Astérisque

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*Astérisque*, tome 75-76 (1980), p. 125-131
<http://www.numdam.org/item?id=AST_1980__75-76__125_0>

LIE ALGEBRAIC METHODS FOR THE CONTROL OF INFINITE DIMENSIONAL NONLINEAR EVOLUTION EQUATIONS

by

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INTRODUCTION. -

Let $\mathcal{A}$ , $\mathcal{B}$ be densely defined operators acting within a Banach space B of $\mathbb{R}^n$ valued functions. We consider systems of the form

(1) $$dv_t/dt = \mathcal{A}(v_t) + u(t)\,\mathcal{B}(v_t)\ ,\quad v_o = \varphi \in B$$

where the control u is Lebesgue measurable with values $|u(t)| \leq 1$ , and unique solutions of (1) are assumed to exist for small $t > 0$ . An observation of (1) is a continuous linear map $g : B \to \mathbb{R}^k$ .

Let $v_t$ denote the reference solution, at time t , corresponding to control $u \equiv 0$ , and $A(t,\varphi) \subset B$ be the set of points attainable at time t by all solutions of (1) . We shall study two questions. The first, that of local controllability, is to derive computable conditions to determine when $g(v_t) \in$ interior $g(A(t,\varphi))$ for small $t > 0$ . This includes the study of $\mathbb{R}^n$ controllability, say for delay equations, but not function space controllability, e.g. see [1], [4] . The second question is that of finite dimensional realizations. Specifically, if $t \to g(v_t(u))$ is a solution $t \to v_t(u)$ of (1), we say g admits a strong differential realization on $\mathbb{R}^k$ . We shall give an example of an observation of a linear controlled parabolic equation which admits a strong (bilinear) differential realization on $\mathbb{R}^2$ . This realization has "singular arcs" which may be studied by high order methods using Lie Theory. This illustrates the difficulties which may occur in "internally

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* This research was supported by NSF grant MCS 76-04419-A01.

controlled" systems of partial differential equations, i.e. systems where the control is not a forcing function or boundary value.

## I.- LIE PRODUCTS OF OPERATORS.

Throughout, the symbol $s$ will be used as the independent variable for a function $v \in B$. The infinitesimal generators we will consider shall be assumed to have the form

$$(2) \qquad (Cv)(s) = f^s((T^s v)(s))$$

where for each $s$, $T^s$ is a closed linear operator acting within $B$ (and these may change with the value $s$) while $f^s : \mathbb{R}^n \to \mathbb{R}^\ell$ is real analytic. We assume, throughout, that our operators generate (at least locally) strongly continuous one parameter semi-groups, i.e., that $dv_t/dt = C(v_t)$, $v_o = \varphi$ has a unique solution for small $t > 0$.

For example, consider the delay equation on $\mathbb{R}^n$ ; $dx(t)/dt = W(x(t-1))$, $x(s) = \varphi(s)$, $-1 \leq s \leq 0$ where $\varphi \in C[-1, 0]$ and $W : \mathbb{R}^n \to \mathbb{R}^n$ is real analytic. Let $\zeta(t, \varphi)$ denote a solution at time $t$ and define $v_t(\varphi)(s) = \zeta(t+s, \varphi)$, $-1 \leq s \leq 0$. The map $t \to v_t(\varphi) \in C[-1, 0]$ is a strongly continuous semi-group with infinitesimal generator

$$(Cv_t)(s) = \begin{cases} d/ds\, v_t(s) = d/dt\, v_t(s), & -1 \leq s < 0 \\ W((Sv_t)(0)) = W(v_t(-1)), & s = 0 \end{cases}$$

where $S$ is the unit shift operator (i.e. $(Sv)(t) = v(t-1)$). Here, referring to (2), $f^s(\lambda) = \lambda$ and $T^s = d/ds$ if $-1 \leq s < 0$ while $f^s(\lambda) = W(\lambda)$ and $T^s = S$ if $s = 0$. In the Banach space $B = C[-1, 0]$, $v_t(\varphi)$ satisfies the equation $d/dt\, v_t(\varphi) = C(v_t)$, $v_o = \varphi$. Let $C$ be an operator of the form (2) and $f'$ denote the derivative of $f$. We define $((DC(v)(s))w)(s) = f'^s((T^s v)(s))(T^s w)(s)$ and the <u>Lie Product</u> of operators $C, \mathcal{D}$ of the form (2) as $([C, \mathcal{D}]v)(s) = ((DC(v)(s))(\mathcal{D}v))(s) - ((D\mathcal{D}(v)(s))(Cv))(s)$. As expected, if $C, \mathcal{D}$ are linear operators then $[C, \mathcal{D}]$ is just the commutator $C\mathcal{D} - \mathcal{D}C$. Furthermore, the above concepts easily generalize to operators of the form $(Cv)(s) = \Sigma_{\nu=1}^m f^s_{\nu,1}((T^s_{\nu,1}v)(s)) \ldots f^s_{\nu,k}((T^s_{\nu,k}v)(s))$; the details and examples of computations can be found in [4], [5]. We introduce the notation $(\mathrm{ad}C, \mathcal{D}) = [C, \mathcal{D}]$ and inductively $(\mathrm{ad}^{k+1} C, \mathcal{D}) = [C, (\mathrm{ad}^k C, \mathcal{D})]$.

