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R. V. PLYKIN
D. A. KAMAEV
A. JU. ZIROV

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ONE-DIMENSIONAL ATTRACTORS OF A-DIFFEOMORPHISMS ON S^2 AND DIFFEMORPHISMS WITH INFINITELY MANY SINKS *

R.V. Plykin, D.A. Kamaev, A.Ju. Zirov

The first part of the present paper gives an account of series of structurally stable diffeomorphisms on S^2 having the spectral decomposition which consists of the connected one-dimensional attractor and four repulsive periodic points. Beginning with the example of such diffeomorphisms given in [10] we shall give a sequence of its modifications. With each diffeomorphism of that series we shall connect a geometric intersection matrix which enables us to calculate a topological entropy. Considering the sequence of values of entropy we shall prove that diffeomorphisms of given series are representatives of a countable set of class of topological conjugation which is not the result of any iteratonsof any finite series of diffeomorphisms (theorem of part 1).

In the second part of the paper the construction of one-dimensional attractor on S^2 is used for investigation of the question about C^1-typicalness of diffeomorphisms with

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infinitely many sources (sinks) on smooth compact manifold of dimension greater than two. The suggested theorem includes a specification of the result of S. Newhouse in dimension greater than two.

1. Diffeomorphisms with One-Dimensional Attractors

All the constructions will be carried out on the plane R^2 that will be turned afterwards into 2-sphere by adding a point at infinity. Let us fix some Cartesian co-ordinate system (x, y) and introduce the following designations. For any segment A of axis Ox $\bar{A}$ and $\overset{+}{A}$ are its left and right end points accordingly. For an integer i $\sigma(i)$ is the sign "+" if i is even and "-" if i is odd. Let L be a fixed smooth curve on the plane with the following properties: L is convex, symmetrically with respect to axis Oy, close enough to semicircle $x^2+y^2=1, y\geqslant 0$ and it has (-1, 0), (1,0) as the end points, moreover its tangents in these points are vertical. For any segment A of the axis Ox let $L^+(A)$ be the curve, obtained from L by similarity transformation centred at origin followed by displacement along axis Ox. Moreover, the end points of $L^+(A)$ are just the same as those of A. The curve $L^-(A)$ is obtained from $L^+(A)$ by reflection in an axis Ox. We say that curves $L^+(A)$ and $L^-(A)$ are based on segment A. For nonintersecting closed segments A, B on Ox (let $\overset{+}{A} < \bar{B}$ for definiteness) we define $\Pi^\sigma(A,B)$ $(\sigma = +, -)$ the closed region bounded by segments A, B and curves $L^\sigma(\bar{A}, \overset{+}{B})$, $L^\sigma(\overset{+}{A}, \bar{B})$ and let be $\Pi^\sigma(A,B) = \Pi^\sigma(B,A)$

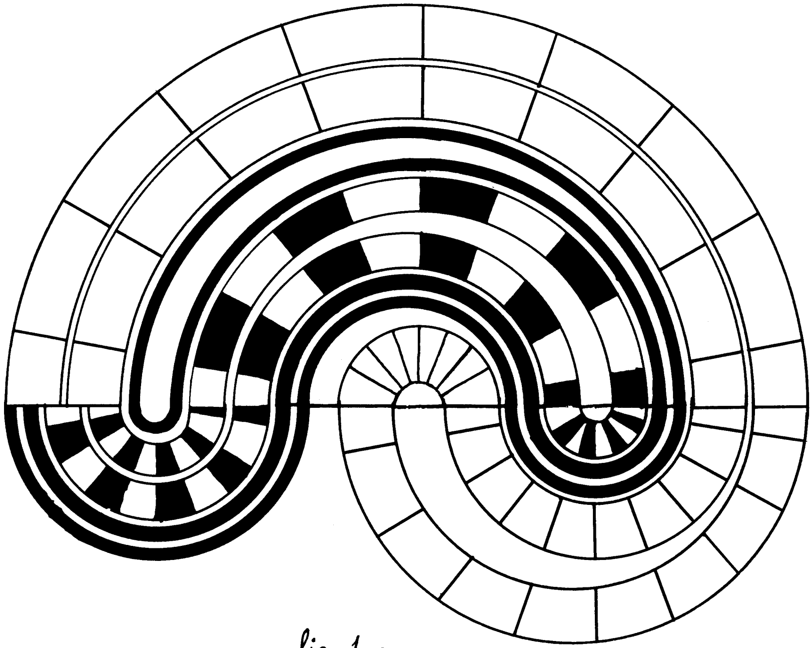


fig 1 a

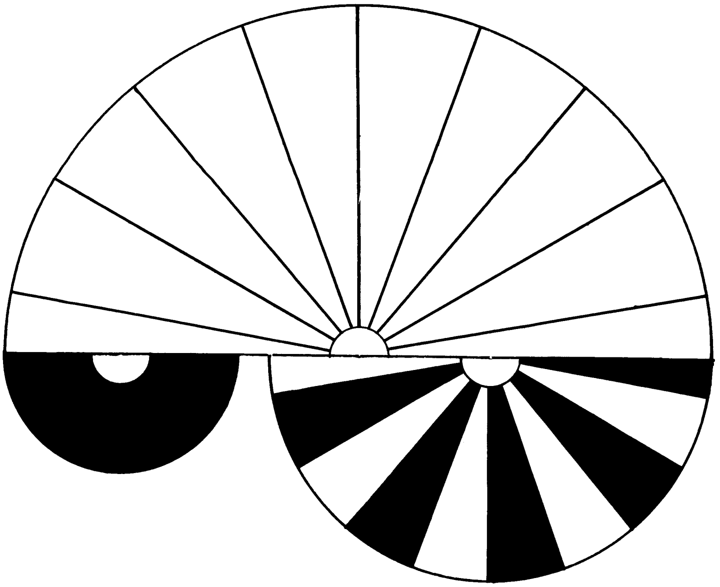


fig 1 B

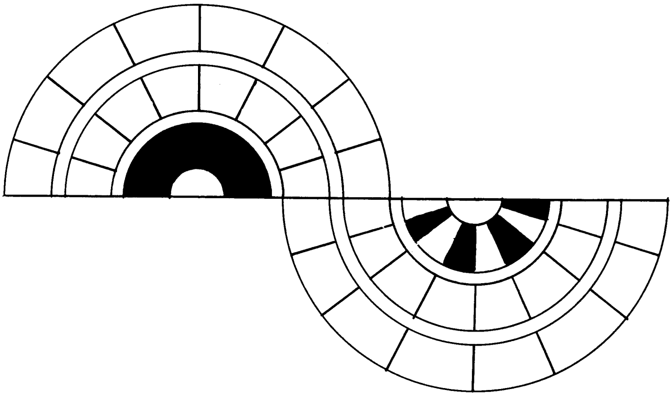


fig 2a

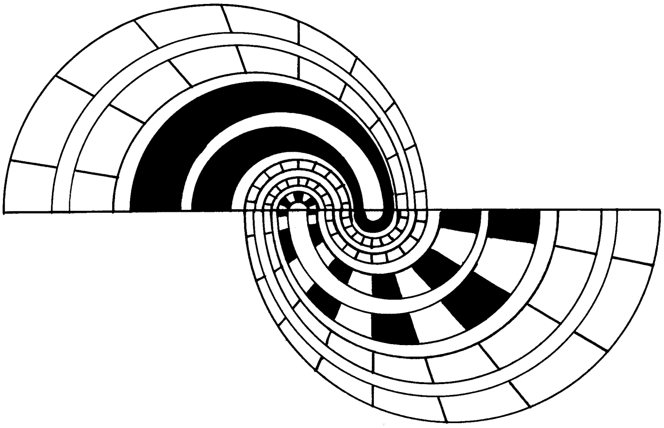


fig 2b

Now we are prepared to describe our construction. Let us lay on axis Ox nonintersecting segments

$$A_1^2,\ A_6^2,\ A_5^1,\ A_6^1,\ A_5^3,\ A_2^3,\ A_9^1,\ A_2^1,\ A_3^2,\ A_4^2,$$
$$A_3^1,\ A_8^1,\ A_3^3,\ A_4^3,\ A_7^1,\ A_4^1,\ A_5^2,\ A_2^2 \qquad (1)$$

ordered by growth of their x-co-ordinates. In addition to that lay segments $A_1^1,\ A_{10}^1,\ A_1^3,\ A_6^3$ that are nonintersecting and ordered by growth of x-co-ordinates too. Require

$$0=\bar{A}_1^2=\bar{A}_1^1<\overset{+}{A}{}_{10}^1<\bar{A}_1^3<\overset{+}{A}{}_1^2,$$
$$\bar{A}_6^2<\overset{+}{A}{}_1^3<\bar{A}_6^3<\overset{+}{A}{}_6^3=\overset{+}{A}{}_6^2,\quad \overset{+}{A}{}_2^2=1$$

Let regions Π_{ip} $(i=1,2,3;\ 1\le p\le S_i$ where $S_1 = S$, $S_2 = S_3 = 5)$ be defined by

$$\Pi_{ip}=\Pi^{\sigma(i+p-1)}\qquad(A_p^i,\ A_{p+1}^i)$$

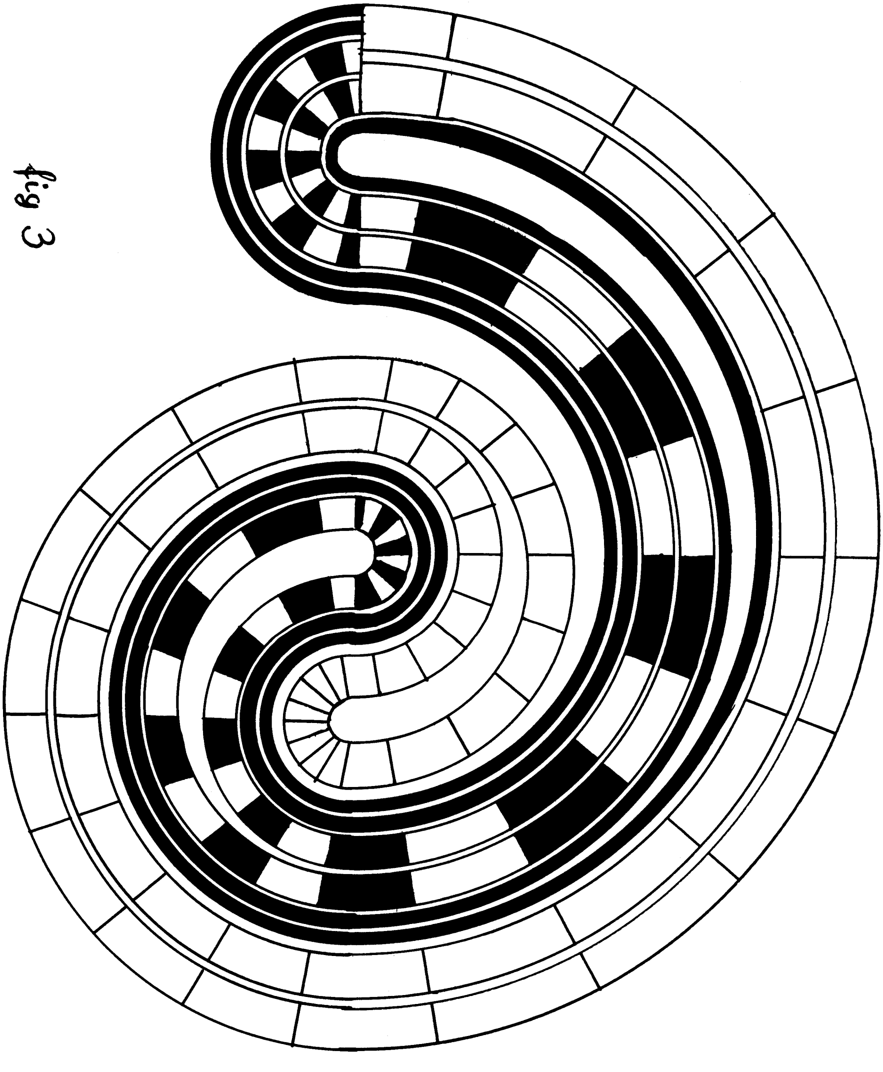
and Π_i' $(i=1,2,3)$, Π' by

$$\Pi_i'=\bigcup_{p=1}^{S_i}\Pi_{ip},\qquad \Pi'=\bigcup_{i=1}^{3}\Pi_i'$$

(see region Π' on the fig. 1a). Subsequently Π' will perform the function of image of region Φ' (see fig. 1b) under diffeomorphism f_1 that will be defined among others below.

