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ON INVARIANTS AND MODULI FOR LINEAR TIME-VARYING SYSTEMS

by

Michel HAZEWINKEL

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## I.- INTRODUCTION.

Consider a linear time-varying dynamical system

$$(1.1) \qquad \dot{x} = Fx + Gu \ , \ y = Hx$$

where $x(t) \in \mathbb{R}^n$ , $y(t) \in \mathbb{R}^p$ , $u(t) \in \mathbb{R}^m$ and where $F$ , $G$ , $H$ are matrices of the appropriate sizes with coefficients which may depend on the time $t$ . To fix the ideas suppose for example that the coefficients of $F$, $G$, $H$ all belong to the field of rational functions over $\mathbb{R}$ . Then it makes perfect sense to consider bases changes of the type $\hat{x} = Sx$ where $S$ is an $n \times n$ matrix also with coefficients in $\mathbb{R}(t)$ with nonzero determinant. Such a base change transforms the equations (1.1) into

$$(1.2) \qquad \hat{x}\cdot = (SFS^{-1} + \dot{S}S^{-1})\hat{x} + SG_u \ , y = HS^{-1}\hat{x}$$

and at least in the algebraic sense one can ask about invariants, moduli and canonical forms just as in the case of non-time varying systems ([3-6]).

Solutions to equations like (1.1) with $u(t) \in \mathbb{R}(t)$ given, certainly exist as vectors with coefficients in some differential extension field (cf. [11], [13], or [14]). They also exist as "functions" albeit as multiple valued functions with poles and branching points if $F, G$ or $u(t)$ have poles, cf. e.g. [9].

The main purpose of the present note is to point out that the results of [5, 6] also go through in a time variable setting like the one discussed just above. In fact, more generally, these results go through for systems

$$(1.3) \qquad \delta x = Fx + Gu, y = Hx$$

where the $F, G, H$ are matrices with coefficients in any differential field $k$

with differentiation operator $\delta$ (for a definition cf. 2.1 below). Examples of such differential fields are

(a) $k = \mathbb{R}(t)$ or $\mathbb{C}(t)$, $\delta = \frac{d}{dt}$

(b) $k$ = real meromorphic functions or complex meromorphic functions, $\delta = \frac{d}{dt}$

(c) $k = \mathbb{R}(\sin t, \cos t, \sin 2t, \cos 2t, \ldots)$, $\delta = \frac{d}{dt}$ .

Thus when one specializes the results for abstract differential fields obtained below to one of these cases one obtains results for "real life" dynamical systems with time variable coefficients.

The techniques used to obtain the results below are basically the same as in [5, 6]. Most of the (minor) difficulties are caused by the fact that differential algebraic geometry is more difficult and certainly far less developped than ordinary algebraic geometry. The present note only outlines the definitions and results. A more complete version is [7] .

## II.- PRELIMINARY REMARKS CONCERNING DIFFERENTIAL ALGEBRA UND DIFFERENTIAL ALGEBRAIC GEOMETRY.

A differential field $k$ is a field together with an additive operator $\delta : k \to k$ which satisfies $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in k$ . Examples were mentioned in the introduction. If the characteristic of $k$ is zero (as in all the examples given) then there exists a differentially closed extension differential field $(k, \delta)$ of $k$ (cf. [14]), i.e. a field such that every polynomial expression in a number of variables and their derivatives has a solution in $K$ . E.g. there will be elements $x_1, x_2 \in K$ such that $x_1^2 + (\delta^3 x_1)(\delta^2 x_2) + x_2^7 = 1$ . If $\mathrm{char}(k) > 0$ the question of existence of a differentially closed extension field is open. But there exist in any case large enough extension fields $K$ to play the rôle of the universal field $\Omega$ of algebraic geometry. Just as an affine algebraic variety over $k$ is the set of solutions in $\Omega^n$ of a set of polynomials in $X_1, \ldots, X_n$ with coefficients in $k$ , one defines an affine differential algebraic (d.a.) variety as the set of points in $K^n$ , $K$ big enough, which satisfy a set of polynomial expressions in their coordinates <u>and their derivatives</u>. There is an obvious Zariski type topology on $K^n$ defined by taking as closed subsets all affine d.a. varieties in $K^n$ and hence an induced topology on affine d.a. varieties. A morphism between d.a.

varieties is a map which can locally be described by means of rational expressions in the coordinates and their derivatives and a d.a. variety is a $T_1$-space which locally looks like an affine d.a. variety. It now requires little imagination to define morphisms of d.a. varieties, d.a. groups, d.a. vectorbundles and d.a. actions of a d.a. group on a d.a. variety. In particular ordinary algebraic varieties, ..., over $k$ are special kinds of d.a. varieties, ..., over $k$. If $G \times V \to V$ is a d.a. action of the d.a. group on the d.a. variety $V$ then an invariant is a d.a. rational function $f : V \to K$ such that $f(gx) = f(x)$ for all $x \in V$, $g \in G$ for which $x$ and $gx$ are both in the domain of $f$. This definition of course agrees with the one of S. Lie in [12], modulo the changes caused by the present algebraic-geometric setting.