Now consider the equation (1) with $\mathcal{A}, \mathcal{B}$ operators of the form (2). Let $\eta_t(\varphi)$ denote the solution, at time $t$, of $dv_t/dt = \mathcal{A}(v_t)$, $v_o = \varphi$, and $D\eta_t(\varphi)$ the differential of the map $\varphi \to \eta_t(\varphi)$. In [4] we showed the

<u>DECOMPOSITION THEOREM</u>. - Assume the maps $t \to \eta_t(\varphi)(s)$ and $t \to D\eta_t(\varphi)(s)$ are real analytic for all $s$ and that $D\eta_t(\varphi) \to id$ in the strong operator topology as $t^+ \to 0$. Then a sufficient condition that the composition $\eta_t(\psi_t(\varphi, u))$ be a solution of equation (1) is that $\psi_t(\varphi, u)$ satisfy

$$(3) \qquad dv_t/dt = u(t)\, \Sigma_{\nu=0}^{\infty} (-t)^{\nu}/\nu!\ (ad^{\nu}\mathcal{A}, \mathcal{B})(v_t)\ ,\ v_o = \varphi\ .$$

Such decomposition theorems provide the key to the applications of Lie theory to differential equations and control systems, [6], [7].

Let $g : B \to \mathbb{R}^k$ be continuous and linear, $C$ an operator of the form (2), $\xi_t(\varphi)$ the solution at time $t$ of $dv_t/dt = C(v_t)$, $v_o = \varphi$ and define $g_*(C_\varphi) = \lim_{t^+ \to 0} d/dt\, g(\xi_t(\varphi))$. Analogous to the case of control systems on manifolds, we associate with (1) the set of operators

$$(4) \qquad \mathcal{J}^1 = \{ (ad^{\nu}\mathcal{A}, \mathcal{B}) : \nu = 0, 1, \ldots \}$$

and let $\mathcal{J}^1(\varphi)$ denote the elements of $\mathcal{J}^1$ evaluated at $\varphi$.

<u>THEOREM</u> (<u>Local Controllability</u>). - <u>The observation</u> $g : B \to \mathbb{R}^k$ <u>of system</u> (1) <u>is locally controllable along the reference solution</u> $\eta_t(\varphi)$ <u>corresponding to</u> $u \equiv 0$ (<u>i.e.</u> $g(\eta_t(\varphi)) \in \text{int}\, g(A(t,\varphi))$ <u>for small</u> $t > 0$) <u>if</u> $\dim.\text{span}\, g_*(\mathcal{J}^1(\varphi)) = k$.

<u>Proof</u> : Form $\hat{B} = B \times \mathbb{R}^k$ ; let $w(t) = g(v_t) \in \mathbb{R}^k$ and $\hat{\mathcal{A}} \begin{bmatrix} v_t \\ w(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} v_t \\ g_*(\mathcal{A} v_t) \end{bmatrix}$, $\hat{\mathcal{B}} \begin{bmatrix} v_t \\ w(t) \end{bmatrix} = \begin{bmatrix} \mathcal{B} v_t \\ g_*(\mathcal{B} v_t) \end{bmatrix}$. The "augmented system" on $\hat{B}$ is

$$(i) \qquad \frac{\partial}{\partial t} \begin{bmatrix} v_t \\ w(t) \end{bmatrix} = \hat{\mathcal{A}} \begin{bmatrix} v_t \\ w(t) \end{bmatrix} + u(t)\hat{\mathcal{B}} \begin{bmatrix} v_t \\ w(t) \end{bmatrix},\ \begin{bmatrix} v_o \\ w(o) \end{bmatrix} = \begin{bmatrix} \varphi \\ g(\varphi) \end{bmatrix}.$$

