [n(\frac{1}{p_2} - 1)]$. (The necessary estimate is $\epsilon > \frac{1}{p_1} - 1 > \frac{1}{p_2} - 1 \geq \frac{1}{n}[n(\frac{1}{p_2} - 1)]$. Note that $b = 1 - \frac{1}{q_1} + \epsilon$.) Consequently $D^\nu (I^\alpha a)^\wedge$ is a continuous function and we only need to check that $D^\nu (I^\alpha a)^\wedge(0) = 0$ if $|\nu| \leq [n(\frac{1}{p_2} - 1)]$. Since a is a (p_1 , q_1 , s)-atom , $\hat{a}(x) = 0(|x|^{s+1})$ as $|x| \to 0$ ((9.1)(i)) and so $(I^\alpha a)^\wedge(x) = |x|^{-\alpha} \hat{a}(x) = 0(|x|^{s+1-\alpha})$ as $|x| \to 0$. Since $s + 1 - \alpha > n\epsilon > [n(\frac{1}{p_2} - 1)]$ it follows that $D^\nu (I^\alpha a)^\wedge(0) = 0$ and we have established that $I^\alpha a$ is a $(p_2 , q_2 , [n(\frac{1}{p_2} - 1)] , \epsilon)$-molecule if $\frac{s}{n} < \epsilon < \frac{s}{n} + \frac{1-\alpha}{n}$ and $\aleph(I^\alpha a) \leq C_{\alpha, p_1, p_2}$.

It is possible to generalize this result and give conditions on kernels k such that the map $f \to k * f$ sends H^{p_1} continuously into H^{p_2} . As a single example in this direction note that if $\hat{k}$ is bounded and

$$\int_{|x|>2R} \int_{|y|<R} |k(x + y) - k(x)|^2 \, dy \, |x|^{n(1+2\epsilon)} dx \leq C \, R^{n(1+2\epsilon)}$$

for all $R > 0$ and some $\epsilon > 0$ then $a \to k * a$ maps $(1 , 2 , 0)$-atoms to $(1 , 2 , 0 , \epsilon)$-molecules, boundedly, and so $f \to$ p.v. $\int k(x - y) f(y) dy$ wil map H^1 continuously into H^1 . For any Riesz kernel, k , we know that

$$|k(x + y) - k(x)| \leq C \, |y| / |x|^{(n+1)} \quad \text{if} \quad |y| < |x|/2$$

and the condition is satisfied for $0 < \varepsilon < 1/n$. Details are left for the reader.

Multipliers. Let $f \in H^p(\mathbb{R}^n)$, $0 < p \leq 1$. It is a direct consequence of the atomic characterization of H^p that $\hat{f}$ is continuous on $\mathbb{R}^n$ and that there is a constant $C > 0$, independent of $f \in H^p$ such that $|\hat{f}(x)| \leq C \|f\|_{H^p} |x|^{n(\frac{1}{p} - 1)}$ (see Proposition (9.14) in Appendix E.) Thus, we may define a multiplier on H^p as a function $m(x)$ that is measurable and such that whenever $f \in H^p$, $m\hat{f}$ is a function that is the Fourier transform of an element of H^p and for which there is a constant $M > 0$, independent of f, such that $\|(m\hat{f})^{\vee}\|_{H^p} \leq M\|f\|_{H^p}$. There is no need to take recourse to a "nice dense subset" of H^p. Furthermore, if we vary f appropriately, we can show that if m is a multiplier on H^p then m is continuous and bounded on $\mathbb{R}^n - \{0\}$ and that there is a constant C, independent of m, such that $|m(x)| \leq C\,M$ (see Proposition (9.21)).

Consequently, if m is a multiplier on some H^p, $0 < p \leq 1$, then m is also a multiplier on L^2, and by any of several interpolation arguments it is also bounded on H^r, $p \leq r \leq 1$; on L^r, $1 \leq r \leq 2$, and then by duality it is also a multiplier on L^r, $2 < r < \infty$; and on BMO. (For an interpolation theorem one can use Theorem 3.5 in [3], or using the atomic and molecular theory one can obtain the interpolation result by elementary calculations for the class of multipliers we describe below, using only the fact that they are linear maps that send p-atoms, boundedly, to p-molecules. Details will be provided elsewhere.)

Let us now state the multiplier theorem that we prove in Appendix D (Theorem 9.26)).

Theorem (4.2). *Suppose t is a positive integer and*

$$(4.3) \qquad R^{2|\beta|-n} \int_{R<|x|\leq 2R} |D^{\beta}_{m}(x)|^2\, dx \leq A^2, \quad 0 \leq |\beta| \leq t,\ R > 0 .$$

Then if $0 < p \leq 1$ *and* $\frac{t}{n} > \frac{1}{p} - \frac{1}{2}$, m *is a multiplier on* $H^p(\mathbb{R}^n)$ *and there is a* $C > 0$, *independent of* m *and* f *such that*

$$\|(m\,\hat{f})^{\vee}\|_{H^p} \leq C\,A\|f\|_{H^p}\ , \text{ for all } f \in H^p\ .$$

We note first that the fact that m is bounded follows from (4.3)(see (9.22)). We will illustrate the proof with a sketch of the one-dimensional case. The proof for $n > 1$ is much more technical.

We have a function m, a positive integer t, a number p, $0 < p \leq 1$ with $t > \frac{1}{p} - \frac{1}{2}$. For all $R > 0$, and integers s such that $0 \leq s \leq t$ we have

$$R^{2s-1} \int_{R<|x|\leq 2R} |D_m^s(x)|^2\, dx \leq A^2\ .$$

We know that m is bounded on $\mathbb{R} - \{0\}$ and set $|m(x)| \leq C\,A$.

Let a be a $(p, 2, t-1)$-atom centered at the origin. We will show that $(m\,\hat{a})^{\vee}$ is a $(p, 2, [\frac{1}{p} - 1], t - \frac{1}{2})$-molecule and that $\aleph((m\,\hat{a})^{\vee}) \leq C\,A$.

For a $(p, 2, t-1)$-atom, a, and a bounded function, m, we only need to establish the estimate

$$\{\|m\,\hat{a}\|_2^{\frac{1}{2}-\frac{1}{p}+t}\ \|D^t(m\,\hat{a})\|_2^{\frac{1}{p}-\frac{1}{2}}\}^{\frac{1}{t}} \leq AC\ . \tag{4.4}$$

To see this we note that from the Plancherel relations (4.4) is the same as

$$\{\|(m\,\hat{a})^{\vee}\|_2^{\frac{1}{2}-\frac{1}{p}+t}\ \|(m\,\hat{a})^{\vee}\ |x|^t\|_2^{\frac{1}{p}-\frac{1}{2}}\}^{\frac{1}{t}} \leq C\,A\ ,$$

which is the size condition for a $(p, 2, [\frac{1}{p} - 1], t - 1/2)$-molecule. To see that moments up to order $[\frac{1}{p} - 1]$ are zero we first note that $(m\,\hat{a})^{\vee} \in L^2$ and $|x|^t\ (m\,\hat{a})^{\vee} \in L^2$ implies that $|x|^s\ (m\,\hat{a})^{\vee} \in L^1$ for $0 \leq s < t - \frac{1}{2}$ and so for $0 \leq s < [\frac{1}{p} - 1]$ since $[\frac{1}{p} - 1] < t - \frac{1}{2}$. But this implies that $D^s\ (m\,\hat{a})$ is continuous and we only need to check that $D^s(m\,\hat{a})(0) = 0$. From (9.1)(i) we have that $\hat{a}(x) = O(|x|^t)$ as $|x| \to 0$ and since m is bounded, $m\,\hat{a}(x) = O(|x|^t)$ as $|x| \to 0$. Since $\hat{a}(0) = 0$ we have that $D^s(m\,\hat{a})(0) = 0$.

Let us now establish (4.4). Let $a = 1 - \frac{1}{p} + \epsilon = t + \frac{1}{2} - \frac{1}{p}$, $b = \frac{1}{2} + \epsilon = t$, $b - a = \frac{1}{p} - \frac{1}{2}$. We rewrite (4.4) as

$$\|m\,\hat{a}\|_2^{\frac{a}{b}}\ \|D^t(m\,\hat{a})\|_2^{1-\frac{a}{b}} \leq C\,A\ . \tag{4.5}$$

Since $\|m\hat{a}\|_2 \leq C\,A\|\hat{a}\|_2 = CA\|a\|_2$ we only need to show that

$$(4.6)\qquad \|D^t(m\hat{a}\|_2 \le C\ A/\|a\|^{\frac{a}{b-a}} = C\ A/\|a\|^{\frac{b}{b-a}-1} ,$$

and consequently, we need to show that for $k + \ell = t$, $0 \le k$, $\ell \le t$,

$$(4.7)\qquad \|(D^k\hat{a})(D^\ell m)\|_2 \le C\ A/\|a\|^{\frac{b}{b-a}-1} .$$

For $k = t$ this is trivial since

$$\|(D^t\hat{a})m\|_2 \le C\ A\|D^t\hat{a}\|_2 \le C\ A/\|\hat{a}\|_2^{\frac{b}{b-a}-1} ,$$

as follows from the fact that a is a molecule and the Plancherel relations. Thus, we may now assume that $0 \le k < t$, $0 < \ell \le t$, $k + \ell = t$. From (9.1) we need the following estimates:

$$(4.8)\qquad (i)\quad |D^k\hat{a}(x)| \le C|x|^{t-k}/\|a\|_2^{\frac{t+1/2}{b-a}-1} ,$$

$$(ii)\quad |D^k\hat{a}(x)| \le C/\|a\|_2^{\frac{k+1/2}{b-a}-1} .$$

We choose K , an integer, so that $2^K \sim \|a\|_2^{\frac{1}{b-a}}$. Then

$$\|(D^k\hat{a})(D^\ell m)\|_2^2 = \sum_{\nu\in\mathbf{Z}} \int_{2^\nu<|x|\le 2^{\nu+1}} |D^k\hat{a}(x)|^2\ |D^\ell m(x)|^2\ dx$$

$$\le C^2 \left[\sum_{-\infty}^{K} \frac{2^{\nu(2t-2k)}}{\|a\|_2^{\frac{2t+1}{b-a}-2}} \int_{2^\nu<|x|\le 2^{\nu+1}} |D^\ell m(x)|^2\ dx \right.$$

$$\left. + \sum_{-\infty}^{K} \frac{1}{\|a\|_2^{\frac{2k+1}{b-a}-2}} \int_{2^\nu<|x|\le 2^{\nu+1}} |D^\ell m(x)|^2\ dx \right]$$

$$\le C^2A^2 \left[\sum_{-\infty}^{K} \frac{2^{\nu(2t-2k)}2^{-\nu(2\ell-1)}}{\|a\|_2^{\frac{2t+1}{b-a}-2}} + \sum_{K}^{\infty} \frac{1}{\|a\|_2^{\frac{2k+1}{b-a}-2}\, 2^{\nu(2\ell-1)}} \right]$$

$$= C^2A^2 \left[\frac{1}{\|a\|^{\frac{2t+1}{b-a}-2}} \sum_{-\infty}^{K} 2^\nu + \frac{1}{\|a\|^{\frac{2k+1}{b-a}-2}} \sum_{K}^{\infty} 2^{\nu(1-2\ell)} \right]$$

$$\leq C^2A^2\left[\frac{\|a\|_2^{\frac{1}{b-a}}}{\|a\|_2^{\frac{2t+1}{b-a}-2}}+\frac{\|a\|_2^{\frac{1-2\ell}{b-a}}}{\|a\|_2^{\frac{2k+1}{b-a}-2}}\right]$$

$$= C^2A^2/\|a\|_2^{2(\frac{t}{b-a}-1)} = C^2A^2/\|a\|_2^{2(\frac{t}{b-a}-1)} .$$

This is the required estimate and the proof if complete.

We complete this section with a description of two extensions of the multiplier theorem.

The main defect in multiplier theorems such as (4.2) is the jump that occurs because of the requirement that t only take integer values. This is a technical defect that is repaired by replacing the condition on the derivative (namely, (4.3)) with an appropriate Lipschitz condition. The Hörmander condition, (4.3), can be interpreted as a requirement that m is, locally, in the potential space $L^{2,t}$; that is, can be represented locally as a Bessel potential of order t of a function in L^2. Such spaces are the integer cases of the Lipschitz-Besov spaces $\Lambda_t^{2,2}$. There are many ways to express this condition locally (all that we have tried have worked!) but the one given below is a handy version for applications.

Let $\Delta_h f(x) = f(x - h)$ and $\Delta_h^{k+1} f = \Delta_h(\Delta_h^k f)$, $f > 1$, $\Delta_h^0 f = f$.

<u>Theorem (4.9).</u> <u>Suppose</u> m <u>is a bounded function,</u> $|m(x)| \leq A$, t <u>is positive and for some integer</u> $\bar{t}$, $\bar{t} > t$,

$$(4.10) \qquad R^{2t-n}\int_{|h|<R/2}|h|^{-2t}\int_{R<|x|<2R}|\Delta_h^{\bar{t}}\, m(x)|^2\, dx\, \frac{dh}{|h|^n} \leq A^2 , \ R > 0 .$$

<u>Then if</u> $0 < p \leq 1$, $\frac{t}{n} > \frac{1}{p} - \frac{1}{2}$, m <u>is a multiplier on</u> $H^p(\mathbb{R}^n)$ <u>and there is a</u> $C > 0$, <u>independent of</u> m <u>and</u> f , <u>such that</u> $\|(m\hat{f})^\vee\|_{H^p} \leq CA\|f\|_{H^p}$ <u>for all</u> $f \in H^p$.

Details of the proof are given in Appendix D (9.45). The discussion preceding the statement of Lemma (9.37) expands upon the statement we made on various formulations of (4.10).

Fractional versions of multiplier theorems are not new. We note in particular a result of R.R. Coifman [5] where he shows that if m is a bounded function on $\mathbb{R}$ and $|\Delta_h^2 m(x)| \leq C\,(|h|/|x|)^{\alpha}$, $|h| < |x|/2$, where $1/2 < p \leq 1$, $\alpha > 1/p - 1/2$, then m is a multiplier on $H^p(\mathbb{R})$. (His result is more general than this, but this is the relevant part for our discussion.) One sees that such an m satisfies (4.10) for any $0 < t < \alpha$ and so (4.9) is a generalization of Coifman's theorem.

We also note that for spaces of homogeneous type that are not locally Euclidean the notion of a derivative defined pointwise is not available, but Lipschitz conditions such as (4.10) always make sense. Thus Taibleson [18], Chapt. VI, Theorem (1.1) gives a multiplier theorem for L^p spaces on local fields using a Lipschitz condition.

As a final comment we note that an essential tool in the proof of Theorem (4.2) and Theorem (4.10) for $n > 1$ is the use of embedding theorems for potential spaces and Lipschitz spaces. These are used explicitly in Lemmas (9.22) and (9.37). The idea behind such results is the Sobolev result which says that a function which is smooth in $L^r(\mathbb{R}^n)$ is also smooth in $L^s(\mathbb{R}^n)$, $s > r$, but has lost $n(\frac{1}{r} - \frac{1}{s})$ degrees of smoothness. (The most elementary version states that a function with $[\frac{n}{2}] + 1$ derivatives in $L^2(\mathbb{R}^n)$ is continuous.)

Using these embedding theorems we can state versions of our multiplier theorems for "Hörmander conditions" with integral exponents $r \neq 2$, $1 \leq r \leq \infty$. For integer values of t we have the following example:

<u>Theorem (4.11)</u>. <u>Suppose</u> t <u>is a positive integer and</u>

$$R^{(|\beta| - \frac{n}{r})} \left[\int_{R<|x|\leq 2R} |D^{\beta} m(x)|^r \, dx \right]^{\frac{1}{r}} \leq A \,, \tag{4.12}$$

$0 \leq |\beta| \leq t$, $R > 0$.

<u>Then if</u> $0 < p \leq 1$ <u>and</u> $\frac{t}{n} > \frac{1}{p} - \frac{1}{\min(2,r)}$, <u>and</u> $1 \leq r \leq \infty$, m <u>is a multiplier on</u> $H^p(\mathbb{R}^n)$ <u>and there is a constant</u> $C > 0$, <u>independent of</u> m <u>and</u> f <u>such that</u> $\|(m\hat{f})^{\vee}\|_{H^p} \leq C A \|f\|_{H^p}$, <u>for all</u> $f \in H^p$.

More details, an equivalent version for integral t and a fractional version is given in Theorem (9.48) and the discussion which precedes it. Note, in particular, that from the condition for $r = \infty$ (usually called a Mihlin condition) down to the condition for $r = 2$ (the Hörmander condition) there is no change in the required smoothness for a multiplier. This result should be compared to the result of Peral and Torchinsky [15] for parabolic spaces.

