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ON GENERIC BIFURCATIONS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS NEAR CLOSED ORBITS

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We shall present some results concerning the generic bifurcations near closed orbits of the second order ordinary differential equations (ODE) on manifolds. Generic bifurcations near critical points of the second order ODE are studied in [5]. The main results obtained in [5] and given here are similar to the results of Sotomayor [7] for 1-parameter families of vector fields (see also [3],[4]) and to the results of Brunovsky [1], [2] for diffeomorphisms.

Definition

A vector field η on the tangent bundle $T(X)$ of a C^{r+1} manifold X is called a second order ODE on X if $D\tau.\eta = 1_{T(X)}$, where $D\tau$ denotes the differential of the natural projection $\tau : T(X) \to X$ and $1_{T(X)}$ is the identical mapping of $T(X)$ onto $T(X)$. A C^r parametrized vector field $\xi : A \times T(X) \to T^2(X) = T(T(X))$ (A is a C^r manifold) is called a parametrized second order ODE on X if the mapping ξ_a ($\xi_a(x) = \xi(a,x)$ for $x \in T(X)$) is a second order ODE for each $a \in A$. We denote the set of such parametrized equations by $H^r(A,X)$.

The condition $D\tau.\xi_a = 1_{T(X)}$ for each $a \in A$ implies that in local charts (U,α), (V,β) the local representative ξ' of $\xi \in H^r(A,X)$ has the following form

$$\xi' : (\mu,x,v) \to (x,v,v,\xi_{\alpha\beta}(\mu,x,v)) ,$$

where $\mu \in \alpha(U) \subset R^1$, $(x,v) \in \beta(V) \times R^n$, $\xi_{\alpha\beta} : \alpha(U) \times \beta(V) \times R^n \to R^n$ is a C^r map (We suppose X,A compact and $\dim X = n$, $\dim = A=1$), what corresponds to the 1-parameter family of second order equations $\ddot{x} = \xi_{\alpha\beta}(\mu,x,\dot{x})$.

We suppose that the set $H^r(A,X)$ is endowed with the Whitney C^r topology.

Let $\xi \in H^r(A,X)$, $a_0 \in A$, γ_{a_0} be a closed orbit of ξ_{a_0} passing through $x_0 \in T(X)$ and let $\sum \subset T(X)$ be an $(2n-1)$-dimensional transverse section to γ_{a_0} at x_0 . Denote by $H=H[\xi,a_0,x_0,\gamma_{a_0},V_1 x(V_2 \cap \sum)] : V_1 x(V_2 \cap \sum) \to \sum$ the parametrized Poincaré mapping, where V_1 , V_2 are neighbourhoods of a_0 and x_0 respectively. Let $H_a : V_2 \cap \sum \to \sum$ be defined via $H_a(x) = H(a,x)$ for $x \in V_2 \cap \sum$. Define $Z_k(H) = \{(a,x) \in V_1 x(V_2 \cap \sum) \mid H_a^k(x) = x ,$ $H_a^j(x) \neq x$ for $0 < j < k\}$.

Definition

We say that $x_0 \in V_2 \cap \sum$ is a quasi-hyperbolic fixed point of H_{a_0} if for some center-manifold W^c , H_{a_0}/W^c is C^r conjugate to one of the following C^r diffeomorphisms

(A) $y_2 = y_1 + \alpha_2 y_1^2 + o(|y_1|^2)$, $\alpha_2 \neq 0$

(B) $y_2 = -y_1 + \alpha_2 y_1^2 + \gamma_1 y_1^3 + o(|y_1|^3)$, $\gamma_1 + \alpha_2^2 \neq 0$

(C) $u_2 = \lambda u_1 + \beta |u_1|^2 u_1 + o(|u_1|^3)$,

u_1 is complex , $\lambda, \lambda^2, \lambda^3, \lambda^4 \neq 1$, $\mathrm{Re}\,\frac{\beta}{\lambda} \neq 0$.

The corresponding closed orbit of ξ_{a_0} passing x_0 is called quasi-hyperbolic.

Denote by $P_1(\xi)$ $(P_2(\xi), P_3(\xi))$ the set of $(a,x) \in V_1 x (V_2 \cap \Sigma)$ such that x is a quasi-hyperbolic fixed point of H_a of the type (A) ((B),(C)).

Theorem.

