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TORAL AUTOMORPHISMS, TOPOLOGICAL ENTROPY AND THE FUNDAMENTAL GROUP

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The theme of this article is representing a discrete dynamical system in a toral automorphism in order to study the influence of the underlying space on the dynamical properties of the system. We study a continuous map $f:X\to X$ of a compact metric space X and assume that X is a finite CW complex so that $\pi_1(X)$ with be finitely generated.

Automorphisms of the n-dimensional torus T^n, particularly the hyperbolic and other ergodic ones, are among the most beautiful dynamical systems. So are automorphisms of nilmanifolds. An endomorphism of T^n is given by an $n \times n$ matrix of integers and an endomorphism of a nilmanifold by an endomorphism of a finitely generated torsion free nilpotent group.

If we put $\pi_1(X) = \Gamma$ we can define a central series $\Gamma = \Gamma_t^0 \supset \Gamma_t^1 \supset \Gamma_t^2 \supset \dots$ inductively by $\Gamma_t^{i+1} = [\Gamma_t^i,\Gamma] \cup$ those of its cosets in Γ_t^i that are of finite order in the quotient group. Then Γ/Γ_t^n is a finitely generated torsion free nilpotent group and $f_*:\Gamma\to\Gamma$ induces an endomorphism (that we shall also call f_*) of Γ/Γ_t^r that determines a nilmanifold endomorphism. If $r = 1$ then $\Gamma/\Gamma_t^r \cong \mathbf{Z}^n$ where n is the first Betti number of X and f_* is the endomorphism of $H_1(X;Z)/\text{torsion}$. We shall denote the corresponding endomorphism of T^n by $\bar{f}_*$ or g .

The quotient homomorphism $\pi_1(X) \to \pi_1(T^n)$ is, since T^n is a $K(\pi,1)$, induced by some map $q: X \to T^n$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & T^n \\ f\downarrow & & \downarrow g \\ X & \xrightarrow[q]{} & T^n \end{array}$$

commutes up to homotopy. In fact all this work holds for any path connected compact metric space X with $\pi_1(X)$ finitely generated provided such a map q exists (compare [6,page 48]). In this thesis [2] Franks obtained the remarkable result that, in such a diagram, provided g is a hyperbolic toral automorphism there is a map $j:X \to T^n$, homotopic to q , such that

$$\begin{array}{ccc} X & \xrightarrow{j} & T^n \\ f\downarrow & & \downarrow g \\ X & \xrightarrow[j]{} & T^n \end{array}$$

commutes. In fact, when g is a general toral endomorphism it is still possible to salvage part of Franks' conclusion. $gq-qf:X \to T^n$ is homotopically trivial and so can be lifted to a map $X \to R^n$. Now $\mathbb{R}^n = E^s \oplus E^t \oplus E^u$ where E^s, E^t, E^u are the direct sums of generalised eigenspaces corresponding to eigenvalues of $\tilde{g}:\mathbb{R}^n \to \mathbb{R}^n$ of absolute value $<,=,>1$. By using only the part of Franks' proof that relates to E^u it is straightforward to modify q to a homotopic map j s.t., when $l=gj-jf:X \to T^n$ is lifted to $\mathbb{R}^n$, $lX \subset E^s \oplus E^t$. Moreover this can be done without assuming that f has a fixed point which can be used as a base point.

<u>Proposition</u> [9]

If $\begin{array}{ccc} X & \xrightarrow{j} & T^n \\ f\downarrow & & \downarrow g \\ X & \xrightarrow[j]{} & T^n \end{array}$ commutes and $j^X=T^n$ then $j\Omega(f)=T^n$ also.

Proof.

If $j\Omega(f) \neq T^n$ choose a periodic point y for g not in $j\Omega(f)$. Then j^{-1} (the g-orbit of y) is a closed f-invariant subset of X disjoint from $\Omega(f)$, which is impossible.