§5. Appendix A. A family of Borel measures on the disk. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane. As in §3, for each $\alpha > 0$ put $\omega_\alpha(z) = \omega(z) = \frac{\alpha}{\pi}(1 - |z|^2)^{\alpha-1}$ for $z \in D$. The "weight function" ω gives rise to a measure on D, which we also denote by ω, defined for each Borel set $E \subset D$ by

$$\omega_\alpha(E) = \omega(E) = \int_E \omega_\alpha(z)d\mu(z) ,$$

where μ is two-dimensional Lebesgue measure.

If $z_0 \in \bar{D}$ (the closed unit disk) and $\epsilon \geq 0$ then

$$B_{z_0}(\epsilon) = \{z \in D : |z - z_0| \leq \epsilon\} = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\} \cap D$$

is called the ball centered at z_0 of radius ϵ.

The main reault of this appendix is the following estimate:

Proposition (5.1). *If* $z_0 \in \bar{D}$, $\epsilon \geq 0$ *then*

$$\omega_\alpha(B_{z_0}(\epsilon)) \sim \begin{cases} (1 - |z_0|)^{\alpha-1}\epsilon^2 , & 0 \leq \epsilon \leq 1 - |z_0| \\ \epsilon^{\alpha+1} , & 1 - |z_0| \leq \epsilon \leq 1 + |z_0| \\ 1 , & \epsilon \geq 1 + |z_0| \end{cases}$$

(The symbol "~" denotes that the ratios of the quantities on the right and left are bounded above and below by positive quantities if either is non-zero, or both quantities are zero.)

We will prove (5.1) later in this appendix. As defined in [8, p.587] a space of homogeneous type is a topological space X endowed with a Borel measure μ and a quasi-distance d (there exists a positive constant K such that

$d(x, y) \le K(d(x, z) + d(z, y)))$. The spheres: $S_x(r) = \{y \in X : d(x, y) < r\}$ form a basis of open neighborhoods of the point $x \in X$. The basic assumption is that there is a positive constant A such that for all $x \in X$ and $r > 0$, $\mu(S_x(2r)) \le A_\mu(S_x(r))$.

Corollary (5.2). D endowed with the measure ω_α and Euclidean distance is a space of homogeneous type.

Proof. The result will follow if we show that

$$(5.3) \qquad \frac{\omega(B_{z_0}(2\epsilon))}{\omega(B_{z_0}(\epsilon))} \sim 1,$$

for all $z_0 \in \bar{D}$, $\epsilon > 0$. Since "≥ 1" is clear we need only show "$\le A$".

Case 1. $\epsilon \ge (1 + |z_0|)/2$. Note that if $\delta \ge 1 + |z_0|$ then $B_{z_0}(\delta) = D$ and hence $\omega(B_{z_0}(\delta)) = \omega(D) = 1$. Thus $\omega(B_{z_0}(2\epsilon)) = 1$ and we need to show that $\omega(B_{z_0}(\epsilon))$ is bounded below.

Subcase A. $\epsilon \ge 1 + |z_0|$. $\omega(B_{z_0}(\epsilon)) = 1$.

Subcase B. $1 - |z_0| \le \epsilon \le 1 + |z_0|$. $\omega(B_{z_0}(\epsilon)) \sim \epsilon^{\alpha+1} \ge ((1 + |z_0|)/2)^{\alpha+1} \ge (\frac{1}{2})^{\alpha+1}$.

Subcase C. $0 < \epsilon \le 1 - |z_0|$.

Sub-subcase (i). $\alpha \ge 1$. $\omega(B_{z_0}(\epsilon)) \sim (1 - |z_0|)^{\alpha-1} \epsilon^2 \ge \epsilon^{\alpha+1} \ge (\frac{1}{2})^{\alpha+1}$.

Sub-subcase (ii). $0 < \alpha \le 1$. $\omega(B_{z_0}(\epsilon)) \sim (1 - |z_0|)^{\alpha-1} \epsilon^2 \ge \epsilon^2 \ge (\frac{1}{2})^2$.

Case 2. $1 - |z_0| \le \epsilon \le (1 + |z_0|)/2$.

$$\omega(B_{z_0}(2\epsilon))/\omega(B_{z_0}(\epsilon)) \sim (2\epsilon)^{\alpha+1}/\epsilon^{\alpha+1} = 2^{\alpha+1}.$$

Case 3. $0 < \epsilon \le (1 - |z_0|)/2$.

$$\omega(B_{z_0}(2\epsilon))/\omega(B_{z_0}(\epsilon)) \sim \frac{(1-|z_0|)^{\alpha-1}(2\epsilon)^2}{(1-|z_0|)^{\alpha-1}\epsilon^2} = 4.$$

Case 4. $(1 - |z_0|)/2 \le \epsilon \le \min [1 - |z_0| , (1 + |z_0|)/2]$. Then $\epsilon/(1 - |z_0|) \sim 1$.
$\omega(B_{z_0}(2\epsilon))/\omega(B_{z_0}(\epsilon)) \sim \epsilon^{\alpha+1}/(1 - |z_0|)^{\alpha-1}\epsilon^2 \sim (\epsilon/(1 - |z_0|))^{\alpha-1} \sim 1$.
This completes the proof of the corollary.

Let

$$(5.4) \qquad \delta(z_0, z) = \begin{cases} |z - z_0|^2 (1 - |z_0|)^{\alpha-1} , & |z - z_0| \le 1 - |z_0| \\ |z - z_0|^{\alpha+1} , & 1 - |z_0| \le |z - z_0| \le 1 + |z_0| . \end{cases}$$

Fix $z_0 \in \overline{D}$, then for $z \in \overline{D}$, $\delta(z_0 , z)$ is a strictly increasing function of $|z - z_0|$, $0 \le |z - z_0| \le 1 + |z_0|$. Our next result shows that δ satisfies a basic regularity property.

Corollary (5.5). If $z_0 \in \overline{D}$ and $0 \le r \le (1 + |z_0|)^{\alpha+1}$ then $\omega(\{z : \delta(z_0 , z) \le r\} \cap D) \sim r$.

Proof. For $0 \le r \le (1 - |z_0|)^{\alpha+1}$, $\{z \in D : \delta(z_0 , z) \le r\}$
$= \{z \in D : |z - z_0|^2 (1 - |z_0|)^{\alpha-1} \le r\} = \{z \in D : |z - z_0| \le r^{\frac{1}{2}} (1 - |z_0|^{\frac{1-\alpha}{2}}\}$
$= B_{z_0}(r^{\frac{1}{2}}(1 - |z_0|^{\frac{1-\alpha}{2}})$. Since, $r^{\frac{1}{2}} (1 - |z_0|)^{\frac{1-\alpha}{2}} \le 1 - |z_0|$,
$\omega(\{z \in D : \delta(z_0 , z) \le r\}) \sim (1 - |z_0|)^{\alpha-1} r(1 - |z_0|)^{1-\alpha} = r$. For
$(1 - |z_0|)^{\alpha+1} \le r \le (1 + |z_0|)^{\alpha+1}$, $\{z \in D : \delta(z_0 , z) \le r\} = \{z \in D : |z - z_0|^{\alpha+1} \le r\}$
$= B_{z_0}(r^{\frac{1}{\alpha+1}})$. Since $(1 - |z_0|) \le r^{\frac{1}{\alpha+1}} \le (1 + |z_0|)$, $\omega(\{z \in D : \delta(z_0 , z) \le r\}$
$\sim (r^{\frac{1}{\alpha+1}})^{\alpha+1} = r$. This completes the proof.

This last result is very suggestive and would expect that the function $\delta(z_0 , z)$ is "almost" a homogeneous metric for the space of homogeneous type. We will now show that this is so.

A "natural" homogeneous metric for our space of homogeneous type is $\overline{d}(z_0 , z) = \inf \{\omega(B_w(\epsilon)) : z_0 , z \in B_w(\epsilon)\}$. It follows from the general theory of such spaces that $\overline{d}$ is a quasi-distance and is homogeneous in the sense that

$\omega(\{z \in D : \bar{d}(z_0, z) \le r\}) \sim r$. We will now show that $\bar{d}(z_0, z) \sim \delta(z_0, z)$.

To see this note that $\delta(z_0, z) \sim \omega(B_{z_0}(|z - z_0|))$. Note also that $z, z_o \in B_{z_0}(|z - z_0|)$. Thus, $\bar{d}(z_0, z) \le \omega(B_{z_0}(|z - z_0|))$. An easy estimats shows that if $z_0, z \in B_w(\epsilon)$ then $B_{z_0}(|z - z_0|) \subset B_w(3\epsilon)$. Thus there is a constant $c > 0$ such that $\omega(B_w(\epsilon)) \ge C^{-1}\omega(B_w(3\epsilon)) \ge C^{-1}\omega(B_{z_0}(|z - z_0|))$. Take the infimum over all such balls $B_w(\epsilon)$ and we get that $\bar{d}(z_0, z) \le \omega(B_{z_0}(|z - z_0|) \le C\,\bar{d}(z_0, z)$ and consequently $\delta(z_0, z) \sim \bar{d}(z_0, z)$.

Before preceeding to a proof of Proposition (5.1) we will give one more easy and important corollary. Note that weight function $\omega_1(z) = 1/\pi$ gives rise to normalized Euclidean measure on D. We will denote this measure in the usual way: $\omega_1(E) = |E|$.

We say that a non-negative function ω on D is in A_q, $q > 1$, if for all $z_0 \in \bar{D}$, $\epsilon > 0$, $B = B_{z_0}(\epsilon)$,

$$(5.6) \qquad \left[\frac{1}{|B|}\int_B \omega(z)d\mu(z)\right]\left[\frac{1}{|B|}\int_B (\omega(z))^{-\frac{1}{q-1}} d\mu(z)\right]^{q-1} \le C .$$

An extensive theory of such weights on $\mathbb{R}^n$ has been developed and a thorough treatment can be found in [6]. We need only the following immediate consequences of (5.6):

Proposition (5.7). If ω is in A_q on D, $f \in L^q(D, \omega(z)d\mu(z))$ and $B = B_{z_0}(\epsilon)$ is any ball with $z_0 \in \bar{D}$ then

$$\frac{1}{|B|}\int_B |f(z)|d\mu(z) \le C\left[\frac{1}{\omega(B)}\int_B |f(z)|^q\,\omega(z)d\mu(z)\right]^{\frac{1}{q}}$$

where C is a constant that is independent of f and B.

Proof. Use Hölder's inequality and (5.6).

Corollary (5.8). ω_α is in A_q if $q > \max\{1, \alpha\}$.

Proof. Let $\omega(B) = \omega_\alpha(B)$ and $\beta = (q - \alpha)/(q-1) > 0$. Let $\tilde{\omega}(B)$ $= \int_B (1 - |z|^2)^{\beta-1} d\mu(z) = \int_B \{(1 - |z|^2)^{\alpha-1}\}^{-\frac{1}{q-1}} d\mu(z) = \frac{\pi}{\beta}\omega_\beta(B)$. We need to show that $\omega(B)(\tilde{\omega}(B))^{q-1} \leq C\,|B|^q$, for $B = B_{z_0}(\varepsilon)$, $z_0 \in \bar{D}$, $\varepsilon > 0$.

Case 1. $\varepsilon \leq |z_0|$. $\omega(B)(\tilde{\omega}(B))^{q-1} \sim (1 - |z_0|)^{\alpha-1}\, \varepsilon^2((1 - |z_0|)^{\frac{1-\alpha}{q-1}}\varepsilon^2)^{q-1} \sim \varepsilon^{2q} \sim |B|$.

Case 2. $1 - |z_0| \leq \varepsilon \leq 1 + |z_0|$. $\omega(B)(\tilde{\omega}(B))^{q-1} \sim \varepsilon^{\alpha+1}\, \varepsilon^{2q-\alpha-1} = \varepsilon^{2q} \sim |B|^q$.

Case 3. $\varepsilon \geq 1 + |z_0|$. $\omega(B)(\tilde{\omega}(B))^{q-1} = 1 \cdot \frac{\pi}{\beta} = \frac{\pi}{\beta} \cdot |B|^q$.

Summary. D *endowed with* ω_α *as a measure and Euclidean distance is a space of homogeneous type. The "balls"* $\{z \in D : \delta(z_0, z) \leq r\}$ *form a natural family of closed neighborhoods about each point* $z_0 \in D$ *with measure on the order of* r, $0 \leq r \leq (1 + |z_0|)^{\alpha+1}$. *The function* $(z_0, z) \to \delta(z_0, z)$ *is equivalent to a homogeneous metric on* D. *The weight* ω_α *is in* A_q, $q > \max\{1, \alpha\}$.

Our penultimate result is the

Proof of Proposition (5.1). If $\varepsilon \geq 1 + |z_0|$ then $B = B_{z_0}(\varepsilon) = D$ and so $\omega(B) = \omega(D) = 1$ so we may assume that $0 < \varepsilon \leq 1 + |z_0|$. From the rotational symmetry of $w(z)$ we may assume that $z_0 = x_0$ is real and non-negative, $0 \leq x_0 \leq 1$.

Case 1. $\varepsilon \leq 1 - x_o$.

For $x_0 = 0$, $\varepsilon \leq 1$, $\omega(B) = 2\alpha \int_0^\varepsilon (1 - r^2)^{\alpha-1}\, rdr = 1 - (1 - \varepsilon^2)^\alpha$. Let $f(y) = 1 - (1 - y)^\alpha$, $0 \leq y \leq 1$, $f(0) = 0$, $f(1) = 1$, $f'(0) = \alpha$. For $0<\alpha\leq 1$, f is an increasing, convex and $\alpha y \leq f(y) \leq y$. For $\alpha \geq 1$, f is increasing, concave and $y \leq f(y) \leq \alpha y$. Thus, for, $0 < \alpha \leq 1$, $\alpha\varepsilon^2 \leq \omega(B) \leq \varepsilon^2$, and for $\alpha \geq 1$, $\varepsilon^2 \leq \omega(B) \leq \alpha\varepsilon^2$. If $x_0 > 0$, $0 \leq \varepsilon \leq 1 - x_0$.

$$\omega(B) = \frac{\alpha}{\pi} \int_0^\varepsilon \int_{-\pi}^{\pi} (1 - |x_0 + \rho e^{i\theta}|^2)^{\alpha-1}\, d\theta\rho d\rho$$

$$= \frac{2\alpha}{\pi}\int_0^{\epsilon}\int_0^{\pi}(1 - x_0^2 - 2x_0\rho\cos\theta - \rho^2)^{\alpha-1}\, d\theta\rho d\rho$$

$$= \frac{2\alpha}{\pi}(1 - x_0^2)^{\alpha-1}\int_0^{\epsilon}\int_0^{\pi}\left(1 - \frac{\rho(2x_0\cos\theta+\rho)}{1-x_0^2}\right)^{\alpha-1} d\theta\rho d\rho .$$

$\rho < \epsilon < 1 - x_0$ so $|2x_0\cos\theta + \rho| < 2x_0 + 1 - x_0 = 1 + x_0$. Consequently,

$$\frac{\rho(2x_0\cos\theta+\rho)}{1-x_0^2} < \frac{\rho}{1-x_0} \quad\text{and}\quad \left|\frac{\rho(2x_0\cos\theta+\rho)}{1-x_0^2}\right| < \frac{\rho}{1-x_0} < 1 .$$

Let $g(y) = (1 - y)^{\alpha-1}$, $-1 < y < 1$.

For $0 < \alpha \le 1$, g is increasing and bounded below by $2^{\alpha-1}$.

For $\alpha \ge 1$, g is decreasing and bounded above by $2^{\alpha-1}$. Let

$$(*) = \int_0^{\pi}\left(1 - \frac{\rho(2x_0\cos\theta+\rho)}{1-x_0^2}\right)^{\alpha-1} d\theta .$$

For $0 < \alpha \le 1$, $\pi 2^{\alpha-1} \le (*) \le \pi\left(1 - \frac{\rho}{1-x_0}\right)^{\alpha-1}$ and for $\alpha \ge 1$, $\pi\left(1 - \frac{\rho}{1-x_0}\right)^{\alpha-1} \le$ $\le (*) \le \pi 2^{\alpha-1}$. Thus, for $0 < \alpha \le 1$,

$$\omega(B) = \alpha 2^{\alpha-1}(1-x_0^2)^{\alpha-1}\int_0^{\epsilon} 2\rho d\rho = \alpha 2^{\alpha-1}(1-x_0^2)^{\alpha-1}\epsilon^2 .$$

$$\omega(B) \le 2\alpha(1-x_0^2)^{\alpha-1}\int_0^{\epsilon}\left(1 - \frac{\rho}{1-x_0}\right)^{\alpha-1}\rho d\rho$$

$$= 2\alpha(1 - x_0^2)^{\alpha-1}\epsilon^2\int_0^1\left(1 - \frac{\epsilon t}{1-x_0}\right)^{\alpha-1} tdt$$

$$\le 2\alpha(1 - x_0^2)^{\alpha-1}\epsilon^2\int_0^1(1 - t)^{\alpha-1}\, tdt$$

$$= \frac{2}{1+\alpha}(1 - x_0^2)^{\alpha-1}\epsilon^2 .$$

For $\alpha \ge 1$ a similar argument yields

$$\frac{2}{1+\alpha}(1 - x_0^2)^{\alpha-1}\epsilon^2 \le \omega(B) \le \alpha 2^{\alpha-1}(1 - x_0^2)^{\alpha-1}\epsilon^2 .$$

Since $1 - x_0^2 \sim 1 - x_0$ this completes the proof if $\epsilon \le 1 - x_0$.

