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EXCURSIONS AND LOCAL TIME

by

John B. WALSH

This article originated with the desire to understand not only Knight and Ray's theorems on the Markov property, but David Williams brilliant explanations of them as well. We wanted to do this without getting too deeply into the specific properties of Bessel processes and conditional Brownian motion. We were aided by Jacques Azema's healthy skepticism. "Why", he asked, when we gave him our naive explanation of this Markov property, "is it strongly Markov ?".

Answering that turned the projected short note into a study of excursion processes. Hopefully, the resulting article still fills the original aim of a simple treatment for the benefit of those generalists who know something about all Markov processes but nothing about any one of them.

Here are the facts at hand. Let X be a diffusion on an interval I, let L_t^x be its local time at the point x and time t, and let T be a terminal time for X. Consider the local time up to time T, and let x vary. This gives a process $\{L_T^x, x \in I\}$. The theorems of Knight and Ray say that, under the proper additional hypotheses, this is a Markov process, usually inhomogeneous, but in some cases, even a diffusion process. This is surprising. We start with a diffusion, X, which is a Markov process, all right, but the parameter is time. We end up with a Markov process in the space variable, and there doesn't seem to be any obvious reason that this Markovianity should transfer from time to space.

These theorems should really be regarded in three stages :

(i) $\{L_T^x, x \in I\}$ is a Markov process ;

(ii) it is even a strong Markov process ;

(iii) and we can calculate its infinitesimal generator.

The first statement is initially the most surprising, but also the most easily explained. The second is already deeper, and the third, which might seem the most mundane, has been the object of considerable ingenuity on the part of several mathematicians. We will deal with the first two stages. We will say a few words on the subject of the infinitesimal generator, but we will not attempt to go deeply into the question.

Our viewpoint and methods are most easily introduced by explaining heuristically why L_T^x is a Markov process. Suppose $x < y < z$. Why should L_T^z and L_T^x be independent given L_T^y ? It is basically a property of the excursions from y. Since X is continuous, each excursion from y lies either entirely above y or entirely below y. The local time at z depends only on the excursions above y, while that at x depends on the excursions below y. But the excursions from y are all independent, so the distributions of L_T^x and L_T^z are independent given the total number of excursions, and this number is essentially measured by L_T^y, the local time at y.

This explanation is nonsense if we take it literally, for there are infinitely many excursions from y, and it is not at all clear what the "number of excursions" and the "independence of the excursions" could mean. Still, it is possible to define these, using Poisson point processes and especially, Ito's idea of the excursion process.

The machinery to do this takes some time to build, but once it is set up, the above argument is easily made completely rigorous (Theorem 2.2).

When we turn to the strong Markov property of L, we will see that it follows from a larger strong Markov property of the excursion process (Theorem 3.4). An unexpected (to us) dividend is that Williams' decomposition of a diffusion into pre - and post-minimum processes falls out as a special case (Theorem 3.5).

Two other points deserve mention. Since one can kill a diffusion at a terminal time T and still have a diffusion, it is sufficient to consider only the case in which $T=\zeta$, the process lifetime. (This would not necessarily be true if we were computing infinitesimal generators, since we would presumably like to relate these to the original process, not the killed process).

Secondly, a point which is not apparent from our heuristic explanation is that both X_o and X_T (or $X_{\zeta-}$ in our case) must be constant. This is not a simplifying assumption, for the Markov property breaks down without it. (This comes from the special nature of the initial and final excursions from x). Ray gets this by conditioning the process on the values of X_o and $X_{\zeta-}$. However, a conditioned diffusion is still a diffusion, so that we may as well assume at the outset that $X_o \equiv a$ and $X_{\zeta-} \equiv z$ a.s.

1 - THE EXCURSION PROCESS

Let X_t be a diffusion on an interval $I \subset \mathbb{R}$, with lifetime $\zeta \leq \infty$. Let δ be the cemetary : $X_t=\delta$ if $t \geq \zeta$. We assume that x is regular, so if $x, y \in I$, then $P^x\{T_y < \infty\} > 0$ and $P^y\{T_x < \infty\} > 0$, where $T_x=\inf\{t > 0 : X_t = x\}$. In particular, each point $x \in I$ is regular for itself, so there exists a <u>local time</u> at x, that is, a continuous additive functional L_t^x, unique up to proportionality with fine support $\{x\}$. Trotter has shown [9] that there is a version such that $(x,t) \to L_t^x$ is a.s. continuous, We will always assume that L_t^x is continuous in (x,t).

In the following, we will suppose that there exists $a \in I$ and $z \in \bar{I}$, (the closure of I in $[-\infty,\infty]$) such that

(A1) $X_o \equiv a$

(A2) Either $X_{\zeta-}$ does not exist a.s. or $X_{\zeta-} = z$ a.s.

Fix x, and define the right continuous inverse τ_t^x of L_t^x by

$$\tau_t^x = \inf\{s > 0 : L_s^x > t\}.$$

The set $\{t : X_t \neq x\}$ is a union of disjoint intervals, called <u>excursion intervals</u>. Since $t \to L_t^x$ increases only when $X_t = x$, it is constant on each excursion interval. If (u,v) is an excursion interval, and if $L_s^x = t$ for $u < s < t$, then $\tau_{t-}^x = u$ and $\tau_t^x = v$, so that all excursion intervals are of the form (τ_{t-}^x, τ_t^x) as t ranges over $[0,\infty)$.

Now let ∂ be a point not in $\mathbb{R} \cup \delta$, and let W be the set of all right-continuous functions w from $\mathbb{R}_+$ to $\mathbb{R} \cup \delta \cup \partial$ which are continuous as long as they are in $\mathbb{R}$, and which are absorbed at ∂ and δ. The function identically equal to x will be denoted by $[x]$. We provide W with the σ-field $\mathcal{W}$ generated by the coordinate functions.

The excursion of X during the interval (τ_{t-}, τ_t) is ξ_t, where,

if
$$\rho_t = \inf\{u>0 : X(\tau_{t-}+u^-) = x\}, \quad \xi_t(s) = \begin{cases} X(\tau_{t-} + s) & \text{if } 0 \leq s < \rho_t \\ \partial & \text{if } s \geq \rho_t \end{cases}$$

Note that $\xi_t(\cdot) \in W$. Then the <u>excursion process</u> $\{\Xi_x(t), t \geq 0\}$ is defined by :

$$\Xi_x(t) = \begin{cases} \xi_t(\cdot) & \text{if } \tau_{t-} < \tau_t \\ \partial & \text{if } \tau_{t-} = \tau_t. \end{cases}$$

It is well-known that the set of points of increase of $t \to L_t^x$ is a.s. identical to $\{t : X_t = x\}$. This implies that there is exactly one excursion in each interval (τ_{t-}, τ_t). On the other hand, it is a priori possible to have more. If, for instance, X has a local minimum at time t, and if $X_t = x$, then L_t^x, being continuous, would not increase at s, but there would still be two excursions, one ending and the second starting at s. For fixed x, this happens with probability zero, but if we fix ω, there will certainly be x for which the above happens.

In order to take account of this possibility, we agree that if t is such that the interval (τ_{t-}^x, τ_t^x) contains two excursions, we will denote the first by

$\Xi_x(t)$, the second by $\Xi_x(t_+)$. (We won't need notation to handle the possibility of more than two excursions in a single excursion interval : it never happens in our situation).

If there is a positive probability of having more than one excursion in one interval, we will call the excursion process degenerate.

Notice that the value of $\Xi_x(t)$ is a function, namely the whole excursion of X during the interval $[\tau_{t-},\tau_t)$. If x is recurrent, then Ξ_x is a non-degenerate Poisson point process (which we will abbreviate ppp in the following) and in general, Ξ_x is a non-degenerate ppp absorbed in the set D of infinite excursions, [6], where

$$D = \{w \in W : w(t) \neq \partial ,\ \forall t\}.$$

(If Ξ is a ppp and T a stopping time for Ξ, the process absorbed at T is defined by

$$\Xi'(t) = \begin{cases} \Xi(t) & t \leq T \\ \partial & t > T. \end{cases}$$

If $A \in \mathcal{W}$, let S_A^x be the first hit of A by Ξ_x :

$$S_A^x = \inf\{t : \Xi_x(t) \in A\}.$$

When there is no danger of confusion, we will write S_A instead of S_A^x. By [6,Theorem 3], if $\Xi_x(t)$ is a ppp absorbed at S_D, there exists a true ppp $\hat{\Xi}_x$ such that Ξ_x has the same distribution as $\hat{\Xi}_x$ absorbed at $\hat{S}_D$ (the first hit of D by $\hat{\Xi}_x$). We can and will assume without loss of generality that Ξ_x is actually equal to $\hat{\Xi}_x$ absorbed at $S_D = \hat{S}_D$.

The process $\hat{\Xi}_x$ is useful because the independence relations for a true ppp are simpler than those for an absorbed ppp. We will indicate quantities defined in terms of $\hat{\Xi}$ by a " ^". Thus, if $A \in \mathcal{W}$

$$N_x^A(t) = \#\{s \leq t : \Xi_x(s) \in A\}$$

and

$$\hat{N}_x^A(t) = \#\{s \leq t : \hat{\Xi}_x(s) \in A\}.$$

If $\hat{N}_x^A(t)$ is a.s. finite, it is a Poisson process and $N_x^A(t)=\hat{N}_x^A(t\wedge S_D)$. Let $n_x(A)$ be the parameter of $\hat{N}_x^A(t)$. We call $n_x(.)$ the <u>characteristic measure</u> of $\hat{\Xi}_x$ (and of Ξ_x).

