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ANALYTIC AND ARITHMETIC THEORY OF POINCARÉ SERIES

by

Dorian Goldfeld

§1. Define $\Gamma = SL_2(\mathbf{Z})$ to be the modular group, and let

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi inz}, \qquad g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi inz},$$

be cusp forms of weight k for Γ which satisfy the modular relations

$$f(\tfrac{az+b}{cz+d}) = (cz+d)^k f(z), \qquad g(\tfrac{az+b}{cz+d}) = (cz+d)^k g(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We are concerned with the general problem of estimating sums of the type

$$(1.1) \qquad \sum_n a_n \overline{b_{n+m}} \qquad (m \in \mathbf{Z}, \quad m \text{ fixed}),$$

and shall show that the solution to this problem is invariably based on the analytic and arithmetic properties of Poincaré series.

The special case, $m = 0$, of (1.1) has for the past forty years been the object of rather extensive research and has its origins in the papers of Rankin and Selberg. In [R] and [S1] it is shown that the L-function

$$L_{f,g}(s) = \sum_{n=1}^{\infty} a_n \overline{b_n}\, n^{-s}$$

has a meromorphic continuation to the entire complex s-plane and satisfies the functional equation

$$R_{f,g}(s) = R_{f,g}(2k-1-s)$$

where

$$R_{f,g}(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)\zeta(2s-2k+2)L_{f,g}(s).$$

Moreover, $L_{f,g}(s)$ is regular except for poles corresponding to the complex zeros of $\zeta(2s-2k+2)$ and a simple pole at $s = k$ with residue

$$\alpha = 12 \frac{(4\pi)^{k-1}}{\Gamma(k)} \langle f,g\rangle$$

where

$$\langle f,g\rangle = \iint_D f(z)\overline{g(z)}\, y^k \frac{dxdy}{y^2}$$

is the usual Petersson inner product (over a fundamental domain D for Γ) for forms of weight k.

The functional equation (1.2) was obtained by analyzing the inner product $\langle f,g\overline{E(*,s)}\rangle$ where

$$E(z,s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\,\sigma z)^s, \qquad \Gamma_\infty = \{\sigma \in \Gamma;\quad \sigma\infty = \infty\}$$

is the non-holomorphic Eisenstein series which satisfies the properties

(1.3) $$E(\sigma z,s) = E(z,s), \qquad \text{for all } \sigma \in \Gamma$$

(1.4) $$\Delta E(z,s) = s(1-s)E(z,s), \qquad \Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

(1.5) $$\pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = \pi^{-(1-s)}\Gamma(1-s)\zeta(2-2s)E(z,1-s).$$

It is of course (1.5) which gives the functional equation (1.2).

In [S2], Selberg returned to the general problem (1.1) and very briefly indicated how to obtain the meromorphic continuation of the function

$$\sum_{n=1}^{\infty} a_n\overline{b_{n+m}}\, n^{-s}.$$

Unfortunately, this function does not satisfy a functional equation and a suitable generalization of (1.2) to the case $m \neq 0$ requires the use of the non-holomorphic Poincaré series

$$P_m(z,s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\,\sigma z)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi|m|\mathrm{Im}\,\sigma z)e^{2\pi i m\,\mathrm{Re}\,\sigma z}$$

where

$$I_v(y) = \sum_{j=0}^{\infty} (\tfrac{1}{2}y)^{2j+r}/j!\,\Gamma(j+v+1)$$

is the modified Bessel function of the first kind which grows exponentially

(1.6) $$\lim_{y\to\infty} \sqrt{y}\, I_v(y) e^{-y} = (2\pi)^{-\frac{1}{2}}.$$

As shown in [NI] and [NE], the Poincaré series $P_m(z,s)$ is similar in behavior to the Eisenstein series $E(z,s)$, and in fact satisfied