## III.- THE D.A. QUOTIENT VARIETY $M^{ar}_{m,n,p} = L^{ar}_{m,n,p}/GL_n$. INVARIANTS.

Let $k$ be any differential field with universal extension $K$. For example $k$ may be the field of rational or meromorphic functions over $\mathbb{R}$ or $\mathbb{C}$, with $\delta = \frac{d}{dt}$. We consider equations

$$\delta x = Fx + Gu, \quad y = Hx \tag{3.1}$$

with $x(t) \in k^n$, $u(t) \in k^m$, $y(t) \in k^p$ and $F, G, H$ matrices of the appropriate sizes with coefficients in $k$. As a rule we shall write $\dot{x}$ instead of $\delta x$.

Let $L_{m,n,p}$ be the d.a. variety of all triples of matrices $(F, G, H)$ of sizes $n \times n$, $n \times m$, $p \times n$ respectively. Let $GL_n$ be the d.a. group of all $n \times n$ invertible matrices. We define a d.a. action of $GL_n$ on $L_{m,n,p}$ by

$$GL_n \times L_{m,n,p} \to L_{m,n,p}, \quad (F, G, H)^S = (SFS^{-1} + \dot{S}S^{-1}, SG, HS^{-1}). \tag{3.2}$$

Note that this is indeed a d.a. $GL_n$-action, but not a morphism of the algebraic variety $GL_n \times L_{m,n,p}$ into the algebraic variety $L_{m,n,p}$. Of course, this action of $GL_n$ corresponds to the transformation $x \to Sx$ in state space in (3.1).

Let $(F, G, H) \in L_{m,n,p}$. We define the $n \times (n+1)m$ matrix $R(F, G)$ by

$$R(F, G) = (G(0) \,\vdots\, G(1) \,\vdots\, \ldots \,\vdots\, G(n)) \tag{3.3}$$

where $G(i)$ is inductively defined by $G(0) = G$, $G(i) = FG(i-1) - \dot{G}(i-1)$, $i=1,2,\dots,n$. More or less dually the matrix $Q(F,H)$ is defined as

$$(3.4) \qquad Q(F,H)^T = (H(0)^T \,\vdots\, H(1)^T \,\vdots\, \dots \,\vdots\, H(n)^T)$$

with $H(0) = H$, $H(i) = H(i-1)F + \dot{H}(i-1)$, $i=1,2,\dots,n$, where the symbol $T$ denotes "transposes". (Note the sign difference).

The triple $(F,G,H)$ is said to be <u>algebraically reachable</u> (abbreviated "ar") if rank $(R(F,G)) = n$ ; the triple $(F,G,H)$ is said to be <u>algebraically observable</u> (abbreviated "ao") if rank $(Q(F,H)) = n$ . These two conditions define open d.a. subvarieties of $L_{m,n,p}$ which we denote $L^{ar}_{m,n,p}$, $L^{ao}_{m,n,p}$ . In addition we define $L^{ar,ao}_{m,n,p} = L^{ar}_{m,n,p} \cap L^{ao}_{m,n,p}$ .

Of course the notions "algebraically reachable" and "algebraically observable" as defined above correspond to the usual geometric notions of reachability and observability in the cases where $k$ is a field of rational or meromorphic function over $\mathbb{R}$ or $\mathbb{C}$ . Indeed the system $(F,G,H)$ is ao iff $Q(F,H)$ has rank $n$ . Because of the nature of the functions involved this happens iff $Q(F(t),H(t))$ has rank $n$ pointwise in $t$ for all $t$ except possibly a set of measure zero and this in turn means that $(F,G,H)$ is completely observable in the usual geometric sense (cf. [16], corollary 8.8). Dually one has that algebraically reachable corresponds to completely reachable in the geometric sense for such differentiable fields.

Let $J_{n,m} = \{(0,1),\dots,(0,m)\,;\,(1,1),\dots,(1,m)\,;\,\dots\,;\,(n,1),\dots,(n,m)\}$ , lexicographically ordered. We use $J_{n,m}$ to label the columns of the matrices $R(F,G)$ by assigning the label $(i,j)$ to the j-th column of $G(i)$. A subset $\alpha \subset J_{n,m}$ is <u>nice</u> if $(i,j) \in \alpha \Rightarrow (i-1,j) \in \alpha$ for all $i,j$ . A nice subset of size $n$ is called as <u>nice selection</u>. Given a nice selection $\alpha$ , a <u>successor index</u> of $\alpha$ is an element $(i,j) \in J_{n,m} \setminus \alpha$ such that $\alpha \cup \{(i,j)\}$ is nice. For every $j \in \{1,\dots,m\}$ and nice selection $\alpha$ there is precisely one successor index $(i,j')$ of $\alpha$ such that $j' = j$ . This successor index will be denoted $s(\alpha,j)$ .

