T. NADZIEJA

A remark on vector fields on open manifolds

Astérisque, tome 50 (1977), p. 315-322
<http://www.numdam.org/item?id=AST_1977__50__315_0>

Société Mathématique de France
Astérisque 50 (1977) p.315-322

A REMARK ON VECTOR FIELDS ON OPEN MANIFOLDS

by

T . NADZIEJA

Summary . A partial answer is given to the question in what circumstances for a vector field X on an open manifold M there exists a neighborhood U in C^0 – Whitney topology , such that for every $Y \in U$ and compact $K \subset M$ the closure of positive semitrajectory $\overline{\bigcup_{t \geqslant 0} Y_t(K)}$ is compact .

Let M be an open differentiable manifold / i.e. non compact without boundary , with a countable basis / and let $X^1(M)$ be the class of all C^1 vector fields on M endowed with C^0-Whitney topology which is given by the neighborhoods of zero

$$\left\{ X \in X^1(M) \quad : \quad \|X(p)\| \leqslant \varepsilon(p) \right\}$$

where ε is a real positive continuous function on M and $\| \ \|$ is a Riemannian complete metric on M .

Throughout the paper X will denote a complete C^1 vector field on M and $\{X_t\}_{t \in R}$ will be the corresponding flow generated by X . The positive semitrajectory of a point $p \in M$ will be denoted by $O_X(p) = \{X_t(p) : t \geqslant 0\}$, its ω – limit set by $\omega^X(p)$ and α – limit set by $\alpha^X(p)$.

Most of the notations and definitions used here are as in [1] , [3]

DEFINITION 1. We say that $X \in X^1(M)$ is C - stable iff there exists a neighborhood U of X such that for every $Y \in U$ and every compact $K \subset M$ the closure of semitrajectory of K $\overline{\bigcup_{t \geqslant 0} Y_t(K)}$ is compact .

EXAMPLES. Let $M = R^2$ and let flow $\{X_t\}_{t \in R}$ be given by

$$\frac{d}{dt}x^1 = x^2 \qquad \frac{d}{dt}x^2 = -x^1$$

The vector field $(x^2 , - x^1)$ is not C-stable. It is easy to see that the vector field on R^2 given by

$$\frac{d}{dt}x^1 = -x^1 \qquad \frac{d}{dt}x^2 = -x^2$$

is C-stable .

We ask : Which conditions imply C-stability of X ?

In this paper we give a partial answer to this question. Proofs and ideas we use in this paper are similar to those used in studying the phenomenon of Ω-explosion / see [2] / .

DEFINITION 2 . A compact invariant subset A of M is called an attractor iff there exists a neighborhood V of A such that for every $p \in V$ its ω-limit set $\omega^X(p)$ is contained in A . The domain of attraction of A is a maximal subset D of M such that for every $p \in D$, $\omega^X(p) \subset A$.

DEFINITION 3 . Let A be an attractor. We say that A is uniformly asymptotically stable if for every neighborhood U of A there exists neighborhood V of A such that $V \subset U$ and $X_t(V) \subset V$ for every $t \geqslant 0$.

Wilson [4] proved that if A is uniformly asymptotically stable and D is a domain of attraction of A then there exists a smooth function $L : D \longrightarrow R^+$ such that :

I. $L|_A = 0$

II. For any $c \in R^+$ $\mathrm{dist}(L^{-1}(c), A) < +\infty$ and

$\lim_{c \to 0} \mathrm{dist}(L^{-1}(c), A) = 0$ here dist denotes the Hausdorff distance .

III. $\lim_{p_n \to \infty} L(p_n) = +\infty$ and $\lim_{p_n \to \partial D} L(p_n) = +\infty$

and $p_n \to \infty$ means that $\mathrm{dist}(p_n, A)$ tends to $+\infty$

IV. If $p \in D \setminus A$ then $\frac{d}{dt} L(X_t(p)) < 0$

The function L is called Lyapunov function .

DEFINITION 4 . A filtration for X is a collection

$$\{M_i : i = 1,2,\ldots\}$$

of submanifolds of M / with boundaries / such that for every i

1. M_i is compact and $M_i \subset \mathrm{Int}\, M_{i+1}$
2. $X_t(M_i) \subset M_i$ for every $t \geqslant 0$
3. X is transverse to the boundary ∂M_i of M_i
4. $\bigcup_{i \in N} M_i = M$

It is clear that if a filtration for X exists then for every compact $K \subset M$ $\overline{\bigcup_{t \geqslant 0} X_t(K)}$ is compact.

The following property which follows immediately from Definition 4 shows that the existence of filtration is an open property .

LEMMA 1 . If $\{M_i : i = 1,2,\ldots\}$ is a filtration for X then there is a C^0-neighborhood U of X such that $\{M_i : i = 1,2,\ldots\}$ is also a filtration for every $Y \in U$.

In virtue of this Lemma the existence of a filtration implies C-stability of X .

LEMMA 2. Let $X \in \mathfrak{X}^1(M)$, $\overline{\bigcup_{t \geq 0} X_t(K)}$ be compact for every compact $K \subset M$ and let A be an attractor with domain of attraction M. Then there is an uniformly assymptotically stable set B such that $A \subset B$.