Now we proceed to describe some construction that will be used to modify region Π' into regions Π^n for any natural n. Then we shall define regions Φ^n and diffeomorphisms $f_n:\Phi^n\to\Pi^n$ for any n. The construction is inductive and looks "on the figure" as follows. Let us consider a part of Π' lying in the region Σ' bounded in bold outline on fig. 1a. The set $Q^1 = \Pi'\cap\Sigma'$ is drawn on fig. 2a and we shall transform it into the region R^1 showed on fig. 2b, so that $R^2\cap\partial\Sigma^1=\Pi'\cap\partial\Sigma'$

fig 3



Let $\Pi^2 = (\Pi' \setminus \Sigma') \cup R^2$ (see fig. 3). A part Q^2 of Π^2 lying in the region Σ^2 bounded in bold outline on fig. 2b is exactly the same as Q^1 on fig. 2a. Consequently we may subject Q^2 to the same transformation to produce region $\Pi^3 = (\Pi^2 - \Sigma^2) \cup R^3$ and so on.

To be more formal let us consider closed regions Q_1^n, Q_2^n, Q_3^n, Q_4^n defined below by the sequence

$$a_1^n,\ a_2^n,\ a_3^n,\ a_4^n,\ a_5^n,\ b_5^n,\ b_4^n,\ b_3^n,\ b_2^n,\ b_1^n$$

of nonintersecting segments on Ox, ordered by growth of x-co-ordinate.

$$Q_1^n = \Pi^+(a_3^n, a_4^n),\quad Q_2^n = \Pi^+(a_1^n, b_5^n) \cup \Pi^-(b_5^n, b_2^n)$$
$$Q_3^n = \Pi^+(a_2^n, a_5^n) \cup \Pi^-(a_5^n, b_1^n),\quad Q_4^n = \Pi^-(b_3^n, b_4^n) \tag{2}$$

The closed region Σ^n bounded by curves $L^+(\bar{a}_1^n, \overset{+}{b}{}_5^n)$, $L^-(\bar{a}_5^n, \overset{+}{b}{}_5^n)$ and segments $[\bar{a}_1^n, \bar{a}_5^n]$, $[\overset{+}{b}{}_5^n, \overset{+}{b}{}_1^n]$ contains the set $Q^n = \bigcup_{i=1}^{4} Q_i^n$ (see fig. 2a). Let us lay on $[\bar{a}_5^n, \overset{+}{b}{}_5^n]$ nonintersecting segments $a_1^{n+1}, a_2^{n+1}, a_3^{n+1}, a_4^{n+1}, a_5^{n+1}$ $b_5^{n+1}, b_4^{n+1}, b_3^{n+1}, b_2^{n+1}, b_1^{n+1}$ so that $\bar{a}_1^{n+1} = \bar{a}_5^n$, $\overset{+}{b}{}_1^{n+1} = \overset{+}{b}{}_5^n$

and let

$$R_1^{n+1} = \Pi^-(b_4^n, a_4^{n+1}) \cup Q_1^{n+1} \cup \Pi^-(a_3^{n+1}, b_3^n),$$

$$R_2^{n+1} = \Pi^+(a_2^n, b_2^{n+1}) \cup Q_2^{n+1} \cup \Pi^-(a_1^{n+1}, b_1^n),$$

$$R_3^{n+1} = \Pi^+(a_1^n, b_1^{n+1}) \cup Q_3^{n+1} \cup \Pi^-(a_2^{n+1}, b_2^n),$$

$$R_4^{n+1} = \Pi^+(a_3^n, b_3^{n+1}) \cup Q_4^{n+1} \cup \Pi^+(a_4^n, b_4^{n+1})$$

(see fig. 2b), where Q_i^{n+1} are defined by (2) with n + 1 instead of n. For $R^{n+1} = \bigcup_{i=1}^{4} R_i^n$ we have $Q^n \cap \partial\Sigma^n = R^{n+1} \cap \partial\Sigma^n$ Thus, given set $Q^1 = \bigcup_{i=1}^{4} Q_i^1$ as in (2) an induction defines sets Q^n, Σ^n $(n \geqslant 1)$ and R^n $(n \geqslant 2)$, moreover, $Q_i^n = R_i^n \cap \Sigma^n$ $(n \geqslant 2)$. Now let the sequence $a_1', a_2', a_3', a_4', a_5', b_5', b_4', \dots, b_1'$ be a part of (1) from A_9' to A_4' inclusive. Then $Q_1' = \Pi_{23}$, $Q_2' = \Pi_{18} \cup \Pi_{17}$, $Q_3' = \Pi_{12} \cup \Pi_{13}$, $Q_4' = \Pi_{33}$ and boundary of Σ^1 is the union of $L^+(\bar{A}_9', \overset{+}{A}_8'), L^-(\bar{A}_3', \overset{+}{A}_4'), [\bar{A}_5', \bar{A}_3'], [\overset{+}{A}_8', \overset{+}{A}_4']$ (a bond outline on fig. 1a). We construct sets Q_i^n, Σ^n and R^n $(n \geqslant 2)$ and define $\Pi_1^{n+1} = (\Pi_1^n \setminus \Sigma^n) \cup R_1^n \cup R_3^n$, $\Pi_2^{n+1} = (\Pi_2^n \setminus \Sigma^n) \cup R_1^n$, $\Pi_3^{n+1} = (\Pi_3^n \setminus \Sigma^n) \cup R_4^n$ by induction, then $\Pi^{n+1} = \bigcup_{i=1}^{3} \Pi_i^{n+1} = (\Pi^n \setminus \Sigma^n) \cup R^n$.

Now let us define sets $\Phi^{(n)} = \bigcup_{i=1}^{3} \Phi_i^n$ and diffeomorphisms $f_n : \Phi^n \to \Pi^n$ $(n \geqslant 1)$. Let $\ell_1 = L^-(0, \overset{+}{A}_2')$, $\ell_2 = L^+(0,1)$, $\ell_3 = L^-(\bar{A}_3^2, 1)$, $\ell_1^n = L^-(\overset{+}{A}_5', \bar{A}_6')$, $\ell_2^n = L^+(\overset{+}{a}_3^n, \bar{a}_4^n)$, $\ell_3^n = L^-(\overset{+}{b}_4^n, \bar{b}_3^n)$ and Φ_i^n be a closed region bounded by a pair of x-segments and curves l_i, l_i^n. It is easy to see that $\Phi^n = \bigcup_{i=1}^{3} \Phi_i^n$ contains Π^n.

Before constructing the diffeomorphisms $f_n : \Phi^n \to \Pi^n$ let us define some partition Γ of Φ^n consisting of line segments. Let the line l_i^n be based on the segment Δ_i^n and choose some point M_i^n in interior of Δ_i^n. For any interior point h of l_i an intersection of a line passing through points M_i^n and h with Φ_i^n If h is an end point of some l_i then let $\gamma_h = [0, 1]$. Thus $\Gamma = \{ \gamma_h \mid h \in \bigcup_{i=1}^{3} \ell_i \}$ is a partition of Φ^n.

Now define $f_n : \Phi^n \to \Pi^n$ so that $f_n(\Phi_i^n) = \Pi_i$

and for any $\gamma_h \in \Gamma$ $f_n(\gamma_h) = \gamma_{h'}$ for some $\gamma_{h'} \in \Gamma$ First define f_n on $\bigcup_{i=1}^{3} l_i$ The boundary of each Π_i^n consists of the pair of line segments and pair of curves m_i and m_i' We can choose designations of them so that $\bigcup_{i=1}^{3} m_i$ is connected and contains the point (0, 0). Let $f|\bigcup_{i=1}^{3} l_i$ be an orientation preserving diffeomorphism uniformly expanding each l_i and mapping it onto m_i with fixed point (0, 0). Define f_n for interior points of Φ^n Let f_n map any γ_h , $h \in \bigcup_{i=1}^{3} l_i$ contracting it onto the connected component of $\gamma_{f_n(h)} \cap \Pi^n$ contains $f_n(h)$

Note that if we make our construction more accurately as in [10] assigning the length to all segments and curves and defining contraction and expantion coefficients we can obtain that the product of those in (0, 0) is greater than 1. We shall need that in the second part of the paper.

Any f may be extended using lemma 1 of part 2 to be diffeomorphism of S^2. We can produce the extention so that $f_n : S^2 \to S^2$ will have exactly four repulsive periodic points with one in each connected component of $S^2 \setminus \Phi^n$ and won't have nonwandering points in $S^2 \setminus \Phi^n$ any more. All these repulsive points are fixed if n is odd. If n is even then two of these are fixed while two others form a periodic orbit. These points are in regions bounded by l_2^n, l_3^n and respective segments of axis Ox. One can prove as in [10] the hyperbolicity of the invariant set $\Lambda_n = \bigcap_{k \geq 0} f_n^k(\Phi^n)$ using criterion of hyperbolicity by M. Hirsch and C. Pugh [4]. Robbins theorem implies structural stability of f_n. One-dimentional compact Λ_n is common frontier of four domains thus being well known Wada's continuum.

Define nonnegative integer 3×3 -matrix $G(f_n) = G_n$ which will be called (following M. Shub and D. Sullivan [5]) geometric intersection matrix of f_n in respect to partition $(\Phi_1^n, \Phi_2^n, \Phi_3^n)$ of Φ^n Element g_{ij}^n of G_n is equal to the number of connected components of $f_n(\Phi_j^n) \cap \Phi_i^n$ that is

$$g_{ij}^n = \operatorname{card} \pi_0 (\Pi_j^n \cap \Phi_i^n)$$

since $f_n(\Phi_j^n) = \Pi_j^n$ This permits to find G_n. It is easy to see that

$$G_1 = \begin{pmatrix} 3 & 0 & 2 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

(see fig. 1a) and

$$G_{n+1} = G_n + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

(see fig. 2a and 2b) implies

$$G_n = \begin{pmatrix} 3 & 0 & 2 \\ 2n+2 & n+2 & n+1 \\ 2n & n+1 & n \end{pmatrix}$$

Matrix G_n has somewhat another geometric meaning if is considered as a solenoid that is to say as an inverse limit of branched manifold and its expanding map (see R. Williams [7]). This construction in our case is obtained as follows. Let us consider an equivalence relation $\sim$ on Φ^n which is generated by partition Γ of Φ^n and a factor space K = $\Phi^n/\sim$ It is easy to see that K is homeomorphic to bunch of cicles $K = \bigvee_{i=1}^{3} S_i^1$ and we enumerate S_i^1 so that $\pi(\Phi_i^n) = S_i^1$ where $\pi : \Phi^n \to K$ is a projection. Since Γ is invariant under f a map $\varphi_n = \pi \circ f_n \circ \pi^{-1} : K \to K$ is well defined. R. Williams' axioms [7] are valid for φ_n in particular φ_n is local homeomorphism in a neighbourhood of any point x $\in$ K except branch point x'. Thus we may introduce 3×3-matrix G_n setting g_{ij}^n = card $(\varphi_n^{-1}(x) \cap S_i^1)$ for any point $x \neq x'$ of S_i^1. Evidently, G_n is just the same matrix as introduced above.

Let X^n denote an inverse limit of spectrum $K \xleftarrow{\varphi_n} K \xleftarrow{\varphi_n} K \leftarrow \dots$ and $\psi_n : X^n \to X^n$ its shift automorphism defined by

$$\psi_n(x_1, x_2, \dots) = (\varphi_n x_1, x_1, x_2, \dots)$$

We claim that for homeomorphism $S : \Lambda_n \to X^n$ defined by

$$S(p) = (\pi p, \pi \circ f_n^{-1}(p), \dots, \pi \circ f_n^{-k}(p), \dots), \quad p \in \Lambda_n$$

the diagram

$$\begin{array}{ccc} \Lambda_n & \xrightarrow{f_n} & \Lambda_n \\ S \downarrow & & \downarrow S \\ X^n & \xrightarrow{\psi_n} & X^n \end{array}$$

commutes.

