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INVERSES FOR NONLINEAR CONTROL SYSTEMS

by

R.M. HIRSCHORN

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ABSTRACT.- A nonlinear control system is invertible if the associated input-output map is injective for nonlinear systems of the form $\dot{x} = A(x) + \sum_{k=1}^{m} u_i B_i(x)$ $y = C(x)$ which evolve on a real analytic manifold we obtain sufficient conditions for invertibility and construct systems which act as inverse systems. In the case of single-input systems our conditions are necessary and sufficient for invertibility. For invertible systems we construct nonlinear systems which act as left-inverses for the original systems.

1.- INTRODUCTION. Consider the system

$$\dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} B_i(x(t)) \quad ; \quad x(0) = x_o \in M$$
$$y(t) = C(x(t))$$

where M is a connected real analytic manifold, $A, B_i \in V(M)$, the real vector space of real analytic vector fields on M, $C : M \to \mathbb{R}^m$ is a real analytic mapping, and $u = (u_1, \dots, u_m)$ is a real analytic control function mapping $[0, \infty)$ into $\mathbb{R}^m$. Let $x(t, u, x_o)$ denote the solution of the above differential equation and set $y(t, u, x_o) = C(x(t, u, x_o))$. The system (*) is said to be invertible at x_o if distinct controls $u \neq \hat{u}$ result in distinct outputs $y(., u, x_o) \neq y(., \hat{u}, x_o)$ and strongly invertible if there exists an open dense submanifold M_o of M such that for all $x_o \in M_o$, the system is invertible at x_o. There is a considerable amount of literal dealing with invertibility for linear control system (cf.[1], [2], [3], [4]) and some partial results are known for more general classes of systems (cf.[5], [6], [7]). The purpose of this paper is to indicate a way in which a standard linear system arguement (see [3]) can be generalized to study the invertibility of certain nonlinear systems.

2.- NONLINEAR INVERTIBILITY AND INVERSE SYSTEMS.

A standard linear test for invertibility involves creating a sequence of systems by differentiating the output map (see [3]). Following this approach we let $y(t)$ denote the output $y(t,u,x_o)$ for the system (*). Differentiating y with respect to t we find that

$$y^{(1)}(t) = d\,C_{x(t)}(A(x(t)) + \sum_{i=1}^{m} u_i(t)\, B_i(x(t))\,) = A\,C(x) + \sum_{i=1}^{m} u_i\, B_i\, C$$

where $\forall X \in V(M)$ and $f: M \to \mathbb{R}^{\ell}$, $X\,F(x) = df_x\, X(x)$ (cf.[8]). Thus we can write $y^{(1)}(t) = AC(x(t)) + D(x(t))\,u$ where $D(x) = [B_1\, C(x)\; B_2 C(x)\; \dots\; B_m C(x)]$ is a $m \times m$ matrix for each $x \in M$. Let $\Gamma_1 = \max_{x \in M} \{\text{rank } D(x)\}$. We assume that the components of C have been reordered so that the submatrix $D_{11}(x)$ of $D(x)$ consisting of the first Γ_1 rows of $D(x)$ has rank Γ_1 for some $x \in M$. Set $M_1 = \{x \in M \mid \text{rank } D_{11}(x) = \Gamma_1\}$. It follows from the real analyticity of the entires of $D(x)$ that M_1 is an open dense submanifold of M. Now let

$$E_o(x) = \left[\begin{array}{c|c} I_{\Gamma_1 \times \Gamma_1} & O \\ \hline F_o(x) & I_{(m-\Gamma_1)\times(m-\Gamma_1)} \end{array}\right]$$

be an $m \times m$ elementary matrix whose entires are real analytic functions on M_1 and with the property that

$$E_o(x)\;D(x) = \begin{bmatrix} D_{11}(x) \\ O \end{bmatrix} .$$

This results in a new system

System (1) :

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i\, B_i(x)\;;\; x \in M_1$$

$$Z_1 = C_1(x) + D_1(x)\,u$$

where $C(x) = E_o(x)\,A\,C(x)$, $D_1(x) = E_o(x)\,D(x)$, and by construction $D_1(x)$ has rank Γ_1 on M_1.

