JACK MORAVA

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THE WEIL GROUP AS AUTOMORPHISMS OF THE LUBIN-TATE GROUP

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Introduction:

Let L be a finite extension of Q_p, with maximal abelian extension L_{ab}; then the canonical monomorphism $\underline{a}$ of [8,XII§3] maps the multiplicative group of L onto an open dense subgroup $W(L_{ab}/L)$ of the Galois group of L_{ab} over L. These modified Galois [or W-] groups can be defined more generally, and behave very much like Galois groups [8, appendix II], but for some purposes they are more convenient.

For example, there is a representation of $W(L_{ab}/L)$ on the Q_p-vector space L, defined by the obvious multiplication map $L^{\times} \times L \to L$.

The trace of this representation defines a p-adic character of $W(L_{ab}/L)$ and therefore [via the natural homomorphism from $W(\overline{Q}_p/L)$ to $W(L_{ab}/L)$] a p-adic character of $W(\overline{Q}_p/L)$. In this note we construct an extension of this character to $W(\overline{Q}_p/Q_p)$ when L is normal over Q_p.

Our construction uses a theorem of Šafarevič: if d is the degree of L over Q_p, and D is a division algebra with center Q_p and invariant $d^{-1} \in Q/Z = Br(Q_p)$, then L can be embedded as a commutative subfield of D; let N(L) be the normaliser of the multiplicative group L^x of L in D^x. The canonical morphism <u>a</u> then extends [8, appendix III] to a canonical isomorphism $\underline{w} : N(L) \xrightarrow{\sim} W(L_{ab}/Q_p)$, and the composition of $\underline{w}^{-1}$ with the reduced trace from D to Q_p defines a character of $W(L_{ab}/Q_p)$ and therefore of $W(\overline{Q}_p/Q_p)$.

In §1 we identify N(L) with a group of "extended automorphisms" of the Lubin-Tate group of L; this action defines a cocycle [and thus a representation) ω of N(L), whose trace is the character described above.

The present work was motivated by the construction of a (topological) spectrum which admits N(L) as a group of automorphisms, such that the representation defined on its $2n^{th}$ homotopy group is the n^{th} tensor power of ω [6]. However, the result of §1 suggests the hope of a constructive proof of the Weil-Šafarevič theorem [which might shed some light on the interpretation of $W(L_{ab}/Q_p)$ as a group of automorphisms [9]] and could therefore be of wider interest.

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§1, proof of the main result

1.1. A continuous homomorphism $\phi : A[[T]] \to A[[T]]$ of commutative formal power series rings will be called an extended endomorphism, if

i) $\phi(T)$ lies in the ideal generated by T, and

ii) the image of the composition $A \hookrightarrow A[[T]] \xrightarrow{\phi} A[[T]]$ lies in A.

Consequently $\phi(\sum_{i\geq 0} a_i T^i) = \sum_{i\geq 0} \phi(a_i)\phi(T)^i$.

Note that the composition of two extended endomorphisms is another, and that the tensor product $\phi \otimes_A \phi$ maps $A[[T\otimes 1, 1\otimes T]]$ to itself by $(\phi \otimes_A \phi)(\sum_{i,j\geq 0} a_{ij} T^i \otimes T^j) = \sum_{i,j\geq 0} \phi(a_{ij})\phi(T)^i \otimes_A \phi(T)^j$.

If $F(X,Y) \in A[[X,Y]]$ is a [one-parameter, commutative] formal group law over A, then the extended endomorphism ϕ of $A[[T]]$ will be called an extended endomorphism of F provided that the diagram

$$\begin{array}{ccc} A[[T]] & \xrightarrow{\Delta_F} & A[[T\otimes 1, 1\otimes T]] \\ \Big\downarrow \phi & & \Big\downarrow \phi \otimes_A \phi \\ A[[T]] & \xrightarrow{\Delta_F} & A[[T\otimes 1, 1\otimes T]] \end{array}$$

[defined by $\Delta_F(T) = F(T\otimes 1, 1\otimes T)$] is commutative.

If $Aut^*(F)$ denotes the group of extended automorphisms of F [under composition] then it follows from i) and ii) that there is an exact sequence

$$1 \longrightarrow Aut_A(F) \longrightarrow Aut_A^*(F) \longrightarrow Aut_{(rings)}(A)$$

with the terminal group consisting of the continuous ring-automorphisms of A; the usual automorphisms of F over A [4,I§2] define the group $Aut_A(F)$.