PROOF. We present in detail an argument from [2] .

Let $B = \overline{\{p \in M \ : \ \alpha^X(p) \cap A \neq \emptyset\}}$. By definition, B is invariant and closed. Let W be the compact neighborhood of A. If $p \in M$ and $\alpha^X(p) \cap A \neq \emptyset$ then there exists $t < 0$ such that $X_t(p) \in W$ hence $p \in \bigcup_{t \geq 0} X_t(W)$ so the set B is compact / being a closed subset of compact $\overline{\bigcup_{t \geq 0} X_t(W)}$ / . It is clear that B is an attractor. We show that B is uniformly asymptotically stable . Let U be the compact neighborhood of A. By definition of the set B, all points of $U \setminus B$ have their α-limit sets empty. For every positive real number r denote $A_r = \bigcap_{0 \leq t \leq r} X_t(U)$. We note that the compact sets A_r are nested . We show that for sufficiently large r , $X_t(A_r) \subset \text{Int } A_1$ for $0 \leq t \leq 1$. Consider the sets

$$V_r = \bigcup_{0 \leq t \leq 1} X_t(A_r) \setminus \text{Int } A_1$$

which are a nested family of compact sets with empty intersection hence there exists r such that $V_r = \emptyset$. For such r and $0 \leq T \leq 1$ $X_T(A_r) \subset A_r$. This implies that $X_T(A_r) \subset A_r$ for every $T \geq 0$. We put $\text{Int } A_r = V$. Then $B \subset V \subset U$ and $X_t(V) \subset V$ for every $t \geq 0$.

THEOREM 1 . Let $X \in \mathcal{X}^1(M)$ and let for every compact $K \subset M$ the closure of positive semitrajectory $\overline{\bigcup_{t \geq 0} X_t(K)}$ and the set $F = \bigcup_{p \in M} \omega^X(p)$ be compact. Then the vector field X is C-stable .

PROOF. F is an attractor with domain of attraction M . By Lemma 2 there is an uniformly asymptotically stable set B with domain of attraction M. Therefore [4] there exists a smooth Lyapunov function L for B. We define $M_1 = L^{-1}([0, 1])$. M_1 is a compact / being a closed , bounded subset of Riemannian manifold with complete metric / submanifold and $X_t(M_1) \subset M_1$ for every $t \geq 0$. Now define a sequence of submanifolds $\{M_i : i = 1,2,\dots\}$ by putting $M_i = X_{-i}(M_1)$. It is then clear that $\{M_i : i = 1,2,\dots\}$ is a filtration for X, and by Lemma 1 , X is C stable .

Suppose that for a vector field X the set $F = \bigcup_{p \in M} \omega^X(p)$ is a union of compact , invariant , isolated subset ω_i i.e.

$$(*) \qquad F = \omega_1 \cup \omega_2 \cup \omega_3 \cup \dots$$

Let $W^s \omega_i = \{p \in M : \omega^X(p) \subset \omega_i\}$ and $W^u \omega_i = \{p \in M : \alpha^X(p) \subset \omega_i\}$ and define in $\{\omega_i : i = 1,2,\dots\}$ the relation

$$\omega_i \leq \omega_j \quad \text{iff} \quad \overline{W^s \omega_i} \cap \overline{W^u \omega_j} \neq \emptyset$$

LEMMA 3. Let ω_1 , ω_2 , ω_3 be any sets appearing in (*) and suppose that there exists a point $p_0 \in \omega_2$ such that $\overline{O_X(p_0)} = \omega_2$ If $\omega_1 \geq \omega_2 \geq \omega_3$ then in every neighborhoods U and V of X and ω_1 respectively there are $Y \in U$ and $p \in V$ such that $\omega^Y(p) \subset \omega_3$

This Lemma is consequence lemma 9 from [2] .

THEOREM 2 . Let X be a vector field , $F = \bigcup_{p\in M} \omega^X(p) = \omega_1 \cup \omega_2 \cup \dots$ be a union of infinitely many compact , invariant , isolated sets and let for every ω_i there exists $p_i \in \omega_i$ such that $\overline{O_X(p)} = \omega_i$. Moreover let $\overline{\bigcup_{t\geq 0} X_t(K)}$ be compact for every compact K . Then X is stable iff no infinite sequence

$$\omega_{i_1} \geq \omega_{i_2} \geq \omega_{i_3} \geq \dots \quad \text{exists .}$$

PROOF. If suffices to show the existence of a filtration for X

We define a set

$$A_1 = \Big\{ \omega_k : \text{there is a sequence } \omega_{i_1}, \dots, \omega_{i_j} \text{ such that } \omega_{i_1} = \omega_1 ,\ \omega_{i_j} = \omega_k \text{ and } \omega_{i_1} \geq \omega_{i_2} \geq \dots \geq \omega_{i_j} \Big\}$$