(3.5) <u>Nice selection lemma</u>. - <u>Let</u> $(F,G,H) \in L^{ar}_{m,n,p}$ . <u>Then there is a nice selection</u> $\alpha \subset J_{n,m}$ <u>such that</u> $\det(R(F,G)_\alpha) \neq 0$ . (<u>Here</u> $R(F,G)_\alpha$ <u>is the</u>

<u>square</u> $n \times n$ <u>matrix obtained from</u> $R(F,G)$ <u>by removing all columns whose index is not in</u> $\alpha$ ).

We now proceed as in [5, 6]. First, note that

$$R(SFS^{-1} + \dot{S}S^{-1}, SG) = SR(F,G) \tag{3.6}$$

(because $(SFS^{-1} + \dot{S}S^{-1})(SG(i)) - (SG(i))^{\cdot} = SFG(i) + \dot{S}G(i) - \dot{S}G(i) - S\dot{G}(i) = S(FG(i) - \dot{G}(i)))$. Let $\alpha$ be a nice selection and let $x = (x_1, \dots, x_m) \in k^{nm} = k^n \times \dots \times k^n$. Using (3.6) one now shows just as easily as in [5, 6] that there exists precisely one triple $(F,G,H) \in L^{ar}_{m,n,p}$ such that $R(F,G)_\alpha = I_n$, $R(F,G)_{s(\alpha,j)} = x_j$ for $j = 1, \dots, m$. It follows that if $U_\alpha = \{(F,G,H) \in L_{m,n,p} \mid \det(R(F,G)_\alpha) \neq 0\}$ then

$$U_\alpha \simeq GL_n \times K^{mn+np}, \quad U_\alpha / GL_n \simeq K^{mn+np}. \tag{3.7}$$

For each nice selection $\alpha$ and $x = (y,z) \in K^{mn+np}$ let $\psi_\alpha(x) = (F_\alpha(x), G_\alpha(x), H_\alpha(x))$ be the unique triple such that $R(F_\alpha(x), G_\alpha(x))_\alpha = I_n$, $R(F_\alpha(x), G_\alpha(x))_{s(\alpha,j)}$ is the j-th component of $y = (y_1, \dots, y_m) \in (k^n)^m$, and such that $H_\alpha(x) = z$.

We now construct the d.a. quotient variety $M^{ar}_{m,n,p}$ as follows ; again as in [6]. For each nice selection $\alpha$ let $V_\alpha = k^{nm} \times k^{np}$ and let $V_{\alpha\beta} = \{x \in V_\alpha \mid \det(R(F_\alpha(x), G_\alpha(x))_\beta) \neq 0\}$. We now glue the $V_\alpha$ together by means of the isomorphisms $\psi_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$, which are defined by $\psi_{\alpha\beta}(x) = y \Leftrightarrow (F_\beta(y), G_\beta(y), H_\beta(y)) = (F_\alpha(x), G_\alpha(x), H_\alpha(x))^S$ where $S = R(F_\alpha(x), G_\alpha(x))_\beta^{-1}$. This defines us a d.a. variety provided we can show that $M^{ar}_{m,n,p}$ is $T_1$. Note that by construction $M^{ar}_{m,n,p} = L^{ar}_{m,n,p}/GL_n$, in any case as sets.

Now let $G_{n,(n+1)m}$ be the d.a. Grassmann variety of n-planes in (n+1)-space. Then by (3.6) $R$ induces a map $g : M^{ar}_{m,n,p} \to G_{n,(n+1)m}$. One now also defines $\tilde{h} : L^{ar}_{m,n,p} \to K^{(n+1)^2}$ by $\tilde{h}(F,G,H) = Q(F,H)\, R(F,G)$. Now note that similarly to (3.6)

$$Q(SFS^{-1} + \dot{S}S^{-1}, HS^{-1}) = Q(F,H)S^{-1}. \tag{3.8}$$

Combining this with (3.6) we see that $\tilde{h}(F,G,H)^S) = \tilde{h}(F,G,H)$, so that $\tilde{h}$ induces a map $h : M^{ar}_{m,n,p} \to K^{(n+1)^2 mp}$. One now shows as in [6] that

$(g,h) : M^{ar}_{m,n,p} \to G_{n,(n+1)m} \times K^{(n+1)^2 mp}$ is injective which proves that $M^{ar}_{m,n,p}$ is $T_1$ and hence a d.a. variety. The maps $g$ and $h$ are d.a. morphisms (defined over $k$).

