IVAN KUPKA

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SOME PROBLEMS IN ACCESSIBILITY THEORY

by

Ivan KUPKA

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0.- In this talk I want to discuss several problems that appear in the theory of accessibility of polysystems. Since very little seems to be known about these problems, I will indicate some ideas about possible solutions. The talk will consist of two parts ; the first one will consider left or right invariant polysystems on Lie groups the second one points out some pathologies in the structure of accessibility sets and discusses the stability of transitivity or non transitivity. The proofs of all the results mentionned will appear elsewhere.

1.- GENERALITIES AND DEFINITIONS.

The setting will be a C^{∞} or C^{ω} (real analytic) connected manifold M whose tangent bundle we denote by TM : the tangent space at $x \in M$ will be T_xM .

Definition 1.- A polysystem F on M will be a subset of TM such that $F(x) = T_xM \cap F$ is a cone in T_xM which is not empty and not reduced to $\{0\}$ for each $x \in M$.

Definition 2.- a) For any $x \in M$, the accessibility set of x under F , denoted by $A(x,F)$, is the set of all $y \in M$ such that there exists an absolutely continuous curve $\varphi : [0,T] \to M$ satisfying the conditions :

1) $\varphi(0) = x$, $\varphi(T) = y$
2) for almost every $t \in [0,T]$, $\frac{d\varphi}{dt}(t) \in F(x(t))$.

b) The boundary $B(x,F)$ of the accessibility set $A(x,F)$ will be the set closure of $A(x,F) \cap$ closure of $(M-A(x,F))$. It is the disjoint union of the accessible boundary $Ba(x,F)$ and the inaccessible boundary $Bi(x,F)$:

$$Ba(x,F) = B(x,F) \cap A(x,F) \qquad Bi(x,F) = B(x,F) \cap (M - A(x,F)) .$$

Definition 3.- A polysystem F is called transitive if for any $x \in M$, $A(x,F) = M$. It is called weakly transitive if $A(x,F)$ is dense in M.

The basic problem of accessibility theory is to find conditions either necessary and sufficient or simply necessary or only sufficient for a polysystem to be transitive or non transitive. For general polysystems such a problem does not make much sense since exemples, which we do not want to discuss here, show that minor changes in a polysystem F will make a non transitive are into a transitive one and vice-versa. So the transitivity property is very unstable. Hence it is advisable to restrict the nature of the systems. For cases of transitivity of general polysystems see [2].

2.- INVARIANT SYSTEMS ON GROUPS AND INDUCED SYSTEMS.

G will denote a connected Lie group, Lie (G) its Lie algebra.

Definition 4.- A polysystem T on G will be called left (resp. right) invariant if for any $x, g \in G$, $F(gx) = g\,F(x)$ (resp. $F(xg)/ = F(x)g$) where the map $T_x G \longrightarrow T_{gx} G$ (resp. $T_x G \longrightarrow T_{xg}(G)$) $u \longrightarrow gu$ (resp. $u \to ug$) is the map induced by the left (resp. right) g-translation.

Remarks .- 1) There is an easy one to one correspondance between left and right invariant polysystems. A left invariant polysystem is transitive if and only if the corresponding right invariant polysystem is transitive.

2) A left or right invariant polysystem F is entirely determined by $F(e)$ where e is the neutral element of G.

Definition 5.- If M is a connected manifold on which G operates on the left, then any right invariant polysystem F on G induces a polysystem F_M on M as follows : if $\Phi : G \times M \to M$ is the action of G on M and $x \in M$,

$F_M(x) = \{d\Phi(e,x)\ (u,0) \mid u \in F(e)\}$.

The following is trivial but important.

Proposition 1. - a) If F is left (resp. right) invariant on G, then $A(e,F)$ is a semi-group.

b) For any $g \in G$, $A(g,F) = g\,A(e,F)$ (resp. $A(g,F) = A(e,F)g$).

c) If G operates on the left on the manifold M and F is right invariant, for any $x \in M$, $A(x,F_M) = A(e,F)x$.

d) F is transitive (resp. weakly transitive) on G if and only if $A(e,F) = G$ (resp. $A(e,F)$ is dense in G).

e) If F is right invariant and if G is transitive on a manifold M, then if F is transitive on G, F_M is transitive on M .

For invariant polysystems on a group G it is easy to give a necessary condition for transitivity.

Theorem 1 (chow). - If an invariant polysystem F is transitive, then $F(e)$ generates the Lie algebra of G .

This gives the following easy corollary.

Corollary. - If F is invariant and symmetric $(-F = F)$ then F is transitive if and only if $F(e)$ generates $Lie(G)$.

The following proposition is well known and technically very useful.