We can generalize this as follows : let f be a positive functional on $W \cup \partial$ with $f(\partial) = 0$. We put

$$N_x f(t) = \sum_{s \leq t} f(\Xi_x(s)).$$

If f is the indicator function of $A \in \mathcal{W}$, this reduces to N_x^A. We could also let f be the local time at y functional, ℓ_y : if $\xi \in W$, $\ell_y(\xi)$ is the total local time at y of ξ. Then $N_x \ell_y(t) = L^y_{\tau_t}$, and $N_x \ell_y(S_D^x) = L^y_\zeta$.

The independence properties of $\hat{N}_x f$ follow from those of the ppp $\hat{\Xi}_x$. First, it is a mixture of Poisson processes, and is thus a process of independent increments. It will be finite iff $\int_W f(\xi)\, n_x(d\xi) < \infty$, in which case its Lévy measure ν will be given by $\nu(A) = n_x\{\xi : f(\xi) \in A\}$. Moreover, we have the following, which we state as a proposition for further reference.

<u>Proposition 1.1</u>.

<u>If</u> f_i, $i=1,\dots,n$ <u>are functionals on</u> W <u>such that</u> $f_i(\xi)\, f_j(\xi) = 0$ <u>for all</u> $\xi \in W$ <u>if</u> $i \neq j$, <u>then the processes</u> $\{\hat{N}_x f_i(t) - \hat{N}_x f_i(0),\ t \geq 0\}$, $i=1,\dots,n$ <u>are independent</u>, <u>and independent of</u> $\hat{N}_x f_i(0)$, $i=1,\dots,n$.

This follows from the corresponding fact for the N^A, so we omit the proof. Note that we can derive the independence properties of the unhatted processes $N_x f$ from this, since $N_x f(t) = \hat{N}_x f(t \wedge S_D)$.

Now let's split our excursion process into two parts. For $A \in \mathcal{W}$, define

$$\Xi_x^A(t) = \begin{cases} \Xi_x(t) & \text{if } \Xi_x(t) \in A \\ \partial & \text{otherwise} \end{cases}$$

Let E_x^+ and $E_x^- \subset W$ be the excursions above and below x respectively :

$$E_x^+ = \{w \in W : w(t) \geq x,\ \forall t < \zeta(w)\};$$

$$E_x^- = \{w \in W : w(t) \leq x,\ \forall t < \zeta(w)\} - \{[x], [\partial], [\delta]\}.$$

Then define

$$\Xi_x^+(t) = \Xi_x^{E_x^+}(t) \ , \ \Xi_x^-(t) = \Xi_x^{E_x^-}(t),$$

and put

$$\mathcal{E}_x^+ = \sigma\{\Xi_x^+(t),\ t \geq 0\} \quad (\stackrel{\text{def}}{=} \sigma\{N_t(A \cap E_x^+), t \geq 0,\ A \in \mathscr{W}\})$$

and
$$\mathcal{E}_x = \sigma\{\Xi_x^-(t),\ t \geq 0\}.$$

We should logically write $\mathcal{E}_x^-$ instead of $\mathcal{E}_x$, but we will be using it so often that simplification is called for.

The initial and final excursions from x play a special role. The initial excursion, $\Xi_x(0)$, is simply the process from time zero until it first hits x. The final excursion, $\Xi_x(S_D)$, is the path from the last time it leaves x on. It is possible that there is no initial of final excursion. In our notation, this is translated by $\Xi_x(0) = \partial$ or $\Xi_x(S_D) = [\partial]$. Because of assumptions (A1) and (A2), we have the following.

<u>Lemma 1.2</u>

<u>Let</u> ξ_i <u>and</u> ξ_f <u>be the initial and final excursions from</u> x. <u>Then each of the following probabilities are either zero or one</u> :

$$P\{\xi_i \in E_x^+\}\ ,\ P\{\xi_i \in E_x^-\},\ P\{\Xi_x(0) = \partial\},\ P\{\xi_f \in E_x^+\},\ P\{\xi_f \in E_x^-\}\ ,\ P\{S_D = \infty\}.$$

<u>Proof</u> : since $X_0 = a$, ξ_i is in E_x^+ or E_x^- according to whether $a > x$ or $a < x$, and ξ_i doesn't exist if $a = x$. The case of ξ_f is only slightly more complicated. If $X_{\zeta-}$ doesn't exist in the extended reals, X must be recurrent, $\zeta = \infty$ and there is no final excursion. If $\zeta < \infty$, then $X_{\zeta-}$ exists and necessarily equals z, so ξ_f is in E_x^+ or E_x^- according to whether $z > x$ or $z < x$. If $z = x$ and if ζ is the first hit of x by X_{t-}, then $\xi_f \in E^+$ if $a > x$, and $\xi_f \in E^-$ if $a < x$. In all other cases in which $z = x$, $\xi_f = [\delta] \in E_x^+$.

2 - THE MARKOV PROPERTY OF LOCAL TIME

We have spent a lot of time setting up our machinery ; it is time to use it.

Proposition 2.1

$\mathcal{E}_x^+$ and $\mathcal{E}_x$ are conditionally independent, given S_D. In particular Ξ_x^+ is conditionally independent of $\mathcal{E}_x$ given S_D.

Proof : except possibly for the initial values, $\hat{\Xi}_x^+$ and $\hat{\Xi}_x^-$ are independent. But by Lemma 1.2, either $\hat{\Xi}_x^+(0) = \partial$ a.s. or $\hat{\Xi}_x^-(0) = \partial$ a.s., so that $\hat{\Xi}_x^+$ and $\hat{\Xi}_x^-$ are truly independent. Turning to the final excursion, note that by Lemma 1.2, either $S_D = S_{D \cap E_x^+}$ a.s. or $S_D = S_{D \cap E_x^-}$ a.s. Thus, S_D is a stopping time for one of the processes Ξ_x^+ or Ξ_x^-, and is independent of the other. Suppose, for instance, that it is independent of $\hat{\Xi}_x^-$. Then, as Ξ_x^- is $\hat{\Xi}_x^-$ absorbed at S_D, it is conditionally independent of $\hat{\Xi}_x^+$, hence of Ξ_x^+, given S_D. qed

Proposition 2.1 gives us the Markov property of the local time. Let ℓ_y the local time functional on W : $\ell_y(\xi)$ is the total local time at y of the excursion ξ. Then $\hat{N}_x \ell_y(t)$ is a process of independent increments and $L_\zeta^y = \hat{N}_x \ell_y(S_D^x)$. But if $x < y$, all the local time at y accrues during the excursions above x. Thus $L_\zeta^y = \hat{N}_x^+ \ell_y(S_D^x) = \sum_s \ell_y(\Xi_x^+(s)) \in \mathcal{E}_x^+$. At the same time, if $v \leq x$, $L_\zeta^v = \sum_s \ell_v(\Xi_x^-(s)) \in \mathcal{E}_x^-$. Thus L_ζ^y and $\{L^v, v \leq x\}$ are conditionally independent given S_D^x by Proposition 2.1. But $S_D^x = L_\zeta^x$! (For $S_D^x = \sup\{t : \tau_t^x < \infty\}$ $= \sup\{t : \exists\, s \ni L_s^x > t\} = L_\zeta^x$). We have proved :

Theorem 2.2

The process $\{L_\zeta^x, x \in I\}$ is a Markov process.

Remarks : the assumptions (A1) and (A2) that X_o and $X_{\zeta-}$ are deterministic are necessary for the above theorem. On the other hand, we did not use the continuity of L_t^x. This only comes into play when we talk about the strong Markov property.

Theorem 2.2 gives the Markov property of the local time, but not its strong Markov property. Just as the Markov property of L^x_ζ follows from Proposition 2.1, which might be thought of as a simple Markov property of the excursion process, its strong Markov property follows from a strong Markov property of the excursion process, which we will prove later, in Theorem 3.4.

We can't expect L^x_ζ to be a time-homogeneous strong Markov process, for we have normalized it rather arbitrarily. Even if we had normalized it carefully, it would still not in general be homogeneous. However, it does have the time-dependent strong Markov property. If we replace x by an $\mathcal{E}^-_x$ - stopping time above, and apply Theorem 3.4 instead of Proposition 2.2, we get

Theorem 2.3

Let $M=\inf\{X_t, t < \zeta\}$. Then the process

$$\{L^x_\zeta, x \geq M, x \in I\}$$

is a time-inhomogeneous strong Markov process.

Theorems 2.2 and 2.3 are known, and we would like to pass on to some refinements. In particular, we would like to look at $\{L^y_T, y \in I\}$ for certain random times T. We must first look at a rather simple idea ; how to tell time in a point process.

To clarify the issue, let's consider the relation between the "real time" of the diffusion X_t and the "excursion time" of $\Xi_x(t)$. If t is a time for which $\Xi_x(t) \neq \partial$, the value of $\Xi_x(t)$ is a function, say $\xi_t(s)$. We might with equal justice call it $\Xi_x(t,s)$, and think of "time" for Ξ_x to be a couple (t,s), where t is the usual parameter of Ξ_x and s is the time during the excursion. In real time, the excursion starts at τ^x_{t-} and ends at τ^x_t, so s runs between 0 and $\tau^x_t - \tau^x_{t-}$, and $\Xi_x(t,s) = X_{\tau^x_t + s}$. Thus the couple (t,s) corresponds to the real time $\tau^x_{t-}+s$. To go back from X to Ξ, remember that L^x_t is the inverse of τ^x_t, so that if e_t is the last exit before t : $e_t = \sup\{s \leq t : X_s = x\}$, then a real time t corresponds to the couple

$\rho(t) = (L_t^x,\ t-e_t)$, and we have $X_t = \Xi_x^+(\rho(t))$.