(1.7) $$P_m(\sigma z,s) = P_m(z,s), \qquad \text{for all } \sigma \in \Gamma$$

(1.8) $$\Delta P_m(z,s) = s(1-s) P_m(z,s)$$

(1.9) $$P_m(z,s) - P_m(z,1-s) = \frac{2\pi^s |m|^{s-\frac{1}{2}} \sigma_{1-2s}(m)}{(2s-1)\Gamma(s)\zeta(2s)} E(z,1-s)$$

where

$$\sigma_w(m) = \sum_{\substack{d|m \\ d>0}} d^w.$$

On the basis of these properties, one obtains the following generalization of the Rankin-Selberg method. Let

$$Z_m(s) = |\pi m|^{\frac{1}{2}-k} \frac{2^{-s}\Gamma(s)}{\Gamma(s-\frac{3}{2}-k)} \sum_{n=1}^{\infty} a_n \overline{b_{n+m}} \left(\frac{n}{2n+m}\right)^s F\left(\frac{s}{2}, \frac{s+1}{2}, s+\frac{3}{2}-k, \frac{m^2}{(2n+m)^2}\right)$$

where $b_u \equiv 0$ for $u \leq 0$, and

$$F(\alpha,\beta,\eta;w) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\Gamma(\alpha+m)\Gamma(\beta+m)\Gamma(\eta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\eta+m)} w^m$$

is the Gauss hypergeometric function which in the special case of (1.10) is just a Legendre function since $\beta-\alpha = \frac{1}{2}$. From the expansion

$$F\left(\frac{s}{2}, \frac{s+1}{2}, s+\frac{3}{2}-k; \frac{m^2}{(2n+m)^2}\right) = 1 + \frac{s(s+1)}{4s+6-4k} \frac{m^2}{(2n+m)^2} + \cdots$$

it is easily seen that $Z_m(s)$ converges absolutely for $\mathrm{Re}(s) > k$. We show that the function $Z_m(s)$ can be continued to a meromorphic function in the entire s-plane which is regular in $\mathrm{Re}(s) \geq k-\frac{1}{2}$ except for simple poles at the points $s = k - \frac{1}{2} + ir_j$ where $\frac{1}{4} + r_j^2$ is an eigenvalue of the Laplace operator for $L^2(\Gamma\backslash H)$. The eigenvalues are discrete and are characterized by the existence of an orthonormal basis of Maass wave forms $\{e_j(z)\}$ satisfying

(1.10) $$e_j(z) = \sum_{n\neq 0} c_j(n) \sqrt{y} K_{ir_j}(2\pi|n|y) e^{2\pi i n x},$$

(1.11) $$\Delta e_j(z) = (\tfrac{1}{4} + r_j^2)e_j(z),$$

(1.12) $$e_j(\sigma z) = e_j(z), \qquad \text{for all} \quad \sigma \in \Gamma,$$

where

$$K_v(y) = \int_0^\infty e^{-\frac{1}{2}y(u+u^{-1})}u^{v-1}\,du$$

is the modified Bessel function of the second kind.

THEOREM (1). The function $Z_m(s)$ can be continued to a meromorphic function of order two which is regular in $\mathrm{Re}(s) \geq k-\frac{1}{2}$ except for simple poles at the points $s = k - \frac{1}{2} + ir_j$ with corresponding residues

$$\alpha_j = \frac{c_j(m)}{2ir_j} \langle f, g\overline{e_j}\rangle.$$

Also, $s = k$ is not a pole, and $Z_m(s)$ satisfies the functional equation

$$Z_m(s) - Z_m(2k-1-s) = \frac{2^{k+\frac{1}{2}}\pi^{2k-\frac{3}{2}}}{(2s+1-2k)} G_m(s) R_{f,g}(s)$$

where

$$G_m(s) = \frac{|m|^{k-\frac{1}{2}-s}\sigma_{2s-2k+1}(m)}{\Gamma(k-s)\Gamma(1-k+s)\zeta(2k-2s)\zeta(2-2k+2s)}$$

is invariant for $s \longrightarrow 2k-1-s$.