1.2. We write $\underline{o}_L$ for the valuation ring of L, and $\hat{\underline{o}}_L$ for the valuation ring of the completion $\hat{L}$ of a maximal unramified extension L_{nr} of L; if $\not{\ell}$ denotes the residue field of $\underline{o}_L$, and $\bar{\not{\ell}}$ is the union of the finite fields, then $\hat{\underline{o}}_L \cong \underline{o}_L \otimes_{W(\not{\ell})} W(\bar{\not{\ell}})$.

If $\pi \in \underline{o}_L$ is a uniformising element, and q is the cardinality of $\not{k}$, then the series

$$\log_\pi(T) = \sum_{i\geq 0} \pi^{-i} T^{q^i}$$

defines a formal group law $F_\pi(X,Y) = \log_\pi^{-1}(\log_\pi(X) + \log_\pi(Y))$ for which the map $\underline{o}_L^\times \ni a \mapsto [a]_\pi(T) = \log_\pi^{-1}(a\cdot\log_\pi(T)) \in \mathrm{Aut}_{\hat{\underline{o}}_L}(F_\pi)$ is a bijection [1]. By "the" Lubin-Tate group of L, we mean the class of formal group laws over $\hat{\underline{o}}_L$ isomorphic to F_π for some (and hence any) choice of π. [If π_0, π_1 are two choices of uniformising element, then [5, lemma 2] there exists an invertible series $\phi_0^1(T) \in \hat{\underline{o}}_L[[T]]$ such that

i) ϕ_0^1 is an isomorphism of F_{π_0} with F_{π_1}, and

ii) if σ is the automorphism of $\hat{L}$ defined by the Frobenius operation $x \mapsto x^q$ on the residue field, then

$$\sigma(\phi_0^1(T)) = \phi_0^1([\pi_0^{-1}\pi_1]_{\pi_0}(T)).]$$

We denote the formal group law over $\bar{\not{k}}$ defined by reducing F_π modulo the maximal ideal $\hat{\underline{m}}_L$ of $\hat{\underline{o}}_L$ by $\bar{F}_\pi$; its height equals the degree of L over Q_p [1, lemma 9].

1.3. Now the ring of endomorphisms of a group law of height d over an algebraically closed field of characteristic p is the valuation ring $\underline{o}_D$ of a division algebra D with center Q_p and invariant $d^{-1} \in Q/Z = \mathrm{Br}(Q_p)$ [4, VI §7.42], and the normalised ordinal valuation of an element of $\underline{o}_D$ is its height as a power series. It follows that the sequence of 1.1 can be continued to the right as

$$1 \longrightarrow \underline{o}_D^\times \longrightarrow \mathrm{Aut}^*_{\bar{\not{k}}}(\bar{F}_\pi) \longrightarrow G(\bar{\not{k}}/F_p) \cong \hat{Z} \longrightarrow 1 :$$

to construct a lifting of the Frobenius endomorphism $\sigma_0(x) = x^p$ of $\bar{\not{k}}$, let $\theta \in \underline{o}_D$ be an endomorphism of height 1 [so $\theta(T) = \theta_0(T^p)$ with θ_0 an invertible series]; then $\theta = \sigma'_0 \circ \theta_0$ has the desired property. This shows moreover that $\mathrm{Aut}^*_{\not{1}}(\overline{F}_\pi)$ is isomorphic to the profinite completion of the multiplicative group $D^\times$ of D under the correspondence which sends the endomorphism ϕ [which can be written as $\phi(T) = \phi_0(T^{p^r})$ with ϕ_0 invertible and $r \geq 0$] to the extended automorphism $\sigma'^r_0 \circ \phi_0$. It suffices to see that the conjugation of a series $a \in \underline{o}_D^\times$ by θ in $\mathrm{Aut}^*_{\not{1}}(\overline{F}_\pi)$ agrees with its conjugation by θ in L, or that $P \circ \sigma'_0 \circ a\, \sigma'^{-1}_0 = a \circ P$, where $P(T) = T^p$; this is an elementary exercise in the composition of power series.

1.4. It follows similarly that if L is a normal extension of Q_p, then $\mathrm{Aut}^*_{\hat{\underline{o}}_L}(F_\pi)$ is a central topological extension of the Galois group $G(L_{nr}/Q_p)$ by the multiplicative group $\underline{o}_L^\times$. To see that the final homomorphism of the sequence in 1.1 is onto, note that if $\pi_0 = \pi$ is a uniformising element and $g \in G(L_{nr}/Q_p)$ then $\pi_1 = g(\pi)$ is another and $\sum_{i\geq 0} a_i T^i \to \sum_{i \geq 0} g(a_i)(\phi_0^1(T))^i$ defines a (noncanonical!) lift of g to an extended automorphism. Since any automorphism of a formal group law over an integral domain of characteristic 0 is determined by its leading coefficient, the group $G(L_{nr}/Q_p)$ acts on the subgroup $\underline{o}_L^\times$ via the canonical homomorphism to $G(L/Q_p)$.

