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A NOTE ABOUT HEDLUND'S THEOREM

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In this note, we propose a new proof of Hedlund's theorem for an arbitrary group without using skew-products.

Let X be a compact space and T a minimal homeomorphism of X. Hedlund proved the following theorem (1)

Theorem

Let f be a continuous function on X. If the sums $\sum_{i=0}^{n-1} f \circ T^{-i}$ are uniformly bounded there exists a continuous function F such that

$$f = F \circ T^{-1} - F$$

One can remark that the sums of iterates appearing in this theorem are the values of a cocycle for the powers of transformation T.
More generally, let X be a compact space and G a group of homeomorphisms of X. An application V from G to the space C(X) of continuous functions on X ($g \mapsto V_g$) is said to be a cocycle if

$$\forall\, h,g \in G \qquad V_{gh} = V_g + V_h \circ g^{-1}$$

Cocycles appear in the study of quasi-invariant probabilities (2) and in the construction of skew-products.

Theorem

Let X be a compact space and G a group of homeomorphisms of X whose action is minimal i.e. every orbit is dense.

Let V be a cocycle on X for the action of G.

If the functions V_g are uniformly bounded there exists a continuous function F such that

$$\forall\, g \in G \qquad V_g = F \circ g^{-1} - F$$

Proof

1. Let a be a point of X and O(a) its orbit under G (which is dense).
Let us define on O(a) a function f by $f(g^{-1}a) = V_g(a)$

We can do it because $V_g(a)$ does not depend on g but only on the point $g^{-1}a$ of O(a) :
If $g^{-1}a = k^{-1}a$, $k = sg$ where s belongs to the group St(a) of all s such that $s(a) = a$.
Then

$$V_{sg}(a) = V_s(a) + V_g(a)$$

But the application from St(a) to R, $s \mapsto V_s(a)$, is a group homomorphism and from the hypothesis of boundedness is the zero homomorphism.
The function f is bounded.

2. Let F the lower semi-continuous function defined by

$$F(x) = \inf \lim_{y \to x,\ y \in O(a)} f(x)$$

F is bounded.
Let us show that the increment of F is V i.e.

$$\forall h \in G \qquad F \circ h^{-1} - F = V_h$$

This relation is true for f for all points in O(a) :

$$f(h^{-1}b) - f(b) = V_{gh}(a) - V_g(a) = V_h(g^{-1}a) = V_h(b) \quad \text{where } b = g^{-1}a$$

Thus $\qquad f(h^{-1}b) = f(b) + V_h(b) \qquad \forall\, b \in O(a) \qquad (1)$

$$F(h^{-1}x) = \inf \lim_{y \to h^{-1}x,\ y \in O(a)} f(y) = \inf \lim_{b \to x,\ b \in O(a)} f(h^{-1}b)$$

with $b = hy$, because h is continuous.
Taking the inf limit in (1)

$$F(h^{-1}x) \geqslant F(x) + V_h(x)$$

and

$$\forall x \in X \quad \forall h \in G \qquad F(h^{-1}x) - F(x) \geqslant V_h(x)$$

Changing h in h^{-1} and x in $h^{-1}x$: $\quad F(hh^{-1}x) - F(h^{-1}x) \geqslant V_h\,(h^{-1}x)$

So $\qquad F(h^{-1}x) - F(x) \leqslant V_h(x) \qquad$ Whence the equality.

3. Let OF be the oscillation function of F

$$OF(c) = \lim \sup_{x,y \to c} F(x) - F(y)$$

OF is a bounded upper semi-continuous function.

$$OF(h^{-1}c) = \sup_{x,y \to c} \lim F(h^{-1}x) - F(h^{-1}y)$$ for h^{-1} is continuous.

But as $F(x) - F(y) = F(h^{-1}x) - F(h^{-1}y) - (V_h(x) - V_h(y))$

Taking the sup limit $OF(c) \leqslant OF(h^{-1}c)$ because the oscillation of the continuous function V_h is zero.

That shows that the function OF is invariant under G.

By minimality, an upper semi-continuous function invariant is a constant.

4. To complete the proof of the theorem it remains to show that the constant function OF is zero. This result comes from the lemma :

Let g a lower semi-continuous function on a topological space E, Og its oscillation. If Og(a) is finite then $\inf_{x \to a} \lim Og(x) = 0$

<u>Proof</u> g being a l.s.c. function

$$Og(x) = \sup_{y \to x} \lim g(y) - \inf_{y \to x} \lim g(y) = \sup_{y \to x} \lim g(y) - g(x)$$

If Og(a) is finite there exists an open neighbourhood V_o of a in which g is bounded. So that for every open neighbourhood $V \subset V_o$, for every ε , there exists an x_V in V with

$$g(x_V) \geqslant \sup_{y \in V} g(y) - \varepsilon$$

Then

$$Og(x_V) \leqslant \sup_{y \quad V} g(y) - g(x_V) \leqslant \varepsilon$$ q.e.d.

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