Thus $f_{*1}:H_1(X;R) \to H_1(X;R)$ being hyperbolic and $jX = T^n$ has implications for the size of $\Omega(f)$ and the way it sits in X. If $jX \neq T^n$ it is some proper g-invariant subset of T^n and such sets have been studied by Hirsch [5] and Franks. For example Franks has shown that, except when it contains a lower dimensional torus, such an invariant subset connot contain any C^2 arcs [3]. On the other hand Hancock has shown how to construct invariant sets containing C^0 arcs [4]. We wonder whether anything can be deduced about $\Omega(f)$ in such cases.

Topological Entropy

Given $\delta > 0$, we say $Y \subset X$ is δ-separated if $y_1, y_2 \in Y$, $y_1 \neq y_2 = d(y_1,y_2) \geq \delta$. For a positive integer k we say $Y \subset X$ is (k,δ)-separated for f if $y_1,y_2 \in Y$, $y_1 \neq y_2 \Rightarrow \exists_i$, $0 \leq i \leq k$ s.t. $d(f^i y_1, f^i y_2) \geq \delta$, [1]. Then put $h(f,\delta) = \limsup_{k\to\infty}(1/K) \log$ (maximal cardinality of a (K,δ)-separated set for f) and define the topological entropy of $f, h(f)=\lim_{\delta\to 0} h(f,\delta)$. The "Entropy Problem for Continuous Maps", which is related to Shubs' Entropy Conjecture [12], can now be stated as follows. What information can be deduced about $h(f)$ from $f^*:H^*(X;R) \to H^*(X;R)$ given only that $f:X\to X$ is continuous ? It turns out that several results on this problem can be viewed in the light of the representation of $f:X\to X$ in the toral endomorphism $g:T^n \to T^n$ described above.

Theorem [10]

$$h(f) \geq \log\lambda \quad \text{where} \quad \lambda = sp(f_{*1}:H_1(X;\mathbb{R}) \to H_1(X;\mathbb{R}))$$

Proof.

We can use the map $q:X \to T^n$ that gives a homotopy commutative diagram without appealing to the commutativity properties of j. Let $\tilde{q}:X \to \mathbb{R}^n$ be the lift to the universal cover. Choose a loop σ in X so that, if $[\sigma]$ is its homology class in $H_1(X;\mathbb{R})$ then $f^k_{*1}[\sigma]$ has the required growth rate λ as k increases. $\tilde{j}\tilde{\sigma}(I)$ is a path joining 0, say, to some $a \in \mathbb{Z}^n \subset \mathbb{R}^n$ and $\tilde{j}\tilde{f}^k\tilde{\sigma}(I)$ is some path whose endpoints differ by the vector $\tilde{g}^k a$, which is a vector of length at least $c\lambda^k$ for some c. Although we know very little else about $\tilde{j}\tilde{f}^k\tilde{\sigma}(I)$ this fact ensures that it contains a set of at least $c\lambda^k/\varepsilon$ points which are ε-separated in $\mathbb{R}^n$, $\varepsilon = \frac{1}{2}$ say. Since $\tilde{j}$ is uniformly continuous like j there is a corresponding δ s.t. by choosing one point in the inverse image of each of these ε-separated points we obtain a δ-separated set, Z', in $\tilde{f}^k\tilde{\sigma}(I) \subset \tilde{X}$. If we choose a set of points in $\tilde{\sigma}(I)$ that is mapped bijectively onto Z' by $\tilde{f}^k$ then the image of this set by the projection $P_X:\tilde{X} \to X$ is a (k,δ)-separated set for $f:X \to X$ (provided we omit one of the points at the end of $\tilde{\sigma}(I)$). In fact the iterates by f of a pair of these points will be δ-separated in X exactly when the corresponding points in $\tilde{X}$ first become δ-separated there. By this technique we have constructed separated sets with growth rate λ which proves the theorem.

