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THETA FUNCTIONS IN POSITIVE CHARACTERISTIC

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1. The theta function of a divisor ; a special case.

The minimal requirement for something which has a claim to being called the theta function Θ_X of a divisor X on an abelian variety A (over a field k), is that Θ_X should be a power series in a finite number of indeterminates, or a quotient of two such power series, and that it should be possible to treat X as the "divisor" of Θ_X much in the same manner as X is the divisor of an element $z \in k(A)$ when X is linearly equivalent to zero. Such theta functions were developed ten years ago in [2] for the case in which k is of characteristic zero ; they were defined in a purely algebraic and local manner (no periods required), and naturally turned out to coincide with classical theta functions when k is the complex field. There seemed to be strong technical difficulties to the extension of the method to characteristic p, but it is now clear that it was only a matter of picking the right end of the rope. This has now been done, and we finally have theta functions also in characteristic p ; typically, no new tool born in the meantime has been necessary. I will now briefly describe the underlying ideas and sketch the method and results ; details, complete proofs, and further developments will be found in a forthcoming paper by V. Cristante ; it is lucky that (Witt) covectors, which I have been using since 1958, have now become popular [3] ; I hope that the same will soon be true of bivectors, of which covectors are only a homomorphic image. Finally, I must apologize for the use of only those concepts with which I am familiar, thus barring sheaves, schemes, spectra, and other complicated simplifications.

Let k be a perfect field of characteristic $p \neq 0$, and let A be an abelian variety of dimension n over k. It may not be useless to say that by that expression I mean a particular set of points of a projective space over the algebraic closure of k ; a point, in turn, means a point with coordinates in that algebraic closure ; on the other hand, if the listener has a different picture in mind, the results will apply equally well to that picture. We could start with a commutative group-variety (without periodic subvarieties) instead of an abelian one, as I did in [2], but the most interesting case arises when A is abelian.

Set $C = k(A)$ = field of rational functions on A, and let $x = \{x_1,\ldots,x_n\}$ be a regular set of parameters of the completion R of the local ring of the identity point $\underline{0}$ on A ; the maximal prime of R will be denoted by R^+ ; thus, $R = k\{x\} = k\{x_1,\ldots,x_n\}$, this being the ring of power series in $x_1,\ldots,x_n$, with coefficients in k and with integral nonnegative exponents. The field C can be canonically embedded in the quotient field of R, and we shall consider it so embedded. We shall also use an affine ring $k[y_1,\ldots,y_m]$ of A, with $m \geqslant n$, such that $C = k(y)$ and that the identity point be at finite distance for y, say at $y = 0$. Let X be a divisor on A ; I shall tacitly assume, whenever a divisor is considered, that none of its components go through $\underline{0}$; naturally, this condition must be eliminated from a complete theory, but the elimination is an easy trick which adds nothing to the substance of the method. If $X \sim 0$ (linearly equivalent to zero), then $X = \operatorname{div} z$ for some $z \in C$; the condition on X entails that $z \in R$, and clearly this z is entitled to be called the <u>theta</u> <u>element</u> of X, and to be denoted by $\Theta_X = \Theta_X(x)$. It is uniquely defined but for a nonzero factor in k, and it can be normalized by requiring that $z \equiv 1 \mod R^+$.

Next step is the case $X \equiv 0$ (algebraically equivalent to zero) ; before describing it I must recall that R is a hyperalgebra over k, with its coproduct $\mathbb{P}$ which maps R algebra-isomorphically into the completed tensor product $R \overline{\times} R$ (over k). A regular set of parameters of $R \overline{\times} R$ is the set

$$\{x \overline{\times} 1, 1 \overline{\times} x\} = \{\ldots, x_i \overline{\times} 1, \ldots, \ldots, 1 \overline{\times} x_i, \ldots\} ;$$

since many copies of R, A, C, etc. will be needed, I would rather index them ; thus, $\mathbb{P}$ may map R into $R_1\overline{\times}R_2$, and this will have regular parameters $\{x_1,x_2\}$; the individual x's , indexed from 1 to n, will never be used specifically again, so that no confusion arises. $\mathbb{P}x$ is a set of regular parameters of $\mathbb{P}R \subset R_1\overline{\times}R_2$, and it is expedient to denote it by x_1+x_2.