<u>Case II</u>. $1 - x_0 \le \epsilon \le 1 + x_0$. Note that $\epsilon \le 2$. We first dispose of the case, $\frac{1}{2} \le \epsilon \le 2$. $\omega(B) \le \omega(D) = 1 = \epsilon^{-(\alpha+1)}\epsilon^{\alpha+1} \le 2^{\alpha+1}\epsilon^{\alpha+1}$. On the other hand, if $\epsilon \ge 1 - x_0$ and $\epsilon \ge \frac{1}{2}$ then $B_{\frac{3}{4}}(\frac{1}{4}) \subset B_{x_0}(\epsilon) = B$ so $\omega(B) \ge \omega(B_{\frac{3}{4}}(\frac{1}{4})) = A_\alpha =$

$= (A_\alpha \epsilon^{-(\alpha+1)})\epsilon^{\alpha+1} = A'_\alpha\ \epsilon^{\alpha+1}$. Thus, we may assume $0 < \epsilon \le \frac{1}{2}$, $1 - x_0 \le \epsilon \le 1 + x_0$. Note that we also have $x_0 - \epsilon \ge 0$ since $x_0 \ge 1 - \epsilon \ge \frac{1}{2}$. Consequently the balls under consideration lie in the region $\mathrm{Re}\, z \ge 0$.

Fix ϵ, $0 \le \epsilon \le \frac{1}{2}$. Let $I_{x_0} = B_{x_0}(\epsilon)$, $1 - \epsilon \le x_0 \le 1$. We need to show that $\omega(I_{x_0}) \sim \epsilon^{1+\alpha}$ with constants that do not depend on x_0 and ϵ.

Consider $I_{1-\epsilon}$. From Case I we have $\omega(I_{1-\epsilon}) \sim (1 - (1 - \epsilon))^{\alpha-1}\ \epsilon^2 = \epsilon^{\alpha+1}$. Now consider the other extreme case, I_1. Let $D_\epsilon = \{1 - \epsilon \le |z| \le 1\}$.

$$\omega(D_\epsilon) = 2\alpha \int_{1-\epsilon}^{1} (1 - r^2)^{\alpha-1}\ r dr = \epsilon^\alpha (2 - \epsilon)^\alpha \sim \epsilon^\alpha .$$

The angular wedge of D_ϵ of aperture $2 \arcsin \epsilon$, centered at $\theta = 0$, contains I_1. Divide D_ϵ into $\left[\frac{2\pi}{2 \arcsin \epsilon}\right]$ wedges, one of which, J, contains I_1. Then, $\left[\frac{2\pi}{2 \arcsin \epsilon}\right]\omega(J) = \omega(D_\epsilon) \sim \epsilon^\alpha$. Consequently, $\omega(I_1) \le \omega(J) \sim$

$\sim \dfrac{\epsilon^\alpha}{\left[\frac{2\pi}{2 \arcsin \epsilon}\right]} \le C\ \epsilon^{\alpha+1}$. On the other hand the wedge of aperture $2 \arcsin \frac{\epsilon}{\sqrt{2}}$,

centered at $\theta = 0$, in the annulus $D_{\epsilon/\sqrt{2}}$ is contained in I_1. Divide the annulus $D_{\epsilon/\sqrt{2}}$ into $\left[\frac{2\pi}{2 \arcsin \frac{\epsilon}{\sqrt{2}}}\right] + 1$ wedges, one of which, L, is contained in I_1. Then $\left[\frac{2\pi}{2 \arcsin \frac{\epsilon}{\sqrt{2}}}\right] + 1\ \omega(L) = \omega(D_{\epsilon/\sqrt{2}}) \sim \epsilon^\alpha$. Thus, $\omega(I_1) \ge \omega(L) \sim$

$\sim \dfrac{\epsilon^\alpha}{\left[\frac{2\pi}{2 \arcsin \frac{\epsilon}{\sqrt{2}}}\right]+1} \ge C\ \ \epsilon^{\alpha+1}$. We conclude that $\omega(I_1) \sim \epsilon^{\alpha+1}$. Recall that

$\omega(I_{1-\epsilon}) \sim \epsilon^{\alpha+1}$.

If $\alpha \ge 1$ it is easy to see that $\omega(I_{x_0})$ is a decreasing function of x_0 on $[1 - \epsilon, 1]$, and so $\omega(_{1-\epsilon}) \ge \omega(I_{x_0}) \ge \omega(I_1)$. We conclude that $\omega(I_{x_0}) \sim \epsilon^{\alpha+1}$.

We now consider the remaining case, $0 < \alpha < 1$. Let J be the "third" of the ball $I_{1-\epsilon}$, that is "closest" to the origin. That is,

$$J = \{z = (1 - \epsilon) + \rho e^{i\theta} :\ 0 \le \rho \le \epsilon,\ \tfrac{2}{3}\pi \le |\theta| \le \pi\} .$$

$$\omega(J) = \frac{2\alpha}{\pi} \int_0^\epsilon \int_{\frac{2\pi}{3}}^{\pi} (1 - |(1 - \epsilon) + \rho e^{i\theta}|^2)^{\alpha-1}\ d\theta\rho d\rho$$

$$\geq C\,(1 - (1-\epsilon)^2)^{\alpha-1}\,\epsilon^2 \sim \epsilon^{\alpha+1}\ .$$

Now let $J_{x_0} = I_1 - 1 + x_0$, the translate of I_1 to x_o . Then $J_{1-\epsilon} \supset J$ so $\omega(J_{1-\epsilon}) \geq \omega(J)$. It is easy to see that $\omega(J_{x_0})$ is increasing, as a function of x , on $[1-\epsilon\ ,\ 1]$ and $I_{x_0} \supset J_{x_0}$, so $\omega(I_{x_0}) \geq \omega(J_{x_0}) \geq \omega(J_{1-\epsilon}) \geq \omega(J) \geq$ $\geq C\epsilon^{\alpha+1}$.

Finally, let K be the region swept out by the balls I_{x_0} , $x_0 \in [1-\epsilon\ ,\ 1]$. Then $I_{x_0} \subset K \subset \{re^{i\theta} : 1 - 2\epsilon \leq r \leq 1\ ,\ |\theta| \leq \arctan \frac{\epsilon}{r}\}$. Thus,

$$\omega(I_{x_0}) \leq \omega(K) \leq \frac{2\alpha}{\pi} \int_{1-2\epsilon}^{1} \left[\int_0^{\arctan \frac{\epsilon}{r}} 1\, d\theta\right] (1 - r^2)^{\alpha-1}\ r\, dr$$

$$= \frac{2\alpha}{\pi} \int_{1-2\epsilon}^{1} (\arctan \tfrac{\epsilon}{r})(1 - r^2)^{\alpha-1}\ r\, dr$$

$$= \frac{2\alpha}{\pi}\,\epsilon \int_{1-2\epsilon}^{1} (1 - r^2)^{\alpha-1}\, dr \leq \frac{2\alpha}{\pi}\epsilon \int_{1-2\epsilon}^{1} (1 - r)^{\alpha-1}\, dr$$

$$= \frac{2}{\pi}\,\epsilon\,(2\epsilon)^{\alpha} = \frac{2^{\alpha+1}}{\pi}\,\epsilon^{\alpha+1}\ .$$

Thus, $\omega(I_{x_0}) \sim \epsilon^{\alpha+1}$, and the proof of proposition (5.1) is complete.

In addition to the estimates in (5.1) we will need a sharpened version for certain limiting cases in the proof of Theorem (3.14) and in Appendix B.

<u>Proposition (5.9)</u>. <u>There is a</u> C_α , $0 < C_\alpha < 1$ <u>so that for all</u> z , ρ_1 , ρ_2 <u>satisfying</u>: $z \in \overline{D}$, $0 < \rho_2 < \rho_1 \leq 1 + |z|$, $\omega_\alpha(B_z(\rho_1)) = 2\omega_\alpha(B_z(\rho_2)) \neq 0$, <u>we have</u> $\rho_2/\rho_1 \leq C_\alpha$.

Proof. Suppose the result is false. Then there is a sequence $\{(z_k\, , \rho_{k1}\, , \rho_{k2})\}_{k=1}^{\infty}$, $0 < \rho_{k2} < \rho_{k1} < 1 + |z_k|$, $z_k \in \overline{D}$, $\omega(B_{z_k}(\rho_{k1})) = 2\omega(B_z\ (\rho_{k2}))$, and $\lim_R \rho_{k1}/\rho_{k2} = 1$. We may suppose further that $0 \leq z_k \leq 1$ and that $\lim_k z_k = z_0$, $\lim_k \rho_{k1} =$ $= \lim_k \rho_{k2} = \rho_0$ exist. We proceed by considering cases.

<u>Case I</u>. $\rho_0 \neq 0$. By the dominated convergence theorem, $\lim_k \omega(B_{z_k}(\rho_{ki})) =$
$= \omega(B_{z_0}(\rho_0)) \neq 0$, $i = 1, 2$. Consequently, $2 = \lim_k \omega(B_{z_k}(\rho_{k1}))/\omega(B_{z_k}(\rho_{k2})) = 1$,
a contradiction.

<u>Case II</u>. $\rho_0 = 0$, $z_0 \neq 1$. It is easy to see that $(1 - |z|^2)^{\alpha-1} =$
$(1 - |z_0|^2)^{\alpha-1} + 0(1)$, $z \in B_{z_k}(\rho_{ki})$ as $k \to \infty$. From this we get that
$\omega(B_{z_k}(\rho_{ki})) \cong \alpha(1 - |z_0|^2)^{\alpha-1}(\rho_{ki})^2$. Consequently, $2 = \lim_k \omega(B_{z_k}(\rho_{ki}))/\omega(B_{z_k}(\rho_{k2}))$
$= \lim_k (\rho_{k1}/\rho_{k2})^2 = 1$, a contradiction. We may now assume that $\rho_0 = 0$, $z_0 = 1$
and $\lim_k (1 - z_k)/\rho_{ki} = \gamma$ exists, $0 \leq \gamma \leq \infty$.

<u>Case III</u>. $\rho_0 = 0$, $z_0 = 1$, $\gamma = \infty$. For k large enough, $\rho_{ki} < 1 - z_k$, and as in the proof of (5.1) Case I, we have,

$$\omega(B_{z_k}(\rho_{ki})) = \frac{2\alpha}{\pi}(1 - |z_k|^2)(\rho_{ki})^2 \int_0^1 \int_0^\pi \left(1 - \frac{t\rho_{ki}(2|z_k|\cos\theta + t\rho_{ki})}{1-|z_k|^2}\right)^{\alpha-1} d\theta t dt .$$

The expression in the integrand in parentheses is bounded above and below by
$1 \pm (\rho_{ki}/(1 - |z_k|))$ which is uniformly $1 + 0(1)$ and so $\omega(B_{z_k}(\rho_{ki})) \cong$
$\cong \alpha(1 - |z_k|^2)^{\alpha-1}(\rho_{ki})^2$. Thus, $2 = \lim_k \omega(B_{z_k}(\rho_{ki}))/\omega(B_{z_k}(\rho_{k2})) = \lim_k (\rho_{k1}/\rho_{k2})^2 =$
$= 1$, a contradiction.

<u>Case IV</u>. $\rho_0 = 0$, $z_0 = 1$, $0 \leq \gamma < \infty$. We change variables sending
$z \to (1 - z)/\rho_{ki}$. Then $(1 - |z|^2)^{\alpha-1} d\mu(z) = z^{\alpha-1}(\rho_{ki})^{\alpha+1}(t - \frac{\rho_{ki}}{2}(t^2 + s^2))^{\alpha-1} dt\, ds$.
Thus,

$$\omega(B_{z_k}(\rho_{ki})) = \frac{\alpha}{\pi} 2^{\alpha-1}(\rho_{ki})^{\alpha+1} \iint_{(t,s)\in I_{(1-z_k)/\rho_{ki}}} (t - \frac{\rho_{ki}}{2}(t^2 + s^2))^{\alpha-1}\, dtdx .$$

$$\cong \frac{\alpha}{\pi} 2^{\alpha-1} \rho_{ki}^{\alpha+1} \iint_{J_\gamma} t^{\alpha-1}\, dtds = A_{\gamma,\alpha}\, \rho_{ki}^{\alpha+1} ,$$

where $I_{(1-z_k)/\rho_{ki}}$ is that part of the disk of radius 1, centered at $(\frac{1-z_k}{\rho_{ki}}, 0)$
that is contained in the circle of radius $1/\rho_{ki}$ centered at $(1/\rho_{ki}, 0)$; and
J_γ is the limiting case, that part of a circle of radius 1 centered at $(\gamma, 0)$
that is contained in the right half plane. Thus, $z = \lim_k \omega(B_{z_k}(\rho_{k1}))/\omega(B_{z_k}(\rho_{k2}))$
$= \lim_k (\rho_{k1}/\rho_{k2})^{\alpha+1} = 1$, a contradiction.

This completes the proof of (5.9).

<u>§6. Appendix B</u>. <u>The Gram Schmidt Estimates</u>. In this appendix we sketch a proof

of Lemma (3.15). Let us note first that the center, z_0, of the molecules and atoms that occur in Theorem (3.13) can be assumed to be real and nonnegative. To see this, note that for ω and δ as given, $\omega(z) = \omega(e^{i\theta}z)$, $z \in D$, and $\delta(z_0, z) = \delta(e^{i\theta}z_0, e^{i\theta}z)$ for $z_0, z \in \overline{D}$, and finally, $\int_D f(z)z^\nu d\mu(z) = 0$ for all $|\nu| \leq s$ if and only if $\int_D f(e^{i\theta}z)z^\nu d\mu(z) = 0$ for all $|\nu| \leq s$. Thus, rotations leave the defining properties of atoms and molecules invariant. (Recall that $z^\nu = (x + iy)^{(\nu_1,\nu_2)} \equiv x^{\nu_1} y^{\nu_2}$.)

The regions E_k that occur in the proof of Theorem (3.13) are either $E_o = B_0 = \{z \in D : |z - z_0| \leq \rho_0\}$ or, for $k \geq 1$, $E_k = B_k - B_{k-1} = \{z \in D : \rho_{k-1} < |z - z_0| \leq \rho_k\}$. There are constants, $0 < A < B < 1$ with $A \leq \rho_{k-1}/\rho_k \leq B$ as follows from (5.1) and (5.9).

The Gran-Schmidt polynomials of Lemma (3.15) can be written $\varphi_\ell^k(z) = \Sigma_{|\nu|\leq s} \beta_{\ell\nu}^k (z - z_0)^\nu$. We will show that there is a constant $C > 0$ independent of z_0, σ, k, ℓ and ν such that $|\varphi_\ell^k(z)| \leq C$, $z \in E_k$, and $|\beta_{\ell\nu}^k| \leq C\rho_k^{-|\nu|}$. When we adopt the argument of §2 we see that the dual basis to the monomials $\{(z - z_0)^\nu\}_{|\nu|<s}$ with respect to the inner product induced by the measure $d\mu(z)/|E_k|$ on E_k, can be written as $\psi_\ell^k(z) = \Sigma_{|\nu|\leq s} \beta_{\nu\ell}^k \varphi_\nu^k(z)$. Thus the estimates for the $\{\psi_\ell^k\}$ follow from the estimates for the $\{\varphi_\ell^k\}$ and the $\{\beta_{\nu\ell}^k\}$. (In what follows we assume that the regions E_k are closed in $\mathbb{R}^2$.)