Definition

A random variable $T \geq 0$ is $\mathcal{E}_x$-identifiable if both $\{T=0\}$ and $\rho(T)$ are $\mathcal{E}_x$-measurable.

Remarks :

1°). Intuitively, T is an instant of real time which we can identify by looking just at the excursions below x. A constant is not in general $\mathcal{E}_x$-identifiable, but times such as T_x, or T_y, for $y \leq x$, are. Indeed, $L_{T_x}^x = 0 = T_x - e_t$, so $\rho(T_x) = (0,0)$. Similarly, so are the last exit time from x, and the time of the process minimum, provided that that minimum is below x. So is $\tau_{t-}^x = \{\inf s : L_s^x \geq t\}$, since $\rho(\tau_{t-}) = (t,0)$.

2°). If T is $\mathcal{E}_x$-identifiable and $X_t \leq x$ a.s. then X_T is $\mathcal{E}_x$-measurable for $X_T = \Xi_x^-(L_T^x,\ T-e_t)$, and both $(L_T^x,\ T-e_t)$ and the process Ξ_x^- are $\mathcal{E}_x$-measurable.

3°). If $X_T \leq x$ and T is $\mathcal{E}_x$-identifiable, then T is $\mathcal{E}_y$-identifiable for all $y > x$. (This is true even without the condition that $X_T \leq x$, for one can show that if T is $\mathcal{E}_x$-identifiable, then $X_T \leq x$ a.s. on $\{T > 0\}$).

Theorem 2.4

Let $S \leq T < \zeta$ be $\mathcal{E}_{x_o}$-identifiable such that $X_S \leq x_o$ and $X_T \leq x_o$ a.s. on $\{T > 0\}$. Then the process $\{L_T^y - L_S^y,\ y \geq x\}$ is an inhomogeneous strong Markov process whose transition function does not depend on x, S or T. Moreover if $S_1 \leq T_1 \leq S_2 \leq \dots \leq T_n$ are $\mathcal{E}_x$-identifiable with $X_{S_i} \leq x_o$ and $X_{T_i} \leq x_o$ a.s. on $\{S_i \neq T_i\}$, then the processes $\{L_{T_j}^y - L_{S_j}^y,\ y \geq x\}$ are conditionally independent given their initial values.

Proof : if $L_T^y - L_S^y = 0$, then either $S=T$ or else $X_t < y$ for all $S < t < T$. In either case, $L_T^v - L_S^v = 0$ for all $v \geq y$, so the process is absorbed at zero.

Let $x_o \leq x < y$ and let ℓ_y be the local time at y functional. Now $X_S, X_T \leq x < y$, so that

$$L^y_T - L^y_S = N_x \ell_y(L^x_T) - N_x \ell_y(L^x_S)$$

Recall that $T < \zeta$, so the final excursion from x does not contribute. This means that we can replace ℓ_y by f_y, where $f_y(\xi) = \ell_y(\xi)$ if $\xi \in W-D$, $f_y(\xi)=0$ otherwise. Furthermore, we can replace N_x by $\hat{N}_x$, again because, as $T < \zeta$, $L^x_T \leq S^x_D$:

$$L^y_T - L^y_S = \hat{N}_x f_y(L^x_T) - \hat{N}_x f_y(L^x_S).$$

But f_y is zero off $E^+_x - D$, which is disjoint from both D and E^-_x, so that by Proposition 1.1, $\hat{N}_x f_y(t)$ is independent of both $\hat{\Xi}^-_x$ and $\hat{\Xi}^D_x$, and hence of $\mathcal{E}_x$ and S^x_D. Since L^x_S and L^x_T are measurable with respect to these, they are independent of $\hat{N}_x f_y$. Now $\hat{N}_x f_y$ is a process of independent increments. Let its transition function be $p_{xy}(t,A)$. Then

$$(2.1)\quad P\{\hat{N}_x f_y(L^x_T) - \hat{N}_x f_y(L^x_S) \in A \mid \mathcal{E}_x\} = p_{xy}(L^x_T - L^x_S, A).$$

Thus the transition function of $(L^{\bullet}_T - L^{\bullet}_S)$ is $p_{xy}(u,A)$.

In order to show the strong Markov property, we replace x above by an $\mathcal{E}_y$-stopping time Y, and remark that by Theorem 3.4, (2.1) still holds on the set $\{Y=x\}$.

Finally, the conditional independence of the $L^y_{T_i} - L^y_{S_i}$ follows from the fact that $\hat{N}_x f_y$ has independent increments :

$$P\{\hat{N}_x f_y(L^x_{T_i}) - \hat{N}_x f_y(L^x_{S_i}) \in A,\ i=1,\dots,n \mid \mathcal{E}_x\} = \prod_{i=1}^{n} p_{xy}(L^x_{T_i} - L^x_{S_i}, A)$$

<u>Theorem 2.5</u>

<u>Let</u> T be <u>$\mathcal{E}_{x_o}$-identifiable</u>. <u>Then, on the set</u> $\{X_T \leq x_o\}$, <u>the processes</u> (i) $\{L^y_T, y \geq x_o\}$ <u>and</u> (ii) $\{L^y_\zeta - L^y_T,\ y \geq x_o\}$ <u>are both inhomogeneous strong Markov processes</u>. <u>Their transition functions do not depend on</u> T <u>or</u> x.

Proof : if the initial value a and the final value z of X are below x, this reduces to a special case of the previous theorem. Thus we may as well assume that $a > x_o$ and $z > x_o$. This means that $L_T^{X_o} > 0$ and $L_\zeta^{x_o} - L_T^{x_o} > 0$ on the set $\{X_T \leq x_o\}$. Let $A \subset W$ be the set of excursions with initial point a, and, as usual, let D be the set of excursions of infinite length. If ℓ_y is the local time at y functional, define

$$f(\xi) = \begin{cases} \ell_y(\xi) & \text{if } \quad \xi \in E_x^+ - A - D \\ 0 & \quad \text{otherwise} \end{cases}$$

and

$$g(\xi) = \begin{cases} \ell_y(\xi) & \text{if } \quad \xi \in E_x^+ \cap A - D \\ 0 & \quad \text{otherwise} \end{cases}$$

If $x_o \leq x < y < z$, the final excursion of X from x doesn't contribute to L_T^y, so that

$$\begin{aligned} L_T^y &= \hat{N}_x g(L_T^x) + \hat{N}_x f(L_T^x) \\ &= \hat{N}_x g(0) + \hat{N}_x f(L_T^x), \end{aligned}$$

this last because $\hat{N}_x g(t) \equiv \hat{N}_x g(0)$, for all excursions after the first start from $x < a$. Furthermore, $\hat{N}_x f(0) = 0$. Now $\hat{N}_x f(t)$ is independent of $\mathcal{E}_x$, while $\hat{N}_x g(0)$ is independent given S_D^x, and, in fact, it is independent of $\mathcal{E}_x$ given that the initial excursion ξ actually reaches x, which is to say, given $\xi \in A-D$. Thus, for $B \subset \mathbb{R}$, let

$$\begin{aligned} U_{xy}(B) &= P\{\hat{N}_x g(0) \in B \mid \xi \in A-D\} \\ &= \frac{1}{n_x(A-D)} \int_{A-D} g(\xi)\, n_x(d\xi), \end{aligned}$$

and let $p_{xy}(t,A)$ be the transition function of $\hat{N}_x f$. Then, evidently

$$(2.2) \qquad P\{L_T^y \in B \mid \mathcal{E}_x\} = U_{xy} * p_{xy}(L_T^y, B), \text{ if } \quad x_o \leq x < y \leq a$$

where "*" is the convolution operator.

The process $L^y_\zeta - L^y_T$ is handled similarly. Let

$$h(\xi) = \begin{cases} \ell_y(\xi) & \text{if } \xi \in E^+_x \cap D - A \\ 0 & \text{otherwise} \end{cases}$$

Notice that $h(\Xi^+_x(t))=0$ except if $t=S^x_D$, when it gives the local time of the final excursion from x. Let e be the last exit time from x:

$$L^y_\zeta - L^y_T = (L^y_\zeta - L^y_e) + (L^y_e - L^y_T) = \hat{N}_x h(S^x_D) + \hat{N}_x f(e) - \hat{N}_x f(t).$$

Note that e is $\mathcal{E}_x$-identifiable and $X_e = x$ on $\{e > 0\}$. If, now, $x_o \leq x < y \leq z$, let ν_{xy} be the distribution of $\hat{N}_x h(S^x_D)$:

$$\nu_{xy}(B) = P\{\hat{N}_x h(S^x_D) \in B \mid S^x_D > 0\}.$$

Then, since $\hat{N}_x h$ and $\hat{N}_x f$ are independent.

$$P\{L^y_\zeta - L^y_T \mid \mathcal{E}_x\} = \nu_{xy} * p_{xy}(L^y_T, B). \tag{2.3}$$

Once again, if we replace x by an $\mathcal{E}_y$-stopping time Y, (2.2) and (2.3) hold on $\{Y=x\}$ by Theorem 3.4. qed

<u>Remarks</u> : in both theorems 2.4 and 2.5 we can replace the fixed point x_o by an $\mathcal{E}_x$-stopping time Y. The proofs go through virtually without change. This can be of interest ; if we choose $T = \inf\{t : X_t = M\}$, the minimum time of the process and let $Y=M$, we can break the local time into the sum of the local times of the pre-and post-minimum processes. This is exactly what David Williams does in [11], and, as he is able to calculate the infinitesimal generator of each of these separately, he gets the generator of the sum by addition.