There is a residual set $H_1^r(A,X) \subset H^r(A,X)$ such that for $\xi \in H_1^r(A,X)$ the following is true :

(1) For all $a \in A$ the second order ODE ξ_a has only hyperbolic and quasihyperbolic closed orbits

(2) If γ_{a_0} is a quasi-hyperbolic closed orbit of ξ_{a_0}, then there is a neighbourhood U of γ_{a_0} such that for each $a \in U$ ξ_a has no quasi-hyperbolic closed orbit in a sufficiently small neighbourhood of ξ_{a_0}

(3) If γ_{a_0} is a closed orbit of ξ_{a_0} passing through x_0, then there is a chart $(V_1 x V_2, h_1 x h_2)$ on $A \times T(X)$ at (a_0, x_0), $h_1(a_0) = 0$, $h_2(x_0) = 0$ such that

(a) $Z_1 = Z_1(H)$ is an 1-dimensional C^r submanifold of $V_1 \times \Sigma$ $(H = H[\xi, a_0, x_0, \gamma_{a_0}, V_1 x (V_2 \cap \Sigma)])$

(b) If $(a_0, x_0) \in P_1(\xi)$, then $(h_1 x h_2)(Z_1(H)) = \{(\mu, y_1, \ldots, y_{2n-1}) \mid \mu = \phi_0(y_1), y_i = \phi_i(y_1), i = 1,2,\ldots,2n-1, y_1 \in J\}$, where J is an open interval, $0 \in J$, $\phi_i \in C^r$, $i=1,2,\ldots,2n-1$, $\phi_0(0) = 0$, $d\phi_0(0)/dy_1 = 0$, $d^2\phi_0(0)/dy_1^2 \neq 0$ and for $\mu > 0$ $\xi_a (a = h_1^{-1}(\mu))$ has exactly two closed orbits γ_1, γ_2 in a neighbourhood of γ_{a_0}, which are hyperbolic. The index of γ_1 is equal to the index of γ_{a_0} and that of γ_2 is equal to the index of γ_{a_0} minus 1. Moreover $V_1 x (V_2 \cap \Sigma) \setminus Z_1(H)$ contains no invariant set

(c) If $(a_0, x_0) \in P_2(\xi)$, then $\bar{Z}_2 = \overline{Z_2(H)}$ is an 1-dimensional C^{r-1} submanifold of $V_1 x \Sigma$, $\bar{Z}_2 \setminus Z_2 = P_2(\xi)$ and $V_1 x (V_2 \cap \Sigma) \setminus (Z_1 \cup Z_2)$ contains no invariant set. This means that if $\mu = h_1(a)$, $\mu_0 = h_1(a_0)$, and τ_0 is a prime period of γ_{a_0}, then for any μ close to μ_0 there is one closed orbit of ξ_a the period of which tends to τ_0 as $\mu \to \mu_0$ and there is another closed orbit of ξ_a for μ close to μ_0, the period of which tends to $2\tau_0$ as $\mu \to \mu_0$

(d) If $(a_0, x_0) \in P_3(\xi)$, then H_{a_0} has the composed focus fixed point, which give rise to a two-dimensional invariant torus.

<u>Sketch of the proof</u>.

Denote by $H_I^r(A,TX)$ the set of all parametrized vector field on $T(X)$ endowed with the C^r Whitney topology. By [3], [4] and [7] the properties (1)-(3) of the theorem are generic in the set $H_I^r(A,TX)$. Therefore the proof of the theorem is based on the following approximation lemmas.

<u>Lemma 1</u> : (see [4, Lemma 6]).

Let $\xi \in H^r(A,X)$, $(a_0, x_0) \in A x\, T(X)$ and let γ_{a_0} be a closed orbit of ξ_{a_0}. Let $V_1 \times V_2$ be a neighbourhood of $(a_0; x_0)$ in $A \times T(X)$ such that the parametrized Poincaré mapping $H = H[\xi, a_0, x_0, \gamma_{a_0}, V_1 x (V_2 \cap \Sigma)]$ is defined. Then there exists a neighbourhood $U(H)$ of H in $C^r(V_1 x (V_2 \cap \Sigma), \Sigma)$ such that for $H_1 \in U(H)$ there is a $\tilde{\xi} \in H_I^r(A,TX)$ such that $\phi^{\tilde{\xi}}(a,x,\tau(a,x)) = H_1(a,x)$ for each $(a,x) \in V_1 x (V_2 \cap \Sigma)$, where $\phi^{\tilde{\xi}}$ is the parametrized flow of $\tilde{\xi}$ and $\tau : V_1 x (V_2 \cap \Sigma) \to \Sigma$ is a C^r function. Moreover $\tilde{\xi}$ depends continuously on H_1.

Lemma 2

Let $\xi \in H^r(A,X)$ and let (a_0,x_0), $\tilde{\gamma}_{a_0}$, $H, U(H), H_1$ are as in Lemma 1. Let $\tilde{\xi}$ be as in Lemma 1 i.e. $\phi^{\tilde{\xi}}(a,x,\tau(a,x)) = H_1(a,x)$ for $(a,x) \in V_1 x (V_2 \cap \Sigma)$. Then there exists an $\eta = \eta(\tilde{\xi}) \in H^r(A,X)$ such that if (a,x) is sufficiently close to (a_0,x_0), then the vector field η_a is differentiably conjugate to $\tilde{\xi}_a$ near $\tilde{\gamma}_a$, where $\tilde{\gamma}_a$ is the closed orbit of $\tilde{\xi}_a$ passing through x.

It is possible to prove this lemma by means of some modification of the proof of Shahshahani's Fundamental Lemma (see [6]).

Now, using this two lemmas and the results of Brunovsky [2] it is possible to prove Theorem.

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