Define $\mathcal{J}^1\begin{bmatrix}\varphi\\ g(\varphi)\end{bmatrix} = \{(\mathrm{ad}^{\nu}\hat{\mathfrak{a}}, \hat{\mathfrak{B}})\begin{bmatrix}\varphi\\ g(\varphi)\end{bmatrix} : \nu = 0, 1, \ldots\}$ and $\pi : \hat{B} \to \mathbb{R}^k$ be projections to $\mathbb{R}^k$, specifically for $\psi \in B$, $y \in \mathbb{R}^k$, $\pi\begin{bmatrix}\psi\\ y\end{bmatrix} = y$. Let $\hat{A}(t,\varphi) \subset \hat{B}$ denote the elements attainable at time $t$ by solutions of (i) corresponding to all admissible controls. The reference solution of (i) corresponding to $u \equiv 0$ is $\begin{bmatrix}\eta_t\\ g(\eta_t)\end{bmatrix} \in \hat{A}(t,\varphi)$. Clearly if $v_t(u)$ denotes a solution of (1) at time $t$ corresponding to control choice $u$, then $\begin{bmatrix}v_t(u)\\ g(v_t(u))\end{bmatrix}$ is a solution of (i). It follows that

$$\pi(\hat{A}(t,\varphi)) = g(A(t,\varphi)) . \tag{ii}$$

Since $\operatorname{span} \mathcal{J}^1\begin{bmatrix}\varphi\\ g(\varphi)\end{bmatrix}$ is the "first order local set of directions" in which one can proceed via solutions of (i), from the inverse function theorem (see [8], chap. I. § 5, in particular corollaries 1, 1s) one has

$$g(\eta_t) = \pi\begin{bmatrix}\eta_t\\ g(\eta_t)\end{bmatrix} \in \operatorname{int} \pi(\hat{A}(t,\varphi)) = \operatorname{int} g(A(t,\varphi)) \tag{iii}$$

for small $t > 0$ if $\dim \operatorname{span} \pi(\hat{\mathcal{J}}^1\begin{bmatrix}\varphi\\ g(\varphi)\end{bmatrix}) = k$. To complete the proof one need only show that $\pi(\hat{\mathcal{J}}^1\begin{bmatrix}\varphi\\ g(\varphi)\end{bmatrix}) = g_*(\mathcal{J}^1(\varphi))$ which is a straightforward calculation using induction. ∎

As an application, we consider a tension controlled vibrating string. Let $\rho$ denote density, and $s = s(t)$ tension. The one dimensional equation of the vibrating string is $\rho\, \partial^2 w/\partial t^2 = \partial/\partial x\, (s(t)\partial w/\partial x)$, where $w$ measures deflection from the rest position. Choose $u_o$ as a nominal value of $s(t)/\rho$, $\mu > 0$ such that $u_o - \mu > 0$ and the control $u(t) = s(t)/\rho - \mu_o$, with $|u(t)| \leq \mu$. Let $\partial w/\partial x = v^1$, $\partial w/\partial t = v^2$ to obtain the first order system

$$\begin{aligned}\partial v^1/\partial t &= \partial v^2/\partial x , & v^1(x) &= \varphi^1(x), \quad 0 \leq x \leq \ell\\ \partial v^2/\partial t &= u_o \partial v^1/\partial x + u(t)\partial v^1/\partial x , & v^2(x) &= \varphi^2(x), \quad 0 \leq x \leq \ell\end{aligned} \tag{i}$$

where we assume the string is clamped at both ends so $\varphi^2(0) = \varphi^2(\ell) = 0$. Take $C_o[0,\ell]$ to be those functions in $C[0,\ell]$ which vanish at $0$ and $\ell$ and $B = C[0,\ell] \times C_o[0,\ell]$. The boundary data $w(t,0) = w(t.\ell) = 0$ is implicit in $B$

if the constants chosen in recovering w from $v^1, v^2$ are properly chosen.

This is assumed. With $\mathcal{A} = \begin{bmatrix} 0 & \partial/\partial x \\ u_o \partial/\partial x & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 & 0 \\ \partial/\partial x & 0 \end{bmatrix}$,

$$\text{(ii)} \qquad \mathcal{J}^1(\varphi) = \{(\mathrm{ad}^\nu \mathcal{A}, \mathcal{B})\varphi : \nu = 0,1,\ldots\} = \left\{ \begin{bmatrix} 0 \\ \partial\varphi^1/\partial x \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \partial^2\varphi^1/\partial x^2 \\ -\partial^2\varphi^2/\partial x^2 \end{bmatrix}, \begin{bmatrix} -2\partial^3\varphi^2/\partial x^3 \\ 2u_o\partial^3\varphi^1/\partial x^3 \end{bmatrix}, \begin{bmatrix} 2^2u_o\partial^4\varphi^1/\partial x^4 \\ -2^2u_o\partial^4\varphi^2/\partial x^4 \end{bmatrix}, \begin{bmatrix} -2^3u_o\partial^5\varphi^2/\partial x^5 \\ 2^3u_o^2\partial^5\varphi^1/\partial x^5 \end{bmatrix}, \ldots \right\}.$$