Let us consider the processes $\{L^y_T - L^y_S,\ y \geq x\}$ for $\mathcal{E}_x$-identifiable S and T. We will show that we can renormalize this in such a way that, after a change of scale, it becomes a diffusion. We will assume that this process is square-integrable. This is automatic if $\zeta < \infty$ a.s., for then L^x_ζ is exponential and has moments of all orders. If $\zeta = \infty$, we can just bound T suitably : $T \leq T_{x_o}$ for some $x_o < x$, will do, for instance.

Let $S \leq T \leq U$ be $\mathcal{E}_x$-identifiable, with $X_S \leq x$, $X_T \leq x$, and $X_U \leq x$ on $\{U > 0\}$.

Let $x \prec y$. By the Markov property, there is a function f such that

$$E\{L_T^y - L_S^y \mid \mathcal{E}_x\} = f(L_T^x - L_S^x).$$

Since the transition function is independent of S and T, we also have

$$E\{L_U^y - L_T^y \mid \mathcal{E}_x\} = f(L_U^x - L_S^x).$$

But, as $L_U^y - L_S^y = (L_U^y - L_T^y) + (L_T^y - L_S^y)$, and the last two both have the same transition function, so this is

$$= f(L_U^x - L_T^x) + f(L_T^x - L_S^x)$$

This implies that for $s,t > 0$, $f(s) + f(t) = f(s+t)$, which means that there is a constant, say $c(x,y)$, such that $f(t) = c(x,y)t$. We can define $c(x,y)$ for each $x,y \in I$; this is consistent since the transition function is independent of x, S and T.

If $x < x' < y$,

$$c(x,y)(L_T^x - L_S^x) = E\{L_T^y - L_S^y \mid \mathcal{E}_x\} = E\{c(x',y)\ (L_T^{x'} - L_S^{x'} \mid \mathcal{E}_x\}$$

$$= c(x',y)\ c(x,x')\ (L_T^x - L_S^x).$$

Thus $c(x',y) = \dfrac{c(x,y)}{c(x,x')}$. Thus, choose $x_o \in I$ and define u by

$$u(x) = \begin{cases} c(x_o,x)^{-1} & \text{if } x \geq x_o \\ c(x,x_o)^{-1} & \text{if } x < x_o \end{cases}$$

and put $M_y = u(y)(L_T^y - L_S^y)$, $y \geq x$.

Then for $x \prec y$

$$E\{M_y \mid \mathcal{E}_x\} = u(y)\ \frac{u(x)}{u(y)}\ (L_T^x - L_S^x) = M_x.$$

Thus $\{M_y, y \geq x\}$ is a martingale.

Let's do the same thing with the variance. There is a function g, independent of S,T,U and x, for which

$$\mathrm{Var}\{u(y)(L_U^y - L_S^y) \mid \mathcal{E}_x\} = g(L_U^x - L_S^x).$$

Now $g(t)$ is strictly positive if $t > 0$. $L^y_U - L^y_T$ and $L^y_T - L^y_S$ are independent given $\mathcal{E}_x$, so their variances add, and this equals

$$= g(L^x_U - L^x_T) + g(L^x_U - L^x_S).$$

Just as above, there must be a $b(x,y) > 0$ such that $g(t) = b(x,y)t$, so

$$\mathrm{Var}\{M_y \mid \mathcal{E}_x\} = b(x,y)\, M_x.$$

If $x < x' < y$

$$\begin{aligned} b(x,y)\, M_x &= E\{(M_y - M_x)^2 \mid \mathcal{E}_x\} \\ &= E\{E\{(M_y - M_{x'})^2 \mid \mathcal{E}_y\} \mid \mathcal{E}_x\} + E\{(M_{x'} - M_x)^2 \mid \mathcal{E}_x\} \\ &= b(x',y)\, E\{M_{x'} \mid \mathcal{E}_x\} + b(x',x) M_x \\ &= (b(x',y) + b(x,x')) M_x \end{aligned}$$

Thus $b(x,y) = b(x,x') + b(x',y)$, so there is a strictly increasing function $\sigma(x)$ so that $b(x,y) = \sigma(y) - \sigma(x)$. Let $s(t) = \inf\{y : \sigma(y) > t\}$ be the inverse of σ. Then we have :

Theorem 2.6

There is a positive function u and a strictly increasing function σ, such that if $x_o \in I$ and if $S \leq T$ are $\mathcal{E}_{x_o}$-identifiable with $X_S \leq x_o$, $X_T \leq x_o$, then $M_y = u(y)(L^y_T - L^y_S)$ is a continuous martingale with associated increasing process

$$\langle M \rangle_y = \int_{x_o}^{y} M_x \, d\sigma(x).$$

Moreover, the process $\{M_{s(t)}, t \geq \sigma(x)\}$ is a diffusion on $[0,\infty)$, absorbed at 0, with infinitesimal generator $\mathcal{G} = \frac{x}{2} \frac{d^2}{dx^2}$.

Proof : we have seen that M is a continuous martingale. Now if $x_o < x < y$

$$\begin{aligned} E\{ \int_x^y M_v \, d\sigma(v) \mid \mathcal{E}_x\} &= \int_x^y E\{M_v \mid \mathcal{E}_x\} d\sigma(v) \\ &= (\sigma(y) - \sigma(x)) M_x = E\{(M_y - M_x)^2 \mid \mathcal{E}_x\}. \end{aligned}$$

Now $\{L^y_T - L^y_S\}$ is an inhomogeneous strong Markov process, hence so is M_y ; when we make the deterministic time change, we find that $\{M_{s(t)}, t \geq s(x)\}$

is still Markov, and moreover, it is a martingale with associated increasing process $\int_{s(x)}^{t} M_{s(v)}dv$. Let's calculate its infinitesimal generator. This is easily done using Ito's formula, for if $f \in C^{(2)}$,

$$\frac{1}{h} E\{f(M_{s(t+h)}) - f(M_{s(t)}) \mid \mathcal{E}_{s(t)}\}$$

$$= \frac{1}{h} E\{\int_t^{t+h} f'(M_{s(v)})dM_{s(v)} \mid \mathcal{E}_{s(t)}\} + \frac{1}{2h} E\{\int_t^{t+h} f''(M_{s(v)})d\langle M\rangle_{s(v)} \mid \mathcal{E}_{s(t)}\}.$$

The stochastic integral has expectation zero, and $d\langle M\rangle_{s(u)} = M_{s(u)}du$. If we let $h \to 0$, we see that

$$\mathfrak{G}f(M_{s(t)},t) = \frac{1}{2} M_{s(t)} f''(M_{s(t)})$$

But $\mathfrak{G}$ does <u>not depend on</u> t, so the process $M_{s(t)}$ is a continuous time-homogeneous strong Markov process, i.e. a diffusion. qed

<u>Remarks</u> : we showed that $M_{s(t)}$ is time-homogeneous by calculating its infinitesimal generator. In fact, had we shown it was homogeneous in some other fashion, we would have found its infinitesimal generator from its stability property. For $L_U^y - L_T^y$ and $L_T^y - L_S^y$ are, after a scale and time-change, both martingales and diffusions, and their sum, $L_U^y - L_S^y$, is the same. The only diffusion which is a martingale and which is stable under sums is the one with infinitesimal generator $cx\frac{d^2}{dx^2}$ for some $c > 0$. (See the appendix).

Continuing in the same vein, note that for $y \geq x$ we can write

$$L_\zeta^y = L_{T_x}^y + (L_{e_x}^y - L_{T_x}^y) + (L_\zeta^y - L_{e_x}^y),$$

where e_x is the last exit time from x. Both e_x and T_x are $\mathcal{E}_x$-identifiable. If $x \geq a$, $x \geq z$ (the initial and final points of X), the first and last terms are zero, hence $L_\zeta^y = L_{e_x}^y - L_{T_x}^y$ is, after the appropriate changes, a diffusion with infinitesimal generator $\mathfrak{G} = \frac{x}{2}\frac{d^2}{dx^2}$. If $a > x$ or $z > x$, however, L_ζ^y will be the sum of several Markov processes, one of which has the generator $\mathfrak{G}$. But the only diffusion which can be added to this so that the sum is a diffusion is one with generator

$$G_a = \frac{x}{2}\frac{d^2}{dx^2} + a\frac{d}{dx}$$

for some real a. If $a=n-1$, this is the infinitesimal generator of the square of a Bessel process, that is the radial part of an n-dimensional Brownian motion. This explains why the Bessel process enters into the theorems of Ray, Knight, and Williams : it is forced by the stability properties.

3 - THE STRONG MARKOV PROPERTY OF THE EXCURSION PROCESS

Let us consider the process $\{\Xi_x^+, x \in I\}$. Note that the parameter is x, rather than t. For a fixed x, the value of Ξ_x^+ is the whole sample path $\{\Xi_x^+(t), t \geq 0\}$. We can consider Ξ^+ as a process taking its values in the canonical space $\hat{W}$ of functions from $[0,\infty)$ to $W \cup \partial$ which equal ∂ at all but a countable number of points.

It might seem that the process Ξ_x^+ takes its values in an immense space, but it turns out to be remarkably regular. We will see below that it is right continuous in a certain sense.