We transform each E_k by a translation of $-z_0$ and a dilation of $\frac{1}{\rho_k}$. For E_0 we get one of the following:

$$S(\epsilon, \frac{1}{r}) = \begin{cases} \{|(x, y)| \leq 1 : x \leq 1 - \epsilon\}, \text{ if } \frac{1}{r} = 0 \\ \{|(x, y)| \leq 1 : |(x, y) - (1 - \epsilon - r, 0)| \leq r\} \text{ if } \frac{1}{r} > 0, \end{cases}$$

for $A \leq t \leq B$, $0 \leq \epsilon \leq 1$, $0 \leq \frac{1}{r} \leq \frac{2}{2-\epsilon}$.

Note that the case $\frac{1}{r} = 0$ never actually occurs. It is a limiting case included so that the domain parametrizing the family of regions is compact. Note also that the dependence on the parameter α resides entirely in the selection of the constants A and B.

The result we are seeking is that the Gram-Schmidt polynomials (and their coefficients) which are orthonormalizations of $\{z^\nu\}_{|\nu|<s}$ (taken in some fixed

order) with respect to the measure $d\mu(z)/|S(\epsilon, \frac{1}{r})|$ (respectively, $d\mu(z)/|T(t, \epsilon, \frac{1}{r})|$) are uniformly bounded, independently of ϵ and $\frac{1}{r}$ (respectively, t, ϵ and $\frac{1}{r}$). Observe that the functions $\{z^\nu\}$ are bounded in absolute value by 1 on each $S(\epsilon, \frac{1}{r})$ and $T(t, \epsilon, \frac{1}{r})$ since each domain is contained in $\overline{D}$. Since the order of each polynomial is bounded it will be enough to show that the coefficients of the polynomials are bounded uniformly.

In either case (for the S 's or the T 's) the situation is the following: There is a collection of subsets of $\mathbb{R}^2$, $\{S(\gamma)\}_{\gamma\in\Gamma}$ where i) Γ is a compact Hausdorff space, ii) Each $S(\gamma)$ has a non empty interior and there is a fixed compact set K so that $S(\gamma) \subset K$ for all $\gamma \in \Gamma$ and iii) $\{S(\gamma)\}_{\gamma\in\Gamma}$ is a continuous family in the sense that $|S(\gamma) \Delta S(\gamma_0)| \to 0$ as $\gamma \to \gamma_0$ in Γ.

The first two conditions are obvious. The proofs of iii) will be sketched at the end of this appendix. We will show first how the required conclusion, on the boundedness of the coefficients, follows from these three conditions.

For $f, g \in L^2(K)$ we define the family of inner products:

$$\langle f, g\rangle^\gamma = \frac{1}{|S(\gamma)|}\int_{S(\gamma)} f(z)g(z)d\mu(z),$$

and assume, for simplicity, that all functions are real-valued.

Let $\{f_j\}_{j=1,\dots,N}$ be a listing of the monomials $\{z^\nu\}_{|\nu|\le s}$. Let $G_0^\gamma = 1$ and

$$G_j^\gamma = \det\begin{bmatrix} \langle f_1, f_1\rangle^\gamma & \cdots & \langle f_1, f_j\rangle^\gamma \\ \vdots & & \vdots \\ \langle f_j, f_1\rangle^\gamma & \cdots & \langle f_j, f_j\rangle^\gamma \end{bmatrix}, \text{ if } j = 1, \dots, N.$$

The j^{th} Gram-Schmidt polynomial on $S(\gamma)$ is given by

$$\varphi_j^\gamma = \det\begin{bmatrix} \langle f_1, f_1\rangle^\gamma & \cdots & \langle f_1, f_j\rangle^\gamma \\ \vdots & & \vdots \\ \langle f_{j-1}, f_1\rangle^\gamma & \cdots & \langle f_{j-1}, f_j\rangle^\gamma \\ f_1 & \cdots & f_j \end{bmatrix} \Big/ \sqrt{G_{j-1}^\gamma\, G_j^\gamma}$$

where the determinant in the numerator is expanded formally in terms of cofactors of the last row. (See the Bateman manuscript [10], Vol. 2, p. 155 for a discussion of this method.)

We see that, from ii), $0 < S(\gamma) < \infty$ so the inner product is well defined for each γ. It follows from iii) that $<f_i , f_j>^\gamma$ is a continuous function of γ and so from i) we get that each $<f_i , f_j>^\gamma$ is a bounded function on Γ. From ii) we see that the $\{f_i\}$ are linearly independent on each $S(\gamma)$ which implies that $G_j^\gamma > 0$. But G_j^γ is a continuous function of the $<f_i , f_\ell>^\gamma$ and so it is continuous and positive on Γ and, hence, bounded below by a position on Γ. From the definition of φ_j^γ above, we see that the coefficients of these Gram-Schmidt polynomials are polynomial functions of the $<f_i , f_\ell>^\gamma$ divided by $\sqrt{G_{j-1}^\gamma \, G_j^\gamma}$ and so are bounded uniformly for $\gamma \in \Gamma$.

To complete the proof we need to see that the families $\{S(\epsilon , \frac{1}{r})\}$ and $\{T(t , \epsilon , \frac{1}{r})\}$ are continuous on their parameter sets. A sketch of this fact completes this appendix.

Observe first that

$$|T(t_1 , \epsilon , \tfrac{1}{r}) \,\Delta\, T(t_2 , \epsilon , \tfrac{1}{r})| \leq 2\pi|t_1 - t_2| ,$$

and

$$|T(t , \epsilon_1 , \tfrac{1}{r_1}) \,\Delta\, T(t , \epsilon_2 , \tfrac{1}{r_2})| \leq |S(\epsilon_1 , \tfrac{1}{r_1}) \,\Delta\, S(\epsilon_2 , \tfrac{1}{r_2})| .$$

Thus, if we show that $|S(\epsilon , \frac{1}{r}) \,\Delta\, S(\epsilon_0 , \frac{1}{r_0})| \to 0$ as $(\epsilon , \frac{1}{r}) \to (\epsilon_0 , \frac{1}{r_0})$ it follows that

$$|T(t , \epsilon , \tfrac{1}{r}) \,\Delta\, T(t_0 , \epsilon_0 , \tfrac{1}{r_0})| \to 0 \quad \text{as} \quad (t , \epsilon , \tfrac{1}{r}) \to (t_0 , \epsilon_0 , \tfrac{1}{r_0}) .$$

To show comintuity for the family $\{S(\epsilon , \frac{1}{r})\}$ we proceed in three steps:

I. Show first that $|S(\epsilon_1 , \frac{1}{r_1}) \,\Delta\, S(\epsilon_2 , \frac{1}{r_2})| = o(1)$ as $\epsilon_0 \to 0$, $0 \leq \epsilon_1 , \epsilon_2 \leq \epsilon_0$. From this it follows immediately that $|S(\epsilon , \frac{1}{r}) \,\Delta\, S(0 , \frac{1}{r_0})| \to 0$ as $(\epsilon , \frac{1}{r}) \to (0 , \frac{1}{r_0})$ for any r_0.

II. We may now assume that $0 < \epsilon_0 \leq 1$. We also assume that $0 \leq \frac{1}{r_0} < \frac{2}{2-\epsilon}$. A straight forward argument shows that $|S(\epsilon , \frac{1}{r}) \,\Delta\, S(\epsilon_0 , \frac{1}{r})| \to 0$ as

$(\varepsilon , \frac{1}{r}) \to (\varepsilon_0 , \frac{1}{r_0})$.

III. The case which remains is $0 < \varepsilon_0 \leq 1$, $\frac{1}{r_0} = \frac{2}{2-\varepsilon_0}$. A somewhat delicate argument which takes into account the fact that $\frac{2}{2-\varepsilon}$ is increasing and concave on $(0 , 1]$ completes the argument. One shows that $|S(\varepsilon , \frac{1}{r}) \Delta S(\varepsilon_0 , \frac{2}{2-\varepsilon_0})| \to 0$ as $(\varepsilon , \frac{1}{r}) \to (\varepsilon_0 , \frac{2}{2-\varepsilon_0})$.

Details are omitted.

§7. Appendix C. Some Calculations for §3. The purpose of this appendix is to establish two claims that occur in §3 in proof of Theorem (3.10) and Theorem (3.13) respectively.

Let $\alpha > 0$ be fixed. Let σ , z_0 , p , q , s and $\{\rho_k\}_{k=0}^n$ be given as in either theorem. The following conditions are satisfied:

i) $0 < \sigma < \frac{1}{2}$

ii) (a) If $0 < \rho_k \leq 1 - |z_0|$ then $\rho_k \sim \left[\frac{2^k\sigma}{(1-|z_0|)^{\alpha-1}}\right]^{\frac{1}{2}}$.

(b) If $1 - |z_0| \leq \rho_k \leq 1 + |z_0|$ then $\rho_k \sim (2^k\sigma)^{\frac{1}{1+\alpha}}$.

(c) $\rho_{k-1} < \rho_k \leq \rho_n = 1 + |z_0| \sim (2^n\sigma)^{\frac{1}{1+\alpha}}$, $1 \leq k \leq n$.

iii) $s \geq \max\{[\frac{1+\alpha}{p} - 2] , [2(\frac{1}{p} - 1)]\}$.

iv) $\varepsilon > \max\{\frac{1}{p} - 1 , \frac{s}{2} , \frac{s+1-\alpha}{1+\alpha}\}$.

v) $0 < p \leq 1$.

Proposition (7.1). If $0 \leq \ell \leq s$ then

$$\sum_{j=k}^{n} \frac{\rho_j^{(\ell+2)}}{2^{j(1+\varepsilon)}} \sim \frac{\rho_k^{(\ell+2)}}{2^{k(1+\varepsilon)}} .$$

Proposition (7.2). If $\ell > s$ then

$$\left\{\sum_{j=0}^{n} \frac{2^j}{\rho_j^{(\ell+2)p}}\right\}^{\frac{1}{p}} \sim \frac{1}{\rho_0^{(\ell+2)}} .$$

Remark. The $\{\rho_k\}$ are Euclidean radii of balls with ω_α measure $2^k\sigma$. Since ω_α and Euclidean measure are mutually comparable in the sense of Coifman and Fefferman [6] the series behave like geometric series so some such result must hold. The point of these two propositions is that the result holds for $0 \leq \ell \leq s$ in the first and for $\ell > s$ in the second.

Proof of (7.1). "$\geq$" is obvious. For "$\leq$" there are two cases.

Case I. $\rho_k \geq 1 - |z_0|$. Then $\rho_j \sim (2^j\sigma)^{\frac{1}{1+\alpha}}$, $k \leq j \leq n$. Since $\epsilon + 1 > \frac{s+2}{1+\alpha} \geq \frac{\ell+2}{1+\alpha}$, we have,

$$\sum_{j=k}^{n} \frac{\rho_j^{(\ell+2)}}{2^{j(1+\epsilon)}} \sim \sum_{j=k}^{n} \frac{(2^j\sigma)^{\frac{\ell+2}{1+\alpha}}}{2^{j(1+\epsilon)}} = \sigma^{\frac{\ell+2}{1+\alpha}} \sum_{j=k}^{n} 2^{j(\frac{\ell+2}{1+\alpha})-(1+\epsilon)} \sim \sigma^{\frac{\ell+2}{1+\alpha}} 2^{k(\frac{\ell+2}{1+\alpha})-(1+\epsilon)} =$$

$$= \frac{(2^k\sigma)^{\frac{\ell+2}{1+\alpha}}}{2^{k(1+\epsilon)}} \sim \frac{\rho_k^{(\ell+2)}}{2^{k(1+\epsilon)}} .$$

Case II. $\rho_k < 1 - |z_0|$. Then $\rho_k \sim \{2^k\sigma/(1-|z_0|)^{\alpha-1}\}^{\frac{1}{2}}$.

For $k \leq j \leq n$ there are two cases:

A. $\rho_j \leq 1 - |z_0|$, $\rho_j \sim \{2^j\sigma/(1-|z_0|)^{\alpha-1}\}^{\frac{1}{2}}$.

B. $1 - |z_0| \leq \rho_j \leq 1 + |z_0|$, $\rho_j \sim (2^j\sigma)^{\frac{1}{1+\alpha}}$.

Estimate for terms satisfying A . Note that $\epsilon > \frac{s}{2} \geq \frac{\ell}{2}$.

$$\sum_{\rho_k \leq \rho_j \leq 1-|z_0|} \frac{\rho_j^{(\ell+2)}}{2^{j(1+\epsilon)}} \leq C \left(\frac{\sigma}{(1-|z_0|)^{\alpha-1}}\right)^{\frac{\ell+2}{2}} \sum_{j=k}^{\infty} 2^{j(\frac{\ell+2}{2}-(1+\epsilon))}$$

$$= C \left(\frac{\sigma}{(1-|z_0|)^{\alpha-1}}\right)^{\frac{\ell+2}{2}} 2^{k(\frac{\ell+2}{2}-(1+\epsilon))} \sim \frac{\rho_k^{(\ell+2)}}{2^{k(1+\epsilon)}} .$$

Estimate for terms satisfying B. Since $\rho_j \geq 1 - |z_0|$ and $\rho_j \sim (2^j\sigma)^{\frac{1}{1+\alpha}}$ we get $2^j \geq C\{(1-|z_0|^{1+\alpha}/\sigma\}$. Thus,

$$\sum_{\rho_j \geq 1-|z_0|} \frac{\rho_j^{(\ell+2)}}{2^{j(1+\epsilon)}} \leq C\,\sigma^{\frac{\ell+2}{1+\alpha}} \sum_{2^j \geq C'\frac{(1-|z_0|)^{1+\alpha}}{\sigma}} 2^{j(\frac{\ell+2}{1+\alpha} - (1+\epsilon))}$$

$$\leq C\,\sigma^{\frac{\ell+2}{1+\alpha}} \left[\frac{(1-|z_0|)^{1+\alpha}}{\sigma}\right]^{(\frac{\ell+2}{1+\alpha} - (1+\epsilon))}$$

$$= C\,\sigma^{1+\epsilon}\,(1-|z_0|)^{(\ell+2)-(1+\epsilon)(1+\alpha)}$$

We need to show that this last term is dominated by

$$C\,\frac{\rho_k^{(\ell+1)}}{2^{k(1+\epsilon)}} \sim C\left[\frac{2^k\sigma}{(1-|z_0|)^{\alpha-1}}\right]^{\frac{\ell+2}{2}} 2^{-k(1+\epsilon)}$$

But, $\sigma^{(1+\epsilon)}\,(1-|z_0|)^{(\ell+2)-(1+\epsilon)(1+\alpha)}\ 2^{k(1+\epsilon)}\ (1-|z_0|^{(\alpha-1)(\frac{\ell}{2}+1)}\ (2^k\sigma)^{-(\frac{\ell+2}{2})}$

$$= \left(\frac{2^k\sigma}{(1-|z_0|)^{\alpha+1}}\right)^{\epsilon-\frac{\ell}{2}} = \left(\frac{2^k\sigma}{(1-|z_0|)^{\alpha+1}} \cdot \frac{1}{(1-|z_0|)^2}\right)^{\epsilon-\frac{\ell}{2}}$$

$\sim\left(\frac{\rho_k}{1-|z_0|}\right)^{2\epsilon-\ell} \leq 1$, and the inequality follows.

Proof of (7.2). The proof proceeds just as for (7.1). "$\geq$" is obvious so we only need to show "$\leq$". Note first that $\ell \geq s+1$ so that $\frac{\ell+2}{1+\alpha} - \frac{1}{p} \geq \frac{s+3}{1+\alpha} - \frac{1}{p} > 0$, and $\ell \geq s+1 > 2(\frac{1}{p} - 1)$.

Case I. $\rho_0 \geq 1 - |z_0|$. Then $\rho_j \sim (2^j\sigma)^{\frac{1}{1+\alpha}}$, $0 \leq j \leq n$.

$$\sum_{j=0}^{n} \left(\frac{\rho_0}{\rho_j}\right)^{(\ell+2)p} 2^j \leq C \sum_{j=0}^{\infty} 2^{-jp(\frac{\ell+2}{1+\alpha} - \frac{1}{p})} \leq C\ .$$

Case II. $\rho_0 < 1 - |z_0|$. Then $\rho_0 \sim (\sigma/(1-|z_0|)^{\alpha-1})^{\frac{1}{2}}$. For $0 \leq j \leq n$ there are two cases:

A. $\rho_j \leq 1 - |z_0|$, $\rho_j \sim (2^j\sigma/(1 - |z_0|)^{\alpha-1})^{1/2}$.

B. $1 - |z_0| \leq \rho_j \leq 1 + |z_0|$, $\rho_j \sim (2^j\sigma)^{\frac{1}{1+\alpha}}$.

Estimates for terms satisfying A .

$$\sum_{\rho_j \leq 1-|z_0|} \left(\frac{\rho_0}{\rho_j}\right)^{(\ell+2)p} 2^j \leq C \sum_{j=0}^{\infty} 2^{-jp(\frac{\ell+2}{2} - \frac{1}{p})} \leq C .$$

Estimates for terms satisfying B. If $\rho_j \geq 1 - |z_0|$ then $z^j \geq C(1 - |z_0|)^{\alpha+1}/\sigma$.)