The fact that segments of invariant partition Γ contracts with coefficients which are less than 1 uniformly implies injectivity of S. To prove that S is onto let us consider for any $(x_1, x_2, \dots) \in X^n$ a family of segments $\alpha_k = f_n^k \circ \pi^{-1}(x_{k+1})$ $(k \geqslant 0)$ containing in elements of Γ Since $\alpha_{k+1} \subset \alpha_k$ and the length of α_k tends to zero as k $\to \infty$, $\bigcap_{k \geqslant 0} \alpha_k$ consists of unique point $p \in \Lambda_n$ Evidently $S(p) = (x_1, x_2 \dots)$ Commutativity of the diagram (2) is a consequence of the commutative diagram

$$\begin{array}{ccc} \Phi^n & \xrightarrow{f_n} & \Phi^n \\ \pi \downarrow & & \downarrow \pi \\ K & \xrightarrow{\psi_n} & K \end{array}$$

Topological entropy of ψ_n is equal to logarithm of the maximal eigenvalue of the matrix G_n (see[12]). Since the spectral decomposition of $\Omega(f_n)$ consists of four periodical points and one attractor Λ_n and $f_n/_{\Lambda_n}$ is conjugate to ψ_n

$$h(f_n/_{S^2}) = h(f_n/_{\Lambda_n}) = h(\psi_n/_{X^n})$$

implies

$$h(f_n) = \log(n+2+\sqrt{(n+1)(n+3)})$$

THEOREM. Diffeomorphisms of the series $\{f_n, n \in N\}$ are representatives of countable many topological conjugacy classes, these classes are not results by itarations of any finite series of diffeomorphisms.

PROOF. The first assertion is a consequence of the fact that values of the entropy of f_n form infinitely sequence.

For the second let us consider any diffeomorphisms $F_1, \dots, F_k$

on S. Let $\mu_i = \exp h(F_i) \quad (i = 1, \dots, K)$.
If for any f_n there exist some F_i such that f_n is conjugate to F_i^p for some integer p, then $\lambda_n = \mu_i^p$ were $\lambda_n = \exp h(f_n)$. It is easy to see that $\lambda_n = 2n + 4 - \alpha_n$ for some $0 < \alpha_n < 1$. Thus our supposition implies that for any integer number of the form $2n + 4$ there exists some integer p such that $|2n + 4 - \mu_i^p| < 1$ for some μ_i belonged to some fixed finite set. This means that any element of the arithmetical progression $2n + 4$ is approximated by elements of finite family of geometrical progressions which is impossible.

2. On Diffeomorphisms with Infinitely Many Sources (Sinks)

Let $\mathrm{Diff}_2(M)$ denote the space of C^r-diffeomorphisms of C^∞ manifold M without boundary $\mathrm{Diff}_2^k(M)$ ($K \leqslant 2$) denote the same set with the uniform C^k-topology.

<u>Definition 1</u>

Some property of the elements of the set $\mathrm{Diff}_2(M)$ is called C^k-typical for $f \in \mathrm{Diff}_2(M)$ if there is a residual subset $\mathcal{B}$ of an open neighbourhood $N(f)$ of f in $\mathrm{Diff}_2^k(M)$ with this property for each element of $\mathcal{B}$.

Perhaps it is worthwhile saying that the notion of the C^k-typical property is the generalization of the C^k-stability. Throughout this paper the notion of C^k-stability is used in its ordinary meaning.

Our aim is to prove the following result.

THEOREM. For any manifold M of dimension greater than two, the set of diffeomorphisms in $\mathrm{Diff}_2(M)$ $(r \geqslant 2)$, for which the property of having infinitely many sources is C^k-typical $(k \geqslant 1)$, is C^0-dense in $\mathrm{Diff}_r(M)$.

Our theorem will be proved as follows. First, sufficient conditions for the appearance of infinitely many sources will be given. This result is due to S.E. Newhouse [1]. Infinitely many sources appearance is based on the fact that tangency points existence for stanble and unstable manifolds is C^k-stable.for some basic set. Secondly, it will be shown that the diffeomorphisms with this property are dense in $\mathrm{Diff}^0_r(M)$, $\dim M \geqslant 3$. Before proving the theorem it is worth saying a few words about notations.

Let us recall (see [1]) that a compact f-invariant set Λ is a basic set for f if it is hyperbolic, topolofically transitive, the periodic points of f are dense in Λ, and Λ has a local product structure. The basic set is non-trivial if it contains more than one orbit. In this case it must be infinite. Given a basic set Λ_f for f, there is a neighbourhood N of f in $\mathrm{Diff}^k_r(M)$ where each $g \in N$ has a unique basic set Λ_g near Λ_f and there is a homeomorphism $h : \Lambda_f \to \Lambda_g$ such that $gh = hf$. For $x \in \Lambda_f$ and $g \in N$, we denote $h(x)$ by x_g.

Let p be a hyperbolic periodic point of f with period ν. Let $\mu_1, \mu_2, \dots, \mu_s, \lambda_1, \dots, \lambda_e$ be the eigenvalues of the derivative D_pf with $|\mu_1| \leq \dots \leq |\mu_s| < 1 < |\lambda_1| \leq \dots \leq |\lambda_e|$ and let be $\mu(p) = |\mu_s|$, $\lambda(p) = |\lambda_e|$

Now we can formulate the propositions discussed above.

THEOREM A. (S.E. Newhouse [1]) Suppose Λ_f is a non-trivial basic set for $f \in \mathrm{Diff}^k_r(M)$ which contains a periodic point p_f such that $\dim W^s(p) = \dim M - 1$ $(\dim W^u(p) = \dim M - 1)$. Let N be a small neighbourhood of f as above such that each $g \in \mathcal{N}$ $g \in N$ has a unique basic set Λ_g near Λ_f Assume there is a neighbourhood $N_1 \subset N$ of f in $\mathrm{Diff}^k_r(M)$ such that if $g \in N$, then $W^u(\Lambda_g)$ and $W^s(\Lambda_g)$ have a point of tangency and

$$\mu(p_g)\ \lambda(p_g) < 1 \qquad (>1)$$

Then there is a residual subset B of N, such that each $g \in B$ has infinitely many sinks (sources). Using our notation the property of having infinitely many sinks (sources) is C^k-typical for f.

THEOREM A' (S.E. Newhouse [1]) Let p be a hyperbolic periodic point of f $\dim W^u(O(p)) = 1$, $\mu(p)\cdot\lambda(p) < 1$ $(\dim W^s(O(p)) = 1$, $\mu(p)\lambda(p) > 1$). Assume x is a point of tangency of $W^s(O(p))$ and $W^u(O(p))$ Then given any neighbourhood U of x in M and N of f in $\mathrm{Diff}^k(M)$, $k \geqslant 1$, there is a $g \in N$ which has a sink in U.

THEOREM B. Given any manifold M, $\dim M \geqslant 3$, the set of C^r-diffeomorphisms, $r \geqslant 2$, for which condition of theorem A holds with $k \geqslant 1$ is dense in $\mathrm{Diff}^0_r(M)$.

Remarks.

1. S.E. Newhouse [1] has shown that on any manifold M, $\dim M \geqslant 2$, there is a diffeomorphism for which conditions of theorem A are satisfied with $k \geqslant 2$.

2. In [9] N.K. Gavrilov has adduced conditions when diffeomorphism f has infinitely many sinks, $f \in \mathrm{Diff}_r(M^2)$, $r \geqslant 2$.

Recall some definitions from general topology and differentiable dynamics.

Let X, Y be metric spaces and X be compact. Let 2^X denote the set of all closed subsets of X.

1. $\{A_n \mid A_n \in 2^X, n \in N\}$ is a sequence of closed subsets of X.

$$\underset{n\to\infty}{Li}(A_n) = \{p \mid \forall n \ \exists p_n \in A_n : p = \lim p_n\}$$

(see [1])

2. The map $F : Y \to 2^X$ is lower semicontinuous, if for any point $y \in Y$ and for any sequence $\{y_n \in Y, n \in N\}$, $y_n \to y$

$$F(y) \subset Li(F(y_n))$$

3. The Hausdorff metric on 2^X is defined by the formula

dist (A, B) = max $\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} (A, b)\}$

where ρ is metric on X (see [8]).

4. Diffeomorphism $f \in \mathrm{Diff}_r(M)$ is a Kupka-Smale diffeomorphism (KS-diffeomorphism) if the following conditions are satisfied

(1) All periodic points of f are hyperbolic.

(2) For any periodic point p $W^s(p)$ and $W^n(p)$ are immersed copies of Euclidean space.

(3) For any periodic points p, q of f $W^s(p)$ and $W^n(q)$ are transversal.

(see [6])

Let KS(N) denote the set of Kupka-Smale diffeomorphisms in $N \subset \mathrm{Diff}_r(M)$.

The lower semicontinuous map is continuous on a residual subset B of Y. The set KS(M) is residual subset of $\mathrm{Diff}_r(M)$.

LEMMA 1. Let $g_n : Y \to 2^X$ be lower semicontinuous map for any $n \in \mathcal{N}$, then the map $g : Y \to 2^X$ given by the formula

$$g(x) = cl\,(\bigcup_{n\in\mathcal{N}} g_n(x))$$

is semicontinuous.

<u>Proof</u>. Recall two elementary properties of Li (see [8])

$$(1)\quad cl\,(Li\,A_n) = Li\,(A_n) = Li\,(cl\,A_n)$$

$$(2)\quad \bigcup_{\kappa\in G} [\mathop{Li}_n A_n(\kappa)] \subset \mathop{Li}_n (\bigcup_{\kappa\in G} A_n(\kappa))$$

where $\kappa \in G$ and G is arbitrary.

Let the seqaence $x_\kappa \in Y$, $\kappa \in \mathcal{N}$ converge to $x \in Y$. From the properties (1)-(2) and the fact that g_n is lower semicontiuous one obtains

$$\mathop{Li}_\kappa g(x_\kappa) = Li\,[cl \bigcup_n g_n(x_\kappa)] = Li\,(\bigcup_n g_n(x_\kappa)) \supset$$

$$\supset \bigcup_n (\mathop{Li}_\kappa g_n(x_\kappa)) \supset \bigcup_n g_n(x)$$

and hence

$$g(x) = cl\,(\bigcup_n g_n(x)) \subset \mathop{Li}_\kappa g(x_\kappa)$$

This completes the proof of lemma 1.

Let Λ be non-trivial basic set of $f \in \mathcal{D}iff^k(M)$, $p \in \Lambda_f$ be hyperbolic periodic point. Let $\mathcal{N}$ be neighbourhood of f in $Diff^k(M)$ described above. For any poin $g \in \mathcal{N}$ H_{p_g} denotesthe homoclinic class of p_g ; that is H_p consists of all periodic points q of g such that $W^u(o(q))$ has a point of transversal intersection with $W^s(o(p))$ and $W^u(o(p))$ has a point of transversal intersection with $W^s(o(q))$. Than H_{pg} contains the periodic points of Λ_g (see [1]). The formula $H(g)=clH_{p_g}$ defines the map $H: KS(\mathcal{N}) \to 2^M$. S(g) denotes the closure of the set of all sources for g and thus it defines the map $S: KS(\mathcal{N}) \to 2^M$.

LEMMA 2 (see [3]) H and S are continuous on a residual subset $\mathcal{B} \subset \mathcal{N}$.

<u>Proof</u>. For any n let's define maps H_n, $S_n : KS(\mathcal{N}) \to 2^M$ by the formulas

$$H_n(g) = H_{P_g} \cap Per_n(g)$$

$$S_n(g) = S(g) \cap Per_n(g)$$

where $Per_n(g)$ is the set of fixed points of g^n. g is KS-diffeomorphism hence the set $Per_n(g)$ is finite and thus the sets $H_n(g)$ and $S_n(g)$ have only finite number of elements. We shall show that H_n and S_n are lower semicontinuous . Than H and S are lower semicontinuous becous $H(g)=cl(\bigcup_n H_n(g))$ and $S(g)=cl(\bigcup_n S_n(g))$ There are residual subsets $\mathcal{B}_S$ and $\mathcal{B}_H$ of $KS(\mathcal{N})$ on which S and H are continuous. Thus to prove the lemma we must set $\mathcal{B} = \mathcal{B}_H \cap \mathcal{B}_S$

(i) The map H_n is continuous on $KS(\mathcal{N})$. Let $g' \in \mathcal{N}$ be a sufficiently close approximation of g. There is the homeomorphism $h: \Lambda_g \to \Lambda_{g'}$ such that $h \circ g = g' \circ h$ and $\rho(x, hx) < \varepsilon$ for any $x \in \Lambda_g$, where $\varepsilon \to 0$ if $g' \to g$. Thus $dist(H_n(g), H_n(g')) \le \varepsilon$ because $H_{pg'} = h(H_{pg})$ and $Per_n(g') \cap \Lambda_{g'} = h(Per_n(g) \cap \Lambda_g)$.