Let g be a bounded continuous function on $\mathbb{R} \cup \delta$ and let h be continuous and of compact support in $(0,\infty)$. Then let f be the functional on W defined by

$$f(\xi) = \int_0^\infty g(\xi(s))\, h(s)ds, \quad \xi \in W. \tag{3.1}$$

Functionals of this type separate points of W ; as h has compact support in $(0,\infty)$, there is an $\varepsilon > 0$ such that $f(\xi) = 0$ unless ξ has a duration of at least ε , so that there are a.s. only finitely many $s \leq t$ for which $f(\Xi_x^+(s)) \neq 0$. Thus, $N_x f(t) = \sum_{s \leq t} f(\Xi_x^+(s))$ is a.s. finite. We will call a finite sum of finite products of functionals of the type (3.1) simple functionals.

Let $D^+ = D \cap E_x^+$ and put

$$\Xi_x^+ = \Xi_x^{E_x^+ - D} + \Xi_x^{D^+}.$$

Define

$$N_x^1 f(t) = \sum_{s \le t} f(\Xi_x^{E_x^+ - D}(s))$$

$$N_x^2 f(t) = \sum_{s \le t} f(\Xi_x^{D^+}(s)).$$

Note that $N_x f = N_x^1 f + N_x^2 f$.

Proposition 3.1

Let $f \ge 0$ be a simple functional and fix ω for which $S_D^x(\omega) > 0$. If t is $\ni$ either $t > S_D^x(\omega)$ or $\tau_{t-}^x(\omega) = \tau_t(\omega)$,

$$\lim_{y \downarrow x} N_y^1 f(t) = N_x^1 f(t)$$

and

$$\lim_{y \downarrow x} N_y^2 f(S_D^y) = N_x^2 f(S_D^x).$$

Proof : if $\tau_{t-}^x(\omega) = \tau_t^x(\omega)$, then $s \to L_s^x$ is strictly increasing at $s=\tau_t^x$. This, along with the continuity of $(s,y) \to L_s^y$ implies that $\tau_t^y \to \tau_t^x$ as $y \to x$. Note that only excursions longer than some minimal length, say $\varepsilon > 0$, contribute to $N_y f$, and there are at most $\frac{1}{\varepsilon}\tau_t^y$ of these. If $||f|| = \sup_\xi |f(\xi)|$, then $N_y^1 f(t) \le \varepsilon^{-1} ||f||\, \tau_t^y$, which is bounded as $y \downarrow x$.

Consider an excursion ξ^x of X above x, which is completed before τ_t^x. If $y > x$ is close enough to x, there are excursions above y contained in ξ^x. Let ξ^y be the longest of these. Since $\tau_t^y \to \tau_t^x$ as $y \to x$, ξ^y will be complete before τ_t^y if y is close enough to x. But clearly $\xi^y(.) \to \xi^x(.)$ boundedly, hence $f(\xi^y) \to f(\xi^x)$. If y is close enough to x, any other excursion ξ' above y which is contained in ξ^x has length less than ε, so that $f(\xi')=0$. Summing over all excursions, we conclude that $N_y^1 f(t) \to N_x^1 f(t)$.

It follows that $\lim_{y \downarrow x} N_y^1 f(t) = N_x^1 f(t)$ for a dense set of $t < S_D^x$. Both sides are continuous in t at S_D^x and are constant on $[S_D^x, \infty)$, so they must be equal for all $t \ge S_D^x$.

Finally, note that $\Xi_y^+(S_D^y)$ is the final excursion above y (if it exists).

The same reasoning shows that

$$N_y^2 f(S_D^y) = f(\Xi_y^+(S_D^y)) \to f(\Xi_x^+(S_D^x)) = N_x^2 f(S_D^x). \qquad \text{q.e.d}$$

Remark : it is possible to topologize $\hat{W}$ in such a way as to have $x \to \Xi_x^+$ a.s. right continuous and $x \to \Xi_x^-$ a.s. left continuous. For instance, if f is a simple functional and if k is continuous and of compact support on $\mathbb{R}_+$, set

$$d_{fk}(\Xi_y^+, \Xi_x^+) = \int_0^\infty \frac{|N_y f(t) - N_x f(t)|}{1 + |N_y f(t) - N_x f(t)|} k(t)dt$$

If we choose a suitable family of f_n and k_n, with $||f_n|| \leq 1$, $||k_n|| \leq 1$, then

$$d(\xi,\eta) = \Sigma\, 2^{-n} d_{f_n k_n}(\xi,\eta)$$

is a metric on $\hat{W}$, and Proposition 3.1 implies that $x \to \Xi_x^+$ is right-continuous in this metric.

Just as important as the a.s. right continuity of Ξ_x^+ is its right continuity in distribution, which is contained in the next result.

Proposition 3.2

Let f be a simple functional. Given that $S_D^x > 0$, S_D^x is an exponential with a parameter $\lambda = \lambda(x)$, and $\{\hat{N}_x^1 f(t),\ t \geq 0\}$ is a process of independent increments, independent of S_D^x and of $\mathcal{E}_x^-$. Furthermore, if $\psi_x(u)$ is its characteristic exponent, then

(i) $x \to S_D^x$ and $x \to \lambda(x)$ are continuous ;

(ii) $x \to \psi_x(u)$ is right continuous for each $u \in \mathbb{R}$

Proof : the fact that S_D^x is exponential, given it isn't zero, is well-known [6]. (We allow $S_D^x \equiv \infty$ as a limiting case of an exponential). Note that $S_D^x = L_\zeta^x$, which is continuous in x, and the continuity of the parameter $\lambda(x)$ follows.

We have already seen that $N_x^1 f$ is a process of independent increments. It is defined from $\hat{\Xi}_x^{E_x^+-D}$, which is independent of $\hat{\mathcal{E}}_x \supset \mathcal{E}_x$ and of $\hat{\Xi}_x^D$, and therefore of S_D^x, the first jump time of Ξ_x^D.

It remains to show that its characteristic exponent is right continuous. This will follow from Proposition 3.1, which involves $N_x f$, not $\hat{N}_x f$.

$$E\{e^{iu\, N_x^1 f(t)} \mid S_D^x > 0\} = E\{E\{e^{iu\, \hat{N}_x^1 f(t \wedge S_D^x)} \mid S_D^x\} \mid S_D^x > 0\}$$
$$= E\{e^{t \wedge S_D^x\, \psi_x(u)} \mid S_D^x > 0\},$$

where we have used the independence of $\hat{N}_x^1 f$ and S_D^x. We can calculate this last quantity directly, since S_D^x, given it is strictly positive, is exponential $\lambda(x)$. It equals

$$(3.2) \qquad \frac{\lambda(x) - \psi_x(u)\, \exp[\psi_x(u) - \lambda(x)]}{\lambda(x) - \psi_x(u)}\,.$$

Choose $t > 0$ so that τ_t^x is a.s. continuous at t. Then $N_y^1 f(t) \to N_x^1 f(t)$ a.s. on $\{S_D^x > 0\}$ as $y \to x$. Since $S_D^y > 0$ for y sufficiently close to x by continuity of S_D^y,

$$E\{e^{iu\, N_y^1 f(t)}\, I_{\{S_D^x > 0\}}\} \longrightarrow E\{e^{iu\, N_x^1 f(t)}\, I_{\{S_D^x > 0\}}\}.$$

From (3.2) we get :

$$\frac{\lambda(y)}{\lambda(y)+\psi_y(u)} + O(e^{-t}) \to \frac{\lambda(x)}{\lambda(x)+\psi_x(u)} + O(e^{-t}).$$

Take t large and note that $\lambda(y) \to \lambda(x)$ as $y \to x$ to see that $\psi_y(u) \to \psi_x(u)$. q e d

The fields $\{\mathcal{E}_x,\ x \in I\}$ increase with x, so we can talk about $\mathcal{E}_x$-stopping times. If Y is an $\mathcal{E}_x$-stopping time, we define the fields $\mathcal{E}_Y$ and $\mathcal{E}_{Y+} = \bigcap_{\varepsilon > 0} \mathcal{E}_{Y+\varepsilon}$ as usual. (In fact, $\mathcal{E}_Y = \mathcal{E}_{Y+}$ but we can't prove this yet). One example of such a time is the process minimum, M : $M = \inf\{X_t,\ t < \zeta\}$. This is an $\mathcal{E}_x$-stopping time, for $M = \inf\{x : L_\zeta^x > 0\}$. It plays a distinguished role in what follows, and we shall need some of its simple properties. Let $m = \inf I$.

Proposition 3.3

If $x > m$, $P\{M=x\} = 0$, and, a.s. on $\{M > m\}$, there exists a unique t such that $X_t = M$.

Proof : if $x \notin I$ and $x > m$, x must be regular for $(-\infty, x)$, so $P^x\{M < x\} = 1$. It follows by the strong Markov property of X that, $P^a\{T_x < \infty, M=x\} = 0$. Now if $M=x$ and $T_x = \infty$, we must have $\liminf\limits_{t \uparrow \zeta} X_t = x$. This can't happen on $\{\zeta = \infty\}$, for X would then have to be recurrent, so that $M = m < x$. It also can't happen on $\{\zeta < \infty\}$, since this would require that $\zeta = \inf\{t : X_{t-} = x\}$, making it impossible for X to ever pass below x, and contradicting the regularity of x. Thus $P\{M = x\} = 0$.

If $z > m$, the fact that $X_{\zeta -} = z$ implies that $M < z$, hence that X must take on its minimum. To see it takes it on exactly once, let $M_t = \inf\limits_{s \le t} X_s$. Apply the above result to the process $\{X_{t+s}, s \ge 0\}$. Since $M_t \in \mathcal{F}_t$, $P\{\inf\limits_{s \ge 0} X_{s+t} = M_t, M_t > m\} = 0$. This being true for all rational t gives the result.