The proof of Theorem (1) is slightly complicated by the fact that the inner product $\langle f, g\overline{P_m}\rangle$ does not converge absolutely for large m. If we look at the Fourier expansion (see [NI]) of the non-holomorphic Poincaré series

(1.13) $$P_m(z,s) = \sqrt{y}\, I_{s-\frac{1}{2}}(2\pi|m|y)e^{2\pi imx} + \frac{2\pi^s|m|^{s-\frac{1}{2}}\sigma_{1-2s}(m)}{(2s-1)\Gamma(s)\zeta(2s)} y^{1-s} + \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} B_\ell(s;m)\sqrt{y}\, K_{s-\frac{1}{2}}(2\pi|\ell|y)e^{2\pi i\ell x}$$

where

$$B_\ell(s;m) = 2 \sum_{c=1}^{\infty} S(\ell,m;c)c^{-1}M_{2s-1}(4\pi c^{-1}(m\ell)^{\frac{1}{2}})$$

$$S(\ell,m;c) = \sum_{\substack{d=1 \\ (d,c)=1}}^{c} e^{2\pi i \frac{a\ell+dm}{c}}, \qquad ad \equiv 1 \pmod c$$

is the Kloosterman sum, and

$$M_v(y(m\ell)^{\frac{1}{2}}) = \begin{cases} J_v(y|m\ell|^{\frac{1}{2}}) & m\ell > 0 \\ I_v(y|m\ell|^{\frac{1}{2}}) & m\ell < 0 \end{cases}$$

it easily follows from (1.6) that

$$|P_m(z,s)| \gg e^{2\pi|m|y} \qquad (y \longrightarrow \infty).$$

Consequently

$$|f(z)g(z)P_m(z,s)| \gg e^{2\pi(|m|-2)y} \qquad (y \longrightarrow \infty)$$

if $a_1b_1 \neq 0$; and, therefore, the inner product $\langle f, g\overline{P_m}\rangle$ does not make sense for $|m| > 1$.

In order to get around this difficulty, define for every $Y > 1$

$$D_Y = \{z \in H;\ \ |z| \geq 1,\ \ \mathrm{Im}\,(z) \leq Y,\ \ 0 \leq \mathrm{Re}(z) \leq 1\}$$

$$A = \{z \in H;\ \ |z| < 1,\ \ 0 \leq \mathrm{Re}(z) \leq 1\}$$

$$P_m^*(z,s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma}{}' (\mathrm{Im}\,\sigma z)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi|m|\mathrm{Im}(\sigma z))e^{2\pi i m\,\mathrm{Re}(\sigma z)}$$

where the prime on the summation symbol means to omit the identity matrix from the sum. Since

$$A = \bigcup_{\sigma\in\Gamma_\infty\backslash\Gamma}{}' \sigma D$$

it is immediate that

$$(1.14) \qquad I_A(s) = \iint_A f(z)\overline{g(z)}\, y^{k+\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi|m|y)e^{2\pi i m x}\,\frac{dxdy}{y^2}$$
$$= \iint_D f(z)\overline{g(z)}\, y^k P_m^*(z,s)\,\frac{dxdy}{y^2}\,.$$

But, for $1 < Y_1 < Y$

$$\iint_A = \int_0^Y \int_0^1 - \int_{Y_1}^Y \int_0^1 = \iint_{D_Y} .$$

Hence

$$(1.15) \quad I_A(s) = \sum_{n=1}^{\infty} a_n \overline{b_{n+m}} \left(\int_0^Y - \int_{Y_1}^Y \right) e^{-2\pi(2n+m)y} I_{s-\frac{1}{2}}(2\pi|m|y) y^{k-\frac{1}{2}} \frac{dy}{y} - I_{D_{Y_1}}(s).$$

Using the transform

$$\int_0 e^{-\gamma y} I_{s-\frac{1}{2}}(ay) y^{\rho} \frac{dy}{y} = \left(\frac{a}{2\gamma}\right)^{s-\frac{1}{2}} \gamma^{-\rho} \frac{\Gamma(2-\frac{1}{2}+\rho)}{\Gamma(s+\frac{1}{2})} F\left(\frac{s-\frac{1}{2}+\rho}{2}, \frac{s+\frac{1}{2}+\rho}{2}, s+\tfrac{1}{2}; \left(\frac{a}{\gamma}\right)^2\right),$$

and letting $Y \longrightarrow \infty$ in (1.15), it follows from (1.14) that

$$(1.16) \qquad Z_m(s+k-1) = \sum_{n=1}^{\infty} a_n \overline{b_{n+m}} \int_0^{\infty} e^{-2\pi(2n+m)y} I_{s-\frac{1}{2}}(2\pi|m|y) y^{k-\frac{1}{2}} \frac{dy}{y}$$