The conditon $X \equiv 0$ means that $\sigma_P X \sim X$ for each $P \in A$; here, σ_P is the translation by P. It also means the following : on $A\times A = A_1\times A_2$ consider the divisors $X_1 = X\times A_2$, $X_2 = A_1\times X$, and $X_{12} = (\mathrm{div}\,\mu)X$ = counterimage of X if $\mu : A_1\times A_2 \longrightarrow A$ is the law of composition ; then $Y = X_{12}-X_1-X_2 \sim 0$ on $A_1\times A_2$. Thus, Y has a theta element in the previous sense, namely an $f(x_1,x_2) \in R_1\overline{\times}R_2$, symmetric in x_1 , x_2 ; we select $f \equiv 1 \bmod (R_1\overline{\times}R_2)^+$, and then it is easily verified that

$$f(x_1+x_3,x_2)f(x_1,x_3) = f(x_1,x_3+x_2)f(x_3,x_2).$$

In other words, f is a symmetric factor set of R into R_m, if R_m denotes the hyperalgebra $k[t]$ (bialgebra really : no inversion) with coproduct $\mathbb{P}t = t \otimes t$ [one indeterminate ; algebraic torus of dimension 1 ; multiplicative straight line]. It produces an extension of R by R_m (or viceversa, depending on the language you use), and it is well known that the only such extension is $R \otimes R_m$ (trivial extension). Therefore f itself is "trivial" as we now say, or associated to 1 as we once said (some H^2 is equal to 1), this meaning that for a suitable $g(x) \in R$ we have $f(x_1,x_2) = g(x_1+x_2)/g(x_1)g(x_2)$. This g(x) is a <u>theta element</u> $\Theta_X(x)$ of X ; it is unique but for a nonvanishing (constant) factor in k, and for a nonzero factor $h(x) \in R$ which satisfies the condition $h(x_1+x_2) = h(x_1)h(x_2)$. Such an h is a <u>multiplicative element</u> of R, and it can exist if and only if R has a block of slope 1 (by now everybody knows this meaning of the word slope, introduced in chapter 5 of [MA] ; anyhow, slope 1 is present if and only if there are points $P \neq \underline{0}$ on A such that $pP = \underline{0}$).

So we now have Θ_X when $X \equiv 0$ (which includes $X \sim 0$) ; it belongs to R ; more generally, if X has poles through $\underline{0}$ it belongs to the quotient field $k\{x\}$ of R ; and it can be chosen in C if and only if $X \sim 0$.

2. Continuation ; general case.

We now relinquish any special condition on X ; we shall denote by C^∞ the perfect closure of C, by $\mathcal{R}^o$ the perfect closure of R, and by $\mathcal{R}$ its completion ; in the notation of [MA], the last three symbols would have been $^\pi\mathcal{R}^o$, $^\pi R$, $^\pi\mathcal{R}$, while C^∞ would have meant the union of the $(p\iota)^{-r}C$ for $r = 1,2,\ldots$, after denoting by ι the identity mapping ; this former C^∞, which contains our present C^∞, is still important, and will be used (and called C') in section 5 ; it is automatically embedded in the quotient field of $\mathcal{R}$ when our C^∞ is so embedded.

Let C_1 be a copy of C, extend A over C_1^∞, and consider the point P of the extension at which the coordinates y assume the values y_1 (copy of y in C_1). It is known that $\sigma_p X-X \equiv 0$ if X denotes also the extension of X over C_1^∞ ; since C_1^∞ is perfect, the discussion of section 1 applies, and $\sigma_p X-X$ has a theta element

$$\varphi(x_1,x) \in C_1^\infty\{x\} \,, \tag{1}$$

which we assume normalized by $\varphi(x_1,0) = 1$. It is not difficult to prove that $\varphi(x_1,x) \in \mathcal{R}^o\{x\} \subset \mathcal{R}\overline{\times}R$, and that we can also require $\varphi(0,x) = 1$. The meaning of X_i, X_{ij} being as in section 1, and that of X_{123} being similar, consider the divisor

$$Y = X_{123}+X_1+X_2+X_3-X_{12}-X_{13}-X_{23}$$

on $A_1\times A_2\times A_3$; it is known that $Y \sim 0$, so that Y has a theta element $F(x_1,x_2,x_3)$ in the quotient field of $C_1 \otimes C_2 \otimes C_3$, normalized by

$$F(0,x_2,x_3) = F(x_1,0,x_3) = F(x_1,x_2,0) = 1.$$

The relation between φ and F is

$$F(x_1,x_2,x_3) = \varphi(x_1,x_2+x_3)/\varphi(x_1,x_2)\varphi(x_1,x_3) = \varphi(x_1+x_2,x_3)/\varphi(x_1,x_3)\varphi(x_2,x_3)$$