This brings us to the promised strong Markov property. In the interest of clarity, we will sacrifice some rigor and speak of conditioning on events whose probability may be zero. The sacrifice is small since one can make this kind of conditioning rigorous by means of regular conditional probabilities.

Theorem 3.4

Let Y be an $\mathcal{E}_x$-stopping time. Then, conditioned on the values of Y and S_D^Y, the process $\Xi_Y^+ = \{\Xi_Y^+(t), t \ge 0\}$ is an absorbed ppp, independent of $\mathcal{E}_{Y+}$, and has the following conditional distributions.

(i) On $\{Y > M$ or $Y=M=m\}$, given that $Y=x$ and $S_D^Y = t$, Ξ_Y^+ is a non-degenerate ppp with the same law as that of Ξ_x^+, given that $S_D^x = t$.

(ii) <u>On</u> $\{Y < M\}$, $S_D^Y=0$ <u>and, given that</u> $Y=x$, $\Xi_Y^+(0)$ <u>has the same law as</u> $\Xi_x^+(0)$, <u>given that</u> $S_D^x=0$.

(iii) <u>On</u> $\{Y=M > m\}$, $S_D^Y=0$, <u>and</u> Ξ_Y^+ <u>is degenerate, consisting of two non-trivial values,</u> $\Xi_Y^+(0)$ (<u>the initial excursion) and</u> $\Xi_Y^+(0+)$, (<u>the final excursion</u>). <u>Given that</u> $Y=x$, <u>these are independent and have the same distributions as those of</u> $\Xi_x^+(0)$ <u>and</u> $\Xi_x^+(S_D^x)$, <u>given that</u> $S_D^x > 0$.

<u>Proof</u> : if $Y \geq \sup_t X_t$, $S_D^Y=0$ and $\Xi_Y^+ \equiv \partial$, so the result is trivially true, so we may assume $Y < \sup_t X_t$. We first consider the case where $Y > M$, and consequently $S_D^Y > 0$.

Suppose Y takes on only the values $\{x_1, x_2, \ldots\}$. If A is a measurable set in $\hat{W}$, the path-space of Ξ^+, and if $\Lambda \in \mathcal{E}_Y^-$, then

$$P\{\Xi_Y^+(\cdot) \in A,\ \Lambda \mid Y = x_i,\ S_D^Y\}$$

$$= P\{\Xi_{x_i}^+ \in A,\ \Lambda \cap \{Y=x_i\} \mid Y=x_i,\ S_D^{x_i}\}\ I_{\{Y=x_i\}}$$

$$= P\{\Xi_{x_i}^+ \in A,\ \Lambda \cap \{Y=x_i\} \mid S_D^{x_i}\}\ P\{Y=x_i \mid S_D^{x_i}\}^{-1}\ I_{\{Y=x_i\}}$$

But $\Lambda \cap \{Y=x_i\} \in \mathcal{E}_{x_i}^-$, while $\Xi_{x_i}^+$ and $\mathcal{E}_{x_i}^-$ are independent given $S_D^{x_i}$, so this is

$$= P\{\Xi_{x_i}^+ \in A \mid S_D^{x_i}\}\ P\{\Lambda \mid Y=x_i,\ S_D^{x_i}\}\ I_{\{Y=x_i\}}$$

$$= P\{\Xi_Y^+ \in A \mid S_D^Y\}\ P\{\Lambda \mid S_D^Y\},$$

which proves the theorem in this case. Notice that this also covers the case where $\{Y=M=m\}$.

Now suppose $Y > M$ is an $(\mathcal{E}_{y+})$-stopping time and define

$$Y_n = \frac{k+1}{2^n} \quad \text{if} \quad k2^{-n} \leq Y < (k+1)2^{-n}, \quad -\infty < k < \infty.$$

Then $Y_n \downarrow Y$, and we have just seen that the theorem holds for each Y_n. If $f_1, \ldots, f_m$ are simple functionals, if $t_1 < t_2 < \ldots < t_n$, and if

$\lambda = (\lambda_{jk})$ are real, set

$$H(x,\lambda) = \prod_{k=1}^{m} e^{i\lambda_{ok} N_x^1 f_k(o)} \prod_{j=1}^{n-1} e^{i\lambda_{jk}(N_x^1 f_k(t_{j+1}) - N_x^1 f_k(t_j))} e^{i\lambda_{jn} N_x^2 f_k(S_D^x)}$$

We can simplifiy this expression by setting $g_j = \Sigma \lambda_{jk} f_k$, which is again a simple functional. Then

$$H(x,\lambda) = e^{iN_x^1 g_o(o)} \left(\prod_{j=1}^{n-1} e^{i(N_x^1 g_j(t_{j+1}) - N_x^1 g_j(t_j))}\right) e^{iN_x^2 g_n(S_D^x)}$$

We can compute $E\{H(x,\lambda) \mid \mathcal{E}_x\}$ explicitly from Proposition 3.2 and verify that there exists a function $K(x,t;\lambda)$ which is right continuous in x and continuous in t, such that, on $\{S_D^x > 0\}$,

$$E\{H(x,\lambda) \mid \mathcal{E}_x\} = K(x,S_D^x\,;\lambda)$$

(If $n=2$, for example

$$K(x,S_D^x,\lambda) = E\{e^{iN_x^1 g_o(0)} \mid S_D^x > 0\}\, E\{e^{iN_x^2 g_n(S_D^x)} \mid S_D^x > 0\}\, e^{t \wedge S_D^x\, \psi_x(1)},$$

where ψ_x is the characteristic exponent of $N_x^1 g_1$).

We can choose the t_j so that $t \to \tau_t^y$ is a.s. continuous at each t_j. Then $H(Y_n,\lambda) \to H(Y,\lambda)$ a.s. by Proposition 3.1. The convergence is bounded since $|H| \leq 1$, and $S_D^{Y_n} \to S_D^Y$ by continuity so

$$(3.3) \qquad K(Y,S_D^Y;\lambda) = \lim_{n\to\infty} K(Y_n, S_D^{Y_n},\lambda)$$

$$= \lim_{n\to\infty} E\{H(Y_n,\lambda) \mid \mathcal{E}_{Y_n}\}$$

$$= E\{H(Y,\lambda) \mid \mathcal{E}_{Y+}\},$$

where the last step follows by Hunt's lemma [3 p.41].

By right continuity in t, (3.3) holds in fact for any choice of the t_j. But the functionals above completely determine the conditional distribution of Ξ_Y^+, so that we have identified the distribution of Ξ_Y^+, given $Y=x$ and $S_D^Y=t$, with that of Ξ_x^+ given $S_D^x=t$. This completes the proof of (i).

Case (ii) is less interesting, so we will omit its proof, and pass on to case (iii).

Suppose, now, that $Y=M > m$. In that case, $S_D^M=0$, and the process X_t has exactly two excursions above M, the initial excursion $\xi = \Xi_M^+(0)$, which lasts until X_t reaches M for the first and only time, and the final excursion $\eta = \Xi_M^+(0+)$, which begins right after.

Let $Y_n \downarrow M$ be as above. We may assume that $S_D^{Y_n} > 0$. Let $\xi_n = \Xi_{Y_n}^+(0)$ and $\eta_n = \Xi_{Y_n}^+(S_D^{Y_n})$, and notice that $\xi_n \to \xi$ and $\eta_n \to \eta$. Thus for any simple f, $N_{Y_n}^1 f(0) = f(\xi_n) \to f(\xi) = N_Y^1 f(0)$, and $N_{Y_n}^2 f(S_D^{Y_n}) = f(\eta_n) \to f(\eta) = N_Y^2 f(\eta)$.

But now, just take $n=1$ to define $H(x,\lambda)$:

$$H(x,\lambda) = e^{iN_x^1 g_o(0)}\, e^{iN_x^2 g(S_D)},$$

and note that the function $K(x,t,\lambda) = \overline{K}(x,\lambda)$ does not depend on t (the distribution of $\Xi_x^+(S_D^x)$ does not depend on the value of S_D^x if $S_D^x > 0$).

Since $S_D^{Y_n} > 0$ we still have :

$$\overline{K}(Y_n,\lambda) = E\{H(Y_n,\lambda) \mid \mathcal{E}_{Y_n}\}.$$

It follows that

$$\overline{K}(M,Y) = E\{H(M,\lambda) \mid \mathcal{E}_{M+}\}$$

which says that, given $M=x, \Xi_M^+(0)$ and $\Xi_M^+(0+)$ have the same joint distribution as $\Xi_x^+(0)$ and $\Xi_x^+(S_D^x)$, given that $S_D^x > 0$.

q e d

The case $Y=M$ is an exceedingly interesting special case of Theorem 3.4 : it is exactly David Williams decomposition of a diffusion into a pre-minimum and a post-minimum process.

Indeed, let Q_t^x be the semigroup of the process X conditioned on $\{T_x < \zeta\}$, then killed at T_x, and let R_t^x be the semigroup of X conditioned on $\{\zeta < T_x\}$. These are well-known: if $f(y) = P^y\{T_x < \zeta\}$, $g(y)=1-f(y)$, and if $(\overline{P}_t)$ is the semigroup of X killed at time T_x, then

$$Q_t^x(y,dz) = f(y)^{-1}\, \overline{P}_t(y,dz)\, f(z)$$

and

$$R_t^x(y,dz) = g(y)^{-1}\,\overline{P}_t(y,dz)\,g(z)$$

Theorem 3.5 (Williams)

Let $U = \inf\{t : X_t = M\}$. Then, given that $M=x > m$, the process $\{X_t, t < U\}$ and the process $\{X_{U+t}, t \geq 0\}$ are independent diffusions with semigroups (Q_t^x) and (R_t^x) respectively.