$$= \iint_D f(z)\overline{g(z)}\, y^k P_m^*(z,s) \frac{dxdy}{y^2} + I_{D_Y}(s)$$

$$+ \sum_{n=1}^{\infty} a_n \overline{b_{n+m}} \int_Y^{\infty} e^{-2\pi(2n+m)y} I_{s-\frac{1}{2}}(2\pi|m|y) y^{k-\frac{1}{2}} \frac{dy}{y}$$

and this holds for any fixed $Y > 1$. Since the second and third terms on the right side of (1.16) are entire functions and $|P_m^*(z,s)| \ll y^{1-\mathrm{Re}(s)}$ as $y \longrightarrow \infty$, it immediately follows that the right side of (1.16) defines an analytic function and gives the analytic continuation of $Z_m(s)$ to the entire complex s-plane.

To determine the poles of $Z_m(s)$, note that if we define

$$\tilde{P}_m(z,s) = \frac{(\pi|m|)^{s-\frac{1}{2}}}{\Gamma(s+\frac{1}{2})} \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\,\sigma z)^s e^{2\pi i m\sigma z}$$

then

$$P_m^*(z,s) - \tilde{P}_m(z,s)$$

is regular for $\mathrm{Re}(s) > 0$. The analytic continuation of $\tilde{P}_m(z,s)$ was given by Selberg in [S2] by expanding it in terms of an orthonormal basis of eigenfunctions $\{e_j\}$ satisfying (1.10) - (1.12). The spectral decomposition of $\tilde{P}_m(z,s)$ is

$$\tilde{P}_m(z,s) = \sum_{j=1}^{\infty} <\tilde{P}_m, e_j> e_j(z) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} <\tilde{P}_m, E(*,w)> E(z,w)\, dw.$$

One easily computes that

$$<\tilde{P}_m, e_j> = \frac{c_j(m)\Gamma(s-\frac{1}{2}-ir_j)\Gamma(s-\frac{1}{2}+ir_j)}{\Gamma(2s)}$$

$$<\tilde{P}_m, E(*,w)> = \frac{\sigma_{2w-1}(m)\Gamma(\frac{s-1-w}{2})\Gamma(\frac{s-2+w}{2})}{\pi^{-w}|m|^{w-\frac{3}{2}}\Gamma(s-\frac{1}{2})\Gamma(w)\zeta(2w)}$$

It follows that the only poles of $\tilde{P}(z,s)$ in $\mathrm{Re}(s) \geqslant \frac{1}{2}$ are at $s = \frac{1}{2} + ir_j$ with corresponding residues

$$\beta_j = \frac{c_j(m)e_j(z)}{2ir_j} .$$

There is no pole at $s = 1$ since the inner product of $\tilde{P}_m$ with a constant function is identically zero. Putting this information back into (1.16) gives

$$Z_m(s+k-1) = \sum_{j=1}^{\infty} c_j(m) \frac{\Gamma(s-\frac{1}{2}-ir_j)\Gamma(s-\frac{1}{2}+ir_j)}{\Gamma(2s)} <f, g\overline{e_j}> + H_m(s)$$

where $H_m(s)$ is regular for $\mathrm{Re}(s) \geqslant \frac{1}{2}$.

The functional equation for $Z_m(s)$ can be easily deduced from (1.9) and the identity

$$I_{s-\frac{1}{2}}(2\pi|m|y) - I_{\frac{1}{2}-s}(2\pi|m|y) = \frac{2\sin(\frac{1}{2}-s)\pi}{\pi} K_{s-\frac{1}{2}}(2\pi|m|y).$$

One then obtains from (1.16) that

$$(1.17) \qquad Z_m(s+k-1) - Z_m(k-s) = Q_m(s) \iint_D f(z)\overline{g(z)}\, y^k E(z,1-s) \frac{dxdy}{y^2}$$

where

$$Q_m(s) = \frac{2\pi^s|m|^{s-\frac{1}{2}}\sigma_{1-2s}(m)}{(2s-1)\Gamma(s)\zeta(2s)} .$$

The Rankin-Selberg method gives

$$(1.18) \qquad \iint_D f(z)\overline{g(z)}\, y^k E(z,1-s) \frac{dxdy}{y^2} = \int_0^\infty \int_0^1 f(z)\overline{g(z)}\, y^{k-s} \frac{dxdy}{y}$$

$$= (4\pi)^{s-k}\Gamma(k-s)L_{f,g}(k-s).$$

On combining (1.17) and (1.18) one immediately obtains the functional equation for $Z_m(s)$.