(ii) The map S_n is lower semicontinuous. If $g \in KS(\mathcal{N})$ then $S_n(g) = \{a_1, \dots, a_k\}$ is hyperbolic set for g. Hence from the stability theorem for hyperbolic set one obtains that for any $\varepsilon > 0$ there is a neighbourhood $\mathcal{N}_2$ of g in $Diff^k_r(M)$ such that $g' \in \mathcal{N}_1$ implies the existence of $\widetilde{S} \subset S_n(g')$ with $dist(\widetilde{S}, S_n(g)) < \varepsilon$, so $S_n(g) \subset \operatorname{Li}_{g' \to g} S_n(g')$. Lemma 2 is proved.

LEMMA 3 Let $g \in \mathcal{N}_1$, then given any periodical point $p \in \Lambda_g$ its neighbourhood U and arbitrary neighbourhood $\mathcal{N}_2$ of g in $Diff^k_r(M)$, there is the diffeomorphism $g' \in \mathcal{N}_2$ which coincides with g on some neighbourhood of Λ_g and has sink $y \in U$

Proof. Let for $q_1, q_2 \in \Lambda_g$ $W^s(q_1)$ and $W^u(q_2)$ have the tangency point x.

(i) It will be shown that thereis arbitrarily small pertrubation g_1 of g outside some neighbourhood of Λ_g such that for any neighbourhood U_1 of point x there is x_1 - the point of tangency $W^s(p_1, g_1)$ and $W^u(p_2, g_1)$, where p_1 and p_2 are periodic points of g_1 and $p_1, p_2 \in \Lambda_g$. For sufficiently small $\alpha, \beta > 0$ and any point $q \in \Lambda_g$ there is stable manifold of size (β, α) (see [4]). Global stable and unstable manifolds are defined by the formulas $W^s(q) = \bigcup_{n \geq 0} g^{-n}(W^s_{g^n(q)})$,

$$W^u(q) = \bigcup_{n \geq 0} g^n(W^u_{g^{-n}(q)}).$$

We need some notations. Let $W^s_n(q)$ and $W^u_n(q)$ denote $\bigcup_{0 \leq k \leq n} g^{-k}(W^s_{g^k(q)})$ and $\bigcup_{0 \leq k \leq n} g^k(W^u_{g^{-k}(q)})$ respectively. It is easy to see that $W^\delta_{n+1}(q) \supset W^\delta_n(q)$, $\delta = u, s$. There are integers $n_s, n_u \geq 1$ such that

$$x \in W^s_{n_s}(q_1), \quad x \bar{\in} W^s_{n_s-1}(q_1);$$

$$x \in W^u_{n_u}(q_2), \quad x \bar{\in} W^u_{n_u-1}(q_2).$$

Now we assert that it is possible to choos sufficiently small disk V which is the neighbourhood of the point x in M, $V \subset U_1$ and $\mathcal{D}^s = V \cap W^s_{n_s}(q_1) \subset W^s_{n_s}(q_1) \setminus W^s_{n_s-1}(q_1)$

$$\mathcal{D}^u = V \cap W^u_{n_u}(q_2) \subset W^u_{n_u}(q_2) \setminus W^u_{n_u-1}(q_2)$$

where $\mathcal{D}^s$ and $\mathcal{D}^u$ are smoothly embedded disks. Let us choos periodical points $p_1, p_2 \in \Lambda_g$ close enough to points q_1 and q_2 for submanifolds $W^s_{n_s}(p_1)$ and $W^u_{n_u}(p_2)$ to be sufficiently close to manifolds $W^s_{n_s}(q_1)$ and $W^u_{n_u}(q_2)$ respectively. Then

$$\mathcal{D}_1^s = V \cap W^s_{n_s}(p_1) \subset W^s_{n_s}(p_1) \setminus W^s_{n_s-1}(p_1)$$
$$\mathcal{D}_1^u = V \cap W^u_{n_u}(p_2) \subset W^u_{n_u}(p_2) \setminus W^u_{n_u-1}(p_2)$$

are smoothly embedded disks in M, Which are sufficiently close to disks $\mathcal{D}^s$ and $\mathcal{D}^u$ respectively. Point x is the point of tangency of $\mathcal{D}^u$ and $\mathcal{D}^s$ hence there is sufficiently C^r-close to identity map diffeomorphism $\omega : M \to M$ such that $\omega(\mathcal{D}_1^s)$ and $\mathcal{D}_1^u$ have a point of tangency $x_1 \in V$ and $\omega|_{M \setminus V}$ =id. From the definition of V one can obtain that $\omega(\mathcal{D}_1^s) \subset W^s(p_1, g_1)$ and $\mathcal{D}_1^u \subset W^u(p_2, g_1)$ where $g_1 = \omega \circ g$. Thus x_1 is the tangency point of $W^s(p_1, g_1)$ and $W^u(p_2, g_1)$.

(ii) Let's show that there is arbitrary small perturbation g_2 of g_1 in $\mathrm{Diff}^k_r(M)$ outside some neighbourhood of Λ_g such that manifolds $W^s(p, g_2)$ and $W^u(p, g_2)$ have a point of tangency x_2 in an arbitrarily small neighbourhood of point x_1 where $p \in \Lambda_g$ is periodical point.

For simplicity let's consider the case when p_1, p_2, p are fixed points. There is the disk $\mathcal{D}_1^s \subset W^s(p)$ arbitrarily close to $W^s_{p_1}$ ($W^s_{p_1}$ is a local stable manifold) because p is homoclinicaly related to p_1. By the same argument there is disk $\mathcal{D}_1^u \subset W^u(p)$ arbitrarily close to $W^u_{p_2}$. Let's choose disk V which is a neighbourhood of point x_1 in M such that

$$V \cap \bigcup_{n>0} g_1^n(\mathcal{D}_1^s) = \phi \quad \text{and} \quad V \cap \bigcup_{n<0} g_1^n(\mathcal{D}_1^u) \neq \phi$$

Let disks $\mathcal{D}^s \subset W^s(p_1) \cap V$ and $\mathcal{D}^u \subset W^u(p_2) \cap V$ be defined as in (i). Then there are disks $\mathcal{D}_2^s \subset \mathcal{D}_1^s$ and $\mathcal{D}_2^u \subset \mathcal{D}_1^u$ such that for some integers $m_s, m_u > 0$ disks $\mathcal{D}_3^s = g_1^{-m_s}(\mathcal{D}_2^s)$

and $\mathcal{D}_3^u = g_1^{m_u}(\mathcal{D}_2^u)$ are arbitrary close to $\mathcal{D}^s$ and $\mathcal{D}^u$, but $g_1^i(\mathcal{D}_2^s) \cap V = \phi$ for $0 \le i \le m_s - 1$ and $g_1^i(\mathcal{D}_2^u) \cap V = \phi$ for $0 \le i \le m_u - 1$. Thus one can obtain the perturbation g_2 as in (i).

(iii) Now we change g_2 to g_3 for $W^u(p_g, g_3)$ to have a tangency point x_3 with $W^s(p_g, g_3)$ near x_2. Finalli move g_3 to g' to introduce a sink y near x_3 using theorem A'. Since the sink y ma may be got arbitrarily close to a certain disk in $W^s(p, g')$ of fixed diameter (depending only on the position of x_3), its orbit under g' will get close to p. The proof of lemma 3 is completed.

<u>Proof of theorem A</u>. Let $g \in \mathcal{B}$ where $\mathcal{B} \subset \mathcal{N}_1$ is a residual subset of $\mathcal{N}_1$ on which maps H, S: $\mathrm{Diff}^K(M) \to 2^M$ are continuous. Fix some topological metric on M. Let P_n be a finite ε_n-net of compact H(g), consisting of periodic points of g, where $\varepsilon_n > 0$ converges to zero. $N_\varepsilon(g)$ denotes the ball of radius ε with its center in g. By the lemma 3, given any n there is a diffeomorphism $g_n \in N_{\varepsilon_n}(g)$ such that for every $B_{\varepsilon_n}(p)$, $p \in P_n$ there is sink $q \in B_{\varepsilon_n}(p)$, $q \in S(g_n)$. Then $\mathrm{dist}(S(g_n), P_n) < \varepsilon_n$. From the continuaty of H, S at the point g, $P_n \to H(g)$ $n \to \infty$ and

$$\mathrm{dist}(S(g_n), H(g_n)) \le \mathrm{dist}(S(g_n), P_n) + \mathrm{dist}(P_n, H(g)) + \mathrm{dist}(H(g), H(g_n))$$

one obtains H(g)=S(g), the set $H(g) = \mathrm{cl}\,H_{p_g}$ being infinite, so this completes the proof of theorem A.

Now we can give the proof of theorem B. We'll consider the case when infinite number of sources appear. We now proceed to state and prove technical lemmas which will be needed for the

the proof of theorem B.

LEMMA 4. Let D be a disk with smooth boundary in the plane R^2 $f: D \to \mathrm{int} D$ and $A: R^2 \to R^2$ - orientation preserving diffeomorphisms, A has a single nonwandering point a, which is a sink $a \in \mathrm{int} f(D)$. Then there are disks D_1, D_2 and diffeomorphism $F: R^2 \to R^2$ such that

$$\mathcal{D}_2 \Supset \mathcal{D} \Supset \mathcal{D}_1 \Supset f(\mathcal{D}),$$

$$F|_{R^2 \setminus \mathcal{D}_2} = A, \quad F|_{\mathcal{D}_1} = f \quad \text{and} \quad \Omega(f) = \Omega(F)$$

(here $U \Subset V$ means $\mathrm{clos} U \subset \mathrm{int} V$)

Proof. (i) Let's finde disks $\mathcal{D}_1$ and $\widetilde{\mathcal{D}}$ with smooth boundaries $\mathcal{D} \Supset \widetilde{\mathcal{D}} \Supset \mathcal{D}_1 \Supset f(\mathcal{D})$ and extention $\bar{f}$ of f on the whole plane such that $\bar{f}: R^2 \to \mathrm{int}\, \widetilde{\mathcal{D}}$, $\bar{f}|_{\mathcal{D}_1} = f|_{\mathcal{D}_1}$ and $\Omega(\bar{f}) = \Omega(f)$. Consider the diffeomorphisms $h: R^2 \to \mathrm{int}\, \widetilde{\mathcal{D}}$ which is equal to the identical on $\mathcal{D}_1$ and set $\bar{f} = f \circ h$. It is obvious that any point x from $R^2 \setminus \mathcal{D}_1$ is wandering becouse $\bar{f}(x) = f \circ h(x) \in f(\mathcal{D}) \subset \mathcal{D}_1$ and $\bar{f}(\mathcal{D}_1) \subset \mathrm{int}\, \mathcal{D}_1$. Hence $\Omega(\bar{f}) \subset \mathcal{D}_1$ and $\Omega(\bar{f}) = \Omega(f)$ because $\bar{f}|_{\mathcal{D}_1} = f|_{\mathcal{D}_1}$. Using the properties of A one can finde disk $\mathcal{D}_2 \Supset \mathcal{D}$ such that $\mathcal{D} \Subset A(\mathcal{D}_2) \Subset \mathcal{D}_2$. It is a consequence of the facts that a is a sink for A and thus A is topologically conjugate to iinear map whith eigenvalues less than 1. and more than -1. Thus there is disk D' such that $A(\mathcal{D}') \subset \mathrm{int}\mathcal{D}'$. From the relation $W^s(a, A) = R^2$ one can obtain some $n \geqslant 0$ with the property $A^{-n}(\mathcal{D}') \Supset \mathcal{D}$, and we set $\mathcal{D}_2 = A^{-(n+1)}(\mathcal{D}')$. Let $\widetilde{\mathcal{D}}_2$ be a disk $\mathcal{D}_2 \Supset \widetilde{\mathcal{D}}_2 \Supset A\mathcal{D}_2$ and $K, \check{K}$ are the rings $\mathcal{D}_2 \setminus \mathrm{int}\, \widetilde{\mathcal{D}}_2$ and $A(\mathcal{D}_2) \setminus \mathrm{int}\, \bar{f}(\widetilde{\mathcal{D}}_2)$. To prove the lemma it remains to extend diffeomorphism $F: R^2 \setminus K \to R^2 \setminus \widetilde{K}$ which is given by the formula $\bar{F} = \begin{cases} Ax, & x \in R^2 \setminus \mathcal{D}_2 \\ \bar{f}(x), & x \in \widetilde{\mathcal{D}}_2 \end{cases}$

to the diffeomorphism F of the whole plane so that $F(K) = \widetilde{K}$ then F satisfy to the conditions of the lemma, because $F(K) = \widetilde{K}$, $K \cap \widetilde{K} = \phi$ and hence all points of $R^2 \setminus \mathcal{D}_2$ are wandering; thus $\Omega(F) = \Omega(\bar{f}) = \Omega(f)$