Note also that

$$\left(\frac{\rho_0}{\rho_j}\right)^{(\ell+2)p} 2^j = \left(\frac{\sigma}{(1-|z_0|)^{\alpha-1}} \cdot \frac{1}{(2^j\sigma)^{1/(1+\alpha)}}\right)^{(\ell+2)p} 2^j$$

$$= \left\{\frac{\sigma}{(1-|z_0|)^{\alpha+1}}\right\}^{\frac{\alpha-1}{\alpha+1} \cdot \frac{(\ell+2)}{2} \circ p} 2^{-jp(\frac{\ell+2}{\alpha+1} - \frac{1}{p})} .$$

Thus, $\displaystyle\sum_{\rho_j \geq 1-|z_0|} \left(\frac{\rho_0}{\rho_j}\right)^{(\ell+2)p} 2^j$

$$\leq C \left\{\frac{\sigma}{(1-|z_0|)^{\alpha+1}}\right\}^{\frac{\alpha-1}{\alpha+1} \cdot \frac{(\ell+2)}{p} \cdot p} \sum_{2^j \geq C \frac{(1-|z_0|)^{\alpha+1}}{\sigma}} 2^{-jp(\frac{\ell+2}{\alpha+1} - \frac{1}{p})}$$

$$\leq C \left\{\frac{\sigma}{(1-|z_0|)^{\alpha+1}}\right\}^{\frac{\alpha-1}{\alpha+1} \cdot \frac{\ell+2}{2} \cdot p} \left\{\frac{\sigma}{(1-|z_0|)^{\alpha+1}}\right\}^{(\frac{\ell+2}{\alpha+1} - \frac{1}{p})p}$$

$$= C \left\{\frac{\sigma}{(1-|z_0|)^{\alpha+1}}\right\}^{(\frac{\ell+2}{2} - \frac{1}{p})p} \sim \left\{\frac{\rho_0}{1-|z_0|}\right\}^{(\ell-2(\frac{1}{p} - 1))p}$$

But $\frac{\rho_0}{1-|z_0|} \leq 1$ and $\ell > 2(\frac{1}{p} - 1)$ and the inequality follows.

Corollary (7.3). If $\ell > s$ then $\left(\frac{\rho_0}{\rho_n}\right)^{\ell+2} \frac{1}{\sigma^{1/p}} \leq C$.

Proof. Look at the last term of the series in (7.2).

§8. Appendix D. Some Miscellany Regarding Atomic $H^p(\mathbb{R}^n)$ and its Dual. The aim of this appendix is to obtain a few facts about $H^p(\mathbb{R}^n)$ and its dual that are easy consequences of the atomic characterization of $H^p(\mathbb{R}^n)$. Within the context of this paper, our purpose is to obtain the result on the local and global behaviour of representatives of the linear functionals in the dual of H^p that we remark upon in the discussion following (2.4) and that we use at the end of the proofs of Theorem (2.9).

Recall that for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\ell = (\ell_1, \dots, \ell_n)$ a multi-index of non-negative integers, $x^\ell = x_1^{\ell_1} \dots x_n^{\ell_n}$. We use the conventions: $0 = (0, \dots, 0)$, $0^0 = 1$, and $\beta = (\beta_1, \dots, \beta_n)$ is also a multi-index of non-negative integers, $\binom{\ell}{\beta} = \binom{\ell_1}{\beta_1} \dots \binom{\ell_n}{\beta_n}$. Recall also that $|\ell| = \ell_1 + \ell_2 + \dots + \ell_n$.

Lemma (8.1). Suppose k is a non-negative integer and φ has the property

$$\int_{\mathbb{R}^n} \varphi(x)\, x^\ell\, dx = \begin{cases} 1, & \ell = 0 \\ 0, & 0 < |\ell| \le k \end{cases}.$$

Then if σ is a point on the unit sphere in $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} \Delta_\sigma^{k+1} \varphi(x)\, x^\ell\, dx = 0, \quad 0 \le \ell \le k.$$

Proof. $(x + h)^\ell = \sum_{\alpha+\beta=\ell} \binom{\ell}{\beta} x^\alpha h^\beta$. Thus, if $0 \le |\ell| \le k$, $\int_{\mathbb{R}^n} \tau_h \varphi(x)dx =$

$$= \int_{\mathbb{R}^n} \varphi(x)(x + h)^\ell\, dx = \sum_{\alpha+\beta=\ell} \binom{\ell}{\beta} h^\beta \int_{\mathbb{R}^n} \varphi(x) x^\alpha\, dx = h^\ell.$$

Recall that $\Delta_\sigma^{k+1} = (1 - \tau_\sigma)^{k+1} = \sum_{s=0}^{k+1} \binom{k+1}{s}(-1)^s\, \tau_{s\sigma}$.

Thus,

$$\int_{\mathbb{R}^n} \Delta_\sigma^{k+1} \varphi(x) x^\ell\, dx = \sum_{s=0}^{k+1} \binom{k+1}{s}(-1)^s \int_{\mathbb{R}^n} \tau_{s\sigma}\, \varphi(x) x^\ell\, dx$$

$$= \sum_{s=0}^{k+1} \binom{k+1}{s}(-1)^s\, s^{|\ell|}\, \sigma^\ell.$$

But this last sum if zero if $0 \le |\ell| \le k$. To see this just apply

$(\frac{\partial}{\partial t})^{\ell} = (\frac{\partial}{\partial t})^{\ell_1} \ldots (\frac{\partial}{\partial t_n})^{\ell_n}$ to $(1 - C^{\sigma \cdot t})^{k+1}$ and evaluate at $t = 0$. This completes the proof of the lemma.

Let $\mathcal{P}_s$ be the collection of polynimials, with complex coefficients, of degree at most s.

This next result is known in several contexts and we include it just for completeness.

Lemma (8.2). The norm

$$\|g\|_{L(\beta,q',s)} = \sup_{Q \subset \mathbb{R}^n} |Q|^{-\beta} \left\{ \frac{1}{|Q|} \int_Q |g - P_Q g|^{q'} \right\} dx^{\frac{1}{q'}}$$

is equivalent to

$$\sup_{Q \subset \mathbb{R}^n} |Q|^{-\beta} \left\{ \inf_{P \in \mathcal{P}_s} \frac{1}{|Q|} \int_Q |g - P|^{q'} \, dx \right\}^{\frac{1}{q'}} .$$

Proof. The polynomial $P_Q g$ is the unique polynomial in $\mathcal{P}_s$ with the property that $\int_Q (g - P)x^{\nu} \, dx = 0$ if $|\nu| \leq s$ and so P_Q is the "Gram-Schmidt" polynomial for g on Q for the monomials up to order s. We discussed this in §2 where we showed that $\sup_{x \in Q} |P_Q g(x)| \leq \frac{C}{|Q|} \int_Q |g| \, dx$. Thus, if $P \in \mathcal{P}_s$ $\left\{ \frac{1}{|Q|} \int_Q |g - P_Q g|^{q'} \, dx \right\}^{\frac{1}{q'}}$ $\leq \left\{ \frac{1}{|Q|} \int_Q |g - P|^{q'} \, dx \right\}^{\frac{1}{q'}} + \left\{ \frac{1}{|Q|} |P - P_Q g|^{q'} \right\}^{\frac{1}{q'}}$. But $P - P_Q g = P_Q(P - g)$, so

$$\left\{ \frac{1}{|Q|} \int_Q |P - P_Q g|^{q'} \, dx \right\}^{\frac{1}{q'}} = \left\{ \frac{1}{|Q|} \int_Q |P_Q(P - g)|^{q'} \, dx \right\}^{\frac{1}{q'}}$$

$$\leq \frac{C}{|Q|} \int_Q |P - g| dx \leq C \left\{ \frac{1}{|Q|} \int_Q |P - g|^{q'} \, dx \right\}^{\frac{1}{q'}} .$$

Thus,

$$\left\{ \frac{1}{|Q|} \int_Q |g - P_Q g|^{q'} \, dx \right\}^{\frac{1}{q'}} \leq (1 + C) \inf_{P \in \mathcal{P}_s} \left\{ \frac{1}{|Q|} \int_Q |P - g|^{q'} \, dx \right\}^{\frac{1}{q'}} .$$

The converse inequality is obvious. This completes the proof.

Corollary (8.3). If $[g] \in L(\beta, q', k)$, $\beta > 0$, $k \geq [n\beta]$, $1 \leq q' \leq \infty$ then g

is continuous.

Proof. It follows trivially that $g \in L(\beta , 1 , k)$. From (8.2) it follows that the restriction of g to any finite ball, Q_0 , is in the Campanato space $\mathcal{L}_k^{(1,(\beta+1)n)}(Q_0)$ (see [2]) and this implies that g is continuous.

Theorem (8.4). *If* $k \geq [n(\frac{1}{p} - 1)]$, $p < 1$ *and* $h \in \mathbb{R}^n$ *then*

$$\Delta_h^{k+1}\delta \in H^{p,\infty,k}(\mathbb{R}^n) \quad \text{and} \quad \|\Delta_h^{k+1}\delta\|_{H^{p,\infty,k}} \leq A|h|^{n(\frac{1}{p}-1)},$$

where A *is independent of* h .

Proof. It will suffice to show that $\Delta_\sigma^{k+1}\delta \in H^{p,\infty,k}(\mathbb{R}^n)$ for some fixed point σ in the unit sphere in $\mathbb{R}^n$. Fix a function φ that satisfies the conditions of Lemma (8.1) with the additional requirement that φ is supported on $\{|x| \leq 1\}$. Let $\varphi_\nu(x) = 2^{n\nu}\varphi(2^\nu x)$. Then

$$\Delta_\sigma^{k+1}\delta = \Delta_\sigma^{k+1}\varphi_0 + \sum_{\ell=0}^{k+1} (-1)^\ell \binom{k+1}{\ell} \sum_{\nu=1}^{\infty} \tau_{\ell\sigma}(\varphi_\nu - \varphi_{\nu-1}) . \tag{8.5}$$

This series clearly converges as a measure, and all terms are supported on a fixed compact set, $\{|x| \leq k + 2\}$. According to Corollary (8.3) elements in $L(\frac{1}{p} - 1 , 1 , [n(\frac{1}{p} - 1)])$ are represented by continuous functions and so (8.5) converges as a linear functional on $L(\frac{1}{p} - 1 , 1 , [n(\frac{1}{p} - 1)])$.

From Lemma (8.1) we conclude that $\Delta_\sigma^{k+1}\varphi_0 = \lambda_0 a_0$ where a_0 is a (p , ∞ , k)-atom and $|\lambda_0| \leq C'$. A trivial calculation shows that $\tau_{\ell\sigma}(\varphi_\nu - \varphi_{\nu-1}) = \lambda_{\ell\nu} a_{\ell\nu}$ where $a_{\ell\nu}$ is a (p , ∞ , k)-atom centered at $\ell\sigma$ and $|\lambda_{\ell\nu}| \leq C' 2^{\nu n/(1-\frac{1}{p})}$. It follows that (8.5) is a decomposition of $\Delta_\sigma^{k+1}\delta$ into (p , ∞ , k)-atoms and so $\Delta_\sigma^{k+1}\delta \in H^{p,\infty,k}(\mathbb{R}^n)$. This completes the proof.

Corollary (8.6). If $[g] \in L(\beta , q' , k)$, $\beta > 0$, $1 \leq q' \leq \infty$, $k \geq [n\beta]$, then there is a constant $A > 0$ independent of g and of $h \in \mathbb{R}^n$ such that

$$|\Delta_h^{k+1} g(x)| \leq A\|g\|_{L(\beta,q',k)} |h|^{n\beta} .$$

Proof. Write $\beta = \frac{1}{p} - 1$, $0 < p < 1$. We see that $[g] \in L(\frac{1}{p} - 1 , 1 , k)$ and $\|g\|_{L(\frac{1}{p}-1,1,k)} \leq \|g\|_{L(\beta,q',k)}$. Thus, we may pair g with $f \in H^{p,\infty,k}$ and obtain $|\langle f , g\rangle| \leq C \|f\|_{H^{p,\infty,k}} \|g\|_{L(\beta,q',k)}$ since $L(\frac{1}{p} - 1 , 1 , k)$ is contained in the dual of $H^{p,\infty,k}$. Observe that $\Delta_h^{k+1} g(x) = (g * \Delta_h^{k+1}\delta)(x)$ and use (8.4). This completes the proof.

Observation (8.7). If $[g] \in L(0 , q' , 0)$, $1 \leq q' \leq \infty$ then $g \in BMO$ and $\|g\|_{BMO} \leq C \|g\|_{L(0,q',0)}$.

Proof. We see that $[g] \in L(0 , 1 , 0)$ and $\|g\|_{L(0,1,0)} \leq \|g\|_{L(0,q',0)}$. But,

$$\|g\|_{BMO} = \left\{ \sup_{Q \subset \mathbb{R}^n} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |g(x) - c| dx \right\} .$$

The result now follows from (8.2) and the definition of $\|g\|_{L(0,1,0)}$.

The following is a very limited extension of (8.7) that is useful in our development. It will be strengthened considerable, later in this appendix.

Remark (8.8). If $[g] \in L(0 , q' , k)$, $k \geq 0$ then the restriction of g to any finite ball is in BMO .

Proof. We argue as in (8.7) and obtain that

$$\sup_{Q \subset \mathbb{R}^n} \inf_{P \in \mathcal{P}_k} \frac{1}{|Q|} \int_Q |g(x) - P(x)| dx < \infty .$$

But S. J. Berman [1] has shown that this implies that the restriction of g to any finite ball is in BMO .

Proposition (8.9). If $[g] \in L(\beta , q' , [n(\beta)])$, $\beta > 0$, $1 \leq q' \leq \infty$, then g is bounded and $g(x) = 0(|x|^{n\beta})$ as $|x| \to \infty$ if $n\beta$ is not an integer; $g(x) = 0(|x|^{n\beta} \log |x|)$ as $|x| \to \infty$ if $n\beta$ is an integer.

Proof. The local boundedness is immediate from (8.3). In particular, there is an $A > 0$ such that

(8.10) $$|g(x)| \leq A \ , \text{ if } |x| \leq 1 \ .$$

Let $k = [n\beta]$. From Corollary (8.6) we obtain

(8.11) $$|\Delta_h^{k+1} g(x)| \leq A|h|^{n\beta} \ , \ x \ , \ h \in \mathbb{R}^n \ .$$

If $k = 0$ then from (8.10) and (8.11) we have, $|g(x)| \leq |g(0)| + |g(x) - g(0)| \leq$ $\leq A + A|x|^{n\beta} = O(|x|^{n\beta})$ as $|x| \to \infty$ and we are done. For $k > 0$ we will show how to "pull back" to a smaller k . We need the following combinatorial identity:

(8.12) $$\Delta_h^k = 2^k \Delta_{h/2}^k + \sum_{\ell=1}^{k} (-1)^{\ell} \binom{k}{\ell} 2^{k-\ell} \Delta_{h/2}^{k+\ell} \ .$$

To see this, observe:

$$\Delta_h^k = (1 - \tau_h)^k = (1 - \tau_{\frac{h}{2}})^k (1 + \tau_{\frac{h}{2}})^k = \Delta_{h/2}^k (2 - (1 - \tau_{\frac{h}{2}}))^k$$

$$\Delta_{h/2}^k (2 - \Delta_{h/2})^k = \Delta_{h/2}^k \sum_{\ell=0}^{k} \binom{k}{\ell} 2^{k-\ell} (-1)^{\ell} \Delta_{h/2}^{\ell}$$

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} 2^{k-\ell} (-1)^{\ell} \Delta_{h/2}^{k+\ell} \ .$$

If we set $G_k(x) = \sum_{\ell=1}^{k} (-1)^{\ell} \binom{k}{\ell} 2^{k-\ell} \Delta_{x/2}^{k+\ell} g(0)$, we have from (8.12) $\Delta_x^k g(0) =$ $= G_k(x) + 2^k \Delta_{x/2}^k g(0)$, and so for every positive integer N we obtain:

(8.13) $$\Delta_x^k g(0) = G_k(x) + 2^k G_k(\tfrac{x}{2}) + \ldots + 2^{Nk} G_k(\tfrac{x}{2^N}) + 2^{(N+1)k} \Delta_{\frac{x}{2^N}}^k g(0) \ .$$

We fix $|x| > 1$ and choose N to be the smallest positive integer such that $k|x| \leq 2^N$. We see that $N \sim \log |x|$ as $|x| \to \infty$ and from (8.11) we have $|G_k(x)| \leq A|x|^{n\beta}$. From (8.13) we obtain:

(8.14) $$|\Delta_x^k g(0)| \leq A|x|^{n\beta} (\Sigma_{\ell=1}^{N} 2^{\ell(k-n\beta)}) + A|x|^k \ .$$

If $n\beta$ is not an integer then $k < n\beta$, the series converges and we have

$|\Delta_x^k g(0)| \le A|x|^{n\beta} + A|x|^k = O(|x|^{n\beta})$ as $|x| \to \infty$. If $n\beta$ is an integer then $k = n\beta$ and $|\Delta_x^k g(0)| \le AN|x|^{n\beta} + A|x|^{n\beta} = O(|x|^{n\beta} \log |x|)$ as $|x| \to \infty$. If $k = 1$ we have $|g(x)| \le |g(0)| + |\Delta_x g(0)|$ and we are done, as with the case $k = 0$. If $k > 1$ we iterate the argument to obtain the desired result. Thus, if $n\beta$ is not an integer we have for $1 \le k - t < k$, $G_{k-t}(\frac{x}{2^\ell}) \le A(\frac{|x|}{2^\ell})^{n\beta} = A2^{-\ell n\beta}|x|^{n\beta}$ and if $n\beta$ is an integer, $G_{k-t}(\frac{x}{2^\ell}) \le A(\frac{|x|}{2^\ell})^{n\beta} \log (\frac{|x|}{2^\ell}) \le A|x|^{n\beta} \log |x| \, 2^{-\ell n\beta} + A|x|^{n\beta} \ell 2^{-\ell n\beta}$; obtaining $|\Delta_x^{k-t} g(0)| \le A|x|^{n\beta}(|x| > 1)$ in the first case and $|\Delta_x^{k-t} g(0)| \le A|x|^{n\beta} \log |x|$ $(|x| > 1)$ in the second case. The result follows.