(not immediate, but not very hard either) ; from this, and from the symmetry of F in x_1,x_2,x_3 follows that $\varphi(x_1,x_2)/\varphi(x_2,x_1)$ is a skew-symmetric bi-multiplicative element of $\mathcal{R}\overline{\times}\mathcal{R}$ (it is the Riemann form of X on the radical part of R ; see section 5) ; as a consequence, there exists a bi-multiplicative element $\chi(x_1,x_2) \in \mathcal{R}\overline{\times}\mathcal{R}$ such that $\Psi(x_1,x_2) = \varphi(x_1,x_2)\chi(x_1,x_2)$ is symmetric ; it is in fact a symmetric factor set of $\mathcal{R}$ (not of R) into R_m . It must again be associated to 1, and again, as in section 1, it provides a <u>theta element</u> $\Theta_X(x)$ of X by the relation $\Psi(x_1,x_2) = \Theta_X(x_1+x_2)/\Theta_X(x_1)\Theta_X(x_2)$. In this general case Θ_X belongs to $\mathcal{R}$ and not necessarily to R ; when it does not belong to R it does not belong to $\mathcal{R}^o$ either. The relation between $\Theta = \Theta_X$ and F is

(2) $F(x_1,x_2,x_3) = \Theta(x_1+x_2+x_3)\Theta(x_1)\Theta(x_2)\Theta(x_3)/\Theta(x_1+x_2)\Theta(x_1+x_3)\Theta(x_2+x_3)$.

This Θ_X is uniquely defined by (2) but for a <u>quadratic exponential</u> factor ; this expression shall mean the product of :

i) a nonzero element of k ;

ii) a multiplicative element of $\mathcal{R}$ (which is necessarily contained in the block of slope 1) ;

iii) an element e ≠ 0 of $\mathcal{R}$ such that $\mathbb{P}e/e\bar{\times}e$ is bimultiplicative (such e's are necessarily contained in the product of the blocks of slopes ≠ 1). The theta element satisfies the usual relation $\Theta_{X+Y} = \Theta_X\Theta_Y$, and it identifies X uniquely.

3. <u>The case of characteristic zero</u>.

The contents of sections 1 and 2 can be applied, with slight modifications, to the case of characteristic zero, and they afford a simplification of the method adopted in section 1 of [2], as they do not use classes of repartitions ($H^o(A, C/\mathcal{O}_A)$ for the connoisseurs). The modifications are the following :

1) R is perfect, hence $\mathcal{R} = \mathcal{R}^o = R$ and $C^\infty = C$;

2) one can select for x a set of integrals of the first kind ; in this case the + of $\varphi(x_1+x_2)$ is a true addition of sets of indeterminates.

In the case of characteristic zero the name "theta functions" is appropriate, since the arguments of which they are functions are canonically selected (see 2 above) ; in the case of characteristic p, on the contrary, $\Theta(x)$ is a special element of $\mathcal{R}$, not a special power series in the x's ; hence the use of the expression "theta element" rather than "theta function".

4. <u>Theta functions in a given hyperalgebra</u>.