(ii) In order to define F we shall consider the standart ring on R^2

$$T = \{(\rho, \varphi) \mid R_0 \le \rho \le R, \quad 0 \le \varphi < 2\pi\}$$

where (ρ, φ) are polar coordinates. Let $S_1 = \{(\rho, \varphi) \mid \rho = R\}$,

$$S_2 = \{(\rho, \varphi) \mid \rho = R\}$$

and map some neighbourhoods $U(K)$ and $U(\widetilde{K})$ onto the neighbourhood $U(T)$ of T using diffeomorphisms h_1, h_2 so that

$$h_1(K) = h_2(\widetilde{K}) = T,$$
$$h_1(\partial \mathcal{D}_2) = h_2(\partial A(\mathcal{D}_2)) = S_1$$
$$h_1(\partial \widetilde{\mathcal{D}}_2) = h_2(\partial \bar{f}(\widetilde{\mathcal{D}}_2)) = S_2$$

Suppose we shall finde diffeomorphism $g : U(T) \to U(T)$ such that $g = h_2 \circ A \circ h_1^{-1}$ on $\{(\rho, \varphi) \mid \rho \ge R\}$ and $g = h_2 \circ \bar{f} \circ h_1^{-1}$ on $\{(\rho, \varphi) \mid \rho \le R_0\}$ that

$$F(x) = \begin{cases} Ax, & x \in R^2 \setminus \mathcal{D}_2 \\ h_2^{-1} \circ g \circ h_1(x), & x \in K \\ \bar{f}(x), & x \in \mathcal{D}_2 \end{cases}$$

will be a extention of F that we need. In order to find diffeomorphism g it is sufficient to finde such diffeomorphisms w and w' that $w|_{\{(\rho,\varphi)|\rho\geq R\}} = A$, $w|_{\{(\rho,\varphi)|\rho\leq r\}} = id$,

$$w'|_{\{(\rho,\varphi)|\rho\geq r'\}} = id, \quad w'|_{\{(\rho,\varphi)|\rho\leq R_0\}} = \bar{f}$$

for some r and r', $R_0 < r' < r < R$ and let

$$g(\rho,\varphi) = \begin{cases} w(\rho,\varphi), & \rho > r' \\ w'(\rho,\varphi), & \rho < r \end{cases}$$

The definition of w' is the same as that of w and we shall give one for w only.

(iii) Let $h_2 \circ A \circ h_1^{-1}(\rho,\varphi) = \Psi(\rho,\varphi) =$

$$= (\Psi_1(\rho,\varphi), \Psi_2(\rho,\varphi))$$

where $\Psi(S_1) = S_1$ that is $\Psi_1(R,\varphi) = R$. Ψ is orientation preserving hence $\frac{\partial\Psi_2}{\partial\varphi}(R,\varphi) > 0$ and there is $\eta > 0$ such that $\det D\Psi(\rho,\varphi) \geq \eta$, $R_0 \leq \rho \leq R$. Let $L = \max_{x\in T}\left|\frac{\partial\Psi_2}{\partial\rho}(x)\right|$,

$$\mathcal{M} = \max_{x\in T}\left|\frac{\partial\Psi_2}{\partial\varphi}(x)\right|, \quad x = (\rho,\varphi)$$

and $0 < \varepsilon < \eta/2\max(LM,L)$. Take r from the $\left(\frac{R+R_0}{2}, R\right)$ so close to R that there is $\delta > 0$, for which

$$\left|\frac{\partial\Psi_1}{\partial\varphi}(\rho,\varphi)\right| < \varepsilon \quad \text{and} \quad \left|\frac{\partial\Psi_2}{\partial\varphi}(\rho,\varphi)\right| \geq \delta \quad \text{when } r\leq\rho\leq R$$

The first condition is satisfied, because $\frac{\partial\Psi_1}{\partial\varphi}(R,\varphi) = 0$ and the second - because $\frac{\partial\Psi_2}{\partial\varphi}(R,\varphi) > 0$ At last

$$\frac{\partial\Psi_1}{\partial\rho}\frac{\partial\Psi_2}{\partial\varphi} \geq \eta + \frac{\partial\Psi_1}{\partial\varphi}\cdot\frac{\partial\Psi_2}{\partial\rho} \geq \eta - \left|\frac{\partial\Psi_1}{\partial\varphi}\right|\left|\frac{\partial\Psi_2}{\partial\rho}\right| \geq$$

$$\geq \eta - \varepsilon L \geq \frac{\eta}{2}$$

hence $\frac{\partial \Psi_1}{\partial \rho} > \frac{r}{2M}$ when $r \le \rho \le R$. Choose constants $\delta > 0$ and $\tau > 1$ so that $\delta \le \frac{\partial \varphi_1}{\partial \rho}(R, \varphi) \le \tau$. Let $\alpha(\varphi) = \frac{1}{\tau} \frac{\partial \varphi_1}{\partial \rho}(R, \varphi)$ then $\frac{\delta}{\tau} \le \alpha(\varphi) \le 1$. Let us define a smooth function $\beta(\rho, \varphi)$ such that

$$\Psi_1(\rho, \varphi) = \Psi_1(R, \varphi) + \frac{\partial \Psi_1}{\partial \rho}(R, \varphi)(\rho - R) + \beta(\rho, \varphi)(\rho - R)^2 =$$
$$= R + \tau\alpha(\varphi)(\rho - R) + \beta(\rho, \varphi)(\rho - R)^2$$

Let's divide segment $(r, R]$ by points

$$r = r_{41} < r_{40} < r_{31} < r_{30} < r_{21} < r_{20} < r_{11} < r_{10} = R$$

on segments $t_4, \Delta_3, t_3, \Delta_2, t_2, \Delta_1, t_1$ so that $r_{40} - r_{41} > \frac{\tau - 1}{\tau}(R - r)$. For any i=1,2,3 we shall define the nondecrease C^∞- function $\lambda_i(\rho)$ with the following properties

1° $\lambda_i(\rho) = 0$, $\rho \le r_{i1}$; $\lambda_i(\rho) = 1$, $\rho \geq r_{i0}$;

2° $\frac{\partial^n}{\partial \rho^n} \lambda_i(\rho) = 0$ for $\rho \bar{\in} t_i$

3° $0 \le \frac{\partial \lambda_i}{\partial \rho} \le \frac{1+\gamma}{r_{i0} - r_{i1}} = \frac{K_i}{R - r}$

where $K_i = (1+\gamma)\frac{R - r}{r_{i0} - r_{i1}}$, $\gamma > 0$.

Let's define the increasing C^∞- function $\mu(\rho)$ such that $\mu(\rho) = \rho$ for $\rho < r$ and $\mu(\rho) = \tau\rho + R(1 - \tau)$ for $\rho > r_{40}$

The existence of such a function follows from relations

$$\mu(r_{40}) = \tau r_{40} + R(1 - \tau) > \tau r + (\tau - 1)(R - r) +$$
$$+ R(1 - \tau) = r = \mu(r_{41})$$

Let's define following maps

$$\Psi^1(\rho, \varphi) = \left(R + \tau\alpha(\varphi)(\rho - R) + \lambda_1(\rho)\beta(\rho, \varphi)(\rho - R)^2, \Psi_2(\rho, \varphi)\right)$$

$$\Psi^2(\rho, \varphi) = \left(R + \tau\left[1 - \lambda_2(\rho)(\alpha(\varphi) - 1)\right](\rho - R), \Psi_2(\rho, \varphi)\right)$$

$$\Psi^3(\rho,\varphi) = \left(R+\tau(\rho-R),\ \varphi+\lambda_3(\rho)\left(\Psi_2(\rho,\varphi)-\varphi\right)\right)$$

$$\Psi^4(\rho,\varphi) = \left(\mu(\rho),\ \varphi\right)$$

It is easy to see that $\Psi' = \Psi$ outside the circle of radius R Ψ^4=id in the circle of radius r and $\Psi^i(\rho,\varphi) = \Psi^{i+1}(\rho,\varphi)$ when $\rho \in \Delta_i$. Thus the lemma will be proved if we find that for i=1,2,3,4 Ψ^i is a diffeomorphism on $\{(\rho,\varphi) \mid r \le \rho \le R\}$ when R-r is sufficiently small.

(iy) Ψ^4 is a diffeomorphism because the function $\mu(\rho)$ is monotonous. If R-r is sufficiently small then the degree of Ψ^i (i=1,2,3) is equal to that of Ψ which is 1. Thus to prove the lemma it remains to prove that $\det\frac{D(\Psi_1^i,\Psi_2^i)}{D(\rho,\varphi)} \neq 0$ when $r \le \rho \le R$ and R-r is sufficiently small. Indeed,

$$\frac{\partial \Psi_1^1}{\partial \rho} = \tau\alpha(\varphi) + \lambda_1'(\rho)\beta(\rho,\varphi)(\rho-R)^2 + \lambda_1(\rho)\frac{\partial\beta}{\partial\rho}(\rho-R)^2 +$$

$$+ 2\lambda_1(\rho)\beta(\rho,\varphi)(\rho-R) \geq \tau + O(\rho-R),$$

$$|\lambda_1'(\rho)\beta(\rho,\varphi)(\rho-R)^2| < K_1|\beta(\rho,\varphi)||\rho-R| = O(\rho-R)$$

then $$\frac{\partial\Psi_2^1}{\partial\varphi} = \tau\alpha'(\varphi)(\rho-R) + \lambda_1(\rho)\frac{\partial\beta}{\partial\varphi}(\rho-R)^2 = O(\rho-R)$$

Thus $$\det\frac{D(\Psi_1^1,\Psi_2^1)}{D(\rho,\varphi)} = \frac{\partial\Psi_1^1}{\partial\rho}\frac{\partial\Psi_2^1}{\partial\varphi} - \frac{\partial\Psi_1^1}{\partial\varphi}\cdot\frac{\partial\Psi_2^1}{\partial\rho} \geq$$

$$\geq \tau\frac{\partial\Psi_2^1}{\partial\varphi} + O(\rho-R) \geq \tau\delta + O(\rho-R) \qquad (1)$$

Further $$\frac{\partial\Psi_1^2}{\partial\rho} = \tau\left[1+\lambda_2(\rho)(\alpha(\varphi)-1)\right] +$$

$$+\tau\lambda_2'(\rho)(\alpha(\varphi)-1)(\rho-R) \geq \tau\left[1+\lambda_2(\rho)(\alpha(\varphi)-1)\right] =$$

$$= \tau(1-\lambda_2(\rho)) + \tau\lambda_2(\rho)\alpha(\varphi) \geq \tau - \tau\lambda_2(\rho) + \delta\lambda_2(\rho) =$$

$$= \delta + \tau(1-\lambda_2(\rho)) - \delta(1-\lambda_2(\rho)) =$$

$$= \delta + (\tau-\delta)(1-\lambda_2(\rho)) \geq \delta \; ; \quad \frac{\partial \Psi_1^2}{\partial \varphi} = -\tau\lambda_2(\rho)\alpha'\cdot(\rho-R) =$$
$$= O(\rho - R)$$

hence

$$\det \frac{\mathcal{D}(\Psi_1^2, \Psi_2^2)}{\mathcal{D}(\rho, \varphi)} = \frac{\partial \Psi_1^2}{\partial \rho} \cdot \frac{\partial \Psi_2^2}{\partial \varphi} + O(\rho - R) \geq \delta\delta + O(\rho - R) \tag{2}$$

Finally, from $\frac{\partial \Psi_1^3}{\partial \varphi} = 0$ one obtaines

$$\det \frac{\mathcal{D}(\Psi_1^3, \Psi_2^3)}{\mathcal{D}(\rho, \varphi)} = \frac{\partial \Psi_1^3}{\partial \rho} \cdot \frac{\partial \Psi_2^3}{\partial \varphi} = \tau\Big(1 - \lambda_3(\rho) + \lambda_3(\rho)\frac{\partial \Psi_2^3}{\partial \varphi}\Big) \geq \tau\big(1 - \lambda_3(\rho) + \lambda_3(\rho)\,\delta\big) > 0 \tag{3}$$

From (1),(2),(3) it follows that Ψ^1, Ψ^2, Ψ^3 are diffeomorphisms when $R-r$ is small enough. The lemma is proved.