<u>Remark (8.15)</u>. <u>If</u> $[g] \in L(\beta, q', k)$, $\beta > 0$, $k > [n\beta]$, $1 \le q' \le \infty$ <u>then</u> $g(x) = O(|x|^k)$ <u>as</u> $|x| \to \infty$.

Proof. We may restrict attention to $q' = 1$. Since $k > [n\beta]$ we have $k > n\beta$. We use (8.14) and we have

$$|\Delta_x^k g(0)| \le A|x|^{n\beta} \, 2^{N(k-n\beta)} + A|x|^k \le A|x|^k \, , \quad |x| > 1 \, .$$

(Recall that $2^N \sim |x|$ as $|x| \to \infty$). Thus from $\Delta_x^{k+1} g(0) = O(\Delta^{n\beta})$ as $|x| \to \infty$ we have obtained $\Delta_x^k g(0) = O(|x|^k)$. Since $k = (k - 1) + 1$, we can argue as in (8.9) to obtain $\Delta_x g(0) = O(|x|^k)$ as $|x| \to \infty$ and hence, $g(x) = O(|x|^k)$ as $|x| \to \infty$.

To obtain these results we have only used the definitions of the various spaces and that part of (2.7) that tells us that $L(\frac{1}{p} - 1, 1, k)$ is embedded continuously in the dual of $H^{p,\infty,k}$. With (8.7) and (8.9) we have enough information on the growth of functions that represent members of $L(\frac{1}{p} - 1, \infty, [n(\frac{1}{p} - 1)])$ to establish that the decompositions of atoms in (2.8) and molecules in (2.9) represent the atom (respectively, the molecule) not only as a function, but also as a linear functional on a $L(1 - \frac{1}{p}, \infty, [n(\frac{1}{p} - 1)])$ space. It is the behaviour at infinity that is important here, since the local behaviour of $[g] \in L(\beta, \infty, k)$ for any $\beta \ge 0$, $k \ge [n\beta]$ is immediate from the definition. Clearly, g is bounded on any finite ball.

Once (2.8) is established we have that any two spaces $L(\frac{1}{p}-1, q_1', s_1)$ and $L(\frac{1}{p}-1, q_2', s_2)$ are isomorphic as spaces of linear functionals on atomic-H^p, provided the indicies (p, q_1, s_1) and (p, q_2, s_2) are admissible indicies for atoms. If we now use the full force of (8.7), (8.8), (8.9), and (8.15) we can test on appropriate atoms and find that the spaces agree as spaces of functions in the following sense:

<u>(8.16).</u> <u>Suppose</u> $[g_1] \in L(\frac{1}{p}-1, q_1', s_1)$ <u>and</u> $[g_2] \in L(\frac{1}{p}-1, q_2', s_2)$ <u>where</u> (p, q_1, s_1) <u>and</u> (p, q_2, s_2) <u>are admissible sets of indicies for</u> p-<u>atoms. Then</u> $[g_1]$ <u>corresponds to</u> $[g_2]$ <u>under the isomorphism of the two spaces as linear functionals on</u> H^p <u>if and only if</u> $[g_1] = [g_2] \bmod \mathcal{P}_{\max[s_1, s_2]}$.

If we simply view the spaces $L(\beta, q', k)$ as collections of functions we have:

<u>(8.17).</u> <u>If</u> $1 \leq q' \leq \infty$ <u>and</u> $k \geq 0$ <u>then</u> $L(0, q', k) = BMO \bmod (\mathcal{P}_k)$. <u>If</u> $1 \leq q' \leq \infty$, $\beta > 0$, $k \geq [n\beta]$ <u>then</u> $L(\beta, q', k) = L(\beta, 1, [n\beta]) \bmod \mathcal{P}_k$.

We omit the proofs. Note also that these identifications are continuous in the obvious sense. For example: $g \in L(0, 1, k)$ if and only if there is a polynomial P in $\mathcal{P}_k$ such that $g - P \in BMO$ and $\|g - P\|_{BMO}$ is equivalent to $\|g\|_{L(0,1,k)}$. Details of statements and proofs can be filled in by the interested reader.

<u>§9.</u> <u>Appendix E.</u> <u>The Multiplier Theorems.</u> We begin with some basic estimates for the Fourier transforms of atoms for $H^p(\mathbb{R}^n)$.

<u>Lemma (9.1).</u> <u>If</u> a <u>is a</u> $(p, 2, s_1)$-<u>atom centered at the origin and</u> $0 \leq |\alpha| \leq s_1$, <u>then</u>

(i) $$|D^{\alpha}\hat{a}(x)| \leq \frac{C|x|^{s_1+1-\alpha}}{\|a\|_2^{\{(\frac{s_1+1}{n}+\frac{1}{2})/(\frac{1}{p}-\frac{1}{2})\}-1}}$$

(ii) $$\| |D^{\alpha}\hat{a}|^2 \|_{r'} \leq \frac{C}{\|a\|_2^{\{(\frac{2|a|}{n}+\frac{1}{r})/(\frac{1}{p}-\frac{1}{2})\}-2}} , \quad \frac{1}{r}+\frac{1}{r'} = 1 , \quad 1 \leq r \leq \infty$$

Proof. Let Q be the ball on which a is supported. Let $d = 1/(\frac{1}{p} - \frac{1}{2})$. The basic estimate that follows from the definition of an atom is

(9.2) $$|Q| \leq \|a\|^{-d} .$$

Let P be any polynomial of degree at most $s_1 - |\alpha|$ in ξ . Then

$$D^{\alpha}\hat{a}(x) = \int_Q a(\xi)(-2\pi i\xi)^{\alpha} e^{-2\pi ix\cdot\xi} d\xi$$

$$= \int_Q a(\xi)(-2\pi i\xi)^{\alpha} [e^{-2\pi ix\cdot\xi} - P(\xi)]d\xi .$$

Choose P to be the Taylor polynomial in ξ of degree $s_1 - |\alpha|$ of $e^{-2\pi ix\cdot\xi}$ about the origin. Then

(9.3) $$\begin{aligned} |D^{\alpha}\hat{a}(x)| &\leq C\ |x|^{s_1+1-|\alpha|} \int_Q |a(\xi)||\xi|^{s_1+1} d\xi \\ &\leq C\ |x|^{s_1+1-|\alpha|} |Q|^{\frac{s_1+1}{n}+\frac{1}{2}} \|a\|_2 . \end{aligned}$$

We introduce (9.2) into (9.3) and (i) follows.

If $r = \infty$ then $r' = 1$ and we have

$$\begin{aligned} \int |D^{\alpha}\ \hat{a}(x)|^2\ dx &= C'\int_Q |\xi^{\alpha}|^2|a(\xi)|^2\ ds \\ &\leq C\ |Q|^{\frac{2|\alpha|}{n}} \|a\|_2^2 . \end{aligned}$$

If $r = 1$ then $r' = \infty$ and we have

$$|D^{\alpha}\hat{a}(x)| \leq C\ (\int_Q |\xi|^{\alpha}\ |a(\xi)|d\xi)^2 \leq C\ |Q|^{\frac{2|\alpha|}{n}+1} \|a\|_2^2 .$$

We interpolate between (9.4) and (9.5) use (9.2) and (ii) follows.

It is interesting to note that the analogue of (9.1)(ii) for differences is also valid.

Lemma (9.6). If a is a $(p\,,\,2\,,\,\bar{t}\,)$-atom centered at the origin then for $0 \leq \ell \leq \bar{t}$

$$\sup_{h\in\mathbb{R}^n} \| |\Delta_h^{\ell}\hat{a}|^2 \|_{r'} \leq \frac{C|h|^{2\ell}}{\|a\|_2^{\{(\frac{2\ell}{n}+\frac{1}{r})/(\frac{1}{p}-\frac{1}{2})-2}} ,$$

$\frac{1}{r} + \frac{1}{r'} = 1\,,\ 1 \leq r \leq \infty\,.$

Proof. This follows exactly as in (9.1)(ii) where, for $k \neq 0$, we use the identity,

$$\Delta_h^{\ell}\hat{a}(x) = \int_Q a(\xi)(1 - e^{2\pi i h\cdot\xi})^{\ell}\, e^{-2\pi i x\cdot\xi}\, d\xi\ ,$$

from which we obtain as before,

$$(9.7)\qquad \left|\frac{\Delta_h^{\ell}\hat{a}(x)}{|h|^{\ell}}\right|^2 \leq (C\int_Q |\xi|^{\ell}|a(\xi)|d\xi)^2 \leq C'|Q|^{\frac{2\ell}{n}}\ \|a\|_2^2\ ,$$

and

$$(9.8)\qquad \int \frac{|\Delta_h^{\ell}\hat{a}(x)|^2}{|h|^{2\ell}}\, dx \leq C\int_Q |\xi|^{2\ell}\ |a(\xi)|^2\ d\xi \leq C\ |Q|^{\frac{2\ell}{n}}\ \|a\|_2^2\ .$$

Several further observations are in order for $(p\,,\,2\,,\,\bar{t})$-atoms centered at the origin. Let ∇ be the gradient.

$$|\Delta_h^{\ell}\hat{a}(x)| \leq C\ |h|^{\ell}|\nabla^{\ell}\hat{a}(x)|$$

$$(9.9)\qquad \leq C\ |h|^{\ell}\ \frac{|\zeta|^{\bar{t}+1-\ell}}{\|a\|_2^{\{(\frac{\bar{t}+1}{n}+\frac{1}{2})/(\frac{1}{p}-\frac{1}{2})\}-1}}\ ,$$

where if $|h| \leq A|x|$ then $|\zeta| = O(|x|)$ and if $|x| \leq A|h|$ then $|\zeta| = O(|h|)$.

Note also:

(9.10) $$\Delta_h^{\ell}\hat{a}(x) \leq \frac{C}{\|a\|_2^{\{\frac{1}{2}/(\frac{1}{p}-\frac{1}{2})\}-1}} ,$$

which follows trivially from (9.1)(ii) with $\alpha = 0$ and $r' = \infty$.

A remarkable amount of information can be derived from these elementary facts. For example, from Lemma (9.1) with $\alpha = 0$ we have for a $(p, 2, s_1)$-atom centered at the origin

(9.11) $$|\hat{a}(x)| \leq C\,|x|^{s_1+1}/\|a\|_2^{\{(\frac{s_1+1}{n}+\frac{1}{2})/(\frac{1}{p}-\frac{1}{2})\}-1}$$

(9.12) $$|\hat{a}(x)| \leq C\,/\|a\|_2^{\{\frac{1}{2}/(\frac{1}{p}-\frac{1}{2})\}-1} .$$

Note that the exponent of $\|a\|_2$ in (9.11) is positive and in (9.12) is negative. We use (9.11) for $\|a\|_2^d \geq |x|^n$ and (9.12) for $\|a\|_2^d \leq |x|^n$ $(d = (\frac{1}{p}-\frac{1}{2})^{-1})$ and we have

(9.13) $$|\hat{a}(x)| \leq C\,|x|^{n(\frac{1}{p}-1)} .$$

<u>Proposition (9.14).</u> <u>If</u> $f \in H^p(\mathbb{R}^n)$ <u>then (in the sense of tempered distribution)</u> $\hat{f}$ <u>is a continuous function and there is a constant</u> $C > 0$, <u>independent of</u> f <u>such that</u> $|\hat{f}(x)| \leq C\,\|f\|_{H^p}\,|x|^{n(\frac{1}{p}-1)}$.

Proof. If we interpret $f \in H^p$ and atoms as tempered distributions, then if $f = \Sigma\lambda_k a_k$ is a decomposition of f in terms of $(p, 2, s)$-atoms, then, clearly, the series also converges as a tempered distribution and, hence, so does $\hat{f} = \Sigma\lambda_k\hat{a}_k$. Strictly speaking (9.13) applies only to atoms centered at the origin, but for other atoms we only need to multiply by an exponential of the form $e^{2\pi i x\cdot\xi_0}$, and so (9.13) holds uniformly for all $(p, 2, s)$-atoms. An atom is in L^1 so $\hat{a}$ is continuous. From this estimate and: $\Sigma|\lambda_k| \leq \{\Sigma|\lambda_k|^p\}^{\frac{1}{p}} < \infty$ we obtain that $\Sigma\lambda_k\,\hat{a}_k(x)$ converges uniformly on compact sets so $\hat{f}$ is a continuous function. Now observe that:

(9.15) $$|\hat{f}(x)| \leq C \; \Sigma |\lambda_k| |x|^{n(\frac{1}{p}-1)} \leq C \{\Sigma |\lambda_k|^p\}^{\frac{1}{p}} |x|^{n(\frac{1}{p}-1)} .$$

But $\|f\|_{H^p} = \inf \{\Sigma|\lambda_k|^p\}^{\frac{1}{p}}$ over all such decompositions and the result follows.

Let us now show that from (9.14) it follows that if m is a bounded multiplier on $H^p(\mathbb{R}^n)$ then m is a bounded function. By a multiplier on H^p we mean, as usual, a measureable function m such that if $f \in H^p(\mathbb{R}^n)$ then there is a constant $M \geq 0$ such that

(9.16) $$\|(m\hat{f})^{\vee}\|_{H^p} \leq M\|f\|_{H^p} .$$

The infimum over all such M is called the norm of the multiplier.

For f a function and $t > 0$ let $f_t(x) = t^{-n/p} f(x/t)$, and extend the map $f \to f_t$ to H^p in the obvious way. It is not difficult to see that

(9.17) $$\|f_t\|_{H^p} = \|f\|_{H^p} , \; f \in H^p .$$

This follows directly from the observation that if a is a (p, q, s)-atom supported on the ball Q, then a_t is a (p, q, s)-atom supported on Q_t, the dilation of Q by t. Note that $|Q_t|^{-p} [\frac{1}{|Q_t|} \int_{Q_t} |a_t|^q \, dx]^{\frac{1}{q}} =$ $|Q|^{-p} [\frac{1}{|Q|} \int_Q |a|^q \, dx]^{\frac{1}{q}}$.