1.5. Now an extended automorphism of $\hat{\underline{o}}_L[[T]]$ maps the ideal $\hat{\underline{m}}_L[[T]]$ to itself, so an extended automorphism ϕ of F_π defines an extended automorphism of $\overline{F}_\pi$, which we will denote by

$$\rho : \mathrm{Aut}^*_{\hat{\underline{o}}_L}(F_\pi) \to \mathrm{Aut}^*_{\not{1}}(\overline{F}_\pi).$$

Since the reduction of a usual automorphism of F_π is a usual automorphism of $\overline{F}_\pi$, we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \underline{o}_L^\times & \longrightarrow & \mathrm{Aut}^*_{\hat{\underline{o}}_L}(F_\pi) & \longrightarrow & G(L_{nr}/Q_p) & \longrightarrow & 1 \\
 & & \Big\downarrow \rho_0 & & \Big\downarrow \rho & & \Big\downarrow & & \\
1 & \longrightarrow & \underline{o}_D^\times & \longrightarrow & (D^\times)^\wedge & \longrightarrow & G(\bar{I}/F_p) \cong \hat{Z} & \longrightarrow & 1.
\end{array}$$

Now the final vertical arrow fits in an exact sequence

$$1 \longrightarrow I(L/Q_p) \longrightarrow G(L_{nr}/Q_p) \longrightarrow G(\bar{I}/F_p) \cong \hat{Z} \longrightarrow 1$$

which defines the inertia group of L over Q_p, and the homomorphism ρ_* is injective since $\bar{F}_\pi$ is of finite height. It follows that ρ_0 is injective, for $I(L/Q_p)$ acts effectively on $\underline{o}_L^\times$.

It will simplify matters to pull our commutative diagram back along the dense embedding $Z \longrightarrow \hat{Z}$: the effect is to replace $(D^\times)^\wedge$ with $D^\times$, $G(L_{nr}/Q_p)$ with the open dense subgroup $W(L_{nr}/Q_p)$, and $\mathrm{Aut}^*_{\hat{\underline{o}}_L}(F_\pi)$ with an open dense subgroup which we will denote Aut^0; the original diagram can be recovered by profinite completion.

1.6. It remains to identify the image of ρ. We observe first that because $\bar{F}_\pi$ has coefficients in $\bar{I}$, the extended automorphism $\rho([\pi]_\pi) = \sigma_0^d = \sigma$ commutes with elements of $\rho_0(\underline{o}_L^\times)$ in $D^\times$. It follows that $\rho_0(\underline{o}_L^\times)$ and σ generate a (normal) subgroup of Aut^0 isomorphic to $L^\times$, and that the image of ρ is therefore contained in the normaliser $N(L)$ of $L^\times$ in $D^\times$. But now the Weyl group of $L^\times$ in $D^\times$ is $G(L/Q_p)$ if L is normal [8, appendix III§7] so we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \Big\downarrow & & & & \\
 & & & & \mathrm{Aut}^0 & \longrightarrow & W(L_{nr}/Q_p) & \longrightarrow & 1 \\
 & & & & \Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & L^\times & \longrightarrow & N(L) & \longrightarrow & G(L/Q_p) & \longrightarrow & 1 \\
 & & & & & & \Big\downarrow & & \\
 & & & & & & 1 & &
\end{array}$$

with exact rows and columns. If $x \in N(L)$ then there is some y in Aut^0 such that $z = y^{-1}x$ lies in L^x, so $x = yz$ lies in Aut^0. This completes the proof of

1.7. <u>proposition:</u> <u>The mormorphism</u> ρ <u>maps an open dense subgroup of</u> $\mathrm{Aut}^*_{\hat{\underline{o}}_L}(F_\pi)$ <u>onto the normaliser</u> $N(L)$ <u>of</u> L^x <u>in</u> D^x.

§2. some corollaries

2.1. If $\delta \in N(L)$, then we write $\bar{\rho}^{-1}(\delta)(T) = \omega(\delta)T$ + higher order terms for the action of the extended automorphism $\bar{\rho}^{-1}(\delta)$ on the formal parameter T; here $\omega(\delta)$ is a unit of $\hat{\underline{o}}_L^x$. The composition $N(L) \longrightarrow W(L_{ab}/Q_p) \longrightarrow W(L_{nr}/Q_p)$ defines an action of $N(L)$ on $\hat{L}$ which we will denote by juxtaposition. With this notation, we have

$$\omega(\delta_0\delta_1) = \delta_1(\omega(\delta_0))\cdot\omega(\delta_1) ;$$

in other words, ω is a crossed antihomomorphism from $N(L)$ to $\hat{\underline{o}}_L^x$. Note that if $\delta \in \underline{o}_L^x$, then $\omega(\delta) = \delta^{-1}$ [7, III§A4].