Several people (including R. Bowen, A.B. Katok [8] and M. Shub) noticed the following improvement on theorem 1. The growth rate of an endomorphism $f_*:\pi_1(X) \to \pi_1(X)$ of a finitely generated group with a given set of generators Δ is defined as $\limsup_{k\to\infty} \max_{\gamma\varepsilon\Delta}(1/k)$ log (word length using Δ of $f_*^k\gamma$). This is independent of Δ and it does not matter if we only know f_*^k up to an inner automorphism (conjugation by an element of Δ, say).

Theorem 2.

$h(f) \geq$ growth rate of $f_*:\pi_1(X)\to\pi_1(X)$

Proof.

Let σ be a loop s.t. $f_*^k[\sigma]$ grows at the given rate. (Here $[\ ]$ denotes homotopy class in $\pi_1(X)$). The number of δ-separated points in $\tilde{f}^k\,\tilde{\sigma}(I)$ is greater than C times the word length of $f_+^k|\sigma|$ for some constant C depending on X, Δ and δ. As before this enables us to find (k,δ)-separated sets in $\sigma(I)$ for $f:X\to X$ whose cardinality grows fast enough to prove the theorem.

Misiurewicz and Przytycki's original proof of their theorem which follows is reminicent of the above techniques.

Theorem 3. [11]

Let $f:T^n\to T^n$ be continuous. Then

$h(f) \geq \log \operatorname{sp} f_{*r}:H_r(T^n;\mathbb{R}) \to H_r(T^n;\mathbb{R})$ for each r.

Proof.

If $g=\overline{f}_{*1}$ is a hyperbolic automorphism of T^n then, by Franks theorem $\exists j \simeq id:T^n\to T^n$ (and therefore surjective) s.t. $jf = gj$. Hence $h(f) \geq h(g)$ which is known to be $\log \operatorname{sp} f_*:H_*(T^n;\mathbb{R}) \to H_*(T^n;\mathbb{R})$

Now suppose that $g=\overline{f}_{*1}$ is an endomorphism and not hyperbolic. In this case, we find $j:T^n\to T^n$ s.t. $jf = gj+1$ where $1:X\to T^n$ has a lift $\tilde{1}:X\to\mathbb{R}^n$ with $\tilde{1}(X)\subset E^s\oplus E^t$. Sp $f_*=g_*$ occurs as $|\mu|$ for some eigenvalue μ in $H_r(T^n;\mathbb{R})$ for some smallest number r. Take an r-dimensional subspace of E^n in which volume is expanded by the factor $|\mu|$. If μ is the product of eignevalues in H_1 some of which are complex this r-space will be rotated as well as expanded. Choose a small ball B in this r-space of r-dimensional Lebesgue measure $a > 0$. $\tilde{g}^k B$ has measure $|\mu|^k a$. Choose $\varepsilon > 0$ and then in $\tilde{g}^k B$ choose a maximal set Y of ε-separated points $\{y_1,\ldots,y_N\}$. number N of such points is at least $|\mu|^k a/\gamma\varepsilon^r$ if $\gamma\varepsilon^r$ is the r-volume of a ball of radius 2ε because if such balls about the points y_i have total measure $<|\mu|^k a$ another point could be chosen to enlarge the ε-separated set. $p_T\tilde{g}^{-k}Y$ is a (k,ε)-separated set for $g:T^n\to T^n$, actually contained in $p_T B$. (Here p_T denotes the projection $R^n\to T^n$). Construct $Z \subset T^n$ to contain one point of $j^{-1}y$ for each $y\varepsilon\, p_T g^{-k}Y$. Then two points $z_1,z_2 \in Z$ have their f-orbits mapped by j to the g-orbits of jz_1,jz_2 except for displacements due to 1 in the s and t directions. We claim that Z is a (k,δ)-separated set for $f:T^n\to T^n$ and prove this by supposing $d(f^i z_1,f^i z_2) < \delta$ for $0\le i\le k$ but $z_1 \ne z_2$. Then $d(jf^i z_1,jf^i z_2) < \varepsilon$ for $0\le i\le k$ and $d(1f^i z_1,1f^i z_2) < \varepsilon$ for $0\le i\le k$ (provided that δ has been chosen using the uniform continuity of 1 as well as that of g). Thus $p_u(jf^i z_1 - f^i z_2) = p_u(g^i jz_1 - g^i jz_2)$ where the projection $p_u R^n\to E^u$ can act on these elements of T^n because they are close to the 0 in T^n and we regard such points as belonging to R^n. Now the right hand side grows steadily to a vector of size $\ge \varepsilon$ before $i = k$ and yet we are assuming that the left hand side remains small. This contradiction shows that Z is (k,δ)-separated for f. Now that we have found a separated set whose cardinality grows with k at the required rate the theorem is proved.

