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# Coexistence, geometrical and asymptotical properties of hyperbolic codimensional one attractors. Application to diffeomorphisms with infinitely many zero-dimentional attractors

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# COEXISTENCE, GEOMETRICAL AND ASYMPTOTICAL PROPERTIES OF HYPERBOLIC CODIMENSIONAL ONE ATTRACTORS. APPLICATION TO DIFFEOMORPHISMS WITH INFINITELY MANY ZERO-DIMENTIONAL ATTRACTORS.

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Dynamical systems with hyperbolic limit sets which admit an order in the sense of Smale-Newhouse are likely to have properties of interest concerned with mutual position and topology of limit sets.

Let M be compact differentiable manifold and f : $M \to M$ be a diffeomorphism. Let $L^-(f)$ $[L^+(f)]$ be a closure of the set $\alpha$-limit $[\omega$-limit$]$ points of diffeomorphism f.

Diffeomorphisms f : $M \to M$ will be referred to as A-diffeomorphism if the hyperbolic set $L(f) = L^-(f) \cup L^+(f)$. In this case $L(f) = L^-(f) = L^+(f)$ is known to be a closure of the set of periodic points and the truly spectral decomposition $L(f) = \Lambda_1 \cup \Lambda_2 \cup \dots \cup \Lambda_n$ into mutually disjunct, invariant, topologic transitive basic sets $\{\Lambda_k\}_{k=\overline{1,n}}$ for which the ratio $\geqslant$ holds good.

On the condition that $\Lambda_i \geqslant \Lambda_j$ is adequate to the existence of the chain $\Lambda_i = \Lambda_{i_1}, \Lambda_{i_2}, \dots, \Lambda_{i_k} = \Lambda_j$ the elements of which are valid $[\text{clos}\, W^u \Lambda_{i_k}] \cap W^s \Lambda_{i_{k+1}} \neq \phi$ (see [1], [2]).

The maximal (minimal) in the sense of $\geqslant$ basic set is an attractor of the source (sink) kind and has the property $W^s(\Lambda) = \Lambda$ $[W^u \Lambda = \Lambda]$.
Codimensional one basic set is certain to be an attractor represented in the form of a finite union of connected subsets which are basic for some iteration f. Zero-dimensional attractors are attractive or repelling periodic orbits.

Definition. Let us say that $x_0$ is a boundary periodic hyperbolic point of diffeomorphism $f : M \to M$ if $\dim W^u(x_0) = 1$ $[\dim W^s(x_0) = 1]$ and one of the components of the connectivity of the set $W^u(x_0) \setminus x_0$ $[W^s(x_0) \setminus x_0]$ has no homoclinic points.

The proof of the following theorem makes use of Smale-Newhouse's order and is analogous to the proof of theorem 1.7 [6].

THEOREM 1. Let $f : M \to M$ be a diffeomorphism having a basic set $\Lambda$ for which $\dim \Lambda = \dim W^s(x) = \dim M - 1$, $x \in \Lambda$ $[\dim \Lambda = \dim W^u(x) = \dim M - 1]$. The set $M \setminus \Lambda$ has a finite number of connectivity components, each of which is region G having the attractor of diffeomorphism f or some of its iteration.

Every components of the linear connectivity of the boundary of region G having at its intersection with some $W^u(x)$, $x \in \Lambda$ $[W^s(x), x \in \Lambda]$ a boundary point of Cantor's discontinuum $\Lambda \cap W^u(x)$ [a boundary point $\Lambda \cap W^s(x)$ if $\dim \Lambda = \dim W^n(x) = \dim M - 1]$ is represented in the form of manifold $W^s(x_0)$ $[W^u(x_0)]$ which corresponds to the boundary periodic point $x_0$.

If $M = S^2$ the number of connectivity components of $S^2 \setminus \Lambda$ is not less than four. For A-diffeomorphism of $S^n$ the hypothesis according to which the rank of the group n - 1 dimensional Čech cogomology of the attractor of codimensional one is not less than $2^{n-2} \cdot 3$.

In the case of $M = S^2$ the set $S^2 \setminus \Lambda$ contains not less than four zero-dimensional attractors of some iteration f.

The mechanism of appearance of zero-dimensional attractors in the presence of one-dimensional attractor is caused by the existence of the contracted loop which is not selfintersecting and is made up of the section of a stable and a section of an unstable manifolds of some point of one-dimensional attractor.

Definition. One-dimensional attractor $\Lambda$ of A-diffeomorphism of the surface is referred to as "loosely arranged" if there is no contracted loop without selfintersecting, formed by a section of stable and a section of unstable manifolds of some point $x \in \Lambda$.

The property of "loose arrangement" as one can see, is not an internal property of the attractors, but the consideration of the inverse images of "loosely arranged" attractors on the universal covering ascertains the regular behaviour on the infinity of the inverse images of stable and unstable manifolds of the attractor points.

The theorems given below develop and generalize the statements of [4], [6], [7] and can serve as basis for further considerations.

Let M be closed surface. It is to be remembered that it can be obtained from the universal covering $\tilde{M}$ through factorization on the group of automorphisms of universal covering isomorphic $\pi_1$ (M).

In case M is different from the sphere the straight line of the metric of the constant curvature on $\tilde{M}$ invariant with respect to some automorphism of universal covering or resptrictively deviated from the invariant straight line is called a rational one; the line which is not rational is called an irrational one.

Let p: $\tilde{M} \to M$ be a mapping of the universal covering.