Furthermore,

(9.18) $$\hat{f}_t(x) = t^{n(1-\frac{1}{p})} \hat{f}(tx) .$$

From (9.16), (9.17) and (9.14) we have

(9.19) $$|m(x)\hat{f}_t(x)| \leq M\|f\|_{H^p} |x|^{n(\frac{1}{p}-1)} .$$

For $x \neq 0$ let $x' = x/|x|$. if we let $t = |x|^{-1}$ and use (9.18) we have

(9.20) $$|m(x)\hat{f}(x') \leq M\|f\|_{H^p} .$$

Fix an $f \in H^p$ such that $\hat{f}(x') \equiv 1$. Simply take a C^{∞} function η on

$[0, \infty)$ that is equal to 1 on $(\frac{1}{2}, 2)$ and is supported on $(\frac{1}{4}, 4)$. Let $\hat{f}(x) = \eta(|x|)$. It is easy to check that $\hat{f}$ is the Fourier transform of a $(p, 2, s, \epsilon)$-molecule for H^p for every possible set of indicies. This establishes:

Proposition (9.21). *If m is a bounded multiplier on $H^p(\mathbb{R}^n)$ with norm M, there is a constant $C > 0$, independent of m such that $|m(x)| \leq C M$ for all $x \neq 0$. Furthermore, m is continuous on $\mathbb{R}^n - \{0\}$.*

In the discussion that follows conditions are imposed on derivatives of m, $D^\alpha m$. For most applications these may be interpreted as ordinary pointwise derivatives. This is, however, not necessary and they only need to be defined as functions that are distributions derivatives of m. The following lemma is crucial for the study of multipliers on $H^p(\mathbb{R}^n)$ if $n > 1$.

Lemma (9.22). *Suppose t is an integer, $t > \frac{n}{2}$, and*

$$R^{2|\beta|-n} \int_{R<|x|\leq 2R} |D^\beta m(x)|^2 \, dx \leq A^2 \tag{9.23}$$

for $0 \leq |\beta| \leq t$ and all $R > 0$. Then there is a C that is independent of m such that if $r = 1$ or $\frac{n}{r} > 2(|\beta| - t) + n$ then

$$\left[\int_{R<|x|\leq 2R} |D^\beta m(x)|^{2r} \, dx\right]^{\frac{1}{r}} \leq C^2 A^2 R^{(\frac{n}{r} - 2|\beta|)}, \quad R > 0. \tag{9.24}$$

When $2(|\beta| - t) + n < 0$ then $|x|^{|\beta|} |D^\beta m(x)| \leq C A$ and $D^\beta m$ is continuous on $\mathbb{R}^n - \{0\}$.

Remark. If $\beta = 0$ we have $n - 2t < 0$ so the lemma implies that m is bounded and continuous on $\mathbb{R}^n - \{0\}$, and the bound of m depends only on the constant A.

Proof. Let η be a radial, C^∞, non-negative function that is bounded by 1, supported on $\{\frac{1}{2} < |x| < 4\}$ and is equal to 1 on $\{1 \leq |x| \leq 2\}$. Let

$f(x) = R^{|\beta|}\eta(x/R)D^\beta m(x)$, $\varkappa = t - |\beta|$, and $g(x) = f(Rx) = R^{|\beta|}\eta(x)D^\beta m(Rx)$ for some fixed $R > 0$. Then $D^\nu g(x) = R^{|\nu|}D^\nu f(Rx)$. It follows from (9.23) that $\|D^\nu g\|_2 \leq C'A$ for $0 \leq |\nu| \leq \varkappa$ for some $C' > 0$, independent of m , ν , $\varkappa$ and R .

Thus, $g \in L^{2,\varkappa}$ and $\|g\|_{L^{2,\varkappa}} \leq C'A$. ($L^{2,\varkappa}$ is variously called a Lebesgue space, Bessel potential space or Sobolev space. Details about these spaces and the Sobolev embedding theorem, that we will use directly below, can be found in Stein [16, Ch. V §2].) It follows from Sobolev's theorem that $L^{2,\varkappa} \subset C_o$ is a continuous embedding if $\varkappa > \frac{n}{2}$ and $L^{2,\varkappa} \subset L^q$ is continuous $\varkappa > \frac{n}{2} - \frac{n}{q} \geq 0$. In particular, $\|g\|_q \leq CA$ if $\varkappa > \frac{n}{2} - \frac{n}{q}$, where C is independent of m and R (but does depend on $\varkappa$). Now recall that $\varkappa = t - |\beta|$ and write $2r = q$. The condition $\varkappa > \frac{n}{2} - \frac{n}{q}$ can be rewritten $\frac{n}{r} > 2(|\beta| - t) + n$. In this case we have (using the standard interpretation for $r = \infty$) :

$$[\int_{R<|x|\leq 2R} |D^\beta m(x)|^{2r}\, dx]^{\frac{1}{r}} \leq R^{-2|\beta|}\, [\int_{\mathbb{R}^n} |f(x)|^{2r}]^{\frac{1}{r}}$$

$$= R^{\frac{n}{r} - 2|\beta|}\, [\int_{\mathbb{R}^n} |g(x)|^{2r}\, dx]^{\frac{1}{r}} = R^{\frac{n}{r} - 2|\beta|}\, \|g\|_{2r}^2 \leq C^2A^2R^{\frac{n}{r} - 2|\beta|}$$

This completes the proof of the lemma.

<u>Notation</u>. <u>If</u> m <u>satisfies the conditions of Lemma (9.22) for some</u> t <u>we say that</u> m <u>satisfies</u> (#) .

The following multiplier theorem requires that m satisfies a smoothness condition in L^2 of integer order; that is, that m satisfies the (#) condition. This condition is usually referred to as a "Hörmander" condition. Some variants will be considered later in this section, but we note that for most applications this result suffices.

While obvious, it is still useful to observe that the "Mihlin" condition:

(9.25) $$\sup_{x\in\mathbb{R}^n} |x|^{|\beta|} |D^\beta m(x)| \leq A \ , \ 0 \leq |\beta| \leq t \ ,$$

implies (#) .

Theorem (9.26). *If* m *satisfies* (#) , $p \leq 1$, $\frac{1}{p} - \frac{1}{2} < \frac{t}{n}$, *then* m *is a multiplier on* $H^p(\mathbb{R}^n)$, *and there is a constant* $C > 0$, *independent of* m *such that the norm of* m *is bounded by* C A (A *the constant in the* (#) *condition*).

Observation. Suppose a is a $(p , 2 , s_1)$-atom, $s \leq s_1$, and $(p , 2 , s , \epsilon)$ is an admissible set of indicies for a molecule. From Proposition (2.3) we obtain that a is a $(p , 2 , s , \epsilon)$-molecule and $\aleph(a) \leq C$, for some constant C independent of a . In particular, if a is centered at the origin and $t = n(\frac{1}{2} + \epsilon)$ is an integer, then from Plancherel we have

(9.27) $$\{\|\hat{a}\|_2^{(\frac{1}{2} - \frac{1}{p} + \frac{t}{n})} \|D^\nu \hat{a}\|_2^{(\frac{1}{p} - \frac{1}{2})}\}^{\frac{n}{t}} \leq C \ ,$$

for all ν , $|\nu| = t$.

The theorem follows from the following proposition:

Proposition (9.28). *Suppose* m *satisfies* (#) *and* a *is a* $(p , 2 , t - 1)$-*atom centered at the origin,* $p \leq 1$, $\frac{1}{p} - \frac{1}{2} < \frac{t}{n}$. *Then* $(m\hat{a})^\vee$ *is a* $(p , 2 , [n(\frac{1}{p} - 1)] , \frac{t}{n} - \frac{1}{2})$-*molecule centered at the origin, and*

(9.29) $$\aleph((m\hat{a})^\vee) \leq C A \ ,$$

where C *depends only on* p , t *and* n .

Proof. Note first that the indicies are admissible for p-atoms and p-molecules. (The only point that requires any care is $t - 1 \geq [n(\frac{1}{p} - 1)]$.) We claim that it will suffice to show that

(9.30) $$(\|m\hat{a}\|_2^{\frac{1}{2} - \frac{1}{p} + \frac{t}{n}} \|D^\nu (m\hat{a})\|_2^{\frac{1}{p} - \frac{1}{2}})^{\frac{n}{t}} \leq C \ , \ |\nu| = t \ .$$

Just as was the case for (9.27) this is equivalent to

(9.31) $$(\|(m\hat{a})^\vee\|_2^{\frac{1}{2} - \frac{1}{p} + \frac{t}{n}} \||x|^t (m\hat{a})^\vee\|_2^{\frac{1}{p} - \frac{1}{2}})^{\frac{n}{t}} \leq C \ ,$$

and so we only need to check that the moments of $(m\hat{a})^\vee$, up to order

$[n(\frac{1}{p} - 1)]$, vanish. Note that $[n(\frac{1}{p} - 1)] < t - \frac{n}{2}$ so if $|\nu| \leq [n(\frac{1}{p} - 1)]$, then $|x|^{|\nu|} (m\hat{a})^{\vee}$ is integrable and consequently $D^{\nu}(m\hat{a})$ is continuous. Thus, to show that $\int (m\hat{a})^{\vee}(x)x^{\nu}\, dx = 0$, $|\nu| \leq [n(\frac{1}{p} - 1)]$ we only need to show that $D^{\nu}(m\hat{a})(0) = 0$. Note that m is bounded (see the remark following Lemma (9.23)) and $\hat{a}(x) = O(|x|^{t})$ as $|x| \to 0$ (see Lemma (9.1)(i) with $\alpha = 0$) and so $(m\hat{a})(0) = \lim_{x\to 0} m(x)\hat{a}(x) = 0$ and we are done if $\nu = 0$. For other values of ν, $D^{\nu}(m\hat{a})(0) = \lim_{|h|\to 0} |h|^{-|\nu|} \Delta^{\nu}_{|h|}(m\hat{a})(0) = \lim_{|h|\to 0} O(|h|^{t-|\nu|}) = 0$. ($\Delta^{\nu}_{|h|} = \Delta^{\nu_1}_{|h|e_1} \cdots \Delta^{\nu_n}_{|h|e_n}$, the "mixed" difference operator, e_k, is the unit vector with coefficient equal to 1 in the k-th coordinate.) This establishes the claim.

As in §2 we will let $a = 1 - \frac{1}{p} + \varepsilon$ and $b = \frac{1}{2} + \varepsilon$. Since $\varepsilon = \frac{t}{n} - \frac{1}{2}$ we have $a = \frac{t}{n} + \frac{1}{2} - \frac{1}{p}$, $b = \frac{t}{n}$ and $b - a = \frac{1}{p} - \frac{1}{2}$. (The use of "a" to represent both an atom and the value $\frac{t}{n} + \frac{1}{2} - \frac{1}{p}$ should cause no confusion in context.)

(9.30) will follow if we can show

$$\|(D^{\alpha}\hat{a})(D^{\beta}m)\|_2 \leq CA/\|a\|_2^{\frac{b}{b-a} - 1}, \quad |\alpha| + |\beta| = t. \tag{9.32}$$

For $\beta = 0$, $|\alpha| = t$ we have

$$\|D^{\alpha}\hat{a})m\|_2 \leq \|m\|_{\infty} \|D^{\alpha}\hat{a}\|_2 \leq C\, \|m\|_{\infty}/\|a\|_2^{\frac{b}{b-a} - 1},$$

which follows from (9.27).

If $0 < |\beta| \leq t$ then $0 \leq |\alpha| < t$ and Lemma (9.1) can be applied.

$$\|(D^{\alpha}\hat{a})(D^{\beta}m)\|_2^2 = \sum_{\ell\in\mathbf{Z}} \int_{2^{\ell}<|x|\leq 2^{\ell+1}} |D^{\alpha}\hat{a}(x)|^2\, |D^{\beta}m(x)|^2\, dx$$

$$\leq C \sum_{-\infty}^{K} \frac{2^{2\ell(t-|\alpha|)}}{\|a\|_2^{2(\frac{b+\frac{1}{2}}{b-a})-2}} \frac{1}{2^{\ell(2|\beta|-n)}} [2^{\ell(2|\beta|-n)} \int_{2^{\ell}<|x|\leq 2^{\ell+1}} |D^{\beta}m(x)|^2\, dx]$$

$$+ C \sum_{K}^{\infty} [\int_{2^{\ell}<|x|\leq 2^{\ell+1}} |D^{\alpha}\hat{a}(x)|^{2r'}\, dx]^{\frac{1}{r'}} [\int_{2^{\ell}<|x|\leq 2^{\ell+1}} |D^{\beta}m(x)|^{2r}\, dx]^{\frac{1}{r}}$$

$$= I_1 + I_2.$$

Estimate for I_1. We have already used Lemma (9.1). Now use (#) (without benefit of Lemma (9.22)) and we have

$$I_1 \le \{CA^2/\|a\|_2^{\frac{2(b+\frac{1}{2})-2}{b-a}}\} \sum_{-\infty}^{K} 2^{\ell n}$$

$$\le CA^2\, 2^{nK}/\|a\|_2^{\frac{2(b+\frac{1}{2})-2}{b-a}} .$$

Choose K so that $2^{nK} \sim \|a\|_2^{\frac{1}{b-a}}$ and we have $I_1 \le C\, A^2/\|a\|_2^{\frac{2b}{b-a}-2}$.

Estimate for I_2. Choice of r. For $|\beta| > \frac{n}{2}$ let $r = 1$. For $0<|\beta| \le \frac{n}{2}$ and $0 < |\beta| < t - \frac{n}{2}$ let $r = \infty$. Note that if $t > n$ these two cases suffice. Note also that if $p \le \frac{2}{3}$ we always have $t > n$. In the remaining case we have $t - \frac{n}{2} \le |\beta| \le \frac{n}{2}$, and we can choose an r so that $1 < r < \infty$ and $0<2|\beta| - \frac{n}{r}<2t-n$. In all cases we have (i) $2|\beta| - \frac{n}{r} > 0$ and (ii) r and β satisfy the conditions of Lemma (9.22).

The result now follows from an application of Lemmas (9.1)(ii) and (9.22).

$$I_2 \le CA^2 \sum_{K}^{\infty} [\int_{2^\ell<|x|\le 2^{\ell+1}} |D^\alpha \hat{a}(x)|^{2r'}\, dx]^{\frac{1}{r'}}\, 2^{\ell(\frac{n}{r}-2|\beta|)}$$

$$\le CA^2\, [\int_{|x|>2^K} |D^\alpha \hat{a}(x)|^{2r'}\, dx]^{\frac{1}{r'}}\, [\sum_{K}^{\infty} 2^{\ell(n-2|\beta|r)}]^{\frac{1}{r}}$$

$$\le CA^2 \| |D^\alpha \hat{a}|^2 \|_{r'}\, 2^{nK(\frac{1}{r}-\frac{2|\beta|}{n})}$$

$$\le CA^2\, \|a\|_2^{-\{\frac{2|\alpha|}{n}+\frac{1}{r}/(b-a)\}+2}\, \|a\|_2^{\{\frac{2|\beta|}{n}-\frac{1}{r})/(b-a)\}}$$

$$= CA^2\, \|a\|_2^{-(\frac{2t}{n}/(b-a))+2} = CA^2/\|a\|_2^{\frac{2b}{b-a}-2} .$$

This completes the proof of the lemma.

We will now consider a variant of the multiplier theorem for fractional orders of smoothness.

For future reference let us fix a function η that is C^∞, radial, non-negative, supported on $(\frac{1}{4}, 4)$ and for some constants $0 < A_1 < A_2$, $A_1 \leq \eta(x) \leq A_2$ for $\frac{1}{2} \leq |x| \leq \frac{5}{2}$. Suppose further that if $\eta_k(x) = \eta(2^{-k}x)$ then $\sum_{k\in U} \eta_k(x) = 1$ if $x \neq 0$. That is, $\{\eta_k\}$ is the usual "nice" partition of unity for $\mathbb{R}^n - \{0\}$. Let $m_k = m\eta_k$.

We say that m satisfies (##) for $t > 0$ if m is bounded, $|m(x)| \leq A$ and for some integer $\bar{t} > t$ and all integers k we have

$$(9.33) \qquad 2^{k(2t-n)} \int_{|h|<2^{k-1}} |h|^{-2t} \int_{2^k<|x|\leq 2^{k+1}} |\Delta_h^{\bar{t}} m(x)|^2 \, dx \, \frac{dh}{|h|^n} \leq A^2 .$$

Remarks. (1) The condition (##) is handy for applications, but for proving theorems a more useful and equivalent variant is that for some integer $\bar{t} > t$ and all integers k

$$(9.34) \qquad \begin{cases} 2^{-kn} \int_{\mathbb{R}^n} |m_k(x)|^2 \, dx \leq A^2 \\ 2^{k(2t-n)} \int_{\mathbb{R}^n} |h|^{-2t} \int_{\mathbb{R}^n} |\Delta_h^{\bar{t}} m_k(x)|^2 \, dx \, \frac{dh}{|h|^n} \leq A^2 \end{cases}$$

The only point that is all delicate is to show that (9.34) implies that m is bounded. This is contained in the proof of Lemma (9.37) that follows, and it is then easy to check that the two variants are equivalent.