The same technique will prove a more general theorem. $H^1(X;R)$ generates by the cup product a subalgebra $\hat{H}(X)$ of $H^*(X)$ which is

$\oplus_i\ H^1(X;R)\cup\ldots\cup H^1(X;R)$ (i times). $\hat{H}(X;R) = j^*H^*(T^n;R)$ where n is the first Betti number of X and $j:X\to T^n$ as above induces an epimorphism $j_*:\pi_1(X)\to\pi_1(T^n)$.

Theorem 4.

If $f:X\to X$ is continuous then $h(f)\geq\log \mathrm{sp}\, f^*\ |\hat{H}(X)$.

Proof.

$\mathrm{sp}\, f^*\ |\hat{H}(X)$ appears as $|\mu|$ for an eignevalue μ of $f^*\ |\ j^*\ H^r(T^n;\mathbb{R})$ for some least r . The eigenvalue μ and its eigenvector v for $g^*\ |H^r(T^n;\mathbb{R})\to H^r(T^n;\mathbb{R})$ correspond to an r-dime - sional g-invariant suspace V of E^u spanned by r eigenvectors of $\tilde{g}:\mathbb{R}^n\to\mathbb{R}^n$ in which r-dimensional Lebesgue measure is increased by $\tilde{g}$ by a factor of $|\mu|$ (or, in the case where some of the relevant eigenvalues of $\tilde{g}$ are complex; to an r-dimensional subspace which is rotated by g while measure is increased by a factor $|\mu|$).

Let σ be a singular cycle in X over $\mathbb{Z}$, $[\sigma]\in H_r(X;\mathbb{Z})$ s.t. $<j_*[\sigma],v>\neq 0$. There is a translation invariant r-form in T^n whose class in de Rham cohomology is v and we choose a singular r-cocycle ω in T^n which, for C^1 simplexes, just integrates this form over the simplex. The evaluation of ω on $j\sigma$ is not affected by lifting each singular simplex from T^n to $\mathbb{R}^n$ and then applying $p_V:\mathbb{R}^n\to V$ (which forgets components in the direction of other eigenvectors). An r-cycle cannot carry any r-dimensional homology in V if its image is of dimension $\leq$ r-1 . So the union of the simplexes in $p_Vj\sigma$ has dimension r . Thus it contains a non-empty open set in V , [7, page 44] and hence also contains (k,ε)-separated sets for $\tilde{g}$ for each k whose cardinality has growth rate $\log|\mu|$. These give rise to finite sets in some simplexes of σ in X that, as in the last proof, are (k,δ)-separated for f which proves theorem 4.

Some technical problems with using this last argument for nilmanifolds remain in the way of showing that whenever X can be mapped to a nilmanifold M found from $\pi_1(X)$ as at the beginning of this article then $j^*H^*(M;R)$ froms a larger part of $H^*(X;R)$ in which $\log sp\, f^*$ still forms a lower bound for the topological entropy of any map f.

Remark :

It is interesting that we seem in each case to be able to find our separated sets in any cycle whose homology class grows at the required rate. From the point of view of applications it is encouraging that when $h(f) \geq$ some lower bound so is $h(f,\delta)$ without δ being too small.

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