LEMMA 5. Given any C^0- neighbourhood U of any diffeomorphism f $\mathrm{Diff}_r(M)$ for a set Σ of n different positive values distinguished from 1, there is a diffeomorphism $f_1 \in U$ having periodical point p in whose neighbourhood f_1 is smoothly conju- to the linear map with a real spectrum and the set of absolute values of elements from this spectrum is Σ.

<u>Proof</u>. (i) M is compact hence there is a nonwandering point of f. Then there is arbitrarily small approximation f_1 of f such that f_1 has periodical point p. We may assume for simplicity that f_1 has a fixed point p.

(ii) We shall show that there is arbitrarily C^1-close to f diffeomorphism f_1 with the derivative $(Df_1)_p : R^n \to R^n$ having a simple spectrum. In some coordinate neighbourhood of p f_1

has the following form $f_1(x)=Ax+\varphi(x)$ where $\varphi\in C^r$, $\varphi(0)=0$ and $(D\varphi)_0=0$, A is a real Jordan form of operator $(Df)_p$. Let $A=\mathrm{diag}(A_1,\dots A_s,B_1,\dots B_t)$, where A_m is Jordan cell corresponding to the real eigenvalue multiple n_m $m=1,\dots s$, B_j is a cell corresponding to complex eigenvalue $\delta_j+i\tau_j$ multiple m_j, $j=1,\dots t$. Let $\varepsilon>0$ be sufficiently small and real values ε_{mk} ($m=\overline{1,s}$, $k=\overline{1,n}$) and δ_{je} ($j=\overline{1,t}$, $e=\overline{1,m_j}$) from the interval $(0,\varepsilon)$ be such that all values $|\lambda_m+\varepsilon_{mk}|$, $|\delta_j+\delta_{je}+i\tau_j|$ are different. Let $E_m=\mathrm{diag}(\varepsilon_{m1},\dots,\varepsilon_{mn_m})$, $F_j=\mathrm{diag}(\delta_{j1},\delta_{j1},\delta_{j2},\delta_{j2},\dots\delta_{jm_j},\delta_{jm_j})$ and $E=\mathrm{diag}(E_1,\dots E_s,F_1,\dots F_t)$. Let $\mu(t)$, $t\in R$ be monotonous C^r-function with properties

(a) $\mu(t)=1$ when $t<\frac{\varepsilon}{2}$

$\mu(t)=0$ when $t>\varepsilon$

(b) $|\mu'(t)|<\frac{3}{\varepsilon}$

Let $f_2(x)=f_1(x)+\mu(|x|)Ex$, then $\mathrm{spec}(Df_2)_c=\{\lambda_m+\varepsilon_{mk},\delta_j+\delta_{je}+i\tau_j\}$ and $|f_1(x)-f_2(x)|\le|\mu(|x|)Ex|=0$ when $|x|\ge\varepsilon$ $|f_1(x)-f_2(x)|\le|Ex|\le\|E\|\,|x|\le\|E\|\,\varepsilon$ *when* $|x|<\varepsilon$, one can obtain that $|f_1(x)-f_2(x)|$ is sufficiently small if ε small enough. Consider

$$\left|\frac{\partial f_{1m}}{\partial x_j}-\frac{\partial f_{2m}}{\partial x_j}\right|=\begin{cases}|\mu'(|x|)\varepsilon_j(x_j)+\mu(x)\,\varepsilon_j|\le 3\varepsilon & \text{when } |x|\le\varepsilon\\ 0 & \text{when } |x|>\varepsilon\end{cases}$$

hence when ε is small enough $\det(Df_2)_x\neq 0$, thus f_2 is diffeomorphism sufficiently C^1-close to f_1 Choosing values ε_{mk} and δ_{je} by the some spetial way one can obtain apply Sternberg's linearization therem and thus we may assume that $f_2(x)=Ax$ where A has a simple spectrum.

(iii) Let's show that there is arbitrarily C - small pertrubation f of f such that $(\mathcal{D}f_3)_p$ has a real spectrum.

Consider the invariant subspace L corresponding to the complex eighnvalue $\sigma+i\tau$ of matrix A, then

$$B(x) = A\big|_L = (\sigma x_1 + \tau x_2,\ -\tau x_1 + \sigma x_2)$$

In polar coordinates the map B has the form $B(\rho,\varphi)=(\alpha\rho,\varphi-\psi)$, where $\alpha=\sqrt{\sigma^2+\tau^2}$, $tg\,\psi=\frac{\tau}{\sigma}$. Let $\varepsilon>0$ and $\lambda(t)$ be a nondecreasing smooth function, which is equal to 0 when $t<\frac{\varepsilon}{2}$ and 1 when $t>\varepsilon$. Let $B_1(\rho,\varphi)=(\alpha\rho,\varphi-\lambda(\rho)\psi)$, it is easy to see that B_1 is diffeomorphism because det $\frac{\mathcal{D}(B_{11},B_{12})}{\mathcal{D}(\rho,\varphi)}=\alpha$. When $\rho<\frac{\varepsilon}{2}$ we have $B_1(\rho,\varphi)=(\alpha\rho,\varphi)$ or in previous coordinates $B_1x=\alpha x$. Extend B_1 on the whole subspace so that outside some neighbourhood of the disk $\{x\in L \mid |x|<\varepsilon\}$ it is equal to f. We shall do the same procedure with other invariant subspaces corresponding to the complex values. Then using the procedure of (ii) one can obtain diffeomorphism f_3 which is smoothly equivalent to the linear map with a simple real spectrum in some neighbourhood of p.

(iy) Let L be an one-dimentional invariant subspace corresponding to eigenvalue λ of the operator A with the simple real spectrum. For any real μ with sgn$\,\mu=$sgn$\,\lambda$ and any $\varepsilon>0$ there is $\delta\in(0,\varepsilon)$ and strictly monotonous smooth function $\nu(t)$, $t\in\mathbb{R}$ such that $\nu(t)=\mu t$ when $|t|<\delta$ and $\nu(t)=t$ when $|t|>\varepsilon$. Then using arbitrary small C^0- perturbation of f_3 one can obtain that $f_3\big|_L=\nu$ in some neighbourhood of p, but other eigenvalues of $(\mathcal{D}f_3)_p$ are the same. Lemma is proved.

LEMMA 6. Let $f_i:E_i\to E_i$ $(i=1,2)$ be C^1-diffeomorphisms of Banach spaces E_i such that

(a) f_1 is Lipschits and for some $r_1 > 0$

$$f_1(E_1(r_1)) \subset \operatorname{int} E_1(r_1)$$

(b) $0 \in E_2$ is a fixed point for f_2.

Let $L(f_2^{-1})$ denotes the Lipschitz constant of $f_2^{-1}|_{E_2(r_2)}$ where $r_2 > 0$. Then for any $\varepsilon > 0$ there is $\delta > 0$ with the following property. If $L(f_2^{-1}) < \delta$ then there is a neighbourhood U of diffeomorphism $f_1 \times f_2 : E_1 \times E_2 \circlearrowleft$ in $\mathrm{Diff}^1_1(E_1 \times E_2)$ such that for any $F \in U$ there is C^1- map $\varphi_F : E_1(r_1) \to E_2$ with Lipschitz constant less than ε and graph φ_F is invariant under F.

<u>Proof.</u> (i) For definitness we shall suppose that $E_1 \times E_2$ has the norm $|(x,y)| = \max(|x|, |y|)$. Let

$$f(x,y) = (f_1(x), f_2(y)) \quad , x \in E_1 \quad , y \in E_2,$$

$$F(x,y) = (f_1(x) + \delta_1(x,y), \; f_2(y) + \delta_2(x,y))$$

where $\delta_i : E_1 \times E_2 \to E_i$ i=1,2 are C^1 maps and

$$\Delta = \max\{\|\delta_1\|_{C^1}, \; \|\delta_2\|_{C^1}\} < 1$$

Consider the space of continuous maps from $E_1(r_1)$ in $E_2(r_2)$ with Lipschitz constant less than 1. We shall denote this space by $\mathcal{H}$ and let $\mathcal{H}$ have uniform topology. $\mathcal{H}$ is a complete metric space. We define the graph transform $\Gamma_F : \mathcal{H} \to \mathcal{H}$ by

$$\Gamma_F(\varphi) = f_2^{-1} \circ [\varphi \circ F_1 \circ (1, \varphi) - \delta_2 \circ (1, \varphi)]$$

Let's prove that Γ_F is well defined map $\mathcal{H} \to \mathcal{H}$ when Δ is C^1-small. First we must show that $\Gamma_F(\varphi)$ is defined for any point of $E_1(r_1)$. This is obvious because

$$F_1(x, \varphi x) = f_1(x) + \delta_1(x, \varphi x)$$

and f_1 maps the ball $E_1(r_1)$ into int $E_1(r_1)$ thus for any $x \in E_1(r_1)$

$F_1(x, \varphi x) \in E_1(z_1)$. Let's evaluate the Lipschitz constant of $\Gamma_F(\varphi)$

$$L(\Gamma_F(\varphi)) = L(f_2^{-1} \circ F_1 \circ (1,\varphi) - \delta_2 \circ (1,\varphi)) \leq L(f_2^{-1}) L(\varphi \circ F_1 \circ (1,\varphi) - \delta_2 \circ (1,\varphi)) \leq$$

$$\leq \delta [L(\varphi) L(F_1) L(1,\varphi) + L(\delta_2) L(1,\varphi)] \leq \delta [L(F_1) + L(\delta_2)] \leq$$

$$\leq \delta [L(f_1) + L(\delta_1) + L(\delta_2)] \leq \delta [L(f_1) + 2\Delta]$$

hence $L(\Gamma_F(\varphi)) < 1$ when δ is sufficiently C^1-small. Thus Γ_F is well defined map of $\mathcal{H}$ into itself.