Proposition 2 ([1]). - a) If F is an invariant polysystem and $F(e)$ generates the Lie algebra Lie (G), then weak transitivity implies transitivity.

b) If F is right invariant, $F(e)$ generates $Lie(G)$ and M is a left G-manifold, then the weak transitivity of F_M implies the transitivity of F_M .

This last proposition enables us to settle the transitivity problem in one case.

Proposition 3.- If G is compact and F is invariant, F is transitive if and only if $F(e)$ generates the Lie algebra of G.

To see this notice that the necessity is just Theorem 1. The sufficiency follows from Proposition 2 and from the following : the closure of $A(e,F)$ is a closed semi-group hence a Lie subgroup of G by the compactness of G. The Lie algebra condition implies that this subgroup is just G. So F is weakly transitive.

No other necessary or necessary and sufficient conditions are known to me. Let us discuss now sufficient or generically sufficient conditions. First, we discuss a special class of polysystems.

For any integer r $0 \leq r \leq \dim \mathrm{Lie}(G)$, let us denote by $\mathrm{Afgr}(\mathrm{Lie}(G), r)$ the set of all affine subvarieties of dimension r in $\mathrm{Lie}(G)$. This space is endowed with a natural topology.

Definition 6.- An invariant polysystem F on a group G is called bilinear if $F(e)$ is an affine variety in $\mathrm{Lie}(G)$.

Remark : It is easy to see that this is really the classical notion of bilinear system. In fact if $\dim F(e) = m$ then there exist elements $A, B_1, \dots, B_m \in F(e)$. Such that $F(e) = \{A + \sum_{j=1}^{m} u_j B_j \mid (u_1, \dots, u_m) \in \mathbb{R}^m\}$. The corresponding equation is $\frac{dx}{dt}(t) = \vec{A}(x(t)) + \sum_{j=1}^{m} u_j(t) \vec{B}_j(x(t))$ where $\vec{A}, \vec{B}_1, \dots, \vec{B}_m$ are the left (or right) invariant vector fields on G generated by $A, B_1, \dots, B_m$.

Definition 7.- The genus of a real Lie algebra L is the smallest of the cardinals of all the subsets of L which generate L as a Lie algebra. If G is a Lie group, the genus of G is the one of $\mathrm{Lie}(G)$. In other words the genus of L is the minimum of all integers m such that there exists a surjective Lie algebra homomorphism φ : Free Lie algebra on $(X_1, \dots, X_m) \longrightarrow L$.

Proposition 5.- Let G be a real Lie group. For any integer r, $r \geq$ genus of G, there exists an algebraic subvariety Σ of $\mathrm{Afgr}(\mathrm{Lie}(G), r)$ of codimension ≥ 1 such that any bilinear F not belonging to Σ is transitive. In particular these systems form on open dense set.

As an example : if G is semi-simple its genus is 2. Hence on G, generically any bilinear system with at least two controls is transitive. This applies in particular to the special linear group.

If $r <$ genus of G, the proposition is definitely false. The set of non transitive bilinear systems has then a non empty interior. V. Jurdjevic and I obtained a general sufficient condition in that case which, in a sense we will not discuss here, is the best possible. I proceed to state our result now, but only in the simple Lie group case where the statement is shorter. Hence we assume that Lie(G) is a real form of a simple complex Lie algebra.

Definition 8.- An element $B \in \text{Lie}(G)$ will be called strongly regular if :

1) ad B is semi-simple. All its eigenvalues except 0 are simple and 0 is an eigenvalue of multiplicity equal to rank Lie(G).
2) If we order the eigenvalues of ad B by the lexicographic order (that is $\lambda \leqslant \mu$ if $\text{Re}\lambda < \text{Re}\mu$ or $\text{Re}\lambda = \text{Re}\mu$ and $\text{Im}\lambda < \text{Im}\mu$) then there is exactly rank Lie(G) indecomposable positive eigenvalues. A positive eigenvalue is called indecomposable if it is not a sum of at least two positive eigenvalues (positive for the lexicographic order !).

This definition calls for several remarks. First the number of indecomposable eigenvalue is always no greater than the rank of Lie(G). For certain B it can be smaller but these are very special B, because of the following : the set of strongly regular elements forms an open dense set in Lie(G) whose complement is a real algebraic variety of codimension .

Before stating the theorem, one should be aware that if $B \in \text{Lie}(G)$, then the set of eigenvalues of ad B is symmetric that is, if β is an eigenvalue -β is one too.

Theorem 2(Jurdjevic-Kupka).- Let G be a real Lie group which is a real form of a complex simple Lie group. An invariant polysystem F on G is transitive provided that :

1) There is a strongly regular element $B \in \text{Lie}(G)$ such that B and $-B$

belong to $F(e)$.

