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Zeta functions and statistical mechanics

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I. ZETA FUNCTIONS

Zeta functions are objects of deep significance in mathematics, and problems related to them are very difficult and fascinating.

In this talk, I shall enumerate a certain number of zeta functions mention possible connections with statistical mechanics, and then indicate some precise results.

I.1. Riemann's zeta function [1]

It is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

for $\operatorname{Re} s > 1$, and has the following properties.

a) ζ extends to a meromorphic function in the entire complex plane with only one pole, which is simple and located at $s = 1$

b) ζ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \frac{\pi s}{2} \zeta(1-s)$$

c) ζ has "trivial" zeros at $z = -2m$ (for integer $m > 0$) . According to the Riemann hypothesis, the other zeros ("nontrivial zeros") are on the line $\operatorname{Re} s = \frac{1}{2}$.

I.2. Weil's zeta function [2]

It counts points in projective algebraic manifolds over finite fields.

More precisely let F_q be a finite field with q elements , $\bar{F}_q$ its algebraic closure, P^n_F the n-dimensional projective space on $\bar{F}$, and M be a non singular algebraic manifold in P^n_F , consisting of the common zeros of a finite family of homogeneous polynomials in $n+1$ variables with coefficients in F . A map $\varphi : P^n_F \to P^n_F$ (Frobenius map) is defined by

$$\varphi(x_0,\ldots,x_n) = (x_0^q,\ldots,x_n^q)$$

and it leaves M invariant. Let $\mathrm{Fix}(\varphi^m|M)$ be the set of fixed points of φ^m restricted to M , and $N_m = \mathrm{card}\,\mathrm{Fix}(\varphi^m|M)$. Then Weil's zeta function is defined by the formal power series

$$\zeta(z) = \exp \sum_{m=1}^{\infty} \frac{N_m}{m} z^m = \prod_{\substack{\gamma:\text{periodic orbit}\\ \text{of } \varphi}} \frac{1}{1-z^{\text{length of }\gamma}}$$

Note that for comparison with Riemann's zeta function one has to put $z = q^{-s}$.

a) ζ extends to a rational function of z . More precisely

$$\zeta(z) = \prod_{k=0}^{2\dim M} [P_k(z)]^{(-1)^{k+1}}$$

where the polynomials P_k have a cohomological intrepretation.

b) There is a functional equation.

c) The polynomial P_k has integer coefficients. Its zeros are on the circle $|z| = q^{-k/2}$.

These properties (Weil conjectures) have been proved by Dwork, Grothendieck and Deligne [3] .

I.3. <u>The Artin-Mazur zeta function for diffeomorphisms</u> [4]

Let M be a differentiable compact manifold. Artin and Mazur have shown that for a dense set of diffeomorphisms f of M (with respect to the C^k topology),

$$\limsup_{n\to\infty} \log \text{card Fix } f^n < +\infty$$

For such diffeomorphisms they defined

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \text{card Fix } f^n$$

a) If f satisfies Smale's Axiom A , then ζ is rational.

This was proved by Guckenheimer and Manning.

I.4. Zeta functions for flows

Smale [5] has suggested to consider the following zeta function for differentiable flows on compact differentiable manifolds

$$Z(s) = \prod_{\gamma:\text{periodic orbit}} \prod_{k=0}^{\infty} (1-e^{-(s+k)\ell(\gamma)})$$

where $\ell(\gamma)$ is the period of γ . In the case of the geodesic flow on a surface of constant negative curvature, this reduces to Selberg's zeta function [6] . Selberg's zeta function is meromorphic, satisfies a functional equation, and its non trivial zeros are on the line $\text{Re } s = \frac{1}{2}$.

Smale asked whether $Z(s)$ would be meromorphic in the case of Axiom A flows.

It should be noted that $Z(s)$, as defined above, does not transform simply under time scaling (multiplication of all $\ell(\gamma)$ by some constant).

In Selberg's case a fixed choice of the constant negative curvature is made, corresponding to a fixed choice of the time scale. It is thus more natural to consider in general

$$\zeta(s) = \prod_{\gamma\text{periodic}} (1-e^{-s\,\ell(\gamma)})^{-1} .$$

We have then

$$\zeta(s) = \frac{Z(s+1)}{Z(s)}$$

In Selberg's case this will again be meromorphic, satisfy a functional equation, and have poles on the line $\text{Re } s = +\frac{1}{2}$ and corresponding zeros on the line $\text{Re } s = -\frac{1}{2}$.

I.5. The Lee-Yang circle theorem [7]

Let Λ be a finite set. If $x,y \in \Lambda$, let $a_{xy} = a_{yx}$ be real in $[-1,+1]$. Write

$$P(z) = \sum_{X \subset \Lambda} z^{\text{card } X} \prod_{x \in X} \prod_{y \notin X} a_{xy}$$

The Lee-Yang circle theorem, very useful in statistical mechanics, states that the zeros of $P(z)$ are all on the circle $|z| = 1$.