<u>Définition</u>.- We call Γ_1 the invertibility index of system (1). The above proceedure can be repeated to produce a sequence of nonlinear systems. Suppose that

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x) \quad ; \quad x \in M_k$$

$$Z_k = C_k(x) + D_k(x) u$$

is the k-th system and has invertibility index Γ_k, and state space M_k, an open dense submanifold of M.

$$Z_k = \begin{bmatrix} \bar{Z}_k \\ \\ \hat{Z}_k \end{bmatrix} = \begin{bmatrix} \bar{C}_k(x) \\ \\ \hat{C}_k(x) \end{bmatrix} + \begin{bmatrix} D_{k_1}(x) & & u \\ & & \\ O & & \end{bmatrix}$$

and differentiating $\hat{Z}_k$ with respect to t we have

$$\hat{Z}_k^{(1)}(t) = A\hat{C}_k(x) + \sum_{i=1}^{m} u_i B_i \hat{C}_k(x) = A\hat{C}_k(x) + D_{k_2}(x) u$$

where D_{k_2} is the matrix with columns $B_i \hat{C}_k$. Set $\hat{D}_k = \begin{bmatrix} D_{k_1} \\ D_{k_2} \end{bmatrix}$ and let $\Gamma_{k+1} = \max_{x \in M_k} \{\text{rank } \hat{D}_k(x)\}$. For simplicity we will assume that components of $\mathbf{C}(x)$ have been reordered so that the submatrix $\hat{D}_{k_1}$ of $\hat{D}_k$ consisting of the first Γ_{k+1} rows of $\hat{D}_k$ has rank Γ_{k+1} for some $x \in M_k$. As above we set $M_{k+1} = \{x \in M_k \mid \text{rank } \hat{D}_{k_1}(x) = \Gamma_{k+1}\}$ and note that M_{k+1} is an open dense submanifold of M_k and hence of M. Finally, let

$$E_k(x) = \left[\begin{array}{c|c} I_{\Gamma_{k+1} \times \Gamma_{k+1}} & O \\ \hline F_k(x) & I_{(m-\Gamma_{k+1}) \times (m-\Gamma_{k+1})} \end{array}\right]$$

be an elementary matrix whose entires are real analytic functions on M_{k+1} and such that

$$E_k(x)\, \hat{D}_k(x) = \begin{bmatrix} \hat{D}_{k_1}(x) \\ O \end{bmatrix} \qquad \text{for all } x \in M_{k+1}$$

This lets us define

System (k+1) :

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x) \; ; \; x \in M_{k+1}$$

$$Z_{k+1} = C_{k+1}(x) + D_{k+1}(x)\, u$$

with invertibility index Γ_{k+1} , $D_{k+1} = E_k \hat{D}_k$, an $C_{k+1} = E_k \begin{bmatrix} \bar{C}_k \\ A\,\hat{C}_k \end{bmatrix}$.

Given system (*) we have constructed a sequence of systems, a sequence of indices $0 \leq \Gamma_1 \leq \Gamma_2 \leq \ldots$ and a sequence of matrix valued functions $E_o(x)$, $E_1(x)$, ... We let α denote the least positive integer k such that $\Gamma_k = m$ or $\alpha = \infty$ if $\Gamma_k < m$ for all $k > 0$ and call α the <u>relative order</u> of the system (*). It is easy to verify that α is well defined (independent of the choice of $E_o(x), E_1(x), \ldots$) and we will show that α is related to the highest order derivative of y used to reconstruct the input from a knowledge of $y(t, u, x_o)$. The following theorems relate the above constructions to the invertibility of the system (*) :

<u>Theorem 1.- If</u> $\alpha < \infty$ <u>then the system</u> (α) <u>constructed above is invertible at</u> x_o <u>for all</u> $x_o \in M_\alpha$. <u>In particular the system</u> (α) <u>is strongly invertible</u>.

<u>Theorem 2.- Consider the system</u> (*) <u>with relative order</u> α . <u>Then if</u> $\alpha = 1$ <u>or if</u> $\alpha > 1$ <u>and for</u> $i \in \{1, 2, \ldots, m\}$

$$B_i \; A^j \; E_k(.) = 0 \quad \text{on} \quad M$$

<u>for</u> $0 \leq k \leq \alpha - 2$ <u>and</u> $0 \leq j \leq \alpha - 2 - k$ <u>the system</u> (*) <u>is invertible at</u> x_o $\forall x_o \in M_\alpha$ <u>and in particular is strongly invertible</u>.

<u>Corollary 1.- For single-input systems</u> (m=1) <u>the condition</u> $\alpha < \infty$ <u>is necessary and sufficient for strong invertibility</u>.

<u>Corollary 2.- Suppose that the system</u> (*) <u>satisfies the hypotheses of theorem 2. Then there exists a matrix function</u> $H_\alpha(x)$ <u>defined on</u> M_α <u>such that</u> $\forall x_o \in M_\alpha$,

$$Z_\alpha(t) = H_\alpha(x(t)) \begin{bmatrix} y^{(1)}(t) \\ y^{(2)}(t) \\ \vdots \\ y^{(\alpha)}(t) \end{bmatrix} = H_\alpha(x(t))\, Y_\alpha(t)$$

and the system

$$(**) \qquad \begin{aligned} \dot{\hat{x}} &= \hat{A}(\hat{x}) + \hat{B}(\hat{x})\,\hat{u} \; ; \quad \hat{x} \in M_\alpha \\ \hat{y} &= \hat{C}(\hat{x}) + \hat{D}(\hat{x})\,\hat{u} \end{aligned}$$