One may now readily check specific initial data and observations for local controllability. For example, if $\varphi^1(x) = (\pi/\ell)\cos(\pi x/\ell)$, $\varphi^2(x) = 0$ the solution with $u(t) \equiv 0$ (i.e. the reference solution) is $w(t,x) = \cos(\sqrt{u_o}\,\pi t/\ell)\sin(\pi x/\ell)$, i.e. at each time $t$, the position of the string is a scalar multiple of $\sin(\pi x/\ell)$. Take as observation $g_1(v_1) = v_t^1(\ell/2)$, $g_2(v_t) = v_t^2(\ell/2)$, i.e. respectively the angle of deflection of the string at $\ell/2$ and the velocity of the point at $\ell/2$. Computing shows rank $g_*(\mathcal{J}^1(\varphi)) = 1$, not 2, hence the theorem does not imply local controllability of this observation. Physically this is expected since the initial data was chosen so that $g_1(v_t)$, the angle of deflection at $\ell/2$, would be zero for all $t$. On the other hand, the velocity of the point at $\ell/2$, i.e. $g_2(v_t)$, can be locally controlled via tension.

One may show that for any integer $k$ there exists initial data $\varphi^1, \varphi^2$ and an $\mathbb{R}^k$ valued observation $g$ which is locally controllable along the reference via tension.

## II.- FINITE DIMENSIONAL DIFFERENTIAL REALIZATIONS.

Consider a special case of equation (1) i.e.

$$\text{(5)} \qquad dv_t/dt = Cv_t + u(t)\,b\,, \quad v_o = \varphi \in B$$

where $C$ is linear and $b \in B$. By differentiating $g(v_t(u))$ with respect to $t$ it follows that if there exists a mapping $C_* : \mathbb{R}^k$ such that

$$g_* C v_t = C_* g(v_t) \tag{6}$$

then a strong differential realization of $g$ exists. If $C$ has $\ker g$ as an invariant subspace, equation (6) may be used to define $C_*$. This rather severe restriction leads one to "change the order of events". Given an equation of the form (8), suppose $C$ has an invariant subspace, $S$, of co-dim $k$. Then any observation $g$ having $S$ as kernel does admit a strong differential realization. A specific example with entails a slightly more general evolutionary equation than (5) is

<u>Example 2.1.</u>- Consider the parabolic partial differential equation

$$\partial v_t/\partial t = \partial^2 v_t/\partial x^2 + u(t)\,(\partial v_t + \partial x + \sin x) \tag{7}$$

$$v_o(x) = 2\cos x + 2 \ .$$

Let $S^1$ be the one sphere parametrized by $-\pi \leqslant x \leqslant \pi$ and consider equation (7) on $S^1 \times [0,\infty]$. Let $B = \{w \in \mathcal{L}_2[-\pi,\pi] : w(-\pi) = w(\pi) = 0\}$, $\mathfrak{A}\, v_t = \partial^2 v_t/\partial x^2$, $\mathfrak{B} v_t = \partial v_t/\partial x$ and $b \in B$ be given by $b(x) = \sin x$. For the observation we choose $g = (g_1, g_2)$ with $g_1(v_t) = \frac{1}{\pi}\int_{-\pi}^{\pi} v_t(x) \sin x \, dx$, $g_2(v_t) = \frac{1}{\pi}\int_{-\pi}^{\pi} v_t(x)\cos x\, dx$. Then $S = \mathrm{span}\{1, \sin \nu x, \cos \nu x : \nu = 2, 3, \ldots\} = \ker g$ while $\mathrm{codim}\, S = 2$ and $\mathfrak{A}, \mathfrak{B} : S \to S$. To compute the strong differential realization, let $v_t(x) = \Sigma_{j=1}^{\infty} \sigma_j(t) \sin jx + \Sigma_{j=0}^{\infty} \gamma_j(t)\cos jx$ and form the equation $d/dt\, g(v_t) = g_* \mathfrak{A} v_t + u(t)\, g_* \mathfrak{B} v_t + u(t)\, g_* b$. Letting $\sigma_1(t) = y_1(t)$, $\gamma_1(t) = y_2(t)$ the differential realization on $\mathbb{R}^2$ is ($\dot{y} = dy/dt$)

$$\begin{aligned} \dot{y}_1 &= -y_1 + u(t)\,(1 - y_2)\ , \quad y_1(0) = 0 \\ \dot{y}_2 &= -y_2 + u(t)\, y_1 \quad\quad\ , \quad y_2(0) = 2\ . \end{aligned} \tag{8}$$

If we consider the problem, for (7), of finding that measurable control $u$ with $|u(t)| \leqslant 1$ such that $g(v_t) = (0,1)$ in minimum time $t$, this leads to the problem of reaching $(0,1)$ in minimum time by a solution of (8). The method of [9, §22] may be used to show that the optimal solution of this latter "bilinear problem" is $y_1(t) \equiv 0$, $y_2(t) = 2e^{-t}$, a singular arc obtained with control $u(t) \equiv 0$.

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