Mackey (unpublished) has suggested that there may be a relation between this result and (c) of I.2. above.

In statistical mechanics of continuous spin systems it is important (in view of applications to constructive field theory) to know if the Fourier Laplace transforms of certain functions have only real zeros. The Riemann hypothesis can also be expressed in this form (see [8] and references quoted there).

The relation between the Lee-Yang circle theorem and zeta functions is at this point only wishful thinking. In what follows we shall indicate more substantial relation between statistical mechanics and zeta functions.

II. ZETA FUNCTIONS OF DIFFEOMORPHISMS AND FLOWS

As we have indicated, the rationality of the zeta function for an Axiom A diffeomorphism has been proved by Guckenheimer [9], and Manning [10]. Their methods are quite different, one using the Lefshetz trace formula, and the other Markov partitions.

II.1. The Lefshetz trace formula

Let x be a fixed point of a diffeomorphism $f : M \to M$. We say that x is hyperbolic if $D_x f : T_x M \to T_x M$ has no eigenvalue of modulus 1. The Lefshetz number of x is then $L(x) = \pm 1$, being the sign of $\det(1 - D_x f)$.

If the diffeomorphism f has only hyperbolic fixed points

$$\sum_{x \in \text{Fix } f} L(x) = \sum_{k=0}^{\dim M} (-1)^k \text{ trace } (f_{*k} : H_k(M,\mathbb{R}) \to H_k(M,\mathbb{R}))$$

where f_{*k} is the action induced by f on the k-th homology group of M with real coefficients. This is the Lefschetz trace formula.

If f satisfies Smale's Axiom A and $x \in \mathrm{Fix}\, f^n$, then x is hyperbolic for $\mathrm{fix}\, f^n$. Therefore

$$\sum_{x \in \mathrm{Fix}\, f^n} L(x,f^n) = \sum_{k=0}^{\dim M} (-1)^k \operatorname{tr} (f_{*k})^n$$

and

$$\exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}\, f^n} L(x,f^n)$$

$$= \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{k=0}^{\dim M} (-1)^k \operatorname{tr} (f_{*k})^n$$

$$= \exp \sum_{k=0}^{\dim M} (-1)^k \left[-\operatorname{tr} \log (1-zf_{*k})\right]$$

$$= \prod_{k=0}^{\dim M} \left[\det(1-zf_{*k})\right]^{(-1)^{k+1}}$$

This shows that if $L(x,f^n) = 1$ for all n, $x \in \mathrm{Fix}\, f^n$, the zeta function is rational. For the general case of Axiom A diffeomorphism we refer to [9]. Notice that in the holomorphic case (as opposed to the differentiable case) $L(x)$ is always $+1$. In particular the left-hand side of the Lefshetz trace formula is just what is needed for the Weil zeta function. The problem there is to define cohomology groups and prove a Lefshetz formula.

Notice also that considerable extensions of the Lefshetz formula have been made, in particular by Atiyah-Bott [11].

II.2. Markov partitions

Bowen [12] following Sinai [13] has proved the existence of Markov partitions for basic sets of diffeomorphisms satisfying the Axiom A of Smale. I won't go here into all the definitions. Let me just say that for an Axiom A diffeomorphism f, the closure of the set of periodic points is a finite union of "basic sets" invariant under f. The existence of Markov partitions implies that for each basic set Λ, there is a *symbolic dynamics*. This means that there is a finite set F , a matrix $(t_{ij})_{(i,j)\in F\times F}$ with entries 0 or 1 and a surjective map $\pi : \Omega \mapsto \Lambda$ where

$$\Omega = \{(\zeta_h)_{h\in\mathbf{Z}} : t_{\zeta_n \zeta_{n+1}} = 1 \text{ for all } n\}$$

such that $\pi\circ\tau = f\circ\pi$ where τ is the shift (to the left) of the sequence (ζ_n) of symbols.

Let us consider the case where π is bijective. This happens for certain basic sets. We have then

$$\text{card Fix } f^n = \text{card Fix } \tau^n$$

The zeta function for f is thus the same as the zeta function for the shift τ . This has been computed by Bowen and Lanford [14] . We have first

$$\text{card Fix } \tau^n = \text{tr } (t^n)$$

where $t = (t_{ij})$. Therefore

$$\exp \sum_{n=1}^{\infty} \frac{z^n}{n} \text{ card Fix } f^n = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr}(t^n)$$

$$= \exp \text{ tr } \sum_{n=1}^{\infty} \frac{z^n t^n}{n}$$

$$= \exp \text{ tr } (-\log(1-zt))$$

$$= \frac{1}{\exp \text{ tr } \log(1-zt)} = \frac{1}{\det (1-zt)}$$

which is indeed rational in z .