where

$$\hat{A} = A - [B_1\ B_2 \ldots B_m]\, D_\alpha^{-1}\, C_\alpha$$

$$\hat{B} = [B_1\ B_2 \ldots B_m]\, \mathbf{D}_\alpha^{-1}\, H^\alpha$$

$$\hat{C} = -D_\alpha^{-1}\, C_\alpha \quad \text{and} \quad \hat{D} = D_\alpha^{-1}\, H_\alpha$$

acts as a left-inverse for the system (*). In particular, if $\hat{u}(t) = (y^{(1)}(t), \ldots, y^{(\alpha)}(t))$ and $\hat{x}(0) = x_o$ then $\hat{y}(., \hat{u}, x_o) = u(t)$.

Corollary 3. - For multivariable time-invariant linear systems $\alpha < \infty$ is a necessary and sufficient condition for strong invertibility and $M_\alpha = M = \mathbb{R}^n$.

We remark that the left-inverse system described in Corollary 2 provides a "practical" way to recover $u(t)$ given $y(t, u, x_o)$.

Proof (Theorem 1) : Since $\Gamma_\alpha = m$ and $D_\alpha(x)$ is by construction a $m \times m$ matrix valued function on M_α of rank Γ_α , we know D_α^{-1} exists on M_α , and the inverse system (**) from Corollary 2 is well defined if we replace $H_\alpha(x)$ by an $m \times m$ identity matrix. Now we set $\hat{u}(t) = Z_\alpha(t, u, x_o)$, and this results in the evolution of the state vector $\hat{x}(t, \hat{u}, x_o)$. A straightforward computation shows that $x(t) = x(t, u, x_o)$ satisfies (**) when $\hat{u} = Z_\alpha$, and thus $\hat{y}(t) = \hat{c}(x(t)) + \hat{D}(x(t))\,\hat{u} = -D_\alpha^{-1}(x)\, C_\alpha(x) + D_\alpha^{-1}(x)\,\hat{u}$. Since $\hat{u} = Z_\alpha = C_\alpha(x) + D_\alpha(x)\, u$, we have $\hat{y}(., Z_\alpha, x_o) = u(.)$. Since u can be recovered from $y(., u, x_o)$ the system (α) is invertible at x_o .

Proof (Theorem 2): The proof used to establish theorem can be repeated here if we can show that $Z_\alpha(t) = H_\alpha(x(t))\ Y_\alpha(t)$ for some $m \times m\alpha$ matrix valued function $H_\alpha(x)$ on M_α. By assumption

$$(***)\qquad \begin{aligned} &B_i\, E_o(.) = B_i\, A\, E_o(1) = \dots = B_i\, A^{\alpha-2}\, E_o(i) = 0 \\ &B_i\, E_1(.) = \dots = B_i\, A^{\alpha-3}\, E_1(.) = 0 \\ &\vdots \\ &B_i\, E_{\alpha-2}(.) = 0 \end{aligned}$$

and by construction

$$Z_1(t) = E_o(x(t))\ y^{(1)}(t) = \begin{bmatrix} E_o^1\, y^{(1)}(t) \\ E_o^2\,(x(t))\ y^{(1)}(t) \end{bmatrix}$$

where E_o^1 is the submatrix of $E_o(x)$ consisting of the first Γ_1 rows and $E_o^2(x)$ is the matrix formed from the last $m-\Gamma_1$ rows. Following the construction of the systems (1), (2), ..., (α), we see that

$$Z_2 = \begin{bmatrix} E_1^1 \\ E_{12}(x) \end{bmatrix} \begin{bmatrix} E_o^1\, y^{(1)} \\ (E_o^2(x)\ y^{(1)})^{(1)} \end{bmatrix} .\ \text{Now}$$

$$\begin{aligned} \frac{d}{dt}\, E_o^2(x(t))\ y^{(1)} &= E_o^2(x)\ y^{(2)} + \{A\, E_o^2\,(x) + \sum_{i=1}^{m} u_i\, B_i\, E_o^2\,(x)\} \\ &= E_o^2\,(x)\ y^{(2)} + A\, E_o^2\,(x)\ y^{(1)} \end{aligned}$$

from (***), and thus

$$Z_2 = \begin{bmatrix} E_1^1 \\ E_1^2\,(x) \end{bmatrix} \begin{bmatrix} E_o^1\, y^{(1)} \\ E_o^2(x)\ y^{(2)} + A\, E_o^2(x)\ y^{(1)} \end{bmatrix} = H_2(x) \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix}$$

One can continue this proceedure to generate $H_\alpha(x)$. To complete the proof one now repeats the steps outlined in the proof of theorem 1 .

Proof (Corollary 1) : See reference [5].

Proof (Corollary 2) : This Corollary is proved in the course of proving Theorem 2.

Proof (Corollary 3) : See [3].

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