2) Let $\mathrm{Lie}(G) \oplus \mathbb{C} = E_o \oplus \oplus_\beta E_\beta$ the decomposition of the complexified Lie algebra of G along the eigenspaces of $\mathrm{ad}\, B$. Then the second requirement is that for each indecomposable β these should exist $A_+(\beta)$, $A_-(\beta) \in F(e)$ such that the component of $A_+(\beta)$ (resp. $A_-(\beta')$ along E_β (resp. $E_{-\beta}$) is non zero. The same should be true if β is the maximal eigenvalue of $\mathrm{ad}\, B$ (maximal for the lexicographic order).

3) If the real part r of the maximal eigenvalue is also an eigenvalue there should exist elements A_+, $A_- \in F(e)$ such that if $A_+(r)$ and $A_-(r)$ are their components in the spaces E_r and E_{-r} respectively then $\mathrm{Kil}(A_+(r), A_-(r)) < 0$. Kil is the killing form of the algebra $\mathrm{Lie}(G)$.

This theorem calls for a few comments. First the condition 1 and 2 are open dense conditions. Condition 3 is different. It is certainly open but not dense unless it is empty,that is, r is not an eigenvalue of $\mathrm{ad}\, B$. It depends on the type of the Cartan algebra $\ker \mathrm{ad}\, B$. The methods of proof of this theorem give other results of similar kind. They also have many corollaries. Let us state one here which has bearing on the problem of accessibility for bilinear systems with one control, that is for invariant polysystems F such that $F(e) \in \mathrm{Afgr}(\mathrm{Lie}(G), 1)$ and G is a real form of a complex simple Lie group.

Corollary.- Let G be as in theorem 2. Let $B \in \mathrm{Lie}(G)$ be a strongly regular element. For any $A \in \mathrm{Lie}(G)$ call $F(A, B)$ the invariant system generated by the affine line $\{A+uB \mid u \in \mathbb{R}\} \subset \mathrm{Lie}(G)$. Then we have the following situation : either 1) the set of all A such that $F(A, B)$ is transitive contains an open dense set in $\mathrm{Lie}(G)$, or 2) the set of all $A \in \mathrm{Lie}(G)$ such that the mean limits :

$$A_+ = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{e^{t\,\mathrm{ad}\, B}(A)}{\| e^{t\,\mathrm{ad}\, B}(A) \|}\, dt \quad , \quad A_- = \lim_{T \to +\infty} \frac{1}{T} \int_{-T}^0 \frac{e^{t\,\mathrm{ad}\, B}(A)}{\| e^{t\,\mathrm{ad}\, B}(A) \|}\, dt$$

exist and are not zero contains an open dense set in $\mathrm{Lie}(G)$ and in the subset where $\mathrm{Kil}(A_+, A_-) < 0$ the set of all A's for which $F(A, B)$ is transitive contains an open dense subset : ($\| \ \|$ is any norm on $\mathrm{Lie}(G)$).

If we consider for a moment the conditions in theorem 2 we see that they are semi-algebraic. Let us recall this concept.

Definition 9. - A subset S in a real algebraic variety E is called semi-algebraic if there exist regular functions $f_1, \ldots, f_m$ on E such that : $S = \{x \mid x \in E \ f_1(x) > 0 \ldots f_n(x) > 0 \ f_{n+1}(x) \geq 0 \ldots f_m(x) \geq 0\}$; $f_1, \ldots, f_m$ do not need be distinct.

Now Kalman's necessary and sufficient condition for the transitivity of linear systems is semi-algebraic. Hence one may ask if there exist necessary and sufficient conditions for the transitivity of bilinear systems which are semi-algebraic. To be more specific, call $T(G, r)$ the subset in the affine grassmannian $\mathrm{Afgr}(\mathrm{Lie}(G), r)$ of all affine varieties of dimension r which give rise to transitive invariant systems. The question then is the following : is $T(G, r)$ a semi-algebraic set ? The answer is no and counterexamples appear for $G = SL(3 ; \mathbb{R})$.

On the other hand the following generalization of the preceeding question could have a positive answer. Let $\mathfrak{R}$ be the field of real rationnal functions of the real algebraic variety $\mathrm{Afgr}(\mathrm{Lie}(G), r)$. Then the new question is : does there exists a finite set $\{g_1, \ldots, g_h\} \subset \mathfrak{R}$, such that $T(G, r)$ is K-semi-algebraic where K is the field generated by $\mathfrak{R}$ and $\exp(g_1), \ldots, \exp(g_n)$. K-semi-algebraic means : there are elements $f_1, \ldots, f_s \in K$ not necessarily distinct, all defined on $T(G, r)$ and such that $T(G, r) = \{x \mid f_1(x) > 0, \ldots, f_t(x) > 0, f_{t+1}(x) \geq 0, \ldots, f_s(x) \geq 0\}$.