Proof : the given processes are just $\Xi_M^+(0)$ and $\Xi_M^+(0+)$ respectively. By Theorem 3.4 (iii) given $M=x > m$, they are independent, with the same distributions as $\Xi_x^+(0)$ and $\Xi_x^+(S_D)$, given S_D^x.

To finish the proof, we need only identify the distribution of these excursions. But the initial excursion above x, given that it ever does reach x, is easily seen to be a diffusion with semigroup (Q_t^x), while the final excursion starts at the last exit time from x. By Theorem 5.1 of [7], this is a diffusion with semigroup (R_t^x).

4 - THE EXCURSION FIELDS

It is interesting to compare the family $\{\mathcal{E}_x, x \in I\}$ of excursion fields with the usual fields $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$ generated by the diffusion. In some senses, the excursion fields are richer, but they share many properties.

Proposition 4.1

The fields $\{\mathcal{E}_x, x \in I\}$ are continuous in x.

Proof : $\mathcal{E}_x \supset \mathcal{E}_{x-\frac{1}{n}}$. But $\mathcal{E}_x$ is generated by Ξ_x^-, and $\Xi_x^- = \lim \Xi_{x-1/n}$ (in the sense of section 3) so $\mathcal{E}_x \subset \bigvee_n \mathcal{E}_{x-\frac{1}{n}}$. To prove right continuity, notice that $\mathcal{E}_x^+$ and $\mathcal{E}_{x+} = \bigcap_n \mathcal{E}_{x+1/n}$ are conditionally independent given S_D^x, which follows from Theorem 3.4 with $Y \equiv x$. Let $\mathcal{G}$ be the class of sets Λ for which

$$P\{\Lambda \mid \mathcal{E}_x\} = P\{\Lambda \mid \mathcal{E}_{x+}\}.$$

If $\Lambda = \Lambda_1 \cap \Lambda_2$, where $\Lambda_1 \in \mathcal{E}_x$, $\Lambda_2 \in \mathcal{E}_x^+$, then

$$P\{\Lambda \mid \mathcal{E}_{x+}\} = I_{\Lambda_1} P\{\Lambda_2 | \mathcal{E}_{x+}\} = I_{\Lambda_1} P\{\Lambda_2 | S_D^x\}$$

$$= I_{\Lambda_1} P\{\Lambda_2 | \mathcal{E}_x\} = P\{\Lambda | \mathcal{E}_x\}.$$

Thus $\Lambda \in \mathcal{G}$. $\mathcal{G}$ is clearly a monotone class, hence $\mathcal{G}$ contains $\mathcal{E}_x \vee \mathcal{E}_x^+$, and in particular, $\mathcal{E}_{x+} \subset \mathcal{G}$.

q e d

Remark : let $\mathcal{U}_{xy}$ be the upcrossing field defined in (10) , i.e. the field generated by all the upcrossings of (x,y), and let $\mathcal{U}_y = \bigvee_{x : x < y} \mathcal{U}_{xy}$.

Then $\mathcal{E}_y = \mathcal{U}_y = \bigcap_{w : w > y} \mathcal{U}_{yw}$.

To see this, recall from (10) that if $x < x' < y < y'$ that $\mathcal{U}_{xy} \subset \mathcal{U}_{x'y} \subset \mathcal{U}_{yw}$. Furthermore, it is clear that $\mathcal{E}_x \subset \mathcal{U}_{x'y} \subset \mathcal{E}_w$. Thus $\mathcal{E}_x \subset \bigvee_{x' : x' < y} \mathcal{U}_{x'y} = \mathcal{U}_y \subset \bigcap_{w' > y} \mathcal{U}_{yw'} \subset \mathcal{E}_w$. Now just let $x \uparrow y$ and $w \downarrow y$ and note that the $\mathcal{E}_x$ are continuous at y.

Davis and Varaiya (2) have defined the notion of the dimension of a family of σ-fields. This is, very roughly, the number of orthogonal martingales supported by the σ-fields. If X is a Brownian motion, for instance, the fields $(\mathcal{F}_t)$ are one-dimensional. This is a consequence of Ito's theorem which says that any $\mathcal{F}_t$-martingale can be written as the sum of a constant plus a stochastic integral with respect to X. However, we have the following, which shows that the excursion fields are richer that $(\mathcal{F}_t)$.

Proposition 4.2

The dimension of the fields $\{\mathcal{E}_x, x \in I\}$ is infinite.

Proof : let a be the initial value of X. Note that there are an infinite number of excursions below a, so that $S_D^a > 0$. a.s. Let $T_1 < T_2 < \ldots < \zeta$ be $\mathcal{E}_a$-identifiable such that $X_{T_j} = a$ and $L^a_{T_j} < L^a_{T_{j+1}}$. We could, for instance, choose $T_n = \inf\{t : L^a_t \geq (1 - \frac{1}{n}) L^a_\zeta\}$. Let $u(x)$ and $\sigma(x)$ be the functions of

Theorem 2.6, and put

$$M_n(x) = u(x)(L^x_{T_{j+1}} - L^x_{T_j}),\ x \geq a.$$

By Theorem 2.4, $\{M_n(x),\ x \geq a\}$, n=1,2,... are martingales which are independent given $\mathcal{E}_a$, and hence are orthogonal. Their increasing processes are

$$(4.1) \qquad \langle M_n \rangle_x = \int_a^x M_n(v)\ d\sigma(v).$$

We extend the M_n to all of I by putting

$$M_n(x) = E\{M_n(a) \mid \mathcal{E}_x\} \quad \text{if} \quad x < a.$$

By (4.1), for any m,n, the measures $d\langle M_n\rangle_x$ and $d\langle M_m\rangle_x$ are equivalent on $a \leq t < \tau$ where $\tau = \inf\{x > a : M_n(x) = 0 \text{ or } M_m(x) = 0\}$. It follows that for any n, there is some (random) interval J such that

$d\langle M_1\rangle_x \sim d\langle M_2\rangle_x \sim \cdots \sim d\langle M_n\rangle_x$ for $x \in J$, hence the dimension is at least n. Since n is arbitrary, we are done.

In general there will be at least one totally inaccessible $\mathcal{E}_x$-stopping time. Set

$$Y = \begin{cases} M & \text{if } M > m \\ \infty & \text{otherwise,} \end{cases}$$

where $M = \inf\{X_t : t < \zeta\}$ and $m = \inf I$. Y is clearly totally inaccessible unless it is identically infinite, for Y has a diffuse distribution by Proposition 3.3, and $\{M > x\}$ is an atom of $\mathcal{E}_x$. We conjecture that Y is the <u>only</u> totally inaccessible time. In fact, we conjecture that if Z is an $\mathcal{E}_x$-stopping time and $P\{Z=Y < \infty\} = 0$, that Z is predictable.

One might think that something like $M_t = \inf\{X_s : s \leq t\}$ would provide another totally inaccessible time, but the following shows that that it doesn't.

<u>Proposition 4.3</u>

<u>Let</u> Z <u>be an</u> $\mathcal{E}_x$<u>-stopping time such that</u> $P\{Z=M\} = 0$. <u>Then</u>

$P\{\exists\, t : X$ <u>has a local minimum at</u> t <u>and</u> $X_t = Z\} = 0$.

Proof : if X had a local minimum at t, and $X_t = z$, then t would be in the interior of some excursion interval (τ^z_{s-}, τ^z_s) (for t can't be a point of increase of L^z), and this interval would contain at least two excursions, one starting and the other ending at t. This would mean that the excursion process Ξ^+_z is degenerate, which contradicts Theorem 3.4.

q e d

There is a line of attack on the conjecture which may be worth mentioning. To prove that all stopping times other than M are predictable, it would be enough to show that all $\mathcal{E}_x$-martingales are continuous except possibly at M. In case X is Brownian motion, the corresponding result for the fields $\mathcal{F}_t$ follows from Ito's representation theorem : each $\mathcal{F}_t$ martingale is a constant plus a stochastic integral, and therefore continuous. Is there some type of representation theorem here ? It is not out of the question. We have all the martingales of the form $\{u(x)(L^x_T - L^x_S),\ x \geq x_o\}$ to integrate with. Perhaps there is some integral representation which involves the local times explicitly.

APPENDIX : A STABILITY PROPERTY

We remarked at the end of section two that the infinitesimal generator of the local time is determined by its stability properties.

This property is doubless known in much greater generality than we will give here, but since we don't know the reference, and since it is relevant and not difficult, we will prove the two results here that we referred to.

Proposition A1

Let $\mathbb{G}$ be the infinitesimal generator of a regular diffusion on natural scale on $[0,\infty)$, absorbed at zero. Suppose that whenever X and Y are independent $\mathbb{G}$-diffusions then $X+Y$ is also a $\mathbb{G}$-diffusion. Then there is $c > 0$ such that

$$\mathbb{G} = \frac{1}{2} cx \frac{d^2}{dx^2}$$

Proposition A2

Let $\mathbb{G}$ be as above and let $\mathbb{H}$ be the infinitesimal generator of a diffusion on $[0,\infty)$. Suppose that whenever X is a $\mathbb{G}$-diffusion and Y is an independent $\mathbb{H}$-diffusion, that $X+Y$ is a diffusion. Then $X+Y$ is also an $\mathbb{H}$ diffusion, and there is a $b \geq 0$ such that

$$\mathbb{H} = \frac{1}{2} cx \frac{d^2}{dx^2} + b \frac{d}{dx}$$

Lemma : let A_t be an adapted continuous process of locally bounded variation, which is locally of the class (D). Suppose that there exist $M,N \leq \infty$ such that for any stopping time T

$$M \leq \limsup_{t \to 0} \frac{1}{t} E\{A_{T+t} - A_T \mid \mathcal{F}_T\} \leq N.$$

Then $\{A_t - Mt\}$ and $\{Nt - A_t\}$ are increasing processes.