An extended automorphism of F_π defines an extended automorphism of $F_\pi \otimes_Z Q$, and it follows that

$$\bar{\rho}^{-1}(\delta)(T) = \log^{-1}_{\delta(\pi)}(\omega(\delta)\cdot\log_\pi(T)) ;$$

consequently the crossed antihomomorphism ω specifies the action of $N(L)$ on $\hat{\underline{o}}_L[[T]]$.

2.2. The 1-cocycle $\delta \mapsto \omega(\bar{\delta}^{-1})$ of $N(L)$ with values in the right $N(L)$ module $(\underline{o}_L^x)^{op}$ [defined by $x^{op}\delta = (\bar{\delta}^{-1}x)^{op}$] defines a class in a continuous cochain cohomology group isomorphic to $H^1_c(W(L_{ab}/Q_p);\hat{\underline{o}}_L^x)$. The Hochschild-Serre spectral sequence of the topological extension

$$E : 1 \longrightarrow W(L_{ab}/L_{nr}) \longrightarrow W(L_{ab}/Q_p) \longrightarrow W(L_{nr}/Q_p) \longrightarrow 1$$

yields an exact sequence

$$\cdots \to H^1_c(W(L_{ab}/Q_p);\hat{\underline{o}}^x_L) \to H^0_c(W(L_{nr}/Q_p);H^1_c(\underline{o}^x_L;\underline{o}^x_L) \cong G(L/Q_p)\text{-}$$
$$\text{invariants of } \mathrm{Hom}_c(\underline{o}^x_L,\underline{o}^x_L) \xrightarrow{d_2} H^2_c(W(L_{nr}/Q_p);\hat{\underline{o}}^x_L) \cong H^2(G(L/Q_p);\underline{o}^x_L)$$

of terms of low degree. The existence of the cocyle ω implies $d_2 = 0$; since $d_2(x) = -x \cup [E]$ [3, theorem 4] it follows that the inclusion $\underline{o}^x_L \to \hat{\underline{o}}^x_L$ induces the zero map from $H^2_c(W(L_{nr}/Q_p);\underline{o}^x_L)$ to $H^2_c(W(L_{nr}/Q_p);\hat{\underline{o}}^x_L)$. A direct proof of this might suggest a construction for ω.

2.3. The isomorphism $G(L_{ab}/Q_p)$ with $\mathrm{Aut}^*_{\hat{\underline{o}}_L}(F_\pi)$ defined in §1.7 respects an implicit proalgebraic group structure, which may be made explicit by observing that $G(L_{ab}/Q_p)$ is isomorphic to the semidirect product $I(L_{ab}/Q_p)\cdot G(\bar{\mathbb{F}}/F_p)$, in which $I(L_{ab}/Q_p)$ is the inertia group of L_{ab} over Q_p. In particular, $I(L_{ab}/Q_p)$ admits a continuous action of $G(\bar{\mathbb{F}}/F_p)$, and may therefore be regarded as a proetale groupscheme over F_p [2,II§5]. On the other hand $\underline{o}^x_D$ is represented by a group of power series with coefficients in $\bar{\mathbb{F}}$, and has an obvious structure as proetale groupscheme defined [a priori] over $\mathbb{F}$, in which the generator of $G(\bar{\mathbb{F}}/\mathbb{F})$ acts on $\underline{o}^x_D$ by π-conjugation in D^x. The maximal compact subgroup $N^0(L)$ of $N(L)$ inherits this structure.

However, if the uniformising element π of $\underline{o}_L$ is chosen to satisfy an Eisenstein equation with coefficients in $\hat{Z}_p$, then $\bar{F}_\pi$ has coefficients in F_p, and $\theta(X) = X^p$ defines an endomorphism of $\bar{F}_\pi$ which maps to an $f^{\underline{th}}$ root of π in $\mathrm{Aut}^*_{\mathbb{F}}(\bar{F}_\pi)$, where $q = p^f$. It follows that $N^0(L)$ is in fact a proetale groupscheme defined over F_p, and is isomorphic as such to $I(L_{ab}/Q_p)$. Consequently the group of F_p-valued points of $I(L_{ab}/Q_p)$ can be identified with the automorphisms of F_π defined over $\hat{Z}_p$, which leads to the corollary: $I(L_{ab}/Q_p)(F_p) \cong \hat{Z}_p^x$

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J. Morava
Department of Mathematics
SUNY at Stony Brook
Stony Brook, New York 11794