There is an easier question than the preceeding : is $T(G, r)$ subanalytic ? I have no idea about the answers to these last two questions. An answer to them would give a good idea of the complexity of the transitivity problem for bilinear sytems. Before ending this part I would like to present a heuristic scheme that could give a method for approaching the preceeding problems and which motivates the second part. One possible way to study the transitivity of invariant systems is to compactify the group G. This can be done in many ways for any semi simple group but in general the compactification is not smooth. Let us then assume that G has a smooth compactification, that is, there exists a smooth compact manifold M on which G operates (on the left say) such that the action has one open dense orbit $\mathcal{O}$ diffeomorphic to G. The remaining set $M-\mathcal{O}$ is called the boundary of the compactification. Given a right invariant polysystem F on G it induces a polysystem F_M on M. The restriction of F_M to $\mathcal{O}$ is isomorphic to the original system F on G. Now the following lemma is fairly obvious.

Lemma.- If $F(e)$ generates $\mathrm{Lie}(G)$ (as a Lie algebra !) then if closure $A(e,F) \cap$ closure $A(e,-F)$ is non empty, F is transitive.

The following generalization, then, of the preceeding lemma, seems plausible : choosing **an** $x_o \in \mathcal{O}$ let $f(x_o, F)$ be the intersection of the closure of $A(x_o, F_M)$ with the boundary of the compactification. If $F(e)$ generates $\mathrm{Lie}(G)$ and belongs to $\mathrm{Afgr}(\mathrm{Lie}(G), r)$, then $F(e)$ belongs to the boundary of $T(G,r)$ if the sets $f(x_o, F)$ and $f(x_o, -F)$ meet in a boundary point.

These considerations lead us to the study of the boundary of accessibility sets.

3.- BOUNDARIES OF ACCESSIBILITY SETS OF REGULAR POLYSYSTEM.

In this part we will consider regular polysystems and their accessibility sets.

Definition 10.- A polysystem F on a manifold M is called C^{∞} (resp. C^{ω}) regular if : (i) for each $x \in M$, $F(x)$ is a strict convex cone with non empty interior ; (ii) if T_oM denotes the open submanifold of TM complement of the zero section, then $F_o = F \cap T_oM$ is a C^{∞} (resp. C^{ω}) submanifold of T_oM ; (iii) if $\pi : T_oM \to M$ denotes the canonical projection then the triple $(F_o, \pi|F_o, M)$ is a sub bundle of the fiber bundle (T_oM, π, M).

The boundaries $B(x,T)$ of the accessibility sets $A(x,F)$ have the following property which is valid in general under very mild conditions on F. We state it here only for regular polysystems.

Proposition 6.- If F is a C^{∞} regular polysystem, then for any $x \in M$, $B(x_o, F)$ is a Lipschitz manifold.

Besides this property the sets $B(x, F)$ can be pathological and if they are smooth their smoothness can be unstable.

Pathology I : There is C^∞ regular polysystem F on any manifold M, $\dim M \geq 3$, such that for some point $x \in M$ the boundary $B(x, F)$ is equal to the accessible boundary $Ba(x, F)$ and is a C^∞-compact manifold, and there exist arbitrarily small perturbations F' of F which are C^∞ regular and such that $B(x, F')$ is not stratifiable.

Pathology II : On any C^ω-manifold M of dimension ≥ 3 there exists C^ω-regular polysystems F such that for some $x \in M$, the inaccessible boundary $Bi(x, F)$ is nowhere subanalytic.

On the other hand I conjecture that if for each $x \in M$ the manifold $F_o(x) = F_o \cap T_x M$ has everywhere positive normal curvature (for some Riemann metric on M ; it does not depend on the metric) then the accessible boundary $Ba(x, F)$ is subanalytic.

Usually the property of transitivity or non transitivity for a regular polysystem is highly unstable under perturbation. Here is a result that characterizes those systems whose transitive or non transitive character is stable under perturbation : for simplicity we assume M compact. Then there is a natural topology on the set of C^∞ regular polysystems, and the following is almost trivial.

Proposition 7.- *The transitive (resp. non transitive) character of a C^∞ regular polysystem F is stable under small C^∞ perturbations if and only if there exists a polysystem F' on M such that : 1) F' is transitive (resp. non transitive) ; 2) for any $x \in M$ the set $F'_o(x) = F'(x) \cap T_o M$ is contained in the interior of $F_o(x) = F(x) \cap T_o M$ (resp. the set $F'_o(x)$ contains $F_o(x)$ in its interior).*

Stably non transitive systems admit the following nice characterization :

Proposition 8.- *A C^∞ regular polysystem F is stably non transitive if and only if there exists a C^1 function $f : M \to \mathbb{R}$ such that for any $v \in F_o = F \cap T_o M$; $df(v) > 0$.*

With this proposition we end our discussion of polysystems.

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Ivan KUPKA
Laboratoire de Math. Pures
B.P.116
38042 ST MARTIN D'HÈRES CEDEX