We shall show that this map is a contruction when δ is sufficiently C^1-small. Let $\varphi, \psi \in \mathcal{H}$ then

$$\| \Gamma_F(\varphi) - \Gamma_F(\psi) \| \leq \delta \| \varphi \circ F_1 \circ (1,\varphi) - \delta_2 \circ (1,\varphi) - \psi \circ F_1 \circ (1,\psi) + \delta_2 \circ (1,\psi) \| \leq$$

$$\leq \delta [L(\delta_2) \| \varphi - \psi \| + \| \varphi \circ F_1 \circ (1,\varphi) - \psi \circ F_1 \circ (1,\psi) \|] \leq$$

$$\leq \delta [\Delta \| \varphi - \psi \| + \| \varphi \circ F_1 \circ (1,\varphi) - \varphi \circ F_1 \circ (1,\psi) \| + \| \varphi \circ F_1 \circ (1,\psi) - \psi \circ F_1 \circ (1,\psi) \|] \leq$$

$$\leq \delta [\Delta \| \varphi - \psi \| + L(F_1) \| \varphi - \psi \| + \| \varphi - \psi \|] \leq$$

$$\leq \delta [1 + 2\Delta + L(f_1)] \| \varphi - \psi \|$$

and one can obtain that $L(\Gamma_F) < 1$ as δ is C^1-small. Thus there is the unique fixed point φ_F of Γ_F by the contracting map theorem. φ_F satisfies the equation

$$f_2^{-1} \circ [\varphi_F \circ F_1 \circ (1, \varphi_F) - \delta_2 \circ (1, \varphi_F)] = \varphi_F$$

hence

$$\varphi_F \circ F_1 \circ (1, \varphi_F) = F_2 \circ (1, \varphi_F)$$

which implies the invariance of gr φ_F under F :

$F(x, \varphi_F x) = (y, \varphi_F y)$ for any $x \in E_1(z_1)$ where $y = F_1(x, \varphi_F x)$

(ii) Next we investigate the differentiability of φ_F. Assume $\mathcal{G}$ is the space of continuous bounded maps $h: E_1(z_1) \to L(E_1, E_2)$ such that $\|h(x)\| < 1$ for any $x \in E_1(z_1)$ where $L(E_1, E_2)$ is the space of continuous linear maps acting from E_1 to E_2. The map

$$\Gamma_F^{\#} : \mathcal{H} \times \mathcal{G} \to \mathcal{H} \times \mathcal{G}$$

is well defined by the formula

$$\Gamma_F^{\#}(\varphi, h)(x) = \left(\Gamma_F(\varphi)(x),\ \mathcal{D}_{\xi}(f_2^{-1})\right)\left\{h(\zeta_2)\mathcal{D}_{\zeta_1}F_1 - \mathcal{D}_{\zeta_2}\delta_2\right\}(1, hx)$$

where $\zeta_1 = (x, \varphi x)$, $\zeta_2 = F_1(\zeta_1)$, $\xi = \varphi(\zeta_2) - \delta_2(\zeta_1)$ if $L(f_2^{-1})$ is small enough. Let $\pi_2 : E_1 \times E_2 \to E_2$ be projection. We shall show that $\pi_2 \Gamma_F^{\#}(\varphi, \cdot) : \mathcal{G} \to \mathcal{G}$ is a contraction. For any $h_1, h_2 \in \mathcal{G}$ we have

$$\|\pi_2 \circ \Gamma_F^{\#}(\varphi, h_1)(x) - \pi_2 \circ \Gamma_F^{\#}(\varphi, h_2)(x)\| \leq$$

$$\leq \|\mathcal{D}_{\xi} f_2^{-1}\| \, \|[h_1(\zeta_2)\mathcal{D}_{\zeta_1}F_1 - \mathcal{D}_{\zeta_1}\delta_2](1, h_1 x) - [h_2(\zeta_2)\mathcal{D}_{\zeta_1}F_1 -$$

$$-\mathcal{D}_{\zeta_1}\delta_2](1, h_2(x))\| \leq L(f_2^{-1})\left(\|h_1(\zeta_2)\mathcal{D}_{\zeta_1}F_1 \circ (1, h_1(x)) -\right.$$

$$- h_1(\zeta_2) \cdot \mathcal{D}_{\zeta_1}F_1 \cdot (1, h_2(x))\| + \|h_1(\zeta_2) \circ \mathcal{D}_{\zeta_1}F_1 \circ (1, h_2(x)) -$$

$$- h_2(\zeta_2) \circ \mathcal{D}_{\zeta_1}F_1 \circ (1, h_2(x))\| + \|\mathcal{D}_{\zeta_1}\delta_2 \circ (0, h_1(x) - h_2(x))\| \leq$$

$$\leq \delta\left(\|h_1(\zeta_2)\| \, \|\mathcal{D}_{\zeta_1}F_1\| \, \|h_1(x) - h_2(x)\| + \|h_1(\zeta_2) - h_2(\zeta_2)\| \cdot \|\mathcal{D}_{\zeta_1}F_1\| \cdot\right.$$

$$\cdot \|(1, h_2(x))\| + \|\mathcal{D}_{\zeta_1}\delta_2\| \, \|h_1(x) - h_2(x)\| \leq$$

$$\leq \delta\left(2\|\mathcal{D}_{\zeta_1}F_1\| + \|\mathcal{D}_{\zeta_1}\delta_2\|\right) \|h_1(x) - h_2(x)\| \leq$$

$$\leq \delta\left(2L(f_1) + 3\Delta\right) \|h_1 - h_2\|$$

Thus $\Gamma_F^{\#}$ is a contruction on the second coordinate provided δ is small. Therefore, by the fiber contruction theorem [4] there is an unique attractive fixed point (φ_F, h_F) of $\Gamma_F^{\#}$. For smooth map φ, $(\varphi, \mathcal{D}\varphi) \in \mathcal{H} \times \mathcal{G}$ and

$$(\Gamma_F^{\#})(\varphi, \mathcal{D}\varphi) = \left(\Gamma_F \varphi,\ \mathcal{D}\Gamma_F(\varphi)\right)$$

it follows from the definition of $\Gamma_F^{\#}$. Indeed

$$\pi_2 \circ \Gamma_F^{\#}(\varphi, \mathcal{D}\varphi)(x) = \mathcal{D}_{\xi}(f_2^{-1})\{\mathcal{D}_{\xi_2}\varphi \ \mathcal{D}_{\xi_1} F_1 - \mathcal{D}_{\xi_1}\delta_2\}(1, \mathcal{D}_x\varphi) =$$

$$= \mathcal{D}_x [f_2^{-1} \circ (F_1 \circ (1,\varphi) - \delta_2 \circ (1,\varphi))] = \mathcal{D}_x \Gamma_F(\varphi)$$

$(\Gamma_F^{\#})^n(\varphi, \mathcal{D}\varphi)$ converges to (φ_F, h_F) as $n \to \infty$. Thus φ_F is the limit of the sequence $\Gamma_F^n(\varphi)$, and h_F is the limit of the sequence of their derivatives $\mathcal{D}[\Gamma_F^n(\varphi)]$, hence $\mathcal{D}\varphi_F = h_F$ (see [6]).

(iii) To prove the lemma it remains to show that φ_F is C^1-close to 0 when $\Delta = \max\{\|\delta_1\|_{\ell}, \|\delta_2\|\}$ is small. For any $x \in E_1(z_1)$ we have

$$|\varphi_F(x)| = |\Gamma_F(\varphi_F)(x)| = |f_2^{-1}(\varphi_F F_1(x, \varphi_F x) - \delta_2(x, \varphi_F x)) -$$

$$- f_2^{-1}(0)| \leq L(f_2^{-1})(|\varphi_F \circ F_1(x, \varphi_F x)| + |\delta_2(x, \varphi_F x)|) \leq$$

$$\leq L(f_2^{-1})(\varphi_F(F_1(x, \varphi_F x)) + \Delta) ,$$

so that

$$|\varphi_F(x)| \leq L(f_2^{-1})(\max_{y \in E_1(z_1)} \varphi_F(y) + \Delta)$$

and

$$\sup_{x \in E_1(z_1)} |\varphi_F(x)| \leq \frac{\Delta}{1 - L(f_2^{-1})}$$

Futher, for any $x \in E_1(z_1)$

$$\|\mathcal{D}_x \varphi_F\| = \|\pi_2 \Gamma_F^{\#}(\varphi_F, \mathcal{D}\varphi_F)(x)\| \leq$$

$$\leq \|\mathcal{D}_{\xi} f_2^{-1}\|\{\|\mathcal{D}_{\xi_2}\varphi_F\|[\|\mathcal{D}_{\xi_1} f_1\| + \|\mathcal{D}_{\xi_1}\delta_1\|] +$$

$$+ \|\mathcal{D}_{\xi_1}\delta_2\|\}\|(1, \mathcal{D}_x\varphi_F)\| \leq$$

$$\leq L(f_2^{-1})\{[L(f_1) + \Delta]\sup_{y \in E_1(z_1)}\|\mathcal{D}_y\varphi_F\| + \Delta\}$$

thus

$$\sup_{x \in E_1(z_1)} \|\mathcal{D}_x \varphi_F\| \leq \frac{\Delta L(f_2^{-1})}{1 - [\Delta + L(f_1)]L(f_2^{-1})}$$

as

$$\mathcal{L}(f_2^{-1}) < \frac{1}{\Delta + \mathcal{L}(f_1)}$$

And this completes the proof of the lemma.

Proof of theorem B.

(i) Let M be a smooth manifold, $\dim M \geqslant 3$, $f: M \to M$ be a C^r-diffeomorphism, $r \geqslant 2$. Using the lemma 5 one obtains that for any C^0-neighbourhood of f there is diffeomorphism f' in this neighbourhood such that

1) f' has periodic point p of period ν,

2) in local coordinates of some neighbourhood of p f' has the form $(f')^\nu(x,y) = (A_1 x, Ay)$, $x \in R^2, y \in R^{n-2}$ where $A_1: R^2 \to R^2$, $A: R^{n-2} \to R^{n-2}$ are linear maps and $\|A_1\| < 1$ while $\|A\| > 1$ is sufficiently large.

A point of R^n we shall denote in two ways; first (x,y) where $x \in R^2$, $y \in R^{n-2}$, and the second (x_1, x_2, y) where $x_1, x_2 \in R^1$. In other words, we shall use two decompositions of R^n is direct sum

$$R^n = R^2 \oplus R^{n-2} = R \oplus R \oplus R^{n-2}.$$

Further for simplicity we shall suppose that the point p is a fixed point ($\nu = 1$).

There is an arbitrarilly small C^0- perturbation f'' of f' which has a form $f''(x,y) = (g(x), Ay)$ in a neighbourhood of the point p which is smaller than one described in 2). Diffeomorphism $g: D^2 \to \operatorname{int} D^2$, where D^2 is two-dimentional disk, has the following properties:

1) g has the basic set $\Lambda \subset \operatorname{int} D^2$, Λ is a sink;

2) g is absolutely structurally stable;

3) $O \in \Lambda$ is the fixed point of g and in some two-dimentional disk $O \in B \subset \operatorname{int} \mathcal{D}^2$ g has the form $g(x_1, x_2) = (\mu x_1, \lambda x_2)$, where $0 < \mu < 1$, $\lambda > 1$, $\lambda\mu > 1$.

Examples of diffeomorphisms whth such properties are given in [10] and in at the beginning of the present paper.

Further we shall denote f'' by f. Thus Λ is the basic set of diffeomorphism f (we don't make distinction between the sets $\Lambda \subset R^2$ and $\Lambda \times \{0\} \subset R^n$, $O \in R^{n-2}$) and the segment γ of the line $\{(x_1, x_2, y) \mid x_1 = 0,\ y = 0\}$ lying in the disk B is in Λ. If $\mathcal{D}^{n-2}$ is sufficiently small nfighbourhood of $O \in R^{n-2}$ then the set $\gamma \times \mathcal{D}^{n-2}$ is local unstable manifold of point O. Every segment $\gamma_t \circ f$ of the line $\{(x_1, x_2, y) \mid x_2 = t,\ y = 0\}$ lying in B is local stable manifolds of the point $Z_t = (0, t, 0) \in \gamma$. Thus the disk B is fibered by the family of the segments γ_t of stable manifolds of points from Λ.

(ii) Now we shall apply the construction due to C.Simon. We shall describe it in R^n keeping in mind that it takes place in a small coordinate neighbourhood of the point $p \in M$.

Let U be a sufficiantly small neighbourhood of Λ in D^2 such that $g(U) \subset \operatorname{int} U$; W_1 - two-dimentional subdisk of B lying in the set $g^{-1}(U) \setminus U$. Consider (n-1)-dimentional subdisk W_2 of the disk $\mathcal{D}^{n-1} = \{(0, x_2, y) \mid (0, x_2) \in B,\ y \in \mathcal{D}^{n-2}\}$, and let $A^{-k}(\pi_y W_2) \cap \pi_y W_2 = \emptyset$ for any $k \geq 0$, where $\pi_y : R^2 \oplus R^{n-2} \to R^{n-2}$ is a natural projection.