We now start from an n-dimensional local equidimensional hyperalgebra R = k{x} over k ; equidimensional means that $p\iota R$ is also of dimension n. If $\mathcal{R}^o$, $\mathcal{R}$ are related to R as in section 2, a nonzero element $\Theta \in \mathcal{R}$ is <u>of type theta</u> on R if the function F of (2) belongs to the quotient field of $R_1 \otimes R_2 \otimes R_3$ (tensor product over k, not completion of ...). This is the same definition adopted in [2], except that Θ is sought in $\mathcal{R}$ rather than R. As in section 3 of [2], and by similar arguments, there exists a smallest subfield C, finitely generated over k, of the quotient field of R, with the property that F belongs to the quotient field of $C \otimes C \otimes C$; the field C inherits $\mathbb{P}$ from R, and is therefore a hyperfield ; in other words, C = k(A) for a suitable commutative group-variety A (a sketchy treatment of hyperfields is given in section 2 of [2] ; a developed theory is contained in [4]). More details about C can be found with the analytic machinery of bivectors : consider the bivector $\{\Theta\} = (\ldots,0,0;\Theta,0,0,\ldots)$; its logarithm exists and is of the type $\log\{\Theta\} = (\ldots,v,v;v,v,\ldots)$, where v is the Artin-Hasse logarithm of Θ . I will next recall the definition of $\mathcal{C}\tilde{R}$: the discrete hyperalgebra $\tilde{R}$ is the dual of R ; $\tilde{\mathcal{R}}$ is the completion of the dual of $\mathcal{R}$; $\mathcal{C}\tilde{\mathcal{R}}$ is the set of the elements d of biv $\tilde{\mathcal{R}}$ which are canonical, namely satisfy $\mathbb{P}d = d\overline{\times}1 + 1\overline{\times}d$ (it is a sort of Dieudonné module) ; $\mathcal{C}\tilde{R}$ is the subset of $\mathcal{C}\tilde{\mathcal{R}}$ formed by those $d = (\ldots,d_{-1};d_o,d_1,\ldots)$ having the property that $d_{-1}R = 0$, or, equivalently, that d_o, viewed as an element of $\mathrm{End}_k\mathcal{R}$, induces an (invariant) derivation in R. For such d's I defined in chapter 5 of [MA] an element d* of $\mathrm{End}_K \mathrm{Biv}\,\mathcal{R}$, where K = vect k, which is not quite a derivation on Biv $\mathcal{R}$ (it turns out to be a component of a covector whose ghost components are derivations on a subring of Biv$\mathcal{R}$). Well, C contains all the components of all the bivectors $d{*}d'{*}\log\{\Theta\}$ (which are really vectors), when d, d' range over $\mathcal{C}\tilde{R}$; if D is the field generated, over k, by these components and by their hyperderivatives, D itself is a hyperfield, and we strongly suspect that C = D ; so far it is only proved that the embedding of D into C is a purely inseparable isogeny. (Added Nov. 78 : C = D now proved).

Two elements of type theta are <u>associated</u> if their ratio is a quadratic exponential ; the <u>dimension</u> of Θ is the dimension of the smallest subhyperalgebra of $\mathcal{R}$ which contains some element associated to Θ ; the transcendency of C over k turns out to be not less than the dimension of Θ ; if it is equal to this dimension, Θ is a theta element on R, and it is nondegenerate when its dimension is n. The discussion in the next section will show that there are theta elements on R if and only if : for each block of slope α and dimension n_α of R, with $0 < \alpha < 1$, R has a block of slope $1-\alpha$ and codimension n_α [this condition reflects the symmetry theorem satisfied by hyperalgebras arising from abelian varieties ; it would be interesting to establish the condition directly within the theory of theta elements, thus supplying a third proof of the symmetry theorem]. Picking theta elements (on R) in $\mathcal{R}$ is equivalent to picking rational composition laws for R ; or also to viewing R as an algebraic group.

Now that we have a theta element defined a priori, we naturally want to know whether $\Theta = \Theta_X$ for some X on A ; the answer is the same as in [2] : X is the only divisor on A (with no component on the degeneration locus in case A is not abelian) such that div F - X×A×A has no component of the form Y×A×A ; this X is very strongly ample on A (if Θ is holomorphic) , namely : $\sigma_P X = X$ only when $P = \underline{0}$; by the way : X is effective (= positive) if and only if Θ is holomorphic, this meaning that $\Theta(x_1+x_2)\Theta(x_1-x_2) \in \mathcal{R} \otimes \mathcal{R}$.