§2. The arithmetic properties of the classical holomorphic Poincaré series of weight k

$$P_m(z) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} \frac{e^{2\pi i m z}}{(cz+d)^k}, \qquad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with Fourier expansion

$$(2.1)\quad P_m(z) = \sum_{n=1}^{\infty} \{\delta_{mn} + 2\pi\left(\frac{n}{m}\right)^{\frac{k-1}{2}} i^k \sum_{c=1}^{\infty} s(m,n;c)c^{-1}J_{k-1}\left(\frac{4\pi}{c}\sqrt{mn}\right)\}e^{2\pi i n z}$$

lead to some interesting identities similar in spirit to the functional equation in Theorem (1). Since $P_m(z)$ must be a cusp form, it can be expanded in terms of a basis $f_1, f_2 \ldots f_n$

$$f_j(z) = \sum_{n=1}^{\infty} a_j(n)e^{2\pi i n z}$$

of the space of holomorphic cusp forms of weight k for Γ. Consequently

$$(2.2)\qquad P_m(z) = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \sum_{j=1}^{h} \frac{a_j(m)}{\langle f_j, f_j\rangle} f_j(z).$$

Equating Fourier coefficients of (2.1) and (2.2)

$$(2.3)\qquad \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \sum_{j=1}^{h} \frac{a_j(m)}{\langle f_j, f_j\rangle} a_j(n)$$

$$= \delta_{mn} + 2\pi(i)^k\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} \frac{s(m,n;c)}{c} J_{k-1}\left(\frac{4\pi}{c}\sqrt{mn}\right).$$

Now, using Barnes' representation for the Bessel function

$$J_{k-1}(x) = \frac{1}{4\pi i}\int_{\alpha-i}^{\alpha+i\infty} \frac{\Gamma\left(\frac{k-1+s}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)} \left(\frac{x}{2}\right)^{-s} ds, \qquad (1-k < \alpha < 0)$$

and multiplying both sides of (2.3) by $a_\ell(m)m^{-w}$, and then summing over all positive integers m, it follows that

$$(2.4)\qquad \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{j=1}^{h} \frac{L_{f_j, f_\ell}(w+k-1)}{\langle f_j, f_j\rangle} a_j(n) - \frac{a_\ell(n)}{n^w} =$$

$$= \frac{(i)^k n^{\frac{k-1}{2}}}{4\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\Gamma(\frac{k-1+s}{2}) n^{-s/2}}{(2\pi)^{s-1} (\frac{k+1-s}{2})}$$

$$\sum_{c=1}^{\infty} c^{s-1} \sum_{\substack{d=1\\(d,c)=1}}^{c} e^{\frac{2\pi i d n}{c}} L_\ell\left(\frac{s+k-1}{2} + w; \frac{a}{c}\right) ds$$

where

$$L_\ell(s;\tfrac{a}{c}) = \sum_{m=1}^{\infty} \frac{a_\ell(m) e^{\frac{2\pi i a m}{c}}}{m^s}, \qquad ad \equiv 1 \pmod c.$$

The disadvantage of (2.3) is the appearance of Kloosterman sums on the right hand side. If in (2.4) we apply the functional equation

$$(2.5)\qquad \left(\frac{c}{2\pi}\right)^s \Gamma(s) L_\ell(s;\tfrac{a}{c}) = (i)^k \left(\frac{c}{2\pi}\right)^{k-s} \Gamma(k-s) L_\ell(k-s, \tfrac{-d}{c})$$

then it is easily seen that the Kloosterman sums are transformed into Ramanujan sums which can be evaluated exactly. Ramanujan's identity

$$\sum_{c=1}^{\infty} c^{-2w} \sum_{\substack{d=1\\(d,c)=1}}^{c} e^{\frac{2\pi i d(m-n)}{c}} = \frac{\sigma_{1-2w}(m-n)}{\zeta(2w)}$$

can then be applied to give

$$(2.6)\qquad \sum_{c=1}^{\infty} c^{-2w} \sum_{\substack{d=1\\(d,c)=1}}^{c} e^{\frac{2\pi i d n}{c}} L_\ell(s; \tfrac{-d}{n}) = \frac{1}{\zeta(2w)} \sum_{m=1}^{\infty} \frac{a_\ell(m)\sigma_{1-2w}(m-n)}{m^s}$$

which is valid for $\text{Re}(w) > 1$ and $\text{Re}(s) > \frac{k+1}{2}$.