Let V_1 be a smoothly imbedded in R^n (n-1)-dimentional disk which is transwersal to $\mathcal{D}^2$ and let the intersection of that disk with $\mathcal{D}^2$ be a smoothly imbedded in $\mathcal{D}^2$ circle S^1, $S^1 \subset \operatorname{int} W_1$. V_2 denotes (n-1)-dimentional disk for which $V_2 \cap V_1 = \emptyset$ and $V_2 \subset \operatorname{int} W_1$. Let K be smoothly imbedded in R^n n-dimentional disk,

$V_1 \cup V_2 \subset \operatorname{int} K$ and $V_1 \cap K = W_1$, $V_2 \cap K = W_2$.

Finally find diffeomorphism $h : R^n \to R^n$ with the following properties:

1) h is identical out of K,

2) $h(V_2) = V_1$ and $h(W_2 \setminus V_2) \cap \mathcal{D}^2 = \emptyset$.

It is easy tosee that Λ is a basic set for $h \circ f$, becouse $h \circ f = f$ in some neighbourhood of Λ.

$S \subset W^u(0, h \circ f)$, indeed, if $(x,y) \in S$, then $h^{-1}(x,y) \in V_2 \subset \mathcal{D}^{n-1}$, hence $h^{-1}(x,y) = (u,v)$ with $u = 0$.

Consider $(h \circ f)^k(x,y) = (h \circ f)^{-k+1} \circ f^{-1}(u,v) =$

$$= (h \circ f)^{-k+1}(0, \lambda^{-1}u_2, A^{-1}v) = f^{-k+1}(0, \lambda^{-1}u_2, A^{-1}v).$$

The last relation can be easily obtained by induction using that $h \circ f = f$ out of K, $K \cap \mathcal{D}^{n-1} = W_2$, $A^{-k}v \in \pi_y W_2$. $(0, \lambda^{-1}u, A^{-1}v) \in W^u(0, f)$, hence $(h \circ f)^{-k}(x,y) \to 0$ when $k \to \infty$. It implies that $(x,y) \in W^u(0, h \circ f)$, thus $S \subset W^u(0, h \circ f)$.

By the same arguments $\gamma_t \cap W_1 \subset W^s(Z_t, h \circ f)$, $Z_t = (0, t, 0) \in \gamma$. This follows from the fact that $x \in W_1$ implies $f(x) \in \mathcal{U}$, $\mathcal{U} \cap W_1 = \emptyset$ and hence $(h \circ f)(x) = f(x)$. At the same time U is f-invariant neighbourhood, hence $(h \circ f)^k(x) = f^k(x)$ for $x \in W_1$. And we obtain that from $x \in W^s(Z_t, f) \cap W_1$ follows $x \in W^s(Z_t, h \circ f)$. S is smoothly imbedded circle in W_1, hence it has tangency with some segment γ_{t_0}, but $S \subset W^u(0, h \circ f)$ and $\gamma_{t_0} \subset W^s(Z_{t_0}, h \circ f)$, hence manifolds $W^u(0, h \circ f)$ and $W^s(\Lambda, h \circ f)$ have a point of tangency.

(iii) It is necessary to prove that the property of tangency is C^k-stable for $k \geq 1$. We shall show that for any diffeomorphism ω from the sufficiently small C^1-neighbourhood the property of tangency of stable and unstable manifolds for basic set takes place. Thus the theorem will be proved.

Let diffeomorphism ω be defined by the formula

$$\omega(x,y) = (gx + \delta_1(x,y),\ Ay + \delta_2(x,y)),$$

where $x \in \mathcal{D}^2$, $y \in \mathcal{D}^{n-2}$ and $\delta_1 : \mathcal{D}^n \to \mathcal{D}^2$, $\delta_2 : \mathcal{D}^n \to \mathcal{D}^{n-2}$ $(\mathcal{D}^n = \mathcal{D}^2 \times \mathcal{D}^{n-2})$ are smooth and sufficiently C^1-small. It was mentioned above that g maps $\mathcal{D}^2$ into int $\mathcal{D}^2$, A was linear map with the norm large enough, hence by the lemma 6 for $f = g \times A$ (setting $E_1 = R^2$, $E_2 = R^{n-2}$ and so on) one can find a smooth map $\varphi : \mathcal{D}^2 \to R^{n-2}$ which is sufficienty C^k-close to the identity and such that its graph $\tilde{\mathcal{D}}_2 = gr\,\varphi = \{(x, \varphi x) \mid x \in \mathcal{D}^2\}$ is invariant under ω.

Let $\xi : \mathcal{D}^2 \times R^{n-2} \to \mathcal{D}^2 \times R^{n-2}$ be new coordinates given by the formula

$$(u,v) = \xi(x,y) = (x,\ y - \varphi(x)).$$

One can obtain $\tilde{\omega}(u,v) = \xi \circ \omega \circ \xi^{-1}(u,v) =$

$$= (gu + \delta_1(u, v + \varphi(u)),\ Av + A \circ \varphi(u) + \delta_2(u, v + \varphi(u)) -$$

$$- \varphi(gu + \delta_1(u, v + \varphi(u)))).$$

It is obvious that $\mathcal{D}^2$ is invariant under $\tilde{\omega}$, because $\mathcal{D}^2 = \xi(\tilde{\mathcal{D}}^2)$ then

$$\omega(u,0) = (g(u) + \delta_1(u, \varphi u), 0) = ([g - \delta_1(I, \varphi)]u, 0)$$

There is a homeomorphism $\eta : \mathcal{D}^2 \to \mathcal{D}^2$, which is sufficiently close to the identity and such that $g + \delta_1(I,\varphi) = \eta \circ g \circ \eta^{-1}$, because g is absolutely structurally stable and $\delta_1(I,\varphi)$ is C^1-small. The point $\eta(0)$ is a fixed point for $\tilde{g}$, $\tilde{\Lambda} = \eta(\Lambda)$ is one-dimentional sink for $\tilde{g}$, $\tilde{\mathcal{U}} = \eta(\mathcal{U})$ is invariant under $\tilde{g}$ neighbourhood of $\tilde{\Lambda}$.

The set $\Lambda_{\omega} = \xi(\tilde{\Lambda}, 0) = \{(x, \varphi x) \mid x \in \tilde{\Lambda}\} \subset \mathcal{D}^2$ is the basic set for ω and $\mathcal{U}_{\omega} = \xi(\tilde{\mathcal{U}}, 0)$ is its invariant neighbourhood in $\mathcal{D}^2$ under ω.

The set $\tilde{W}_1 = \tilde{\mathcal{D}}^2 \cap K$ is smoothly embedded in R^n disk which is sufficiently close to W_1 when δ_1 and δ_2 are C^1-small enough. Thus we can consder that

$$W_1 \subset int(\eta \circ g^{-1}(\mathcal{U}) \setminus \eta(\mathcal{U})) = int(\tilde{g}^{-1}(\tilde{\mathcal{U}}) \setminus \tilde{\mathcal{U}}),$$

because η is small. W_1 and U_{ω} are close to $\tilde{W}_1$ and $\mathcal{U}$ respectively and we obtain that

$$\tilde{W}_2' \subset int(\omega^{-1}(\mathcal{U}_{\omega}) \setminus \mathcal{U}_{\omega}).$$

It is easy to see that each segment of the stable manifold $W^s(\Lambda_{\omega}, \omega)$ of the family fibering the disk $\tilde{W}_1$ is also a segment of the stable manifold $W^s(\Lambda_{h \circ \omega}, h \circ \omega)$,

$$\Lambda_{h \circ \omega} = \Lambda_{\omega}.$$

Consider the map $\omega - T$, where

$$T(x, y) = (\mu x_1, \lambda x_2, Ay)$$

for $(x, y) \in B \times R^{n-2}$ we have

$$(\omega - T)(x, y) = (\delta_1(x, y), \delta_2(x, y)),$$

hence Lipschitz constant $L(\omega - T)$ is small because δ_1, δ_2 are C^1-cmall. There is function $x_1 = \psi(x_2, y)$, $(0, x_2, y) \in \mathcal{D}^{n-1}$ such that

$$gr\,\psi = \{(\psi(x_2, y), x_2, y) \mid (0, x_2, y) \in \mathcal{D}^{n-2}\} = \bigcap_{K \geq 0} \omega^K(\mathcal{D}^n).$$

It follows from the stable manifold theorem for a point (see [4], [6]).

$p_\omega = (\eta(0), \varphi\cdot\eta(0)) \in \mathcal{D}^n$ is a fixed point for ω, gr ψ is the local unstable manifold $W^u_{loc}(p_\omega, \omega)$. We can treat that function ψ is sufficiently C^1-small, because it smoothly depends on ω in C^1-topology and ω is C^1-close to T. In such a case choosing ω C^1-close to f we can obtain that the set $\tilde{W}_2 = gr\,\psi \cap K$ is smoothly embedded in R^{n-1} (n-1) - disk which is C^1-close to W_2. As in (ii) we can obtain that the disk $h(\tilde{W}_2)$ is smoothly embedded in $W^u(p_\omega, h\circ\omega)$.

Thus, if ω is C^1-close to f then

1) disks W_1 and $\tilde{W}_1$ are close to each other in C^1-topology and are smoothly embedded,

2) smoothly embedded in R^n disks W_2 and $\tilde{W}_2$ are C^1-close to each other, hence (n-1)-disks $h(W_2)$ and $h(\tilde{W}_2)$ are close too,

3) disks W_1 is transversal to $h(W_2)$ and their intersection is the circle S smoothly embedded in R^n.

Therfore if the diffeomorphism ω is close to f in Diff (M) then

1) disks $\tilde{W}_1$ and $h(\tilde{W}_1)$ are transversal,

2) their intersection is smoothly embedded in M circle $\tilde{S}^1$ which is C^1-close to S.

From the preceding remarks it is clear that $\tilde{S}^1 \subset W^u(p_\omega, h\circ\omega)$ the disk $\tilde{W}_1$ being fibered by the smooth familyof segments of stable manifolds ofpoints of $\Lambda_{h\circ\omega}$, alike the (ii) one can find that $W^s(\Lambda_{h\circ\omega}, h\circ\omega)$ has a point of tangency with $W^u(\Lambda_{h\circ\omega}, h\circ\omega)$ Finally, notice that values $\mu(p_{h\circ\omega})$ and $\lambda(p_{h\circ\omega})$ are approximat-elyequal to μ and min(λ, $\|A\|$) respectively and $\|A\|$ is large enough, hence $\mu(p_{h\circ\omega})\lambda(p_{h\circ\omega})>1$ and the theorem B has been proved.

REFERENCES

[1] S.E. Newhouse : "Diffeormorphisms with infinitely many sinks", Topology, vol. 12, 1974, 9-18

[2] S.E. Newhouse : "Non-density of axiom A(a) on S^2" Global Analysis, Proc. Symp. in Pure Math. vol. XIV (1970), 191-203

[3] C. Pugh : "An improved closing lemma and a general density theorem". Am. J. Math. 89 (1967), 1010-1022

[4] M. Mirsch and C. Pugh : "Stable manifolds and hyperbolic sets", Global Analysis, Proc. Symp. in Pure Math., vol. XIV (1970), 133-165

[5] M. Shul and D. Sullivan : "Homology theory and dynamical systems" Topology, vol. 14 (1975), 109-132

[6] Z. Nitecki : "Differentiable Dynamics" MIT press, (1971)

[7] R.F. Williams : "One dimensional nonwandering sets" Topology, 6(1967) 473-478

[8] К.Куратовский, "Топология",т.2, Москва, "Мир",1969.

[9] Н.К.Гаврилов, "О трехмерных динамических системах, близких к системам с негрубой гомоклинической кривой", Мат. заметки,т.14,№5,1973,687-696.

[10] Р.В.Плыкин, "Источники и стоки А-диффеоморфизмов поверхностей" Мат.сборник,т94/136/,№2/6/, 1974, 243-264.

[11] А.Ю.Жиров,Д.А.Камаев,Р.В.Плыкин, "Одномерные плоские соленоиды и некоторые их приложения", ДАН СССР,т.232,№4,1977, 756-759

[12] А.Ю.Жиров,Ю.И.Устинов, "Топологическая энтропия одномерных соленоидов Вильямса",Мат.заметки,т.14,№6,1973,859-866.

R.V.Plykin

D.A.Kamaev

A.Ju.Zirov

Institute MIFI
Chair of Applied Mathematics
Joliot-Curie Street 1
Moscow, USSR