To take into account the fact that in general π is not bijective, Manning [10] introduces several shifts τ_α and shows that

$$\text{card Fix } f^n = \sum_\alpha (-1)^{\ell_\alpha} \text{ card Fix } \tau_\alpha^n$$

with ℓ_α integer even or odd. From this results immediately that ζ is rational. One shows that the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \text{ card Fix } f^n$$

is $\exp[-P(0)]$, where $P(0)$ is the topological entropy of f restricted to Λ. In fact $e^{-P(0)}$ is a simple pole of ζ .

II.3. Generalized zeta-functions

Symbolic dynamics described above is reminiscent of the statistical mechanics of one-dimensional lattice spin systems. The symbols are the possible values of the spin at one point, and $\operatorname{tr}(t^n)$ is "the partition function for a system of length n with periodic boundary conditions, and no interaction". The analogy between symbolic dynamics and statistical mechanics has first been exploited by Sinai [15]. Here it suggests to replace $\operatorname{tr}(t^n)$ by a "partition function with interaction".

If $f : \Lambda \to \Lambda$ is any map and φ a complex-valued function we are led to writing formally

$$\zeta(\varphi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \operatorname{Fix} f^n} \prod_{k=0}^{n-1} \varphi(f^k \zeta) \quad .$$

Similarly, if (f^t) is a flow on X , and A a complex-valued function, we write

$$\zeta(A) = \prod_{\gamma} \left[1 - \exp \int_0^{\ell(\gamma)} A(f^t x_\gamma)\, dt\right]^{-1}$$

where the product extends over the periodic orbits γ of the flow, $\ell(\gamma)$ is the prime period of γ , and x_γ a point of γ .

Interesting results are obtained for Axiom A diffeomorphisms and flows. We state the main facts [16] .

Let f be the restriction to a basic set of an Axiom A diffeomorphism and assume that it is topologically mixing. Let A be real Hölder continuous. Then $z \to \zeta(z\, e^A)$, at first defined for small $|z|$, extends to a meromorphic function in a disk $|z| < R(A)$. This function has no zero in the disk, and only one pole, simple and located at $\exp[-P(A)]$. Here $P(A)$ is the "topological pressure" of A (see [17]) and $e^{-P(A)} < R(A)$.

Let (f^t) be the restriction to a basic set of an Axiom A flow. Let A be real Hölder continuous. Then $\zeta(A-s)$ converges for sufficiently large Re s. It extends to a meromorphic function in a region

$$\{s : \operatorname{Re} s > P(A)\} \cup \{s : |s - P(A)| < \Gamma(A)\}$$

where it has no zero and only one pole, located at $P(A)$. Here $P(A)$ is the topological pressure of A for the flow (f^t) and $\Gamma(A) > 0$.

The result about diffeomorphisms is proved along the lines of II.2. using a Markov partition, Manning's ideas, and replacing the matrix t by the transfer matrix $\mathcal{L}$ of statistical mechanics. For flows one uses the same techniques and Bowen's symbolic dynamics for flows [18] .

There are now examples showing that $z \to \zeta(ze^A)$ (diffeomorphisms) and $s \to \zeta(A-s)$ (flows) cannot always be extended meromorphically to the entire complex plane. These examples, due to Gallavotti (unpublished) in particular give a negative answer to Smale's question of section I.4.

II.4. The real analytic case

In spite of the above counterexamples one can ensure the meromorphy of the functions $z \to \zeta(ze^A)$ and $s \to \zeta(A-s)$ by imposing suitable real analyticity conditions to f and A . The idea is to obtain that the transfer matrix $\mathcal{L}$ be a trace class operator on a space of analytic functions, so that one can use Grothendieck's Fredholm theory (see [19]), and express ζ in terms of Fredholm determinants. The following two results are proved in [20] .

1) Let M be a connected compact real-analytic manifold, and $f : M \to M$ a real analytic map which is expanding (i.e. such that $||Tfu|| > \theta ||u||$ with $\theta > 1$ for some Riemann metric on M). Furthermore let φ be a complex-valued real analytic function on M .Then

$$z \mapsto \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}\, f^n} \prod_{k=0}^{n-1} \varphi(f^k x)$$

extends to a meromorphic function in the entire complex plane.

2) Let $\ell(\gamma)$ be the length of the closed geodesics on a compact manifold of constant negative curvature. (The $\ell(\gamma)$ are also the periods of periodic orbits for the geodesic flow on this manifold, this flow is know to

satisfy Axiom A). Then the function

$$s \to \prod_{\gamma} (1-e^{-s\,\ell(\gamma)})^{-1}$$

extends to a meromorphic function in the entire complex plane.

This generalizes a result of Selberg mentioned in section I.4.

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