(3.9) Corollary. - $M^{ar}_{m,n,p}$ is an irreducible quasi projective d.a variety. It is the quotient of $L^{ar}_{m,n,p}$ by $GL_n$ in the category of d.a. varieties.

One also verifies with no trouble that $M^{ar}_{m,n,p}$ in addition enjoys the pleasant quotient property that $M^{ar}_{m,n,p}(k') = L^{ar}_{m,n,p}(k')/GL_n(k')$ for all intermediate differential fields $k \subset k' \subset K$.

Let $\pi : L^{ar}_{m,n,p} \to M^{ar}_{m,n,p}$ be the natural projection. Then $M^{ar,ao}_{m,n,p}$ the image of $L^{ar,ao}_{m,n,p}$ is an open d.a. subvariety of $M^{ar}_{m,n,p}$ and one shows as in [6] that the morphism $h$ above is injective on $M^{ar,ao}_{m,n,p}$. Its image is readily described. An $(n+1) \times (n+1)$ block matrix with blocks of size $p \times m$

$$A = \begin{bmatrix} A_{o,o} & \cdots & A_{o,n} \\ \vdots & & \vdots \\ A_{n,o} & & A_{n,n} \end{bmatrix}$$

is of the form $h(F,G,H)$ for some triple $(F,G,H) \in L^{ar,ao}_{m,n,p}$ is and only if the following two conditions hold : (i) $\operatorname{rank}(A) = n = \operatorname{rank}(A')$, where $A'$ is the matrix obtained from $A$ by removing the last column and row of blocks, and

(ii) $\qquad A_{i+1,j} - A_{i,j+1} = \dot{A}_{i,j}$ for all $i,j \in \{0,1,\ldots,n-1\}$.

(3.10) Corollary. - $M^{ar,ao}_{m,n,p}$ is a quasi-affine d.a. variety.

(3.11) Corollary. - Every differential invariant of $GL_n$ acting on $L_{m,n,p}$ is a rational function in the entries of the matrix $h(F,G,H) = Q(F,H)\, R(F,G)$ and their derivatives.

Note that $L_{m,n,p}$, $M^{ar}_{m,n,p}$, $M^{ar,ao}_{m,n,p}$ are defined by polynomials involving no derivatives, and hence are ordinary algebraic varieties reinterpreted within the context of d.a. varieties. On the other hand the definitions of $L^{ar}_{m,n,p}$, $L^{ar,ao}_{m,n,p}$ do involve derivatives and so do the projection map $\pi : L^{ar}_{m,n,p} \to M^{ar}_{m,n,p}$, the embedding $h : M^{ar,ao}_{m,n,p} \to K^{(n+1)^2 mp}$ and hence the description of $M^{ar,ao}_{m,n,p}$ as a quasi affine d.a. subvariety. Note that if $k$ is one of the "function differential fields" mentioned in the introduction then $M^{ar,ao}_{m,n,p}$ is a certain space of functions which satisfy certain differential equations.

## IV.- CANONICAL FORMS, UNIVERSAL FAMILIES, LIMITS.

We can be brief about the matter of existence or nonexistence of global continuous canonical forms. On the one hand there exist of course the local canonical forms $c_{*\alpha} : U_\alpha \to U_\alpha$ for every nice selection $\alpha$ defined by $c_{*\alpha}(F, G, H) = (F, G, H)^S$, $S = R(F, G)_\alpha^{-1}$. On the other hand the same examples and constructions used in [5, 6] show that global continuous canonical forms on $L^{ar,ao}_{m,n,p}$ exist if and only if $m = 1$ or $p = 1$. This is not completely immediate from the corresponding result in the non-time-varying case, because, a priori, the canonical form of a non-time varying linear system could be timevarying in the present setting. There are similar analogues of all the other results of [5, 6] pertaining to canonical forms. E.g., there is a continuous canonical form on $L^{ar}_{m,n,p}$ (resp. $L^{ao}_{m,n,p}$) if and only if $m = 1$ (resp. $p = 1$).

Let us also note that $L^{ar}_{m,n,p} \to M^{ar}_{m,n,p}$ is a locally trivial principal d.a. $GL_n$ fibre bundle over $M^{ar}_{m,n,p}$, in complete analogy with the situation in the non-time-varying case.

It is also true that $M^{ar}_{m,n,p}$ is a fine moduli space for a suitable notion of families of time-varying linear dynamical systems. And finally one also has "degeneracy" or "partial completeness" results analogous to those of [8].

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