Due to our asumption there is no infinite sequence

$$\omega_{i_1} \geq \omega_{i_2} \geq \omega_{i_3} \geq \dots$$

and since $\overline{\bigcup_{t\geq 0} X_t(K)}$ is compact for every compact K , A_1 is finite and $C_1 = \bigcup_{\omega_i \in A_1} \omega_i$ is compact, therefore there is a neighborhood V of C_1 such that for every $p \in V$ $\omega^X(p)$ is contained in C_1 . Hence C_1 is an attractor with a domain of attraction $D_1 = \bigcup_{\omega_i \in A_1} W^s\, \omega_i$. Let $B_1 = \{ p \in D_1 : \alpha^X(p) \cap C_1 \neq \emptyset \}$ The set B_1 is compact invariant / see Lemma 2 / and $B_1 \subset D_1$ by definition of the relation $\leq$. In the some way as in Lemma 2 we may show that B_1 is uniformly asymptotically stable . Let L_1 be a Lyapunov function for B_1 and $M_1 = L_1^{-1}([0, 1])$.

Starting with M_1 we construct a filtration $M_1, M_2, M_3, \dots$ by induction . Suppose that the submanifolds $M_1, M_2, M_3, \dots M_k$ are already done . To define M_{k+1} , put $N_k = X_{-1}(M_k)$.

We choose a set ω_{i_0} not contained in N_k and define

$$A_{k+1} = \{ \omega_k : \text{there is a sequence } \omega_{i_1}, \dots, \omega_{i_j} \text{ such that } \omega_{i_1} = \omega_{i_0} \text{ and } \omega_k = \omega_{i_j} \text{ and } \omega_{i_1} \geqslant \omega_{i_2} \geqslant \dots \geqslant \omega_{i_j} \}$$

Again

$$C_{k+1} = \bigcup_{\omega_i \in A_{k+1}} \omega_i \cup B_k \quad \text{is an attractor.}$$

Let B_{k+1} be the uniformly asymptotically stable set such that $C_{k+1} \subset B_{k+1}$ and let L_{k+1} be a Lyapunov function for B_{k+1}.

Put $M_{k+1} = L_{k+1}^{-1}([0, c_k])$ where c_k is choosen such that $N_k \subset M_{k+1}$

The sequence $M_1, M_2, M_3, \dots$ of compact submanifolds is a filtration for X , for it is clear that $X_t(M_i) \subset M_i$ for $t \geqslant 0$ and X is transverse to ∂M_i . To verify that $\bigcup_{i=1} M_i = M$, take $p \in M$, by assumption $\omega^X(p) \neq \emptyset$, hence there exists k such that p is contained in domain of attraction of B_k . By our construction $X_{-i}(M_k) \subset M_{k+i}$ so there exists l such that $p \in M_l$, hence $\bigcup_{i=1} M_i = M$.

Suppose now that there exists an infinite sequence $\omega_{i_1} \geqslant \omega_{i_2} \geqslant \omega_{i_3} \geqslant \dots$. We will show then , that in every neighborhood of the vector X there is a vector field Y and a point $p \in M$ such that $\omega^Y(p) = \emptyset$. Since the closure of the semitrajectory of every compact set is compact ,we may choose from the sequence $\omega_{i_1}, \omega_{i_2}, \dots$ a sequence $\tilde{\omega}_{i_1} \geqslant \tilde{\omega}_{i_2} \geqslant \tilde{\omega}_{i_3} \geqslant \dots$ such that if only $k > l+1$ then

$$\overline{W^s \tilde{\omega}_{i_k}} \cap \overline{W^u \tilde{\omega}_{i_l}} = \emptyset$$

Then choose for every $\tilde{\omega}_{i_k}$ a neighborhood V_{i_k} of $\tilde{\omega}_{i_k}$ such that $V_{i_k} \cap V_{i_l} = \emptyset$ if $k \neq l$. By Lemma 3 there are a vector field Y_1 and $p_{i_1} \in V_{i_1}$ such that $\omega^{Y_1}(p_{i_1}) \subset \tilde{\omega}_{i_3}$, $Y_1 = X$ off V_{i_1} and $\sup_{p \in M} \| Y_1(p) - X(p) \|$ is arbitrarily small. Similarly we change the vector field Y_1 on V_{i_3} so that for this new vector field Y_2 $\omega^{Y_2}(p_{i_1}) \subset \tilde{\omega}_{i_4}$ and then repeat this procedure for V_{i_4}, V_{i_5}, ... In this way we get a vector field Y which is arbitrarily near to X and such that $\omega^{Y}(p_{i_1}) = \emptyset$. This proves the necessity part of our theorem.

REFERENCES

1. P.Hartman Ordinary differential equations Wiley 1964
2. Z.Nitecki M. Shub Filtrations, decompositions and explosions Amer. Jour. of Math. 4 1975
3. S. Smale Differentiable dynamical systems Bull. Amer. Math. Soc. 73 1967
4. F.W.Wilson Smoothing derivatives of functions and applications Trans. Amer. Math. Soc. 139 1969

Tadeusz Nadziejá
Instytut Matematyczny, Uniwersytet Wrocławski
Pl. Grunwaldzki 2/4 50 234 Wrocław