Proof : we claim the two processes are sub-martingales. The lemma follows from this, for each process will then be the sum of an increasing process and a continuous martingale. But the processes are both of finite variation, hence the martingales must be constant, and the processes themselves increasing.

Let $s \leq t$ and define stopping times $T_o \leq T_1 \leq \ldots$ by induction as follows : $T_o=0$, T_{n+1} is $\mathcal{F}_{T_n}$-measurable with the property that $T_{n+1} = T_n$ if $T_n=t$, $T_{n+1} > T_n$ otherwise, $T_{n+1} \leq t$, and

$$(M-\varepsilon)(T_{n+1}-T_n) \leq E\{A_{T_{n+1}}-A_{T_n} \mid \mathcal{F}_{T_n}\} \leq (N+\varepsilon)(T_{n+1}-T_n).$$

If $\lim T_n < t$, define $T_\omega = \lim T_n$, and define $T_{\omega+1}, T_{\omega+2},\ldots$ and so on, thru the countable ordinals. There exists a countable ordinal β such that $T_\beta = t$ a.s.

Now $A_t-A_s = \sum_{\alpha<\beta} A_{T_{\alpha+1}}-A_{T_\alpha}$ so

$$E\{A_t-A_s \mid \mathcal{F}_s\} = E\{\sum_{\alpha<\beta} E\{A_{T_{\alpha+1}}-A_{T_\alpha} \mid \mathcal{F}_{T_\alpha}\} \mid \mathcal{F}_s\}$$

$$\leq (N+\varepsilon)(t-s)$$

Thus $\{(N+\varepsilon)t-A_t\}$ is a sub martingale for all $\varepsilon > 0$, hence $\{Nt-A_t\}$ is a sub martingale. The fact that A_t-Mt is a submartingale follows in the same way. qed

Now let X be a G-diffusion. If X is on natural scale it is a local martingale. Let $\langle X\rangle_t$ be its associated increasing process. If $a < x < b$, let $\tau = \inf\{t : X_t=a \text{ or } b\}$ and define

$$h(x) = \limsup_{t \to 0} \frac{1}{t} E^x\{\langle X\rangle_{t\wedge\tau}\}$$

$$g(x) = \liminf_{t \to 0} \frac{1}{t} E^x\{\langle X\rangle_{t\wedge\tau}\}$$

Now $E^x\{\langle X\rangle_{t\wedge\tau}\} \leq E^x\{\langle X\rangle_\tau\} = E^x\{X_\tau^2-x^2\} \leq (b-a)^2$, so $\langle X\rangle_{t\wedge\tau}$ is integrable. Moreover, recall that

$$\lim_{t \to 0} \frac{1}{t} P^x\{|X_t-x| > \varepsilon\} = 0,$$

so that the definitions of h and g are independent of a and b.

Lemma 2

Let G be as in Theorem A1. Then h is increasing and sub-additive while g is increasing and superadditive. Both are finite everywhere.

Proof : let X and Y be independent G-diffusions with initial values x and y respectively. Then X and Y are orthogonal martingales, so $\langle X+Y\rangle = \langle X\rangle + \langle Y\rangle$. Let $r > x+y$ and put $\tau = T_r$. Then

$$h(x) \le h(x+y) = \limsup \frac{1}{t} E^{x,y}\{\langle X+Y\rangle_t\}$$

$$= \limsup \frac{1}{t} E^{x,y}\{\langle X\rangle_t + \langle Y\rangle_t\}$$

$$\le \limsup \frac{1}{t} E^{x}\{\langle X\rangle_t\} + \limsup \frac{1}{t} E^{y}\{\langle Y\rangle_t\}$$

$$= h(x) + h(y)$$

Thus h is increasing and sub-additive, and will be finite everywhere if $h(x) < \infty$ for any $x > 0$. But apply Lemma 1 to $A_t = \langle X\rangle_{t\wedge\tau}$. If $h(x) \equiv \infty$, we could take M as large as we wished, which would imply that $A_t \equiv \infty$, a contradiction. The proof for g is similar.

q e d

We are ready to prove Proposition A1. Let $a < x < b$, let τ be the first exit from $[a,b]$, and apply Lemma 1 to $A_t = \langle X\rangle_{t\wedge\tau}$. Since $a \le y \le b \Rightarrow h(a) \le h(y) \le h(b)$, lemma 1 implies that $h(b)t - \langle X\rangle_{t\wedge\tau}$ and $\langle X\rangle_{t\wedge\tau} - h(a)t$ are increasing processes. In other words, $h(a)dt \le d\langle X\rangle_{t\wedge\tau} \le h(b)dt$. Then $g(x) = \liminf \frac{1}{t} E^{x}\{\langle X\rangle_{t\wedge\tau}\} \ge h(a)$, so that $h(a) \le g(x) \le h(x)$. Let $a \uparrow x$ to see that

$$h(x-) \le g(x) \le h(x)$$

Thus, if h is continuous at x, y, and $x+y$, lemma 2 implies

$$h(x+y) \le h(x)+h(y) = g(x)+g(y) \le g(x+y) = h(x+y)$$

In other words, h is additive, so $h(x) = cx$ for some $c > 0$. It follows easily that $d\langle X\rangle_t = c\, X_t\, dt$.

To calculate G, let $f \in C^{(2)} \cap D(G)$ and use Ito's formula :

$$E^x\{f(X_t)-f(x)\} = E^x\{\int_0^t f'(X_s)dX_s\} + \frac{1}{2}E^x\{\int_0^t f''(X_s)d\langle X\rangle_s\}$$

$$= \frac{c}{2}E^x\{\int_0^t X_s f''(X_s)ds\}.$$

Thus

$$\mathbf{G}f(x) = \lim_{t \to 0} \frac{c}{2t} E^x\{\int_0^t X_s f''(X_s)ds\} = \frac{c}{2} xf''(x),$$

which proves Proposition A1.

We can prove Proposition A2 by similar methods. Let X be a $\mathbf{G}$-diffusion and Y an independent $\mathbf{H}$-diffusion. Y and $X+Y$, being diffusions, are semi-martingales and can be written uniquely in the form of a martingale plus a process of locally bounded variation with initial value zero.

Let $r, y \geq 0$, let $r > x+y$, and put $\tau = T_r$. Define

$$m(y) = \limsup_{t \to 0} \frac{1}{t} E^y\{Y_{t\wedge\tau} - y\}$$

$$n(x+y) = \limsup_{t \to 0} \frac{1}{t} E^{x,y}\{(X+Y)_{t\wedge\tau} - x - y\}$$

$$= \limsup_{t \to 0} \frac{1}{t} E^{x,y}\{Y_{t\wedge\tau} - y\} = m(y)$$

where we have used the fact that the above definitions are independent of r. Setting x and y alternately equal to zero, we see $n(x) = m(0) = m(y)$. So m is constant, say $m \equiv b$.

Now Y, being a diffusion can be written in the form $Y = M_t + V_t$, where M_t is a martingale, V_t a process of locally bounded variation. The reader can check that V_t satisfies the hypotheses of Lemma 1. From the lemma we conclude first that b is finite, then that $V_t = bt$.

Next, define $\langle Y \rangle_t = \langle M \rangle_t$. As X and M are orthogonal, $\langle X+Y \rangle = \langle X \rangle + \langle Y \rangle$. If $x \geq 0$, set

$$j(y) = \limsup_{t \to 0} \frac{1}{t} E^y\{\langle Y \rangle_{t\wedge\tau}\}$$

$$k(x+y) = \limsup_{t \to 0} \frac{1}{t} E^{x,y}\{\langle X+Y \rangle_{t\wedge\tau}\}$$

$$= \lim_{t \to 0} \frac{1}{t} E^x\{\langle X \rangle_{t\wedge\tau}\} + \limsup_{t \to 0} \frac{1}{t} E^y\{\langle Y \rangle_{t\wedge\tau}\}$$

$$= \frac{cx}{2} + j(y)$$

As in the previous proof, Lemma 1 shows j and k can't be infinite everywhere, so they are everywhere finite, and if $a = j(0)$,

$$j(x) = \frac{c}{2}x+a = k(x),$$

and that $d\langle Y\rangle_t = (\frac{c}{2}Y+a)dt$, $d\langle X+Y\rangle = (\frac{c}{2}(X+Y)+a)dt$. We can calculate the infinitesimal generator as before. Both Y and $X+Y$ have the generator

$$\mathbb{H} = (\frac{c}{2}x+a)\frac{d^2}{dx^2} + b\frac{d}{dx}.$$

But Y, being positive, can't have a negative drift at zero, so $b=m(0) \geq 0$. We claim that $a=0$. Suppose not. The origin is either absorbing, reflecting, or sticky for Y. It can't be reflecting, for if it were, we would have $m(0) = \infty$. It can't be absorbing, for then we would have $d\langle X+Y\rangle = (\frac{c}{2}(X+Y) + a\, I_{\{Y>0\}})dt$, which is not a function of $X+Y$. Similarly, if it were sticky, $d\langle X+Y\rangle$ would depend on whether or not $Y=0$. Thus a must equal zero, and we are done.

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