On combining (2.4), (2.5), and (2.6) we get

$$(2.7)\qquad \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{j=1}^{h} \frac{L_{f_j, f_\ell}(w+k-1)}{\langle f_j, f_j\rangle} a_j(n)$$

$$= \frac{a_\ell(n)}{n^w} + \frac{(2\pi)^{2w} n^{\frac{k-1}{2}}}{\zeta(2w)} \sum_{m=1}^{\infty} \frac{a_\ell(m)\sigma_{1-2w}(m-n)}{m^{\frac{k+1}{2}-w}} I_w\left(\frac{m}{n}\right)$$

where

$$(2.8)\qquad I_w(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\Gamma(\frac{k+1}{2} - w - s)\Gamma(\frac{k-1}{2}+s)}{\Gamma(w+\frac{k-1}{2}+s)\Gamma(\frac{k+1}{2}-s)} x^s\, ds.$$

Since $k \geqslant 12$, the right side of (2.7) converges absolutely for $1 < \mathrm{Re}(w) \leqslant 2$ and $\alpha < -\mathrm{Re}(w)$, say.

The integral in (2.8) can be computed as follows. If $x < 1$

$$(2.9)\qquad I_w(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(k-w+m)}{\Gamma(k+m)\Gamma(w-m)} x^{\frac{k+1}{2} - w + m}$$

$$= \frac{\Gamma(k-w)}{\Gamma(k)\Gamma(w)} x^{\frac{k+1}{2} - w} F(k-w, 1-w, k; x)$$

since $\Gamma(z)$ has poles at $z = -m$ with residue $\frac{(-1)^m}{m!}$. Similarly, for $x > 1$

$$(2.10)\qquad I_w(x) = \frac{\Gamma(k-w)}{\Gamma(k)\Gamma(w)} x^{-\frac{k-1}{2}} F(k-w, 1-w, k; x^{-1}).$$

Moreover, by continuity, these results are also valid when $x = 1$; and, in fact, using Gauss' formula

$$F(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

we have

$$(2.11)\qquad I_w(1) = \frac{\Gamma(k-w)\Gamma(2w-1)}{\Gamma(2)^2\Gamma(k-1+w)}.$$

It, therefore, follows from (2.7), (2.9), (2.10), (2.11) and the functional equation for the Riemann zeta function that

$$\left(\frac{\pi}{n}\right)^{k-1}\Gamma(k-1)\sum_{j=1}^{h}\frac{R_{f_j,f_\ell}(w+k-1)}{\langle f_j,f_j\rangle}a_j(n) = \Gamma(w+k-1)\zeta(1-2w)\frac{\Gamma(1-2w)}{\Gamma(1-w)}\frac{a_\ell(n)}{n^{w+k-1}}$$

$$+ \Gamma(k-w)\zeta(2w-1)\frac{\Gamma(2w-1)}{\Gamma(w)}\frac{a_\ell(n)}{n^{k-w}}$$

$$+ \frac{\Gamma(w+k-1)\Gamma(k-w)}{\Gamma(k)}\sum_{m<n}\frac{a_\ell(m)\sigma_{1-2w}(m-n)}{n^{k-w}} F(k-w,1-w,k;\tfrac{m}{n})$$

$$+ \frac{\Gamma(w+k-1)\Gamma(k-w)}{\Gamma(k)}\sum_{m>n}\frac{a_\ell(m)\sigma_{1-2w}(m-n)}{m^{k-w}} F(k-w,1-w,k;\tfrac{n}{m}).$$