5. The abelian case and the Riemann form.

For a deeper discussion we must make full use of the tools provided by [MA], in particular those of chapter 6 ; let us go back to the case discussed in sections 1 and 2, where $\Theta = \Theta_X$ for some X on A. Let us denote by C' the union of the fields $(p\iota)^{-r}C$ described at the opening of section 2 ; theorems 6.12, 6.13, 6.14 (and others) of [MA] provide, for each $d \in \mathcal{C}\tilde{R}$, an element z_d of vect C', uniquely determined, but for a summand in K = vect k, by the following property : for each prime divisor Y on A^∞ (inverse limit of $A \xleftarrow{p\iota} A \xleftarrow{p\iota} A\ldots$), let x_Y be a representative of X at Y ; then all the components of the vector $d{*}\log\{x_Y\} - z_d$ belong to the local ring of Y on A^∞. After a suitable choice of the arbitrary summand it can be proved (easily) that for suitable elements $\eta_d \in \mathcal{C}R$ and $c_d \in$ biv k (this being the quotient field of K) we have

(3) $$z_d = d*\log\{\Theta\} + \eta_d - c_d .$$

The mapping $d \longrightarrow \eta_d$ is K-linear and commutes with π (Frobenius) and t (shift) ; in particular, if d has slope 0 also η_d must have slope 0 ; but $\mathcal{C}R$ contains no element of slope 0 (those elements all come from the separable = totally disconnected = etale block of a hyperalgebra) ; hence $\eta_d = 0$ if d has slope 0, a fact which shows that for such d's, the bivector $d*\log\{\Theta\} - c_d$ is in vect R. Assume instead that d has no direct summand of slope 0 ; more precisely, in what follows d will range over $\mathcal{C}\tilde{R}_r$, where $\tilde{R}_r$ is the radical part of $\tilde{R}$, made up of all the blocks of slope $\neq 0$; then $\eta_d \in \mathcal{C}R_r$ (and this R_r is made up of blocks of slope $\neq 1$), and we would like to know more about it. If χ has the meaning of section 2, define the elements ζ_d, ξ_d of $\mathcal{C}R$ (actually of $\mathcal{C}R_r$) by : $\zeta_d = (d \otimes 1)*\log\{\chi\}$, and $\xi_d = (1 \otimes d)*\log\{\chi\}$; consider also the operators $\alpha = \lim_{r\to\infty} p^r(p\iota)^{-r}$, and $\beta = \lim_{r\to\infty} p^{-r}(p\iota)^r$, and remember that $\beta z_d \in \mathcal{C}\mathcal{R}_r$ is an old acquaintance, namely $\varphi_X d$, where φ_X is the Riemann form of X (see chapters 6 and 7 of [MA] ; do not confuse with the mapping of A into its dual denoted by φ_X in Lang's book : this mapping I had christened λ_X in 1954, and I haven't changed since). Application of β to (3) gives $\varphi_X d = \beta(d*\log\{\Theta\}-c_d)+\eta_d$, while application of α gives $0=\alpha(d*\log\{\Theta\})+\eta_d$. On the other hand, from the definition of Θ_X and from (1) we can derive that $\alpha(d*\log\{\Theta\}) = \xi_d$ and that $\beta(d*\log\{\Theta\}-c_d) = \zeta_d$. So $\eta_d = -\xi_d$, and $\varphi_X d = \zeta_d+\eta_d = \zeta_d-\xi_d$. We conclude that (3) gives the decomposition of the mapping

(4) $$(d,d') \longrightarrow d'*(z_d+\zeta_d)$$

into the alternating part

(5) $$(d,d') \longrightarrow d'*\varphi_X d$$

and the "symmetric" part

(6) $$(d,d') \longrightarrow d'*d*\log\{\Theta\} .$$

The word " symmetric " is in quotations because mappings (4) and (6) are not K-bilinear ; with this limitation, due to the imperfect nature of d* as a derivation, (5) is a holomorphic differential of dimension 2, while (6) is a metric.

We can now go back to a question left open in section 4 ; given a nondegenerate theta element Θ, when is the group-variety A abelian ? The discussion of this section provides the answer. To begin with, the equidimensionality of R rules out the possibility of periodic group-varieties ; hence A can only be the extension of an abelian variety by multiplicative lines (tori) ; the answer can thus be sought by "counting" p-division points, which are lacking on multiplicative lines. Equivalently, let f be the dimension of the block of slope 1 of R ; then, in order that A be abelian it is necessary and sufficient that when d ranges over the elements of $\mathcal{C}\tilde{R}$ of slope 0, the vectors $d*\log\{\Theta\} = z_d$, modulo vect C, span a K-module with f free generators ; when this is the case we say that Θ is an abelian theta element.

