Astérisque

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*Astérisque*, tome 40 (1976), p. 177-179

<http://www.numdam.org/item?id=AST_1976__40__177_0>

Société Mathématique de France
Astérisque 40 (1976) p.177-179

# ERGODIC PROPERTIES OF A COMPLEX CONTINUED FRACTION TRANSFORMATION

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Recently R. Kaneiwa, J. Tamura and the author of this paper gave an alternative proof of the theorem of Perron [2] : For any complex number $z$ not belonging to Eisenstein's field $Q(\sqrt{-3})$ there exist infinitely many integers $p$, $q$ in $Q(\sqrt{-3})$ such that

$$\left| z - \frac{p}{q} \right| < \frac{1}{\sqrt[4]{13}\,|q|^2} .$$

If $z = \frac{1}{2}(\xi + \sqrt{\xi+4})$, where $\xi = \frac{1}{2}(1+\sqrt{-3})$, the constant $\sqrt[4]{13}$ cannot be improved. They proved it by making use of the following simple continued fraction expansion for complex numbers. (Perron's original proof had been based on a lemma on Cassini's curve.) Every complex number $z$ can be uniquely written in the form $z = u\xi + v\bar{\xi}$, where $u$ and $v$ are real. We put $[z] = [u]\xi + [v]\bar{\xi}$, where in the right-hand side $[x]$ is the largest rational integer not exceeding a real number $x$, and define a complex continued fraction algorithm as follows;

$$(*) \quad \begin{cases} T^n z = \dfrac{1}{T^{n-1}z} - \left[\dfrac{1}{T^{n-1}z}\right] \ (n \geq 1), \quad T^0 z = z - [z] , \\[2ex] a_n = a_n(z) = \left[\dfrac{1}{T^{n-1}z}\right] \ (n \geq 1), \quad a_0 = a_0(z) = [z] . \end{cases}$$

These procedures terminate; i.e. $T^n z = 0$ for some $n \geq 0$, iff $z$ belongs to $Q(\sqrt{-3})$. Hence every complex number z can be expanded in the form

$$z = a_0 + \frac{1|}{|a_1} + \dots + \frac{1|}{|a_n + T^n z} \qquad (n \geq 0),$$

provided $T^k z \neq 0$ for all $k < n$. Further for any complex number $z$ we have [1]

$$z = \lim_{n\to\infty} \left(a_0 + \frac{1|}{|a_1} + \quad + \frac{1|}{|a_n}\right).$$

By means of this equality the algorithm (✱) well-defines a complex continued fraction expansion which is a natural extension of the real one.

In metrical theory of real Diophantine approximations the study of the continued fraction transformation is of special importance. We shall investigate in this line the ergodic properties of the remainders $T^n z$ appearing in the algorithm (✱). Put $X = \{u\xi + v\bar{\xi} : 0 \leq u, v < 1\}$ and define a transformation T of X onto itself by

$$Tz = \frac{1}{z} - \left[\frac{1}{z}\right] \qquad (z \in X),$$

which is clearly non-singular with respect to Lebesgue measure. Then we have the following theorem : There exists a finite T-invariant measure equivalent to Lebesgue measure under which T is exact (in Rohlin's sense).

## REFERENCES

[1] R. Kaneiwa, I. Shiokawa and J. Tamura, A proof of Perron's theorem on Diophantine approximation in Eisenstein's field $Q(\sqrt{-3})$, to appear.

[2] O. Perron, Über einen Approximationzzatz von Hurwitz und über die Approximation einer komplexen Zahl durch Zahlen des Körpers der dritten Einheitswurzeln, Sitz. d. Bayer (1931), 129 - 154.

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