THEOREM (2). Let $g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi inz}$ be a cusp form of weight k for Γ. Define

$$B_m(s) = \frac{\Gamma(s)\Gamma(2k-1-2s)}{\Gamma(k-1)\Gamma(k-s)} \zeta(2k-1-2s) \frac{b_m}{m^s}$$

$$Y_m(s) = \sum_{n<m} \frac{b_n \sigma_{2s+1-2k}(m-n)}{m^s} F(s,s+1-k,k;\tfrac{n}{m})$$

$$+ \sum_{n>m} \frac{b_n \sigma_{2s+1-2k}(n-m)}{n^s} F(s,s+1-k,k;\tfrac{m}{n})$$

Then

$$(\frac{\pi}{m})^{k-1} \sum_{j=1}^{h} \frac{R_{f_j,g}(s)a_j(n)}{\langle f_j,f_j\rangle} = B_m(s) + B_m(2k-1-s) + \frac{\Gamma(s)\Gamma(2k-1-s)}{\Gamma(k)\Gamma(k-1)} Y_m(s).$$

The functional equation for $R_{f_j,g}(s)$ immediately implies

$$Y_m(s) = Y_m(2k-1-s),$$

but this could just as easily have been obtained from the Gauss transformation

(2.12) $$F(\alpha,\beta,\gamma;z) = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma;z).$$

Curiously, (2.12) can also be used to give a novel proof of (1.2), the functional equation of the Rankin-Selberg zeta function.

As an example of a special case of Theorem (2), we put $k = 12$, $s = 10$ and

$$g(z) = \sum_{n=1}^{\infty} \tau(n)e^{2\pi inz}$$

to be the Ramanujan cusp form of weight twelve. It follows that

$$\frac{10!L_{g,g}(13)}{(4\pi)^{11}\langle g,g\rangle} \tau(m) = (\frac{24\zeta(3)}{11} m + \frac{1}{m^2}) \tau(m)$$

$$+ \frac{24m}{11} \sum_{n<m} \tau(n)\sigma_{-3}(m-n)(5-6\tfrac{n}{m})$$

$$+ \frac{24m}{11} \sum_{n>m} \tau(n)\sigma_{-3}(n-m)(\tfrac{m}{n})^{10} (5-6\tfrac{n}{m}).$$

§3. Using the methods of §1 it is possible to derive explicit formulae relating partial sums of the type (1.1) with sums going over the eigenvalues of the Laplacian. For example, we can obtain

THEOREM (3). *Let* $x \longrightarrow \infty$ *and* $\varepsilon > 0$ *be fixed. Then*

$$\sum_{n=1}^{\infty} a_n \overline{b_{n+m}} \, e^{-\frac{n}{x}} \ll x^{k-\frac{1}{2}+\varepsilon}$$

where the $\ll$*-symbol depends only on* ε *and* m.

The proof of Theorem (3) uses the fact that there are no eigenvalues in $(0,\frac{1}{4}]$ for the group Γ. This, however, may not be the case for arbitrary groups; although it is conjectured that there are no eigenvalues in $(0,\frac{1}{4})$ for any congruence subgroup of $SL_2(\mathbb{Z})$.

Let Γ' be any fixed congruence subgroup of $SL_2(\mathbb{Z})$, and let

$$\lambda_j' = \frac{1}{4} + r_j'^2 \qquad (\lambda_0' = 0 < \lambda_1' \leqslant \lambda_2' \leqslant \ldots)$$

be the eigenvalues of the Laplacian in $L^2(\Gamma' \backslash H)$. Put

$$a = \max_j |\mathrm{Re}\ ir_j|.$$

THEOREM (4). *Let* $x \longrightarrow \infty$ *and* $\varepsilon > 0$ *be fixed. If* a_n, b_n *are the* n^{th} *Fourier coefficients, respectively, of two cusp forms of weight* k *for* Γ', *then*

$$\sum_{n=1}^{\infty} a_n \overline{b_{n+m}} \, e^{-\frac{n}{x}} \ll \begin{cases} x^{k-\frac{1}{2}+a} & a > 0 \\ x^{k-\frac{1}{2}+\varepsilon} & \text{otherwise}. \end{cases}$$

The $\ll$*-symbol depends only on* ε, m *and* Γ'.

In the case that $a > 0$, it is possible in many cases to replace the upper bound in Theorem (4) by an asymptotic relation. We also remark that all of our Theorems remain valid if the $\{b_n\}$ are taken to be Fourier coefficients of Eisenstein series.

§4. BIBLIOGRAPHY

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