6. Conclusion.

This exposition starts with an article of faith (about Θ having to be a power series, albeit with non-integral exponents, as we have later seen) which I am about to abjure, with a warning that the following free-wheeling considerations are more than wishful thinking but less than a description of work accomplished.

For a general setting we start from a group G which I choose to call "analytic", and which is the candidate for being the "completion" of a commutative algebraic group A (this A is assumed to be an abelian variety in the description which follows) ; "completion" is the accepted word, but a very poor choice for something which is usually smaller than A. Anyhow, since groups are only dimly present, while hyperalgebras of "analytic" functions on groups are very much present, it is better to speak of $C = k(A)$ and of R = functions on G : in C we select an array (= order) S such that $\mathbb{P}S$ is a subring of the quotient field of $S \otimes S$; on S we place a suitable topology T_o, and denote by R the T_o-completion of S (naturally T_o comes from a metric) ; we then seek a "universal covering" $\mathcal{G}$ of G, which in terms of rings of functions means a "maximal embedding" of R into a hyperalgebra $\mathcal{R}$. Essentially, $\mathcal{G}$ must give enough information on the universal covering of A, which algebraically seems to be the maximal covering of A whose ramification arises only from inseparability. In the first five sections S has been the local ring of $\underline{0}$ on A and T_o has been its natural local topology ; however, the same R, which I will now revert to calling ${}^{\pi}R$, can be reached, as explained in chapter 6 of [MA] (modulo some silly mistake), by taking for S the intersection S_p of the local rings of all the p-division points on A, and for T_o the π-topology (this S_p is an array in C if k is not too small) ; hence the use of ${}^{\pi}R$ rather than R. I will give a list of four possible selections of R and $\mathcal{R}$, of which number 1 is the one just described, i.e. the one adopted in the preceding sections :

1) S is S_p ; T_o is the π-topology (see chapter 6 of [MA]) ; R is ${}^{\pi}R$, and $\mathcal{R} = {}^{\pi}\mathcal{R}$ is the completion of the direct limit ${}^{\pi}R \xrightarrow[\pi]{} {}^{\pi}R \xrightarrow[\pi]{} \ldots$ This ${}^{\pi}\mathcal{R}$ is the completed tensor product ${}^{\pi}\mathcal{R}_{\pi} \mathbin{\overline{\times}} {}^{\pi}\mathcal{R}_r$, where ${}^{\pi}\mathcal{R}_r$ is the radical part of ${}^{\pi}\mathcal{R}$ (slopes < 1, and certainly > 0), while ${}^{\pi}\mathcal{R}_{\pi}$ is the logarithmic, or toroidal, part (slope 1). If f = sep codim A = dim ${}^{\pi}\mathcal{R}_{\pi}$, it is not idle to remark that ${}^{\pi}\mathcal{R}_{\pi}$ is isomorphic and homeomorphic to the hyperalgebra of certain measures on $\mathbb{Q}_p^f$ with values in k (at least when k is algebraically closed) ; the topology is that of uniform convergence on balls of bounded radius (k is taken to be discrete). This interpretation of ${}^{\pi}\mathcal{R}_{\pi}$, as well as similar interpretations in the cases which follow, are the object of [5].

2) S is S_p ; T_o is the t-topology ; R is ${}^{t}R$, and $\mathcal{R} = {}^{t}\mathcal{R}$ is the completion of the direct limit ${}^{t}R \xrightarrow[t]{} {}^{t}R \xrightarrow[t]{} \ldots$ Now ${}^{t}\mathcal{R} = {}^{t}\mathcal{R}_r \mathbin{\overline{\times}} {}^{t}\mathcal{R}_t$, where ${}^{t}\mathcal{R}_r \cong {}^{\pi}\mathcal{R}_r$ and ${}^{t}\mathcal{R}_t$ is the separable, or etale, part of ${}^{t}\mathcal{R}$ (slope 0) ; it is isomorphic and homemorphic to the hyperalgebra of continuous functions on $\mathbb{Q}_p^f$, with values in k ; the topology is that of uniform convergence on compacts.