THEOREM2. Given A-diffeomorphism f : $M \to M$ of the closed surface having a one-dimensional attractor . The property of the "loose arrangement" is equal to the following: for any x, y $\in p^{-1}\Lambda$ the intersection of lines $\tilde{W}^s(x)$, $\tilde{W}^u(y)$ covering $W^s(px)$ and $W^n(py)$ accordingly consists of not more than one point.

THEOREM 3. Let $\Lambda$ is a "loosely arranged" attractor of A-diffeomorphism of the closed surface. Then the line $\tilde{W}^s(x)$ [$\tilde{W}^n(x)$] lying on the universal covering $\tilde{M}$ which is covering the manifold $W^s(px)$ [$W^u(px)$] having no boundary periodic point is restrictively deviated from some irrational straight line. If $W^s(\rho x)$ $[W^u(\rho x)]$ has a boundary periodic point and the inclusion $W^s(\rho x) \subset \Lambda$ $[W^u(\rho x) \subset \Lambda]$ is valid, the line $\tilde{W}^s(x)$ $[\tilde{W}^u(x)]$ is included into the asymptotic polygon formed by the lines $\tilde{W}^s(x) = \tilde{W}^s(x_1)$, $\tilde{W}^s(x_2)$, ..., $W^s(x_n)$,... $[\tilde{W}^u(x) = \tilde{W}^u(x_1)$, $\tilde{W}^u(x_2)$, ..., $W^u(x_n)$,...$]$,

which contain the inverse image of boundary periodic points and which at the same time are restrictively deviated from irrational straight lines and $\tilde{W}^s(x_k)$ and $\tilde{W}^s(x_{k+1})$ $[\tilde{W}^u(x_k)$ and $\tilde{W}^u(x_{k+1})]$ have a similar asymptotic bahaviour in one of the directions defined by them.

In case the inclusion $W^s(\rho x) \subset \Lambda$ $[W^u(\rho x) \subset \Lambda]$ doesn't take place only one of the continuity components of the set $\tilde{W}^s(x) \setminus x$ $[\tilde{W}^u(x) \setminus x]$ is restrictively deviated from some irrational ray and goes into infinity.

THEOREM 4. Let $q : M_1 \to M$ be a two sheeted covering of the non-oriented surface M by the oriented surface $M_1$. If $\Lambda$ is a "loosely arranged" attractor of A-diffeomorphism $f : M \to M$, there will be two "loosely arranged" attractors $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ of covering diffeomorphism $f_1 : M_1 \to M_1$ or its iteration $f_1^2$ so the $q\tilde{\Lambda}_i = \Lambda$, $i = 1, 2$.

THEOREM 5. The number of different "loosely arranged" attractors of A-diffeomorphism of the surface $S_n^m$, $m = 0, 1, 2$ obtained from the oriented surface of n kind by means of cutting out m disks and patching of the cuttings by Moebius strips doesn't exceed

$$n + \max\left\{\left[\frac{n+m}{2}\right] - 1,\ 0\right\}.$$

Corollary. A-diffeomorphisms of the surfaces $S_0^m$, m = 0, 1, 2 have no "loosely arranged" attractors.

A-diffeomorphisms of the surfaces $S_n^m$, M = 0, 1, 2, n = 0, 1 have zero-dimensional attractors in their spectral decompositions.

In conclusion let us state the application of the structure stable diffeomorphisms of two-dimensional sphere which have one-dimensional attractors to the diffeomorphisms having an infinite number of zero-dimentional attractors introduced by Newhouse [3].

Definition. Some property of the elements of the set $\mathrm{Diff}_r(M)$ is called $C^k$-typical of f $\mathrm{Diff}_r(M)$ $(k \geqslant 1,\ r \geqslant 1)$ if there is a residual subset B of an $C^k$-neighbourhood M(f) of f in $\mathrm{Diff}_r M$ with this property for each element of B.

THEOREM 6 (A.Ju. Žirov, D.A. Kamaev, R.V. Plykin)
The set of diffeomorphisms in $\mathrm{Diff}_r M$, dim M $\geqslant 3$ for which the property of having infinitely many zero-dimensional attractors is $C^k$-typical where $k \geqslant 1$, is $C^0$-dense in $\mathrm{Diff}_r M$.

The complete proof of this theorem is given in the appendix of the paper which is a part of collective report on the activity of the seminar on topology and dynamical systems in Obninsk Branch of Moscow Engineering-Physics Institute in 1976.

## REFERENCES

1 S. Smale
Differentiable Dynamical Systems, Bull. Am. math. Soc. 73 (1967), 747-817.

2 S.E. Newhouse
Hyperbolic Limit Sets, Trans. Am. math. Soc. 167 (1972), 125-150.

3 S.E. Newhouse
Diffeomorphisms with Infinitely Many Sinks, Topology 13 (1974), 9-18.

4 В.В.Гринес О топологической сопряженности диффеоморфизмов двумерного многообразия на ориентируемых базисных множествах I , Труды Московского Математического Общества, т. 32, I975

5 А.Ю.Жиров, Д.А.Камаев, Р.В.Плыкин Одномерные плоские соленоиды и некоторые их приложения, ДАН СССР, т. 232, №4,I977.

6 Р.В.Плыкин Источники и стоки А-диффеоморфизмов поверхностей, Мат.сборник, 94 /I36/ : 2 /6/, I974, 243 - 264.

7 Р.В.Плыкин О топологии базисных множеств диффеоморфизмов Смейла, Мат. сборник, 84 /I26/ /I97I/ 30I - 3I2.

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