(2). If t is an integer the conditions (#) and (##) are equivalent. To see this note that the second version of (##) is equivalent to

$$(9.35) \qquad 2^{k(2|\beta|-n)} \int_{\mathbb{R}^n} |(D^\beta m_k)(x)|^2 \, dx \leq A^2$$

for all $k \in \mathbb{Z}$ and for $\beta = 0$ and all $|\beta| = t$ (t an integer), using a Plancherel argument. It now follows that (##) is equivalent to requiring (9.35) for $0 \leq |\beta| \leq t$ and from that condition to (#) is immediate.

(3). Many other variants of (#) and (##) can also be used and may be simpler to apply in particular situations. For example, (9.35) can be replaced by conditions such as

$$(9.36) \qquad 2^{k(2r-n)} \int_{\mathbb{R}^n} |h|^{-2(t-\ell)} \int_{\mathbb{R}^n} |\Delta_h^{\bar{t}} D^{\beta} m_k(x)|^2 \, dx \, \frac{dh}{|h|^n} \leq A^2$$

for all $k \in \mathbf{Z}$, $|\beta| = \ell < t$, where $\bar{t}$ is an integer, $\bar{t} > t - \ell$.

We need a variant of Lemma (9.22) for the (##) condition.

<u>Lemma (9.37).</u> <u>Suppose</u> m <u>satisfies</u> (##) <u>for some</u> $t > \frac{n}{2}$. <u>Then there is a constant</u> C <u>that is independent of</u> m <u>such that if</u> $r = 1$ <u>or</u> $\frac{n}{r} > n - 2(t-s)$ <u>and</u> $\bar{s} > s > 0$, $\bar{s}$ <u>an integer then</u>

$$(9.38) \qquad 2^{k(2s-\frac{n}{r})} \int_{\mathbb{R}^n} |h|^{-2s} \left[\int_{2^k < |x| \leq 2^{k+1}} |\Delta_h^{\bar{s}} m(x)|^{2r} \, dx \right]^{\frac{1}{r}} \frac{dh}{|h|^n} \leq C^2 A^2 .$$

<u>Furthermore,</u> m <u>is bounded and continuous on</u> $\mathbb{R}^2 - \{0\}$ <u>and</u> $\|m\|_\infty \leq CA$.

<u>Proof.</u> Let $g_\nu(x) = m_\nu(2^\nu x) = \eta(x) m(2^\nu x)$. Then

$$(9.39) \qquad \begin{cases} \int_{\mathbb{R}^n} |g_\nu(x)|^2 \, dx \leq A^2 \\ \int_{\mathbb{R}^n} |h|^{-2t} \int_{\mathbb{R}^n} |\Delta_h^{\bar{t}} g(x)|^2 \, dx \, \frac{dh}{|h|^n} \leq A^2 \end{cases}$$

Consequently, $g_\nu \in \Lambda_t^{2,2}$ and $\|g_\nu\|_{\Lambda_t^{2,2}} \leq C A$. ($\Lambda_\alpha^{pq}$ is variously called a Besov or Lipschitz space. Details about these spaces and the Besov-Taibleson embedding theorems (which gives various continuous inclusions between the Lipschitz and Bessel potential spaces and among the Lipschitz spaces) can be found in Stein [16, Ch. V §5].) We know that $\Lambda_t^{2,2} = L^{2,t} \subset L^{\infty, t-\frac{n}{2}} \subset \mathbf{C}_0$ and that the embeddings are continuous. Thus g_ν is bounded, continuous and $\|g_\nu\|_\infty \leq CA$. From this it follows that m is continuous on $\mathbb{R}^n - \{0\}$ and $\|m\|_\infty \leq CA$.

We may assume, without loss of generality, that $\bar{t}$ is the smallest integer greater than t.

If $\frac{n}{r} > n - 2(t-s)$ then $g_\nu \in \Lambda_t^{2,2} \subset \Lambda_{t-\frac{n}{r}+\frac{n}{2r}}^{2r,2} \subset L^{2r}$ since $t - \frac{n}{2} + \frac{n}{2r} > 0$. Since the embeddings are continuous $g_\nu \in L^{2r}$, $\|g_\nu\|_{2r} \leq CA$ and so

$$(9.40) \qquad 2^{-\nu\frac{n}{r}} \left[\int_{2^\nu < |x| \leq 2^{\nu+1}} |m(x)|^{2r} \, dx \right]^{\frac{1}{r}} \leq C A .$$

(If $r = 1$ this is immediate from (##).)

Observe, again, that $g_\nu \in \Lambda^{2r,2}_{t-\frac{n}{2}+\frac{n}{2r}}$. Thus, if $0 < s \leq t - \frac{n}{2} + \frac{n}{2r}$, then $g \in \Lambda^{2r,2}_s$ (the inclusion is continuous) so that if $\bar{s}$ is an integer and $\bar{s} > s$

$$(9.41) \qquad \int_{\mathbb{R}^n} |h|^{-2s} \left[\int_{\mathbb{R}^n} |\Delta_h^{\bar{s}} g_\nu(x)|^{2r} dx\right] \frac{dh}{|h|^n} \leq C^2 A^2 .$$

(If $r = 1$ this follows from $g_\nu \in \Lambda^{2,2}_t \subset \Lambda^{2,2}_s$.) From (9.41) we obtain

$$(9.42) \qquad 2^{\nu(2s-\frac{n}{r})} \int_{\mathbb{R}^n} |h|^{2s} \left[\int_{\mathbb{R}^n} |\Delta_h^{\bar{s}} m_\nu(x)|^{2r} dx\right]^{\frac{1}{r}} \frac{dh}{|h|^n} \leq C A^2 .$$

This implies

$$(9.43) \int_{|h|<\frac{2^{\nu-1}}{t}} |h|^{-2s} \left[\int_{2^\nu<|x|\leq 2^{\nu+1}} |\Delta_h^{\bar{s}} m(x)|^{2r} dx\right]^{\frac{1}{r}} \frac{dh}{|h|^n} \leq C A^2 2^{\nu(\frac{n}{r}-2s)} .$$

But if we use (9.40) we easily obtain

$$(9.44) \qquad \begin{aligned} &\int_{|h|\geq\frac{2^{\nu-1}}{t}} |h|^{-2s} \left[\int_{2^\nu<|x|\leq 2^{\nu+1}} |\Delta_h^{\bar{s}} m(x)|^{2r} dx\right]^{\frac{1}{r}} \frac{dh}{|h|^n} \\ &\leq C A^2 2^{\nu\frac{n}{r}} \int_{|h|\geq\frac{2^{\nu-1}}{t}} |h|^{-2s} \frac{dh}{|h|^n} \leq CA^2 2^{\nu(\frac{n}{r}-2s)} . \end{aligned}$$

This completes the proof of the Lemma.

Theorem (9.45). If m satisfies (##) , $p \leq 1$, $\frac{1}{p} - \frac{1}{2} < \frac{t}{n}$, then m is a multiplier on $H^p(\mathbb{R}^n)$, and there is a constant $C > 0$, independent of m , such that the norm of f is bounded by CA (A the constant of (##)).

Proof. We may assume that t is not an integer (the "integer case" was done in (9.26)) and we may fix $\bar{t} = [t] + 1$.

We will show that if a is a $(p , 2 , \bar{t})$-atom centered at the origin then $(m\hat{a})^\vee$ -molecule centered at the origin and $\mathfrak{N}((m\hat{a}^\vee) \leq CA$, C independent of a . Just as in the proof of Proposition (9.28) this will follow if we can show that

$$(9.46)\qquad \{\|(m\hat{a})\|_2^{\frac{1}{2}-\frac{1}{p}+\frac{t}{n}}\ ([\int|h|^{-2t}\int|\Delta_h^{\bar{t}}(m\hat{a})(x)|^2\,dx\,\frac{dh}{|h|^n}]^{\frac{1}{2}})^{\frac{1}{p}-\frac{1}{2}} \leq C\,A\,.$$

We use the identity

$$(9.47)\qquad \Delta_h^{\bar{t}}(fg) = \sum_{k+\ell=t}\binom{\bar{t}}{\ell}(\tau_{-kh}\Delta_h^{\ell}f)(\Delta_h^k g)$$

Thus we need to check that for $k+\ell=\bar{t}$

$$\int|h|^{-2t}\int|\tau_{-kh}\,\Delta_h^{\ell}\hat{a}(x)|^2\,|\Delta_h^k m(x)|^2\,dx\,\frac{dh}{|h|^n} \leq \frac{CA}{\|a\|_2^{\frac{2b}{b-a}-2}}\,,$$

which is the analogue of (9.23).

If $k=0$ this follows from $\|m\|_\infty \leq C\,A$ (see 9.37) and the estimate

$$\int|h|^{-2t}\int|\Delta_h^{\ell}\hat{a}(x)|^2\,dx\,\frac{dh}{|h|^n} \leq C\,/\|a\|_2^{\frac{2b}{b-a}-2}\,,$$

which follows from Plancherel, Fubini and Proposition (2.3).

We now consider $k+\ell=\bar{t}$, $0<k\leq\bar{t}$, $0\leq\ell\leq\bar{t}$, and choose K so that $2^{nK}\sim\|a\|_2^{\frac{1}{b-a}}$.

$$\int_{h\in\mathbb{R}^n}|h|^{-2t}\int_{x\in\mathbb{R}^n}|\tau_{-kh}\,\Delta_h^{\ell}\hat{a}(x)|^2\,|\Delta_{\ell}^{k}m(x)|^2\,dx\,\frac{dh}{|h|^n}$$

$$=\sum_{\nu\in\mathbf{Z}}\int_{h\in\mathbb{R}^n}\cdots\int_{2^\nu<|x|\leq 2^{\nu+1}}\cdots dx\,\frac{dh}{|h|^n} = \sum_{-\infty}^{K}\int_{|h|\leq 2^{\nu+1}}\cdots\int_{2^\nu<|x|\leq 2^{\nu+1}}\cdots dx\,\frac{dh}{|h|^n}$$

$$+\sum_{-\infty}^{K}\int_{|h|>2^{\nu+1}}\cdots\int_{2^\nu<|x|\leq 2^\nu+1}\cdots\,dx\frac{dh}{|h|^n}+\sum_{K}^{\infty}\int_{h\in\mathbb{R}^n}\int_{2^\nu<|x|\leq 2^{\nu+1}}\cdots\,dx\,\frac{dh}{|h|^n}$$

$$= P_1+P_2+P_3\,.$$

<u>Estimate for P_3</u>. This proceeds exactly as for the estimate for I_2 in (9.28). Choose r as in that estimate. Then

$$P_3 \leq \sum_{-\infty}^{K}\sup_{h\in\mathbb{R}^n}\left[\int\left|\frac{\tau_{-kh}\Delta_h^{\ell}\hat{a}(x)}{|h|^{\ell}}\right|^{2r'}dx\right]^{\frac{1}{r'}}\int_{h\in\mathbb{R}^n}|h|^{-2(t-\ell)}\left[\int|\Delta_{\ell}^{k}m(x)|^{2r}\,dx\right]^{\frac{1}{r}}\frac{dh}{|h|^n}\,.$$

Now use Lemma (9.6) and Lemma (9.37) and the estimate follows exactly as before.

Estimate for P_1. This mimics the estimate for I_1 in (9.28).

$$P_1 \le \sum_{-\infty}^{K} \sup_{\substack{|h|\le 2^{\nu+1} \\ x\in(2^\nu,2^{\nu+1}]}} \left|\frac{\tau_{-kh}\Delta_h^{\ell}\hat{a}(x)}{|h|^{\ell}}\right| \int_{h\in R^n} |h|^{-2(t-\ell)} \int_{2^\nu<|x|\le 2^{\nu+1}} |\Delta_h^k m(x)|^2 \, dx \frac{dh}{|h|^n} .$$

In the first term of each summand we have $|h| \le 2|x|$ and so we use (9.9). In the second use (9.37) with $r = 1$. The estimate follows as before.

Estimate for P_2. We reduce this to two other estimates.

$$P_2 \le \int_{|x|\le 2^K} \cdots \int_{|h|>|x|} \cdots \frac{dh}{|h|^n} dx$$

$$= \int_{|h|\le 2^K} \cdots \int_{|x|<|h|} \cdots dx \frac{dh}{|h|^n} + \int_{|h|>2^K} \cdots \int_{|x|<2^K} \cdots dx \frac{dh}{|h|^n}$$

$$= P_4 + P_5 .$$

Estimate for P_4. Since $|x| < |h|$ we use (9.9) and obtain

$$|\tau_{-kh} \Delta_h^{\ell}\hat{a}(x)| \le C|h|^{\bar{t}+1}/\|a\|_2^{\{(\frac{t+1}{n}+\frac{1}{2})/(b-a)\}-1} .$$

Thus,

$$P_4 \le C \|m\|_\infty^2/\|a\|_2^{\{(2\frac{(\bar{t}+1)}{n}+1/(b-a)\}-2} \int_{|h|\le 2^K} |h|^{2(\bar{t}+1-t)} \frac{dh}{|h|^n}$$

$$\le C \|m\|_\infty^2 \frac{2^{nK2(\frac{\bar{t}+1-t}{n})}}{\|a\|_2^{\{(2\frac{(t+1)}{n}+1)/(b-a)\}-2}}$$

$$\le C \|m\|_\infty^2/\|a\|_2^{\{(\frac{2t}{n}+1)/(b-a)\}-2} .$$

Estimate for P_5. We use Lemma (9.6) with $r = 1$.

$$|\tau_{-kh} \Delta_h^{\ell}\hat{a}(x)| \le C |h|^{\ell}/\|a\|_2^{\{(\frac{\ell}{n}+\frac{1}{2})/(b-a)\}-1} .$$

Thus,

$$P_5 \leq [C\ \|m\|_\infty^2\ 2^{nK}/\|a\|_2^{\{(\frac{\ell}{n}+\frac{1}{2})/(b-a)\}\cdot 2-2}]\int_{|h|>2^K} |h|^{-2(t-\ell)}\ \frac{dh}{|h|^n}$$

$$\leq C\ A^2\ \frac{2^{nK(\frac{2(\ell-t)}{n}+1)}}{\|a\|_2^{\{2(\frac{\ell}{n}+\frac{1}{2})/(b-a)\}-2}} \leq C\ A^2/\|a\|_2^{\frac{2b}{b-a}-2} .$$

This completes the proof of the Theorem.

There is an occasional use of conditions such as (#) or (##) for L^r - norms with r not equal to 2. A convenient formulation is the following:

Let m be a measureable function on $\mathbb{R}^n$ and define the functions m_k as we did in the discussion which preceeded the statement of Lemma (9.37). We say that m satisfies (###) for r, $1 \leq r \leq \infty$ and $t > n/\min\{r, 2\}$ if m is bounded, $|m(x)| \leq A$, and for some integer $\bar{t} > t$ and all $k \in \mathbb{Z}$,

$$2^{k(t-\frac{n}{r})}\ [\int_{h\in\mathbb{R}^n} |h|^{-2t}\ [\int_{x\in\mathbb{R}^n} |\Delta_h^{\bar{t}} m_k(x)|^r\ dx]^{\frac{2}{r}}\ \frac{dh}{|h|^n} \leq A .$$

If t is an integer this condition on the "difference" can be replaced with one on the derivative:

$$2^{k(t-\frac{n}{r})}\ [\int_{x\in\mathbb{R}^n} |D^\beta m_k(x)|^r\ dx]^{\frac{1}{r}} \leq A ,$$

for all $|\beta| = t$.

These conditions require that m "locally satisfy an $\Lambda_t^{r,2}$ -condition" in the first instance and a "local $L^{r,t}$ -condition" in the second. (These are not equivalent if $r \neq 2$, but the continuous inclusions $L^{r,t+\epsilon} \subset \Lambda_t^{r,2} \subset L^{r,t-\epsilon}$ for all $\epsilon > 0$, give us the room we need and, for the purposes of the multiplier theorem, they are equivalent.) To use these conditions we reduce them to the L^2 -case. If $r \geq 2$ the conditions drops to the corresponding condition for $r = 2$, without loss in the smoothness index t, since the integration with respect to x takes place on an annulus. One restricts the integration with respect to h to $|h| \leq 2^{k-1}$ and then uses $|m(x)| \leq A$ for the other part of the integral. If $1 \leq r \leq 2$ one "lifts" to $r = 2$ using the embedding theorems for the Λ_{α}^{pq} -spaces (as in the proof of Lemma (9.37)) with a subsequent loss in smoothness from t to $t - \frac{n}{r} + \frac{n}{2}$. Using these observations we obtain the following multiplier theorem as a corollary of (9.45).

Theorem (9.48). If m satisfies (###), $p \leq 1$, $\frac{1}{p} - \{\frac{1}{\min[2,r]}\} < \frac{t}{n}$, then m is a multiplier on $H^p(\mathbb{R}^n)$.

Details are left to the reader.

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