3) S is S_p ; T_o is the $p\iota$-topology (remember that $p\iota = \pi t$) ; R is R, and $\mathcal{R} = \mathcal{R}$ is the completion of the direct limit $R \xrightarrow[p\iota]{} R \xrightarrow[p\iota]{} \ldots$ Now $\mathcal{R} = \mathcal{R}_{\pi} \mathbin{\overline{\times}} \mathcal{R}_r \mathbin{\overline{\times}} \mathcal{R}_t$, with $\mathcal{R}_{\pi} \cong {}^{\pi}\mathcal{R}_{\pi}$, $\mathcal{R}_r \cong {}^{\pi}\mathcal{R}_r \cong {}^{t}\mathcal{R}_r$, $\mathcal{R}_t \cong {}^{t}\mathcal{R}_t$.

4) S is S_q for a prime $q \neq p$ (usually called ell) ; T_o is the $q\iota$-topology, where a basis for neighbourhoods of 0 consists of the maximal primes of S_q ; R can be called R_q, and $\mathcal{R} = \mathcal{R}_q$ is the completion of the direct limit $R_q \xrightarrow[q\iota]{} R_q \xrightarrow[q\iota]{} \ldots$ If n = dim A, $\mathcal{R}_q$ is isomorphic and homeomorphic to the hyperalgebra of continuous functions on $\mathbb{Q}_q^{2n}$, with values in k ; the topology is

that of uniform convergence on compacts.

If k has characteristic zero the only two known possibilities, barring special fields such as the complex field, are :

1') (similar to 1) S is the local ring of $\underline{0}$; R is its completion ; $\mathcal{R} = R$ (see section 3).

4') Same as 4, q being any prime.

After deciding on an $\mathcal{R}$, a theta element on R (or element of type theta as the case may be) is simply a $\Theta \in \mathcal{R}$ such that the F of formula (2) is the ratio of two elements of $R \otimes R \otimes R$; notice the $\otimes$ rather than $\bar{\times}$. Naturally now F must be written as

$$((\iota \mathbin{\bar{\times}} \mathbb{P})\mathbb{P}\Theta)(\Theta \mathbin{\bar{\times}} \Theta \mathbin{\bar{\times}} \Theta)/(\mathbb{P}\Theta \mathbin{\bar{\times}} 1)(1 \mathbin{\bar{\times}} \mathbb{P}\Theta)(sc_{12}(1 \mathbin{\bar{\times}} \mathbb{P}\Theta)) .$$

The field C is then retrieved as the quotient field of the smallest subring U of R having the property that the quotient field of $U \otimes U \otimes U$ contains F. The existence of such Θ's, for instance in case 4', is essentially due to the fact that the only crossed product of $\mathbb{Q}_q$ by the multiplicative group of nonzero elements of k is the direct product. Knowledge of $\Theta = \Theta_X$ must entail knowledge of the restriction ρ_X of the Riemann form of X to $\mathcal{R} \mathbin{\bar{\times}} \mathcal{R}$; the recipe is as follows : find a bimultiplicative element $\chi \in \mathcal{R} \mathbin{\bar{\times}} \mathcal{R}$ such that $\mathbb{P}\Theta/(\Theta \mathbin{\bar{\times}} \Theta)\chi \in C^{\infty} \mathbin{\bar{\times}} \mathcal{R}$; then ρ_X is simply $\chi/sc\chi$, or its reciprocal according to taste. This is ρ_X viewed as a skew-symmetric bimultiplicative element ; in order to view it as a bilinear element one must take its "logarithm" according to some suitable definition of the term.

Examples : in case 1 (the object of this exposition) χ is the χ of section 2, and the logarithm is log{ } ; in case 4, χ is given by $\chi(r,v) = \Theta(0)\Theta(v+r)/\Theta(v)\Theta(r)$ for $v \in \mathbb{Q}_q^{2n}$ and $r \in \mathbb{Z}_q^{2n}$; the logarithm is the inverse of a standard homomorphism of $\mathbb{Q}_p$ into the group of q^{∞}-th roots of 1 in the algebraic closure of k.

It is now only fair to ask whether Mumford's thetas [6] fit into this scheme. The work of comparison is a tall order, except that in 1970-71, when I still refused to consider thetas which were not power series, I devoted some time and effort to the construction of (illegal) theta elements which fall under case 4 above ; they turned out to be very similar to Mumford's thetas as described in section 8 of [6], modulo the fact that I had not selected q = 2. Thus, the answer to the question should be yes.

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